Iranian Journal of Fuzzy Systems
Volume 17, Number 5, (2020), pp. 127-136
University of Sistan and Baluchestan

Fuzzy decision in testing hypotheses by fuzzy data: Two case studies

A. Parchami[1]

[1] *Department of Statistics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran*

parchami@uk.ac.ir

Abstract

In testing hypotheses, we may confront with cases where data are recorded as non-precise (fuzzy) rather than crisp. In such situations, the classical methods of testing hypotheses are not capable and need to be generalized. In solving the problem of testing hypotheses based on fuzzy data, the fuzziness of the observed data leads to the fuzzy p-value. This paper has been focused to calculate fuzzy p-value based on fuzzy data using the extension principle. Also, considering that p-value method is the most widely used / popular approach for testing hypotheses among different sciences users, two fuzzy p-value-based case studies have been provided in this paper. The first case study is discussed on "the fuzzy data from a speedometer camera" and the second is deliberate about "the lifetime of produced batteries in a factory" and both of them have been solved by a novel approach considering other studies found in the literature.

Keywords: Testing hypotheses, fuzzy decision, fuzzy p-value, extension principle, fuzzy significance level, fuzzy data.

1 Introduction and preliminaries

During the past four decades, several scientific researches have been published in various fields of testing hypotheses using fuzzy set theory, see e.g. [23] and Blanco-Fernndez et al. [1]. To solve the problem of hypotheses testing, the following approaches developed till now in fuzzy environment:
(i) Neyman-Pearson Lemma (e.g. see [24]),
(ii) Bayesian (e.g. see [2]),
(iii) Likelihood ratio (e.g. see [25]),
(iv) Minimax (e.g. see [22]),
(v) p-value (e.g. see [7]), and
(vi) Confidence interval (e.g. see [4]).

In the sequence, we briefly review some p-value-based researches on testing hypotheses in fuzzy environment. For the first time, Filzmoser and Viertl [7] investigated the problem of testing hypotheses and were developed by others such as [18, 19]. Another efficient p-value-based method for testing fuzzy hypotheses is presented in [20] on the basis of the probability measure of fuzzy event. Some statistical properties of the fuzzy p-value are discussed in [12] to evaluate the compatibility of the observed data with the assumed hypothetical model. Useful functions from R software packages *FPV* and *Fuzzy.p.value* are proposed in [15, 16, 17] for testing hypotheses in a fuzzy environment. Different fuzzy set approaches to statistical parametric / nonparametric tests are discussed in [14, 13, 21]. Several approaches for the goodness-of-fit test by fuzzy data by fuzzy p-value are proposed in [9, 10]. The main goal of this paper is to prove the presented formulas of fuzzy p-value in [7] by extension principle approach and investigate testing hypotheses based on fuzzy data by two practical results.

This paper is organized as follows. Some preliminary concepts and literatures are presented in Section 1. In Section 2, a fuzzy p-value is provided using Zadehs extension principle for testing hypotheses based on fuzzy data. Comparing fuzzy p-value and fuzzy significance level is discussed in Section 3 to achieve a fuzzy decision for test. Two practical results on speedometer camera fuzzy data and the lifetime of battery are presented in Section 4 and Section 5, respectively. Our study is concluded in the final section.

Corresponding Author: A. Parchami
Received: September 2019; Revised: December 2019; Accepted: April 2020.

In the sequence of this section, we are going to introduce preliminary concepts and definitions which are needed in the next sections. Any $A \in F(R)$ is called a fuzzy set on R and any $T \in F_T(R)$ is called a fuzzy triangular number. Meanwhile, the fuzzy set T of the real line is said to be triangular fuzzy number, and denoted by $T(a,b,c)$, if its membership function is

$$T(x) = \begin{cases} \frac{x-a}{b-a} & if \quad a \leq x < b, \\ \frac{c-x}{c-b} & if \quad b \leq x < c, \\ 0 & elsewhere, \end{cases}$$

where $a, b, c \in R$ and $a \leq b \leq c$. Let us denote the δ-cuts and the support of fuzzy set A by $A_\delta = \{x \in R : A(x) \geq \delta\}$ and $Supp(A) = \{x \in R : A(x) > 0\}$, respectively.

2 Testing hypotheses based on fuzzy data: The p-value approach

In a nonrandomized test, the space of possible values of the test statistic T is decomposed into the rejection region and its complement (the acceptance region). Depending on the precise hypotheses H_0 and H_1, the rejection region usually takes one of the following forms

$$a) \;\; T \leq t_l \qquad b) \;\; T \geq t_r \qquad c) \;\; T \notin (t_1, t_2), \tag{1}$$

where t_l or t_r , or t_1 and t_2 are quantiles of the distribution of T [3].

Let us consider the case of having fuzzy data $\tilde{\mathbf{x}}_1, ..., \tilde{\mathbf{x}}_n \in F(R)$ which are to be used for a testing hypotheses. Using extension principle, one can obtain the membership function of a continuous test statistic $\mathbf{t} = h(\tilde{\mathbf{x}}_1, ..., \tilde{\mathbf{x}}_n)$. In this case p-value, as a function of $\mathbf{t}$, is a fuzzy set on $[0, 1]$, which we denote it by $\mathbf{P}$.

The main problem: Let we take a random sample of size n from $f_\theta(x), \theta \in \Theta$, but instead of crisp data, we observe the fuzzy data $\tilde{\mathbf{x}}_1, ..., \tilde{\mathbf{x}}_n \in F(R)$. The main problem studied in this work is testing hypotheses

$$H_0 : \; \theta \in \Theta_0, \qquad vs. \qquad H_1 : \; \theta \in \Theta_1,$$

by p-value approach based on the fuzzy observations $\tilde{\mathbf{x}}_1, ..., \tilde{\mathbf{x}}_n \in F(R)$.

Theorem 2.1. *Suppose that $f_\theta(x), \theta \in \Theta$ be a continuous random variable. In testing hypotheses with fuzzy data, for any critical region of the forms (a)-(c) indicated in Eq. (1), the membership function of fuzzy p-value are as follows:*

$$a) \quad \boldsymbol{P}(p) = \boldsymbol{t}(q_T(p)), \tag{2}$$

$$b) \quad \boldsymbol{P}(p) = \boldsymbol{t}(q_T(1-p)), \tag{3}$$

$$c) \quad \boldsymbol{P}(p) = \begin{cases} \boldsymbol{t}(q_T(\frac{1-p}{2})) & if \quad t_l \geq m, \\ \boldsymbol{t}(q_T(\frac{p}{2})) & if \quad t_r \leq m, \end{cases} \tag{4}$$

in which $p \in [0,1]$, $\boldsymbol{t}$ is the membership function of fuzzy test statistic and function $q_T(p)$ is the p-th quantile of T distribution under the boundary of H_0. In case (c), m is the median of the distribution of T, and also t_l and t_r are the first and the end points of $Supp(\boldsymbol{t})$.

Proof. The crisp p-value in case (a) is equal to $p = Pr_{H_0}(T \leq t)$ in which t is the crisp observed value of test statistic T. Considering the continuity property of random variable X, the test statistic T will be continuous [19]. Therefore, the p-value is strictly increasing function of t and is one to one function. Hence, considering Zadehs extension principle we have

$$\begin{aligned} \mathbf{P}(p) &= \sup_{t:\; p=Pr_{H_0}(T \leq t)} \mathbf{t}(t) = \mathbf{t}(t)|_{p=Pr_{H_0}(T \leq t)} \\ &= \mathbf{t}(t)|_{p=F_T(t)} = \mathbf{t}(t)|_{t=F_T^{-1}(p)} \\ &= \mathbf{t}(q_T(p)), \qquad p \in [0,1], \end{aligned}$$

where $F_T(.)$ is the c.d.f (cumulative distribution function) of T under the null hypothesis and $F_T(.)^{-1}$ is its inverse which is equivalent to the quantile of T denoted by $q_T(.)$ under the null hypothesis. Note that, the one to one property for $p = F_T(t)$ is causes the existence of the inverse of c.d.f in case (a).

The crisp p-value in case (b) is equal to $p = Pr_{H_0}(T \geq t) = 1 - F_T(t)$ under the null hypothesis. Note that, the one to one property for p causes to conclude the one to one property for $1 - p = F_T(t)$, and so there exists the inverse of c.d.f in case (b). Therefore, by Zadehs extension principle

$$\mathbf{P}(p) = \sup_{t:\; p=Pr_{H_0}(T \geq t)} \mathbf{t}(t) = \mathbf{t}(t)|_{p=Pr_{H_0}(T \geq t)}$$

$$\begin{aligned} &= \mathbf{t}(t)|_{1-p=F_T(t)} = \mathbf{t}(t)|_{t=F_T^{-1}(1-p)} \\ &= \mathbf{t}(q_T(1-p)), \quad p \in [0,1], \end{aligned}$$

The crisp p-value in case (c) is equal to

$$p = 2\ min\{Pr_{H_0}(T \geq t), Pr_{H_0}(T \leq t)\} = \begin{cases} 2\ Pr_{H_0}(T \geq t) & \text{if} \quad t \geq m, \\ 2\ Pr_{H_0}(T \leq t) & \text{if} \quad t < m, \end{cases}$$

and so the proof of case (c) is similar to the proof of cases (a) and (b). □

Theorem 2.2. *Under assumptions of Theorem 2.1, in testing hypotheses with fuzzy data the δ-cuts of fuzzy p-value for any critical region of the forms (a)-(c) indicated in Eq. (1) are as follows:*

$$a) \quad \boldsymbol{P}_\delta = [Pr(T \leq t_1(\delta)), Pr(T \leq t_2(\delta))], \tag{5}$$

$$b) \quad \boldsymbol{P}_\delta = [Pr(T \geq t_2(\delta)), Pr(T \geq t_1(\delta))], \tag{6}$$

$$c) \quad \boldsymbol{P}_\delta = \begin{cases} [2Pr(T \geq t_2(\delta)), 2Pr(T \geq t_1(\delta))] & \textit{if} \quad t_l \geq m, \\ [2Pr(T \leq t_1(\delta)), 2Pr(T \leq t_2(\delta))] & \textit{if} \quad t_r \leq m, \end{cases} \tag{7}$$

in which $\delta \in [0,1]$ and $\boldsymbol{t}_\delta = [t_1(\delta), t_2(\delta)]$.

Proof. [1] Regarding to Theorem 1 and considering strictly increasing property of F_T, one can obtain the δ-cuts of $\mathbf{P}$ in case (a) as follows

$$\begin{aligned} \mathbf{P}_\delta &= \{p \in [0,1]:\ \mathbf{P}(p) \geq \delta\} \\ &= \{p \in [0,1]:\ \mathbf{t}(q_T(p)) \geq \delta\} \\ &= \left\{p \in [0,1]:\ \mathbf{t}\left(F_T^{-1}(p)\right) \geq \delta\right\} \\ &= \left\{p \in [0,1]:\ F_T^{-1}(p) \in [t_1(\delta), t_2(\delta)]\right\} \\ &= \{p \in [0,1]:\ p \in [F_T(t_1(\delta)), F_T(t_2(\delta))]\} \\ &= [Pr(T \leq t_1(\delta)), Pr(T \leq t_2(\delta))], \quad \delta \in (0,1]. \end{aligned}$$

For case (b), considering strictly decreasing property of F_T, similarly we have

$$\begin{aligned} \mathbf{P}_\delta &= \{p \in [0,1]:\ \mathbf{P}(p) \geq \delta\} \\ &= \{p \in [0,1]:\ \mathbf{t}(q_T(1-p)) \geq \delta\} \\ &= \left\{p \in [0,1]:\ \mathbf{t}\left(F_T^{-1}(1-p)\right) \geq \delta\right\} \\ &= \left\{p \in [0,1]:\ F_T^{-1}(1-p) \in [t_1(\delta), t_2(\delta)]\right\} \\ &= \{p \in [0,1]:\ 1-p \in [1-F_T(t_2(\delta)), 1-F_T(t_1(\delta))]\} \\ &= [Pr(T \geq t_2(\delta)), Pr(T \geq t_1(\delta))], \quad \delta \in (0,1]. \end{aligned}$$

The proof of case (c) is obvious by considering the proof of cases (a) and (b). □

Note 1. It must be highlighted that the source of the presented formulas in Theorem 2 are previously presented by Filzmoser and Viertl in 2004. They defined the fuzzy p-value by similar formulas and they have been proved some properties for their defined fuzzy p-value [26]. But, here from another point of view, we prove / obtain their formulas on the basis of the extension principle by minor revisions in case (c).

3 Fuzzy decision based on fuzzy p-value and fuzzy significance level

If the p-value is fuzzy, it is more appropriate to consider a fuzzy significance level for the problem. Therefore, we need a criterion for comparing two fuzzy subsets, fuzzy p-value P and fuzzy level S. There are many ways to carry out this comparison, see, for example [5, 6, 27]. We use Yuans approach in Definition 3.1, since it is reasonable and has appropriate properties [28].

Definition 3.1. *Let $A, B \in F(R)$, be normal and convex. Let*

$$\Delta_{AB} = \int_{a_{A\delta}^{+} > a_{B\delta}^{-}} (a_{A\delta}^{+} - a_{B\delta}^{-})\, d\delta + \int_{a_{A\delta}^{-} > a_{B\delta}^{+}} (a_{A\delta}^{-} - a_{B\delta}^{+})\, d\delta, \tag{8}$$

[1] This proof is presented by late Prof. N. R. Arghami.

where $a^{+}_{A\delta} = \sup\{x:\ x \in A_\delta\}$, $a^{-}_{A\delta} = \inf\{x:\ x \in A_\delta\}$, *and* A_δ *is the* δ*-cut of* A, $\delta \in (0,1]$. *Then the degree of "A is grater than B" is defined to be*

$$D(A \succ B) = \frac{\Delta_{AB}}{\Delta_{AB} + \Delta_{BA}}. \tag{9}$$

Definition 3.2. *In testing hypotheses problem* $D(\boldsymbol{P} \succ \boldsymbol{S})$ *is called the degree of acceptance of* H_0 *and* $D(\boldsymbol{S} \succ \boldsymbol{P}) = 1 - D(\boldsymbol{P} \succ \boldsymbol{S})$ *the degree of rejection of* H_0.

4 Practical results on speedometer camera

The task of a speedometer camera mounted in a highway of Kerman is to control the speed of cars in transit. In addition recording the specifications of each vehicle, the speedometer camera can also estimate the speed of the vehicle in three different computational methods (in km/h). According to the program given to the camera, there are only random data recorded from the vehicles travelling daily on this highway. To do so, the camera records the characteristics of a car every half hour and estimates its speed in three different ways. In this study, two approaches / strategies for recording data are considered:

Strategy 1 (recording data in the form of precise numbers): In this method, the average of the three registered numbers is considered as the estimated speed of the vehicle.

Strategy 2 (recording data in the form of non-precise / fuzzy numbers): In this method, the set of three registered numbers is considered as a triangular fuzzy number for the estimated speed of the vehicle. The middle of the registered number is considered as the core of the triangular fuzzy number. Also, the smallest and the largest registered numbers are respectively considered as the first and the end points of the triangular fuzzy number support. In Strategy 2, the information / data are recorded non-precisely and therefore, the inferences and conclusions based on them can not be made by classical statistical methods.

Therefore, the extracted data from the camera was recorded in the form of precise / real numbers (Strategy 1) as well as non-precise / fuzzy numbers (Strategy 2) during one week which are summarized in Table 1. The membership functions of 28 fuzzy numbers for vehicles speed are depicted in Figure 1 which are gathered only during the day of 19 June 2017. Moreover, the membership functions of some statistics (such as fuzzy mean, fuzzy variance and fuzzy standard deviation) are drawn in Figure 1 and Figure 2.

Table 1: The means of samples from vehicles speed between 18 and 24 June 2017 by two different strategies.

Date	Sample size	Mean of speed (Strategy 1)	Mean of speed (Strategy 2)
2017-06-18 day	28	94.59	T(89.4, 94.6, 98.4)
2017-06-18 night	20	82.77	T(76.9, 82.8, 86.5)
2017-06-19 day	28	93.64	T(89.1, 93.6, 98.1)
2017-06-19 night	20	86.34	T(81.3, 86.3, 90.0)
2017-06-20 day	28	96.34	T(91.9, 96.3, 100.4)
2017-06-20 night	20	81.42	T(75.1, 81.4, 85.1)
2017-06-21 day	28	93.71	T(88.5, 93.7, 98.0)
2017-06-21 night	20	86.48	T(79.6, 86.5, 90.7)
2017-06-22 day	28	95.87	T(90.2, 95.9, 100.8)
2017-06-22 night	20	96.91	T(91.2, 96.9, 100.7)
2017-06-23 day	28	85.51	T(79.6, 85.5, 89.7)
2017-06-23 night	20	76.49	T(72.2, 76.5, 81.3)
2017-06-24 day	28	83.90	T(76.4, 83.9, 88.4)
2017-06-24 night	20	84.22	T(78.7, 84.2, 87.8)

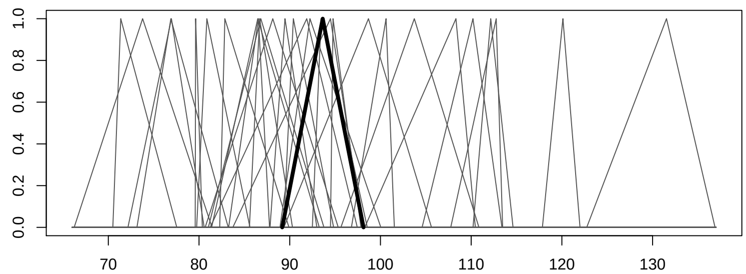

Figure 1: The membership functions of 28 fuzzy-valued observed data for speed which are gathered during the day of 19 June 2017 and the membership function of their fuzzy mean (bold line).

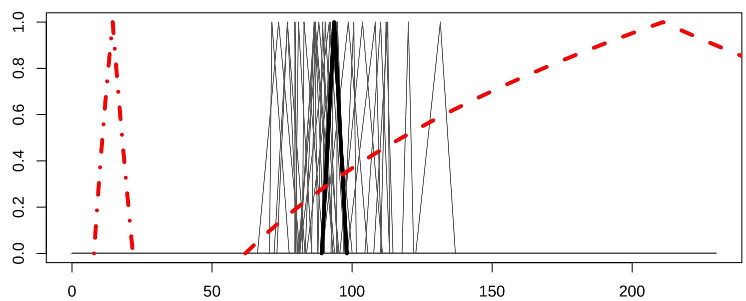

Figure 2: The membership functions of the fuzzy variance (dash line) and the fuzzy standard deviation (dot-dash line) for 28 fuzzy data which are gathered during the day of 19 June 2017.

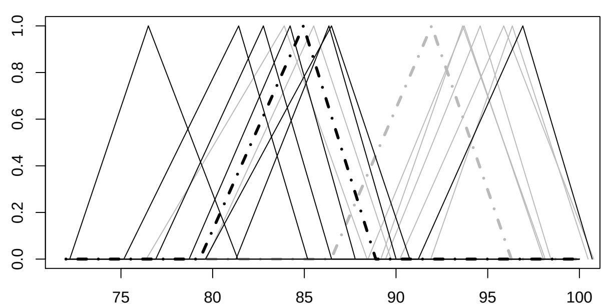

Figure 3: The membership functions of fuzzy means for fuzzy speeds in each day (gray lines) and each night (black lines). The membership functions of total fuzzy means for 7 days (gray dot-dash line) and 7 nights (black dot-dash line).

The membership functions of fuzzy means for fuzzy speeds are separately shown in Figure 3 for each day and night. Meanwhile, one can compare the membership functions of two total fuzzy means for 7 days and 7 nights in Figure 3.

According to the Iranian freeways rules, on highways the maximum allowed speed for a variety of vehicles is 95 km/h during the day and 85 km/h at night. Due to the lower maximum allowed speed of the night rather than the day, there has recently been a claim by one of the high-ranking officers that there is an increase in the number of overtakes during the night rather than to the day, and on that basis, more vehicles control are proposed throughout the night at the site. Regarding to the fact that increasing the variance of vehicles speed is equivalent to increasing the number of overtakes, a test for comparing day and night vehicle variances was proposed to check the validity of the presented claim. So, the main goal in this study is to answer this question: Does the vehicles speed variance during the day (from

5 am to 7 pm) is significantly less than the vehicles speed variance during the night (from 7 pm to 5 am)? In other words, the goal is testing the following hypotheses at significant level 0.05,

$$H_0: \ \sigma_D^2 = \sigma_N^2$$
$$H_1: \ \sigma_D^2 < \sigma_N^2$$

in which σ_D^2 and σ_N^2 are vehicles speed variances during the day and night, respectively. The normality assumption for the vehicles speed variance during the day (and similarly, at night) is acceptable due to the large sample size and the nature of the random variable. It should be mentioned that because of the importance of correctly recording speed, in this project the speeds of vehicles are estimated by the camera by three different methods. Therefore, we will continue testing the above hypotheses based on two different data recording strategies:

In Strategy 1, the p-value easily can computed based on the observed precise data as follows

$$p-value = P_{H_0}(F \leq f) = 1.382 \times 10^{-6}$$

in which the test statistic $F = \frac{\sigma_N^2}{\sigma_D^2}$ has Fisher distribution with degrees of freedom 195 and 139 and σ_D^2 and σ_N^2 are variance samples of the random speeds during day and night. Also, the crisp observed value of F in the first strategy is $f = 0.482$. Therefore, by comparing p-value and significance level, the null hypothesis obviously rejected at significance level 0.05.

The fuzzy p-value is a function of the observed fuzzy-value test statistics, **f**, which is a fuzzy number in Strategy 2. Therefore, first of all we obtain the membership functions of fuzzy variances and then the membership function of **f**, which are drawn using Package *FuzzyNumbers* [8] in Figure 4 and Figure 5 based on the cuts of observations.

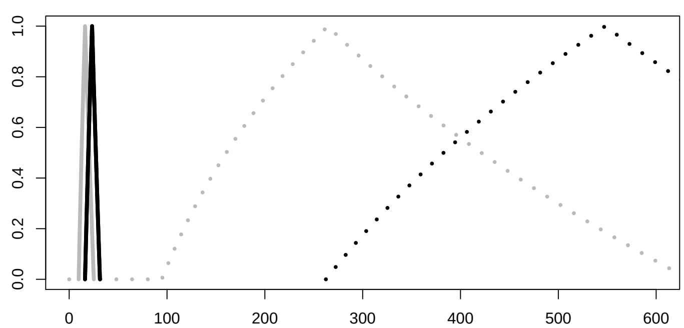

Figure 4: The membership functions of fuzzy variances (dotted lines) and fuzzy standard deviations (continuous lines) for all 7 days (gray) and all 7 nights (black).

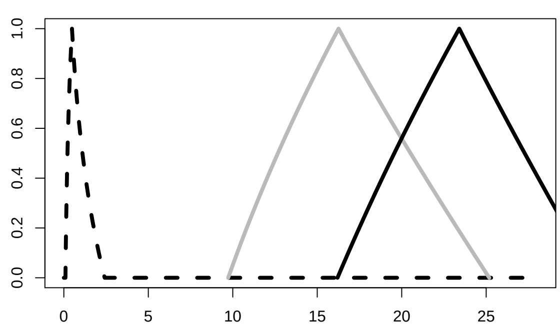

Figure 5: The membership function of the fuzzy test statistic f (dash line) and the membership function of total fuzzy standard deviations for days (gray line) and for nights (black line).

Regarding to the alternative hypothesis, the rejection region is of the form (1.a) and considering Eq. (5) the δ-cuts of the fuzzy p-value $\mathbf{P}$ are intervals of the form

$$\mathbf{P}_\delta = \left[P_{\sigma_D^2=\sigma_N^2}\left(F \leq f_1(\delta)\right)\ ,\ P_{\sigma_D^2=\sigma_N^2}\left(F \leq f_2(\delta)\right)\right] = \left[\Phi\left(f_1(\delta)\right)\ ,\ \Phi\left(f_2(\delta)\right)\right],$$

in which $\mathbf{f} = [f_1(\delta), f_2(\delta)]$ for all $\delta \in (0, 1]$. Moreover, $\Phi(.)$ is the c.d.f. of the Fisher with degrees of freedom 195 and 139. Hence, one can construct the fuzzy p-value by its δ-cuts in Figure 6. After computing $\Delta_{\mathbf{PS}} = 0.522$ and $\Delta_{\mathbf{SP}} = 0.062$, the null hypothesis is accepted with degree $D(\mathbf{P} \succ \mathbf{S}) = 0.894$ at the significance level 0.05.

As discussed in this study, two different strategies are considered in recording data which lead the scientist to the different results in testing hypotheses. But according to the author belief, Strategy 2 has two advantages rather than Strategy 1:

Firstly, the dispersion between the three registered numbers does not disappeared when the triangular fuzzy data is recorded in Strategy 2, but the dispersion between the three registered numbers has been omitted in Strategy 1. For example, two different speed vectors (50, 55, 60) and (42, 103, 20) have the same effect / worth in Strategy 1, but are different in Strategy 2.

Secondly, in fuzzy case, the estimation fuzziness of the device is maintained / saved until the end of the calculation and finally gives "the degree of acceptance" for the accepted hypothesis to the practitioner. In other words, fuzzy decision making is done in Strategy 2 based on fuzzy / non-precise data which is more justifier than crisp decision of classical test in Strategy 1.

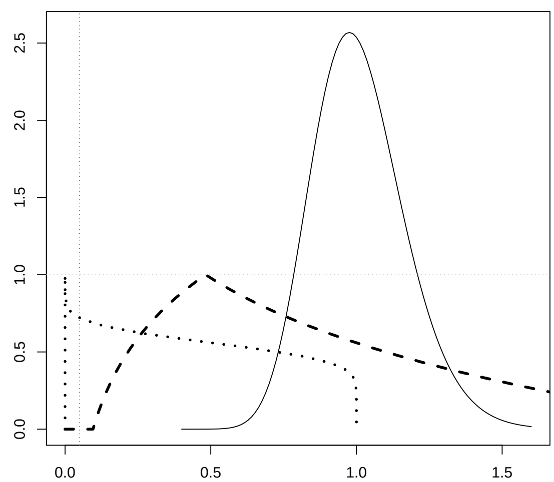

Figure 6: The membership function of fuzzy test statistic $\mathbf{f}$ (dash line), the membership function of the fuzzy p-value (dotted line) with regard to the p.d.f of test statistic (continuous line) and test significance level (red dotted line).

5 Practical results on the lifetime of battery

The lifetime X of battery (in term of hour) produced by a factory is distributed normally with mean $\mu = 1300$ and variance $\sigma^2 = 900$ hours. After doing some preferences and corrections in assembly line, a random sample of size $n = 10$ lamps is taken. Consider this fact that the lifetime of a battery does not end at a moment and its useful life / capability will down slowly over the time. Therefore the lifetime of a battery is a non-precise number, and our observations are recorded using triangular fuzzy numbers as follow:

T(1257,1261,1278), T(1251,1287,1302), T(1315,1346,1372), T(1306,1330,1348), T(1298,1329,1349), T(1288,1301,1320), T(1298,1317,1333), T(1241,1269,1284), T(1325,1353,1369), T(1301,1337,1355).

After applying changes and corrections into the factory assembly line, a new question is presented: Whether the lifetime mean of the produced batteries is increases after doing preferences and corrections? In other words, we wish to test the null hypothesis $H_0:\ \mu = 1300$ versus the alternative hypothesis $H_1:\ \mu > 1300$. The test statistic $T = \bar{X} = \frac{\sum_{i=1}^{10} X_i}{10}$ has normal distribution $N\left(1300,\ \frac{30^2}{10}\right)$ and one can easily calculate its observed value as triangular fuzzy number T(1288,1313,1331) based on extension principle. In this case, the rejection region is of the form (1.b) and considering Eq. (6) the δ-cuts of the fuzzy p-value $\mathbf{P}$ are intervals of the form

$$\mathbf{P}_\delta \quad = \quad \left[P_{\mu=1300}\left(\bar{X} \geq t_2(\delta)\right)\ ,\ P_{\mu=1300}\left(\bar{X} \geq t_1(\delta)\right)\right]$$

$$= \left[\int_{\frac{t_2(\delta)-1300}{\frac{30}{\sqrt{10}}}}^{\infty} (2\pi)^{-\frac{1}{2}} \exp(-\frac{z^2}{2})\ dz\ ,\ \int_{\frac{t_1(\delta)-1300}{\frac{30}{\sqrt{10}}}}^{\infty} (2\pi)^{-\frac{1}{2}} \exp(-\frac{z^2}{2})\ dz \right],\ \delta \in (0,1],$$

in which $\mathbf{t} = T(1288, 1313, 1331)$ and thus the first and end points of δ-cut $\mathbf{t}$ is equal to $\mathbf{t}_1(\delta) = 1288 + 25\delta$ and $\mathbf{t}_2(\delta) = 1331 - 18\delta$ for all $\delta \in (0, 1]$, respectively. Therefore, we can construct the fuzzy p-value by its δ-cuts in Figure 7 using Package *FuzzyNumbers*, see [8]. If we want to test H_0 versus H_1 at significance level approximately 0.05, which is denoted here with $\mathbf{S} = T(0, 0.05, 0.1)$, then $\Delta_{\mathbf{PS}} = 0.46$ and $\Delta_{\mathbf{SP}} = 0.06$. Therefore, the null hypothesis is accepted with degree $D(\mathbf{P} \succ \mathbf{S}) = 0.86$ at level approximately 0.05.

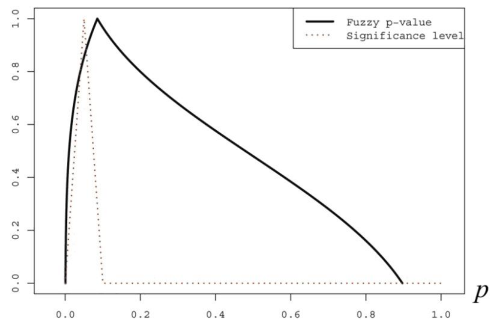

Figure 7: The membership functions of fuzzy p-value and fuzzy significance level.

After the null hypothesis acceptance, another question has been presented by quality control engineers: Whether the lifetime mean of the produces batteries has any significant changes after doing preferences and corrections, or not? In other words, we must test the hypothesis $H_0 : \mu = 1300$, against the hypothesis $H_1 : \mu \neq 1300$. In this case, the rejection region is of the form (1.c). Note that $m \in Supp(\mathbf{t})$ and the fuzzy p-value can not presented by Theorem 1. Hence we are face with a non-decision problem in this example (see Section 3 of [7] for more details). But, if we suppose that the test statistic is observed as $\mathbf{t} = T(1300, 1313, 1321)$, then $t_l \geq m$ and so considering Eq. (7), the δ-cuts of the fuzzy p-value $\mathbf{P}$ are intervals of the form

$$\mathbf{P}_\delta = \left[2\int_{\frac{t_2(\delta)-1300}{\frac{30}{\sqrt{10}}}}^{\infty} (2\pi)^{-\frac{1}{2}} \exp(-\frac{z^2}{2})\ dz\ ,\ 2\int_{\frac{t_1(\delta)-1300}{\frac{30}{\sqrt{10}}}}^{\infty} (2\pi)^{-\frac{1}{2}} \exp(-\frac{z^2}{2})\ dz \right],\ \delta \in (0,1],$$

where the first and end points of δ-cut $\mathbf{t}$ is equal to $\mathbf{t}_1(\delta) = 1300+13\delta$ and $\mathbf{t}_2(\delta) = 1321-8\delta$, respectively. Now, one can construct the fuzzy p-value by its δ-cuts as depicted in Figure 8. Testing the above hypotheses at fuzzy significance level $\mathbf{S} = T(0, 0.15, 0.3)$, lead the user to calculate $\Delta_{\mathbf{PS}} = 0.45$, $\Delta_{\mathbf{SP}} = 0.14$ and therefore the null hypothesis is acceptable with degree $D(\mathbf{P} \succ \mathbf{S}) = 0.76$.

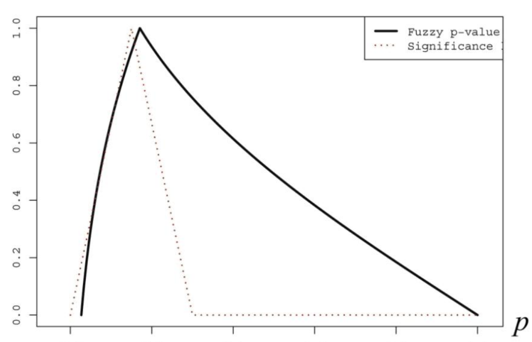

Figure 8: The membership functions of fuzzy p-value and fuzzy significance level.

6 Conclusions and future research directions

At the first of conclusions part, a brief discussion in need to show why we need record information by non-precise data rather than precise common data. In statistical data collection, we may face to situations where information / data entry is accompanied by uncertainty / fuzziness. Some applied instances are: (1) measuring the amount of gas exiting the crater of an volcano per hour, (2) the amount of interest / satisfaction of a worker from his / her job, (3) the length of time a refrigerator safely holds from a specific food / fruit, (4) monthly income of a Taxi driver, (5) the degree of satisfaction / utility of life, and (6) the threshold of patient tolerance. In such cases, recording data in the form of a fuzzy number can help keep more hidden information from the observations. In testing hypotheses based on fuzzy data one can consider the p-value approach in which the entrance of fuzzy data into the statistical computations leads to creating the fuzzy p-value. For such situation, the comparison of the fuzzy p-value and the fuzzy significance level are also discussed in this paper. Moreover, two case studies have been investigated and applied with fuzzy p-value approach in this paper based on a novel approach in comparison of the other studies in the literature. The first case study is about the observed data from a speedometer camera, and the second one is on the lifetime of produced batteries in a manufacture.

It must be noted that the proposed fuzzy decision making have two advantages in comparison of the classical common p-value-based method: (i) proper storage of data using fuzzy numbers to avoid losing the useful information, and (ii) maintain the fuzziness of the problem until the end of the calculation to obtain “the degree of acceptance” in the test.

Regarding to the high application of p-value method in testing hypothesis, discussion on other real-data studies by fuzzy p-value approach are some applied potential topics for future work. One of the key problems in Statistics is to get information about the form of the population from which a sample is drawn. Although Hesamian and Taheri [11], Grzegorzewski and Jedrej [9] and Grzegorzewski and Szymanowski [10] proposed goodness-of-fit tests for fuzzy data, but still there are not too many tools in fuzzy Statistics that help under fuzzy data from the unknown distribution. So, the study of goodness-of-fit testing using fuzzy data by other methods like simulation, defuzzification, extension principle and considering the fuzzy-valued p.d.f. are other potential topics for future research.

Acknowledgement

This paper is dedicated to the memory of Prof. N. R. Arghami (1949, 2017) to show my gratitude. Hereby, I would like to thanks Prof. N. R. Arghami for sharing his pearls of wisdom during my educations.

References

[1] A. Blanco-Fernández, M. R. Casals, A. Colubi, N. Corral, M. Garca-Brzana, M. A. Gil, G. González-Rodriguez, M. T. López, M. A. Lubiano, M. Montenegro, A. B. Ramos-Guajardo, S. de la Rosa de Sa, B. Sinova, *Random fuzzy sets: A mathematical tool to develop statistical fuzzy data analysis*, Iranian Journal of Fuzzy Systems, **10** (2013), 1-28.

[2] M. R. Casals, M. A. Gil, P. Gil, *On the use of Zadehs probabilistic definition for testing statistical hypotheses from fuzzy information*, Fuzzy Sets and Systems, **20** (1986), 175-190.

[3] G. Casella, R. L. Berger, *Statistical inference*, Brooks/Cole Publishing, Belmont, 1990.

[4] J. Chachi, S. M. Taheri, R. Viertl, *Testing statistical hypotheses based on fuzzy confidence intervals*, Austrian Journal of Statistics, **41** (2012), 267-286.

[5] T. Deoeux, M. H. Masson, P. A. Hébert, *Nonparametric rank-based statistics and significance tests for fuzzy data*, Fuzzy Sets and Systems, **153** (2005), 1-28.

[6] D. Dubois, H. Prade, *Possibility theory*, Plenum Press, New-York, 1988.

[7] P. Filzmoser, R. Viertl, *Testing hypotheses with fuzzy data: The fuzzy p-value*, Metrika, **59** (2004), 21-29.

[8] M. Gagolewski, J. Caha, *Fuzzy numbers package: Tools to deal with fuzzy numbers in R*, *R* package version 0.4-1 (2015). https://cran.r-project.org/web/packages=FuzzyNumbers

[9] P. Grzegorzewski, A. Jedrej, *Chi-square goodness-of-fit test for vague data*, in: Proceedings of the Eleventh International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems - IPMU 2006, Paris (2006), 952-956.

[10] P. Grzegorzewski, H. Szymanowski, *Goodness-of-fit tests for fuzzy data*, Information Sciences, **288** (2014), 374-386.

[11] G. Hesamian, S. M. Taheri, *Fuzzy empirical distribution function: Properties and application*, Kybernetika, **49** (2013), 962-982.

[12] O. Hryniewicz, *Statistical properties of the fuzzy p-value*, International Journal of Approximate Reasoning, **93** (2018), 544-560.

[13] C. Kahraman, C. E. Bozdag, D. Ruan, A. F. Özok, *Fuzzy set approaches to statistical parametric and nonparametric tests*, International Journal Of Intelligent Systems, **19** (2004), 1069-1087.

[14] C. Kahraman, Ö. Kabak, *Fuzzy statistical decision-making: Theory and applications*, Chapters in Springer International Publishing Switzerland, **343** (2016).

[15] A. Parchami, *Fuzzy. p. value: Computing fuzzy p-value*, *R* package version 1.0, (2016). URL: https://CRAN.Rproject.org/package=Fuzzy.p.value.

[16] A. Parchami, *FPV: Testing hypotheses via fuzzy p-value in fuzzy environment*, *R* package version 0.5, (2017). URL: https://CRAN.R-project.org/package=FPV.

[17] A. Parchami, *Fuzzy decision making in testing hypotheses: An introduction to the packages "FPV" and "Fuzzy.p.value" with practical examples*, Iranian Journal of Fuzzy Systems, (2019). DOI: 10.22111/IJFS.2019.4878

[18] A. Parchami, S. M. Taheri, M. Mashinchi, *Fuzzy p-value in testing fuzzy hypotheses with crisp data*, Statistical Papers, **51** (2010), 209-226.

[19] A. Parchami, S. M. Taheri, M. Mashinchi, *Testing fuzzy hypotheses based on vague observations: A p-value approach*, Statistical Papers, **53** (2012), 469-484.

[20] A. Parchami, S. M. Taheri, B. Sadeghpour-Gildeh, M. Mashinchi, *A simple but efficient approach for testing fuzzy hypotheses*, Journal of Uncertainty Analysis and Applications, Springer, **4**(2) (2016), 1-16.

[21] A. Parchami, S. T. Taheri, B. Sadeghpour Gildeh, M. Mashinchi, *Testing fuzzy hypotheses: A new p-value-based approach*, C. Kahraman and O. Kabak (eds.), Fuzzy Statistical Decision-Making: Theory and Applications, Springer International Publishing Switzerland, **343** (2016), 155-173.

[22] A. Parchami, S. M. Taheri, R. Viertl, M. Mashinchi, *Minimax test for fuzzy hypotheses*, Statistical Papers, accepted (2017). DOI: 10.1007/s00362-017-0926-4.

[23] S. M. Taheri, *Trends in fuzzy statistics*, Austrian Journal of Statistics, **32** (2003), 239-257.

[24] S. M. Taheri, J. Behboodian, *Neyman-pearson lemma for fuzzy hypotheses testing*, Metrika, **49** (1999), 3-17.

[25] H. Torabi, J. Behboodian, *Sequential probability ratio test for fuzzy hypotheses testing with vague data*, Austrian Journal of Statistics, **34** (2005), 25-38.

[26] R. Viert, *Statistical methods for fuzzy data*, Wiley, Chichester, 2011.

[27] X. Wang, E. E. Kerre, *Reasonable properties for the ordering of fuzzy quantities (II)*, Fuzzy Sets and Systems, **118** (2001), 387-405.

[28] Y. Yuan, *Criteria for evaluating fuzzy ranking methods*, Fuzzy Sets and Systems, **43** (1991), 139-157.