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An improved similarity measure of generalized trapezoidal fuzzy numbers and its application in multi-attribute group decision making

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Abstract

Generalized trapezoidal fuzzy numbers (GTFNs) have been widely applied in uncertain decision-making problems. The similarity between GTFNs plays an important part in solving such problems, while there are some limitations in existing similarity measure methods. Thus, based on the cosine similarity, a novel similarity measure of GTFNs is developed which is combined with the concepts of geometric distance, center of gravity, area and perimeter after analyzing the limitations of previous methods. Then comparative analysis is conducted with existing similarity measures, and the results show that the novel similarity measure has better distinguishability and lower invalidity. Furthermore, a general process, which combines the new similarity measure of GTFNs with compromise methods, is developed to deal with multi-attribute group decision making (MAGDM) problem. Finally, we combine fuzzy VIKOR with the general process as illustrated example, which proves the superiority of the developed similarity measure in solving MAGDM problem.

Keywords: Generalized trapezoidal fuzzy numbers, similarity measure, MAGDM, fuzzy VIKOR, cosine similarity.

1 Introduction

Problems with uncertainty have attracted wide attention of several research fields, especially the fields of evaluation and decision making [39, 41, 46, 40]. Researchers have introduced many uncertainty theories to process and transform imprecision information, such as fuzzy sets [1, 52, 16, 5], rough sets [42, 37, 43], hybrid soft sets [44, 33, 23], and so on. Particularly, fuzzy set theory [60] is an effective tool for dealing with the uncertain information, which has been developed rapidly. Many scholars have proposed various kinds of fuzzy numbers, such as triangular fuzzy number, trapezoidal fuzzy number, interval 2-type fuzzy number, intuitionistic fuzzy number, generalized trapezoidal fuzzy numbers (GTFNs), and so on. Compared with other forms of fuzzy numbers, GTFNs can act as a proxy of many forms of numbers, including real numbers, interval numbers, triangular fuzzy number, and trapezoidal fuzzy numbers. Importantly, it loosens restrictions on the regularity condition of fuzzy number and makes it more flexible to deal with uncertain information [13, 56].

GTFNs have been widely applied in many fields, including fuzzy risk analysis [45, 14, 21, 38, 29, 11, 26, 54], fault diagnosis [64, 53], pattern recognize [56], decision making [8, 35, 30], and many other fields. It is inevitable and vital to compare fuzzy numbers or to measure the similarity between GTFNs for solving these problems [25, 12]. Thus, measuring the similarity between GTFNs has attracted many scholars to research it from different points of view in recent years. It is obvious that many factors can affect the similarity, e.g., the positions, the sizes, the shapes, and so on. So, existing similarity measures are usually defined on the different characteristics of GTFNs, such as geometric distance, perimeter, center of gravity (COG), area, etc. These similarity measures, to some extent, can estimate the degree of similarity between two GTFNs, but there still have some limitations, such as, some methods fail to give correct similarity between crisp-valued fuzzy numbers, the distinguishability is not good enough, the results are not robust enough due to the introduction of adjustable parameters. Thus, a new similarity measure of GTFNs is developed

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by combing the cosine similarity of center of gravity (COG) with geometric distance, area and perimeter, which could overcome the shortcomings of existing methods. We also compare it with some existing similarity measures through 30 sets of GTFNs to highlight its advantages.

The superiority of the method should not only be reflected in the indexes of distinguishability and validity, but also in the application of solving practical problems. Multi-attribute group decision making (MAGDM) is an important part of modern decision science, which has been applied in various areas such as project evaluation, personnel evaluation, investment decision making, and so on. In order to solve such problems, a lot of techniques have been developed. Among these techniques, compromise decision-making methods are very representative. Its inevitable to measure the distances between alternatives and ideals when using such compromise methods. Therefore, by analyzing the characteristics of the MAGDM problem, the reason why the proposed similarity measure can be applied in solving process is explained. Then, the general process of an integrated compromise decision-making method based on our similarity measure is described. Finally, its combination with fuzzy VIKOR is proposed as an example, and a relevant numerical example is given to verify its effectiveness. More importantly, this example also proves the superiority of our novel similarity measure in solving practical problems.

The rest of the paper is structured as follows. After the basic concept of GTFN, relative definitions, and arithmetic operators are introduced in Section 2, the existing similarity measures between GTFNs are briefly reviewed in Section 3. In Section 4, an improved similarity measure of GTFNs is developed and the comparative experiment is also conducted. The general process of an integrated compromise method based on the developed similarity measure to solve MAGDM problem is proposed in Section 5, and its combination with fuzzy VIKOR is conducted in Section 6. Section 7 gives some conclusions and future research directions.

2 Preliminaries

The basic concept of GTFNs and some relevant operators are introduced in this section. Assume $A = (a_1, a_2, a_3, a_4; h_a)$ is a GTFN, where a_1,a_2,a_3,a_4 are real numbers such that $0 \le a_1 \le a_2 \le a_3 \le a_4 \le 1$, and $0 \le h_a \le 1$ shows the height of A. Fig.1 shows the membership function $\mu_A(x)$ of A, which is expressed as Eq. (1). The GTFN A is a normal fuzzy number and denoted as $A = (a_1, a_2, a_3, a_4)$ if $h_a = 1$. A is a generalized interval fuzzy number if $a_1 = a_2$ and $a_3 = a_4$, then A is a crisp interval number if $a_1 = a_2$, $a_3 = a_4$ and $h_a = 1$. A is a generalized triangular fuzzy number if $a_1 < a_2 = a_3 < a_4$, then A is a regular triangular fuzzy number if $a_1 < a_2 = a_3 < a_4$ and $h_a = 1$. A is a real number if $a_1 = a_2 = a_3 = a_4$ and $h_a = 1$.

$$\mu_A(x) = \begin{cases} \frac{h_a\cdot(x-a_1)}{a_2-a_1}, & a_1 < x < a_2 \\ h_a, & a_2 \le x \le a_3 \\ \frac{h_a\cdot(a_4-x)}{a_4-a_3}, & a_3 < x < a_4 \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

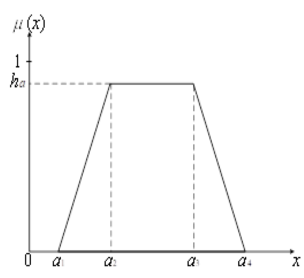

Figure 1: The membership function of a GTFN

Assume that there are two GTFNs A and B, where $A = (a_1, a_2, a_3, a_4; h_a)$, $B = (b_1, b_2, b_3, b_4; h_b)$, and $0 \le a_1 \le a_2 \le a_3 \le a_4 \le 1$ and $0 \le b_1 \le b_2 \le b_3 \le b_4 \le 1$. The multiplication operator $\otimes$ and the additional operator $\oplus$ between A and B defined by [54] are shown as Eqs.(2) and (3).

$$A \otimes B = (a_1, a_2, a_3, a_4; h_a) \otimes (b_1, b_2, b_3, b_4; h_b) = (a_1b_1, a_2b_2, a_3b_3, a_4b_4; \min(h_a, h_b)), \tag{2}$$

$$\begin{aligned} A \oplus B &= (a_1, a_2, a_3, a_4; h_a) \oplus (b_1, b_2, b_3, b_4; h_b) \\ &= (a_1 + b_1 - a_1b_1, a_2 + b_2 - a_2b_2, a_3 + b_3 - a_3b_3, a_4 + b_4 - a_4b_4; \min(h_a, h_b)). \end{aligned} \tag{3}$$

3 Literature review

Many concepts have been introduced by researchers to measure the similarity between GTFNs. Existing similarity measures can be divided into two main categories according to these concepts: one is purely based on existing concepts of similarity, such as Jaccard index or Dice index; the other is to combine different concepts of distance with COG, area, perimeter or other numerical indexes.

For the first type of similarity measures between GTFNs, a new similarity measure between GTFNs has been developed using the Jaccard index by Hwang and Yang [22]. Ye [55] extended the Dice similarity for measuring the similarity between GTFNs. The principle of this type of similarity measures is to treat each GTFN as a set of real numbers, and then calculate the similarity between any two sets. However, this method does not make a full use of the information of GTFNs.

Unlike the first type of similarity, the second type of similarity integrates more information about GTFNs, such as location, shape, size, and so on. This type of similarity measures can be traced back to the method proposed by Chen and Chen [45]. They proposed a similarity measure between GTFNs with different heights through the concepts of geometric distance and COG. This COG-based similarity measure has been proved to be successful in most situations. However, when two GTFNs have the same COG points, it cannot correctly handle it. Thus, Yong, et al. [56] used the radius of gyration points instead of COG points. Considering the computational complexity of radius of gyration, Wei and Chen [49] proposed a new method, which combines the concepts of geometric distance, height, and perimeter. However, the conclusion of computing the GTFNs with the arithmetic operators proposed by Wei and Chen is wrong. Thus, Xu, et al. [54] developed new arithmetic operators and combined the concepts of COG and geometric distance to develop a new similarity measure. Based on the method described by Wei and Chen [49], Hejazi, et al. [21] presented a new similarity measure, which is based on the concepts of geometric distance, height, area, and perimeter. When the heights of two GTFNs are equal to zero, the similarity cannot be evaluated by the method of Hejazi, et al. [21]. To overcome this limitation, Patra and Mondal [38] gave a new similarity measure. Later, Khorshidi and Nikfalazar [26] added the concept of perimeter and defined a new distance of COG points to overcome the drawbacks of the work of Patra and Mondal [38]. Chutia and Gogoi [11, 12] combined the concepts of geometric distance, left height, right height, radius of gyration, values or ambiguity to measure the similarity of GTFNs with different left heights and right heights, whose definition is different from previous.

Some scholars believe that it is more reasonable to replace geometric distance with exponential distance. Wen, et al. [50] developed a new similarity measure between nonstandard GTFNs by introducing the exponential distance of COGs. In view of the shortcomings of this method, Zuo, et al. [64] presented an enhanced similarity measure by transforming geometric distance into exponential distance, and then combining it with the concepts of area and perimeter. Later, Li and Zeng [29] combined the exponent distance of the span and center width with the similarity of perimeter to measure the similarity of GTFNs. They also introduced two indexes for evaluating the performance of different similarity measures. Based on the works of Wen, et al. [50] and Zuo, et al. [64], Xie, et al. [53] combined the exponential distance with the height, the center width, and the span of the GTFNs.

Since previous similarity measures have been reviewed, and their limitations have been analyzed in the literatures [29, 26, 53]. These three studies, as the most recent study, have showed their proposed similarity measures works better than others. In addition, the similarity measure of Chen and Chen [45] is almost the earliest and most representative one. Therefore, we only discuss and compare our proposed similarity measure with these four methods in this paper.

(1) The similarity measure of Chen and Chen [45].

$$S_{cc}(A,B)=\left(1-\frac{\sum_{i=1}^{4}|a_i-b_i|}{4}\right)\times(1-|x_A^*-x_B^*|)^{B(L_4,L_B)}\times\frac{\min(y_A^*,y_B^*)}{\max(y_A^*,y_B^*)}, \tag{4}$$

where (x_A^*,y_A^*) and (x_B^*,y_B^*) are the COGs of A and B , respectively. We just give the formulas of x_A^* and y_A^*. The calculations of x_B^* and y_B^* are the same.

$$y_A^*=\begin{cases}\frac{\left(2+\frac{a_3-a_2}{a_4-a_1}\right)\times h_a}{6}, & a_1\neq a_4\\ \frac{h_a}{2}, & a_1=a_4\end{cases} \tag{5}$$

$$x_A^*=\frac{y_A^*\times(a_3+a_2)+(h_a-y_A^*)\times(a_4+a_1)}{2\times h_a}, \tag{6}$$

$$L_A=a_4-a_1,\cdot L_B=b_4-b_1, \tag{7}$$

$$B\left(L_A, L_B\right)=\begin{cases}0, & L_A+L_B=0 \\ 1, & L_A+L_B>0\end{cases} \tag{8}$$

(2) The similarity measure of Khorshidi and Nikfalazar [26].

$$S_{bn}(A,B)=\left(1-\frac{\sum_{i=1}^{4}\left|a_i-b_i\right|}{4}\times d(A,B)\right)\times\left(1-\frac{\left(|S(A)-S(B)|+|h_a-h_b|+\frac{|P(A)-P(B)|}{\max(P(A),P(B))}\right)}{3}\right), \tag{9}$$

where $\frac{|P(A)-P(B)|}{\max(P(A),P(B))}=0$ when $\max(P(A),P(B))=0$, $d(A,B)=\sqrt{\frac{\left(x_A^*-x_B^*\right)^2+\left(y_A^*-y_B^*\right)^2}{1.25}}$ is the distance between COGs of A and B, in which x_A^* and y_A^* can be defined as Eqs. (10) and (11), x_B^* and y_B^* are computed as the same.

$$y_A^*=\begin{cases}\frac{h_a\times\left(\frac{a_3-a_2}{a_4-a_1}+2\right)}{6}, & \text{if } a_4\neq a_1 \\ \frac{h_a}{2}, & \text{if } a_4=a_1\end{cases} \tag{10}$$

$$x_A^*=\begin{cases}\frac{y_A^*\times(a_3+a_2)+(a_4+a_1)\times(h_a-y_A^*)}{2h_a}, & \text{if } h_a\neq 0 \\ \frac{a_4+a_1}{2}, & \text{if } h_a=0\end{cases} \tag{11}$$

The areas of A and B are denoted as $S(A)$ and $S(B)$. $S(A)$ can be computed as Eq. (12). The calculation of $S(B)$ is similar to $S(A)$.

$$S(A)=\frac{1}{2}\times\left[(a_4-a_1)+(a_3-a_2)\right]\times h_a. \tag{12}$$

The perimeters of A and B are denoted as $P(A)$ and $P(B)$. $P(A)$ can be computed as Eq. (13). The calculation of $P(B)$ is similar to it.

$$P(A)=\sqrt{\left(a_1-a_2\right)^2+h_a^2}+\sqrt{\left(a_3-a_4\right)^2+h_a^2}+\left(a_4-a_1\right)+\left(a_3-a_2\right). \tag{13}$$

(3) The similarity measure of Li and Zeng [29]. Based on the concepts of exponent distance and shape difference indexes such as center width, span and perimeter, Li and Zeng [29] proposed the following similarity measure:

$$S_{lz}(A,B)=se\times sp, \tag{14}$$

in which,

$$se=\begin{cases}e^{-|a_1-b_1|}, & a_1=a_4 \text{ and } b_1=b_4 \\ e^{-(k+h+s+z)}, & \text{otherwise}\end{cases} \tag{15}$$

where k is the span difference, h is the center width difference, s is the minimum value of difference between left point and right point of peak value, z is the center difference for GTFNs A and B, respectively.

$$k=\left|(a_4-a_1)-(b_4-b_1)\right|, \tag{16}$$

$$h=\left|(a_3-a_2)-(b_3-b_2)\right|, \tag{17}$$

$$s=\min\left(\left|a_2-b_2\right|,\left|a_3-b_3\right|\right), \tag{18}$$

$$z=\left|\left(a_4+a_1\right)/2-\left(b_4+b_1\right)/2\right|. \tag{19}$$

And

$$sp=\frac{\min(P(A),P(B))+\varepsilon}{\max(P(A),P(B))+\varepsilon}, \tag{20}$$

where ε is a modified parameter, which avoids the case where the denominator or molecule is equal to zero. It satisfies $\varepsilon\in(0,0.1]$.

(4) The similarity measure of Xie et al. [53]. Based on the concepts of exponent distance and some shape indexes including center width, height and span, Xie et al. [53] proposed the following similarity measure:

$$S_x(A,B)=se\times sw, \tag{21}$$

in which

$$se = \begin{cases} e^{-|a_1-b_1|}, & a_1 = a_4 \text{ and } b_1 = b_4 \\ e^{-\frac{k+z+h+lr}{w}}, & \text{otherwise} \end{cases} \tag{22}$$

where k is the span difference, z is the max distance of the two left endpoints or right endpoints, h is the center width difference, lr is the max distance of two left points or right points between the core of A and B, w is the maximum span of A and B, respectively. k and h can be computed as Eqs. (16) and (17), others can be computed as follows:

$$z = \max\left(|a_1 - b_1|, |a_4 - b_4|\right), \tag{23}$$

$$w = \max\left(a_4 - a_1, b_4 - b_1\right), \tag{24}$$

$$lr = \max\left(|a_2 - b_2|, |a_3 - b_3|\right), \tag{25}$$

and

$$Sw = \frac{\min\left(h_a, h_b\right)}{\max\left(h_a, h_b\right)}. \tag{26}$$

As mentioned before, literatures [29, 26, 53] have analyzed the shortcomings of the previous similarity measure. Now, some associated drawbacks of their works are listed as follows.

The performance of some similarity measures is not good enough, which is reflected in validity, distinguishability, and the robustness of results: **1)Validity**. Some similarity measures fail in certain cases. For example, when $h_a = h_b = 0$, the similarity measures (1) and (4) fail to compute. **2)Distinguishability**. The distinguishability of some methods is not good enough. For the similarity measure (4), suppose $A = (0.10, 0.20, 0.30, 0.40; 1.00)$, $B = (0.10, 0.25, 0.25, 0.40; 1.00)$ and $C = (0.10, 0.21, 0.23, 0.40; 1.00)$ are three GTFNs. According to Eqs. (21)-(26), the similarity between A and B is 0.606531, which is equal to the similarity between A and C. However, B and C are obviously two different GTFNs. **3)Robustness**. The results of some methods are not robust enough due to the introduction of adjustable parameters. Different values of lead to different results in the similarity measure (3). For example, $A = (0.10, 0.20, 0.30, 0.40; 1.00)$ and $B = (0.10, 0.25, 0.25, 0.40; 1.00)$ are two GTFNs, the similarity between A and B is 0.829551 when $\varepsilon = 0.01$, while it is 0.830548 when $\varepsilon = 0.09$. Moreover, similarity measure (3) did not give the optimal value of ε.

Thus, an improved method to measure the similarity between GTFNs so as to cover above shortages is necessary. These existing similarity measures only make use of the position information of COGs, but do not make a full use of their direction information. Cosine similarity is a classical tool to solve this defect. The cosine similarity is the most widely used vector similarity measurement method, which is a classical measure method in information retrieval [30]. Therefore, we introduce the concept of cosine similarity to improve the similarity measure between two COGs in the next section. Then, we develop new similarity measure between GTFNs based on geometric distance, area, perimeter, and the cosine similarity of COGs.

4 The novel similarity measure between GTFNs

In this section, an improved similarity measure between GTFNs combing the cosine similarity of COGs with geometric distance, area, and perimeter is developed. In addition, some properties are also discussed. Lastly, a comparative experiment is conducted to verify the performance of our proposed similarity measure.

4.1 Proposed similarity measure

Supposed two GTFNs $A = (a_1, a_2, a_3, a_4; h_a)$ and $B = (b_1, b_2, b_3, b_4; h_b)$, where $0 \le a_1 \le a_2 \le a_3 \le a_4 \le 1$ and $0 \le b_1 \le b_2 \le b_3 \le b_4 \le 1$. The COGs of A and B are denoted as (x_A^*, y_A^*) and (x_B^*, y_B^*), their areas are $S(A)$ and $S(B)$, and their perimeters are $P(A)$ and $P(B)$, respectively. The similarity between A and B is defined as

$$S(A, B) = \left(1 - \frac{\sum_{i=1}^{4} |a_i - b_i|}{4}\right) \times d_{AB} \times sp_{AB}, \tag{27}$$

where $sp_{AB} = \frac{\min(S(A),S(B))+\min(P(A),P(B))}{\max(S(A),S(B))+\max(P(A),P(B))}$, and $SP_{AB} = 1$ when $\max(S(A), S(B)) + \max(P(A), P(B)) = 0$. $S(A)$ and $P(A)$ are able to be computed as Eqs. (12) and (13), respectively. $S(B)$ and $P(B)$ are computed in a similar way.

$$d_{AB} = \begin{cases} 1 - \sqrt{\frac{\left(x_A^* - x_B^*\right)^2 + \left(y_A^* - y_B^*\right)^2}{1.25}}, & \text{if } x_A^* = y_A^* = 0 \text{ or } x_B^* = y_B^* = 0 \\ \frac{x_A^* \times x_B^* + y_A^* \times y_B^*}{\sqrt{x_A^{*2} + y_A^{*2}} \times \sqrt{x_B^{*2} + y_B^{*2}}}, & \text{otherwise} \end{cases} \tag{28}$$

where x_A^* and y_A^* can be computed as Eqs. (10) and (11). Similarly, x_B^* and y_B^* are able to be calculated in this way.

The similarity $S(A,B)$ proposed in this paper has following properties, which are also proved in references [29, 64, 53]:

Property1. $0 \le S(A,B) \le 1$.

Proof. As $0 \le a_1 \le a_2 \le a_3 \le a_4 \le 1$ and $0 \le b_1 \le b_2 \le b_3 \le b_4 \le 1$, $0 \le |a_i - b_i| \le 1$ is undoubted, then $0 \le \sum_{i=1}^4 |a_i - b_i| \le 4$, thus $0 \le \left(1 - \frac{\sum_{i=1}^4 |a_i-b_i|}{4}\right) \le 1$ is satisfied.

If $a_4 \ne a_1$, as $0 \le a_1 \le a_2 \le a_3 \le a_4 \le 1$, $0 \le \frac{a_3-a_2}{a_4-a_1} \le 1$ is obtained. $0 \le h_a \le 1$, so $0 \le y_A^* = \frac{h_a \times \left(\frac{a_3-a_2}{a_4-a_1}+2\right)}{6} \le \frac{1}{2}$. If $a_4 = a_1$, $0 \le y_A^* = \frac{h_a}{2} \le \frac{1}{2}$ is obvious. Therefore, $0 \le y_A^* \le \frac{1}{2}$. Similarly, $0 \le y_B^* \le \frac{1}{2}$.

If $h_a \ne 0$, as $0 \le a_1 \le a_2 \le a_3 \le a_4 \le 1$, $0 \le h_a \le 1$ and $0 \le y_A^* \le \frac{1}{2}$, $0 \le y_A^* \times (a_3+a_2) + (a_4+a_1) \times (h_a - y_A^*) \le 2$ is able to be proved, therefore $0 \le x_A^* = \frac{y_A^* \times (a_3+a_2)+(a_4+a_1)\times(h_a-y_A^*)}{2h_a} \le 1$. Else, as $0 \le a_1 \le a_2 \le a_3 \le a_4 \le 1$, $0 \le x_A^* = \frac{a_4+a_1}{2} \le 1$ is satisfied. Overall, $0 \le x_A^* \le 1$ is undoubted. Similarly, $0 \le x_B^* \le 1$.

So, when $x_A^* = y_A^* = 0$ and $x_B^* = y_B^* = 0$, $d_{AB} = 1 - \sqrt{\frac{(x_A^*-x_B^*)^2+(y_A^*-y_B^*)^2}{1.25}}$, where $0 \le (x_A^*-x_B^*)^2 + (y_A^*-y_B^*)^2 \le 1.25$, hence $0 \le d_{AB} = 1 - \sqrt{\frac{(x_A^*-x_B^*)^2+(y_A^*-y_B^*)^2}{1.25}} \le 1$ is proved. Otherwise, $d_{AB} = \frac{x_A^* \times x_B^* + y_A^* \times y_B^*}{\sqrt{x_A^{*2}+y_A^{*2}} \times \sqrt{x_B^{*2}+y_B^{*2}}}$, in which, $0 \le x_A^*, x_B^* \le 1$ and $0 \le y_A^*, y_B^* \le \frac{1}{2}$. $0 < x_A^* \times x_B^* + y_A^* \times y_B^* \le 1.25$ and $0 < \sqrt{x_A^{*2}+y_A^{*2}} \times \sqrt{x_B^{*2}+y_B^{*2}} \le 1.25$ are true, so $0 \le d_{AB} = \frac{x_A^* \times x_B^* + y_A^* \times y_B^*}{\sqrt{x_A^{*2}+y_A^{*2}} \times \sqrt{x_B^{*2}+y_B^{*2}}} \le 1$. Overall, $0 \le d_{AB} \le 1$.

$\min(S(A),S(B)) \le \max(S(A),S(B))$ and $\min(P(A),P(B)) \le \max(P(A),P(B))$ are true, therefore, $0 \le sp_{AB} = \frac{\min(S(A),S(B))+\min(P(A),P(B))}{\max(S(A),S(B))+\max(P(A),P(B))} \le 1$ is established.

All in all, $S(A,B) \in [0,1]$ is satisfied. □

Property2. $S(A,B) = 1 \Leftrightarrow A = B$.

Proof. (1) $A = B \Rightarrow S(A,B) = 1$

If $A = B$, then $a_i = b_i$, $i = 1,2,3,4$, $h_a = h_b$, $x_A^* = x_B^*$ and $y_A^* = y_B^*$ are obtained, then $1 - \frac{\sum_{i=1}^4 |a_i-b_i|}{4} = 1 - \frac{0}{4} = 1$,

$$d_{AB} = \frac{x_A^* \times x_B^* + y_A^* \times y_B^*}{\sqrt{x_A^{*2}+y_A^{*2}} \times \sqrt{x_B^{*2}+y_B^{*2}}} = \frac{x_A^* \times x_A^* + y_A^* \times y_A^*}{\sqrt{x_A^{*2}+y_A^{*2}} \times \sqrt{x_A^{*2}+y_A^{*2}}} = \frac{x_A^{*2}+y_A^{*2}}{x_A^{*2}+y_A^{*2}} = 1,$$

$$S(A) = \frac{1}{2} \times [(a_4-a_1)+(a_3-a_2)] \times h_a = \frac{1}{2} \times [(b_4-b_1)+(b_3-b_2)] \times h_b = S(B),$$

$$\begin{aligned} P(A) &= \sqrt{(a_1-a_2)^2+h_a^2} + \sqrt{(a_3-a_4)^2+h_a^2} + (a_4-a_1) + (a_3-a_2) \\ &= \sqrt{(b_1-b_2)^2+h_b^2} + \sqrt{(b_3-b_4)^2+h_b^2} + (b_4-b_1) + (b_3-b_2) = P(B) \end{aligned},$$

Thus, $sp_{AB} = \frac{\min(S(A),S(B))+\min(P(A),P(B))}{\max(S(A),S(B))+\max(P(A),P(B))} = 1$.

Finally, $S(A,B) = \left(1 - \frac{\sum_{i=1}^4 |a_i-b_i|}{4}\right) \times d_{AB} \times sp_{AB} = 1 \times 1 \times 1 = 1$ is verified.

(2) $S(A,B) = 1 \Rightarrow A = B$

As proved in **Property1**, $0 \le \left(1 - \frac{\sum_{i=1}^4 |a_i-b_i|}{4}\right) \le 1$, $0 \le d_{AB} \le 1$ and $0 \le sp_{AB} \le 1$ are satisfied. Thus, if $S(A,B) = 1$, $\left(1 - \frac{\sum_{i=1}^4 |a_i-b_i|}{4}\right) = d_{AB} = sp_{AB} = 1$ is undoubted. This implies that $a_i = b_i, \cdot i = 1,2,3,4$, $P(A) = P(B)$ and $S(A) = S(B)$. According to $S(A) = \frac{h_a}{2} \times [(a_4-a_1)+(a_3-a_2)] = S(B) = \frac{h_b}{2} \times [(b_4-b_1)+(b_3-b_2)]$, we can see that $h_a = h_b$. Therefore, A and B are the same. □

Property3. $S(A,B) = S(B,A)$.

Proof.

$$1 - \frac{\sum_{i=1}^4 |a_i-b_i|}{4} = 1 - \frac{\sum_{i=1}^4 |b_i-a_i|}{4},$$

$$\frac{\min(S(A),S(B))+\min(P(A),P(B))}{\max(S(A),S(B))+\max(P(A),P(B))} = \frac{\min(S(B),S(A))+\min(P(B),P(A))}{\max(S(B),S(A))+\max(P(B),P(A))},$$

$$\frac{x_A^* \times x_B^* + y_A^* \times y_B^*}{\sqrt{x_A^{*2} + y_A^{*2}} \times \sqrt{x_B^{*2} + y_B^{*2}}} = \frac{x_B^* \times x_A^* + y_B^* \times y_A^*}{\sqrt{x_B^{*2} + y_B^{*2}} \times \sqrt{x_A^{*2} + y_A^{*2}}}$$

and $\sqrt{\frac{(x_A^*-x_B^*)^2+(y_A^*-y_B^*)^2}{1.25}} = \sqrt{\frac{(x_B^*-x_A^*)^2+(y_B^*-y_A^*)^2}{1.25}}$ are obvious, thus $S(A,B) = \left(1 - \frac{\sum_{i=1}^4 |a_i-b_i|}{4}\right) \times d_{AB} \times Sp_{AB} = \left(1 - \frac{\sum_{i=1}^4 |b_i-a_i|}{4}\right) \times d_{BA} \times Sp_{BA} = S(B,A)$ is undoubted. Therefore, we obtain that $S(A,B) = S(B,A)$. □

4.2 The comparative experiment

We use the concepts of distinguishability and invalidity rate proposed by Li and Zeng [29] to estimate the performances of similarity measures. The distinguishability has also been used by Xie et.al [53] to reflect the performance differences of similarity measures. Suppose there are n sets of data to participate in the comparison.

Definition 4.1. ***(Distinguishability)*** *Sort a set of data in descending order, and then take the sum of the top 20%, denoted by $S_{h_1}, S_{h_2}, \cdots, S_{h_{n/5}}$, subtract the sum of the bottom 20%, denoted by $S_{l_1}, S_{l_2}, \cdots, S_{l_{n/5}}$, and the average is the final result. Let η denotes the distinguishability:*

$$\eta = \frac{\sum_{i=1}^{n/5} S_{h_i} - \sum_{i=1}^{n/5} S_{l_i}}{n/5} \times 100\%. \tag{29}$$

The larger the distinguishability is, the better the discreteness is, and the higher recognition rate is.

Definition 4.2. (Invalidity) *Because some similarity measures may be incomplete, then there exist some data set which similarity measure cannot be calculated, or be negative. Such value of similarity measure is invalid. Suppose that there are m groups of data whose similarity measures are invalid, and let λ denotes invalidity rate:*

$$\lambda = \frac{m}{n} \times 100\%. \tag{30}$$

Using these two indexes, through 30 sets of sample data from reference [54], the proposed similarity measure can be compared with some existing similarity measures reviewed in Section 3, as illustrated in Table 1. In Table 2, we compute the similarity values using different methods, the value of two evaluation indexes η and λ are also given for comparison. The character '♯' indicates that the similarity measure is invalid.

Table 1: 30 sets of typical GTFNs

No.	A	B	No.	A	B
1	(0.10,0.20,0.30,0.40;1.00)	(0.10,0.25,0.25,0.40;1.00)	16	(0.10,0.20,0.30,0.40;0.00)	(0.10,0.20,0.30,0.40;0.00)
2	(0.10,0.20,0.30,0.40;1.00)	(0.50,0.65,0.65,0.80;1.00)	17	(0.10,0.25,0.25,0.40;0.80)	(0.10,0.25,0.25,0.40;0.60)
3	(0.10,0.20,0.30,0.40;1.00)	(0.30,0.45,0.45,0.60;1.00)	18	(0.10,0.25,0.25,0.40;0.80)	(0.30,0.45,0.45,0.60;0.60)
4	(0.10,0.20,0.30,0.40;1.00)	(0.10,0.25,0.25,0.40;0.20)	19	(0.10,0.25,0.25,0.40;0.80)	(0.30,0.45,0.45,0.60;0.00)
5	(0.10,0.20,0.30,0.40;1.00)	(0.20,0.35,0.35,0.50;0.00)	20	(0.10,0.25,0.25,0.40;0.01)	(0.10,0.25,0.25,0.40;0.00)
6	(0.10,0.20,0.30,0.40;1.00)	(0.50,0.50,0.50,0.50;0.00)	21	(0.10,0.25,0.25,0.40;0.80)	(0.30,0.30,0.30,0.30;0.60)
7	(0.10,0.20,0.30,0.40;1.00)	(0.10,0.20,0.30,0.40;1.00)	22	(0.10,0.25,0.25,0.40;0.80)	(0.30,0.30,0.30,0.30;0.00)
8	(0.10,0.20,0.30,0.40;1.00)	(0.50,0.60,0.70,0.80;1.00)	23	(0.10,0.20,0.30,0.40;0.80)	(0.20,0.35,0.35,0.50;0.00)
9	(0.10,0.20,0.30,0.40;1.00)	(0.30,0.40,0.50,0.60;1.00)	24	(0.10,0.20,0.30,0.40;0.00)	(0.20,0.35,0.35,0.50;0.00)
10	(0.10,0.20,0.30,0.40;1.00)	(0.10,0.20,0.30,0.40;0.80)	25	(0.10,0.20,0.30,0.40;0.00)	(0.30,0.30,0.30,0.30;0.00)
11	(0.10,0.20,0.30,0.40;1.00)	(0.20,0.30,0.40,0.50;0.00)	26	(0.30,0.30,0.30,0.30;0.00)	(0.50,0.50,0.50,0.50;0.00)
12	(0.10,0.20,0.30,0.40;1.00)	(0.30,0.30,0.30,0.30;0.40)	27	(0.40,0.40,0.40,0.40;0.40)	(0.50,0.50,0.50,0.50;0.20)
13	(0.10,0.20,0.30,0.40;0.80)	(0.20,0.30,0.40,0.50;0.40)	28	(0.00,0.00,0.00,0.00;0.00)	(0.00,0.00,0.00,0.00;0.00)
14	(0.10,0.20,0.30,0.40;0.80)	(0.20,0.30,0.40,0.50;0.00)	29	(0.00,0.00,0.00,0.00;0.00)	(0.00,0.00,0.00,0.00;0.01)
15	(0.10,0.20,0.30,0.40;0.01)	(0.10,0.20,0.30,0.40;0.00)	30	(0.00,0.00,0.00,0.00;0.00)	(1.00,1.00,1.00,1.00;1.00)

From the Table 2, we can draw some conclusions as follows:

(1) The results calculated in literatures [45] and [53] have high invalidity rate, which means their similarity measures are failed in some cases.

(2) For the similarity measure proposed by Li and Zeng [29], different values of ε lead to different similarity measure results and different distinguishability. This makes the method not robust enough.

(3) Distinguishability η of our proposed similarity measure is the highest value among these five measure methods, and the invalidity λ is zero. That means compared with other four methods, our proposed similarity measure has higher recognition rate, can better avoid phenomenon of indistinguishability, and also has the lowest invalidity. Namely, the proposed similarity measure has better performance.

Table 2: Different similarity measures and the values of evaluation indexes

No.	S_{cc}[45]	$S_{lz}(\varepsilon=0.01)$[29]	$S_{lz}(\varepsilon=0.09)$[29]	S_{kn}[26]	S_x[53]	S
1	0.835714	0.829551	0.830548	0.970010	0.606531	0.921193
2	0.308571	0.411943	0.412438	0.830894	0.042144	0.491649
3	0.548571	0.614547	0.615285	0.935154	0.159880	0.708733
4	0.167143	0.288091	0.306415	0.450714	0.121306	0.229209
5	♯	0.196311	0.214951	0.337097	0.000000	0.111882
6	♯	0.001766	0.015387	0.239100	0.000000	0.000000
7	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
8	0.360000	0.449329	0.449329	0.856892	0.069483	0.537552
9	0.640000	0.670320	0.670320	0.964223	0.263597	0.767328
10	0.800000	0.835732	0.840989	0.865017	0.800000	0.827730
11	♯	0.206376	0.225972	0.337097	0.000000	0.111882
12	0.439714	0.213423	0.226998	0.501727	0.038789	0.252841
13	0.405000	0.499800	0.511936	0.697772	0.256709	0.480479
14	♯	0.246941	0.268698	0.433011	0.000000	0.155700
15	♯	0.998367	0.998556	0.995447	0.000000	0.994909
16	♯	1.000000	1.000000	1.000000	♯	1.000000
17	0.750000	0.798259	0.806257	0.855738	0.750000	0.786348
18	0.480000	0.535089	0.540450	0.823466	0.197698	0.585530
19	♯	0.211001	0.229211	0.436089	0.000000	0.160308
20	♯	0.998910	0.999036	0.995797	0.000000	0.996315
21	0.760000	0.418543	0.428525	0.763356	0.119910	0.527100
22	♯	0.003459	0.029897	0.351264	0.000000	0.000000
23	♯	0.234897	0.255593	0.433011	0.000000	0.155700
24	♯	0.778801	0.778801	0.991056	♯	0.900000
25	♯	0.010453	0.083169	0.663685	♯	0.000000
26	♯	0.818731	0.670320	0.964223	♯	0.800000
27	0.450000	0.458004	0.450762	0.756969	0.452419	0.434144
28	♯	1.000000	1.000000	1.000000	♯	1.000000
29	0.000000	0.333333	0.818182	0.663333	0.000000	0.000000
30	0.000000	0.001830	0.005828	0.000000	0.000000	0.000000
λ	46.47%	0	0	0	16.67%	**0**
η	0.822857	0.902136	0.877230	0.714122	0.721790	**0.966756**

5 The applications of new similarity measure of GTFNs in MAGDM problems

The description of MAGDM problems is presented first to explain why the new similarity measure of GTFNs can be used in MAGDM problems. Then, we propose the general process of integrated approach for MAGDM problems based on compromise methods with new similarity of GTFNs.

5.1 The description of MCGDM problems

A MAGDM problem is consisted of several steps [17]: The decision makers firstly compare alternatives on each criterion according to their experience. By assembling their preference information, a judgment matrix (preference relation) is construct. And then the priorities of alternatives are determined. Such problems have attracted widespread attention in the field of management practice and research, such as supplier or project selection, medical treatment, and so on. As the complexity and uncertainty of the group decision-making environment, some scholars have focused on the preference function and consensus reaching process recently [17, 32, 19, 62, 9, 18]. Besides, the fuzzy set [61, 48, 27], grey system theory [32, 15, 2], and rough set theory [57, 63] have been widely applied in dealing with information so as to reduce the distortion of uncertain information in solving MAGDM problems.

A MAGDM problem with m alternatives, K criteria and n evaluators is able to be described as follows. Suppose there are m potential candidates, $A = (A_1, A_2, \cdots, A_m)$, which are qualified by K criterion, $C = (C_1, C_2, \cdots, C_K)$. Evaluations of the performance of each evaluator $D_i (i = 1, 2, \cdots, n)$ for each alternative $A_j (j = 1, 2, \cdots, m)$ on C_k criteria $(k = 1, 2, \cdots, K)$ are $\tilde{\mathbf{R}}_i = (\tilde{x}_{ijk})_{m\times K} (i = 1, 2, \cdots, n)$. The importance assigned by the evaluator D_i to criterion C_k is $\tilde{\omega}_{ik}$. The weight of the D_i is able to be expressed as $\lambda_i (i = 1, 2, \cdots, n)$, which satisfies $\lambda_i \in [0, 1]$ and $\sum_{i=1}^{n} \lambda_i = 1$. How to select the optimal alternative according to the above information?

The MAGDM problems solving process includes three steps [58, 4, 10]: (1) Ranking phase. Each evaluator provides evaluation information for each alternative on attributes. (2) Aggregation phase. Based on the set of individual decisions, a collective preference decision is obtained. And (3) Exploitation phase. Based on the collective preference decision, a given method is used to obtain the optimal alternative(s). For the first stage, most scholars focus on the forms of evaluation information by decision makers, such as crisp numbers, interval numbers, linguistic terms, and so on. Further, many of them use fuzzy numbers to express the linguistic terms, including hesitant fuzzy number, interval type-2 fuzzy number, intuitionistic fuzzy numbers, and so on. As mentioned before, the evaluation information is conveyed by GTFNs. For the second stage, some methods have been used, include simple additive weighting, novel aggregation operators, evidential theory, and so on. This stage is not the focus of this paper, so the existing weighted averaging operators of GTFNs are adopted to solve this integration problem. For the third stage, many techniques have been proposed. These approaches can be classified into four main categories [20, 6]]: (1) multi-attribute utility methods such as AHP [28] and ANP, (2) outranking methods including Preference Ranking Organization Method for Enrichment of Evaluations (PROMETHEE) [31], Elimination and Choice Translating Reality Method (ELECTRE) [47], and so on, (3) compromise methods, such as VIse Kriterijumska Optimizacija kompromisno Resenje (VIKOR) [59, 34, 51, 7], Technique for Order Performance by Similarity to Idea Solution (TOPSIS) [3, 24], and so on, (4) Others. Our proposed novel similarity measure can be integrated into this stage to rank the alternatives. It is especially suitable for combination with the third method. Thus, we will show how the proposed similarity is combined with compromise methods to deal with the MAGDM problem.

5.2 A combination with compromise methods

Compromise methods represent the degree of closeness to the ideal solution through an aggregating function [54]. According to the nature of methods, the general process of integrated approach for MAGDM problem based on compromise methods with new similarity of GTFNs is described as follows (In order to illustrate a general and typical decision process simply, the weights of the decision maker are assumed to be known):

Step 1. *Construct the initial decision-making matrix.*

The initial decision-making matrix can be construct as follows:

$$\tilde{\mathbf{R}}_i = \begin{array}{c} \\ A_1 \\ \vdots \\ A_j \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{ccccc} C_1 & \cdots & C_k & \cdots & C_K \end{array} \\ \begin{bmatrix} \tilde{x}_{i11} & \cdots & \tilde{x}_{i1k} & \cdots & \tilde{x}_{i1K} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \tilde{x}_{ij1} & \cdots & \tilde{x}_{ijk} & \cdots & \tilde{x}_{ik} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \tilde{x}_{im1} & \cdots & \tilde{x}_{imk} & \cdots & \tilde{x}_{imK} \end{bmatrix} \end{array}$$

where $\tilde{x}_{ijk}$ is the evaluation of alternative $A_j (j = 1, 2, \cdots, m)$ provided by $D_i (i = 1, 2, \cdots, n)$ on the criteria $C_k (k = 1, 2, \cdots, K)$.

Step 2. *Normalize the decision-making matrix.*

Transfer linguistic terms for rating alternatives on each criterion into GTFNs. Linguistic terms, complementary terms and their corresponding GTFNs are shown in Table 3. The criteria are divided into benefit criteria set F_1 and cost criteria set F_2 according to the principle of criteria category. F_1 and F_2 satisfy $F_1 \cup F_2 = C$ and $F_1 \cap F_2 = \varnothing$, where $\varnothing$ is the empty set. The initial decision matrix $\tilde{\mathbf{R}}_i$ is normalized as follows:

$$\tilde{\tilde{x}}_{jk} = \begin{cases} \tilde{x}_{ijk}, & \text{for benefit criteria } C_k \in F_1 \\ (\tilde{x}_{jk})^C, & \text{for cost criteria } C_k \in F_2 \end{cases} \tag{31}$$

where $(\tilde{x}_{jk})^C$ is the complement of $\tilde{x}_{jk}$, which satisfies $(\tilde{x}_{ijk})^C \in L^C$. Therefore, we form n normalized decision matrixes $\tilde{\tilde{\mathbf{R}}}_i = \left(\tilde{\tilde{x}}_{ijk}\right)_{m\times K}$.

Table 3: 7-number linguistic terms set and their corresponding GTFN for rating alternatives [28]

Linguistic terms (L)	Complementary terms (L^c)	GTFN
Very Poor (VP)	Very Good (VG)	(0.00, 0.00, 0.10, 0.20; 1.00)
Poor (P)	Good (G)	(0.10, 0.20, 0.20, 0.30; 1.00)
Medium Poor (MP)	Medium Good (MG)	(0.20, 0.30, 0.40, 0.50; 1.00)
Fair (F)	Fair (F)	(0.40, 0.50, 0.50, 0.60; 1.00)
Medium Good (MG)	Medium Poor (MP)	(0.50, 0.60, 0.70, 0.80; 1.00)
Good (G)	Poor (P)	(0.70, 0.80, 0.80, 0.90; 1.00)
Very Good (VG)	Very Poor (VP)	(0.80, 0.90, 1.00, 1.00; 1.00)

Step 3. *Aggregate the criteria weights.*

Suitable linguistic variables for the criteria importance weights are expressed by each evaluator and are further transferred into GTFNs, which as shown in Table 4.

Table 4: Linguistic terms and their corresponding GTFN for weighting criteria importance

Linguistic terms	GTFN
Very Low (VL)	(0.00, 0.10, 0.20, 0.30; 1.00)
Low (L)	(0.10, 0.20, 0.30, 0.40; 1.00)
Medium (M)	(0.30, 0.40, 0.50, 0.60; 1.00)
High (H)	(0.50, 0.60, 0.70, 0.80; 1.00)
Very High (VH)	(0.70, 0.80, 0.90, 1.00; 1.00)

Let $\omega_{ik} = \left(\omega_{k1}, \omega_{k2}, \omega_{ik3}, \omega_{ik4}; \omega_{ik}^h\right)$, $i = 1, 2, \cdots, n, \cdot k = 1, 2, \cdots, K$ be the weight to criteria C_k assigned by the evaluator D_i. Hence, by extending the integration method proposed by Chen, et al. [8], the aggregated fuzzy weight ω_k of criteria C_k assessed by the group of n evaluators can be calculated as

$$\omega_k = \left(\omega_{k1}, \omega_{k2}, \omega_{k3}, \omega_{k4}; \omega_k^h\right), \tag{32}$$

where $\omega_{k1} = \min_i\{\omega_{ik1}\}$, $\omega_{k2} = \frac{1}{n}\sum_{i=1}^n \omega_{ik2}$, $\omega_{k3} = \frac{1}{n}\sum_{i=1}^n \omega_{ik3}$, $\omega_{k4} = \max_i\{\omega_{ik4}\}$, $\omega_k^h = \frac{1}{n}\sum_{i=1}^n \omega_{ik}^h$.

Step 4. *Build the weighted normalized decision matrix.*

Multiply weights vector $\mathbf{w} = (w_1, w_2, \cdots, w_K)^T$ of evaluators with normalized matrix $\tilde{\tilde{\mathbf{R}}}_i = \left(\tilde{\tilde{x}}_{ijk}\right)_{m\times K}$ by using the relative arithmetical operators (2) and (3). The weighted normalized fuzzy decision matrix is

$$\mathbf{R} = (x_{jk})_{m_N K}, j = 1, 2, \cdots, m; k = 1, 2, \cdots, K, \tag{33}$$

where $x_{jk} = \left(\lambda_1 \otimes w_k \otimes \tilde{\tilde{x}}_{1jk}\right) \oplus \left(\lambda_2 \otimes w_k \otimes \tilde{\tilde{x}}_{2jk}\right) \oplus \cdots \oplus \left(\lambda_n \otimes w_k \otimes \tilde{\tilde{x}}_{nk}\right)$.

Step 5. *Obtain the fuzzy negative and positive ideal solutions.*

By considering the finite set of criteria, the fuzzy positive ideal solution (FPIS) and fuzzy negative ideal solution (FNIS) can be obtained from the weighted normalized fuzzy decision matrix $\mathbf{R} = (x_{jk})_{m_N K}$, where the obtained elements x_{jk} are GTFNs expressed as $x_{jk} = (x_{jk1}, x_{jk2}, x_{jk3}, x_{jk4}; h_{jk})$.

Then, FPIS A^+ and FNIS A^- can be defined respectively as:

$$A^+ = \left(x_1^+, \cdots, x_k^+, \cdots, x_K^+\right), \tag{34}$$

$$A^- = \left(x_1^-, \cdots, x_k^-, \cdots, x_K^-\right). \tag{35}$$

According to definition of FPIS A^+ and FNIS A^- proposed by Chen et al. [8], x_k^+ and x_k^- can be calculated by

$$x_k^+ = \left(\max_j\{x_{jk4}\}, \max_j\{x_{jk4}\}, \max_j\{x_{jk4}\}, \max_j\{x_{jk4}\}; \max_j\{h_{jk}\}\right), \tag{36}$$

$$x_k^- = \left(\min_j \{x_{jk1}\}, \min_j \{x_{jk1}\}, \min_j \{x_{jk1}\}, \min_j \{x_{jk1}\}; \min_j \{h_{jk}\} \right), \tag{37}$$

$j = 1, 2, \cdots, m; k = 1, 2, \cdots, K$.

Step 6. *Measure the closeness to the ideal solution based on similarity/ distance.*

In this step, the closeness/ distance of candidate to FPIS and FNIS on each attribute are measured by our proposed similarity measure. Then, the overall closeness/ distance of the candidate is obtained.

Step 7. *Rank the alternatives.*Rank the overall closeness (distance) in descending (ascending) order.

Step 8. *Propose a compromise solution.*Propose a compromise solution according to some rules.

6 A numerical example

VIKOR proposed by Opricovic [36] is a classical compromise method for solving multi-attribute decision making problems, which has been widely extended into solving MAGDM problems in recent years. This method gives the best and compromise solutions with both maximum group utility and the minimum individual regret of opponent when applied to various decision issues. The new similarity of GTFNs is combined with fuzzy VIKOR to solve the MAGDM problem in this section.

Steps 1-5. are the same as in Section 5.2.

Steps 6. *Compute the group utility G_j, individual regret R_j and general VIKOR index Q_j.*

$$G_j = \sum_{k=1}^{K} \frac{1 - S_{jk}^+ \left(x_k^+, x_{jk}\right)}{1 - S_k \left(x_k^+, x_k^-\right)}, \tag{38}$$

$$R_j = \max_k \frac{1 - S_{jk}^+ \left(x_k^+, x_{jk}\right)}{1 - S_k \left(x_k^+, x_k^-\right)}, \tag{39}$$

$$Q_j = v \left(\frac{G_j - \min_j G_j}{\max_j G_j - \min_j G_j} \right) + (1 - v) \left(\frac{R_j - \min R_j}{\max_j R_j - \min_j R_j} \right), \tag{40}$$

where $S_{jk}^+ \left(x_k^+, x_{jk}\right)$ is the similarity between x_k^+ and x_{jk}, $S_k \left(x_k^+, x_k^-\right)$ is the similarity between x_k^+ and x_k^-, which can be calculate by Eqs. (36) and (37). $v(0 \le v \le 1)$ is the weight of the strategy of the maximum group utility. Usually, $v = 0.5$. The best candidate is determined with the minimum of general VIKOR index Q_j.

Steps 7. *Rank the alternatives.*

Rank the alternatives by sorting the values for G_j, R_j and Q_j in ascending order.

Steps 8. *Propose a compromise solution.*

If the following two conditions are satisfied, the alternative can be denoted as a promising solution:

$Cond_1$: Acceptable advantage is no less than the threshold:

$$Q\left(A^{(2)}\right) - Q\left(A^{(1)}\right) \ge \frac{1}{m-1}, \tag{41}$$

where $A^{(1)}$ and $A^{(2)}$ is the alternative ranked first and second by Q_j, respectively. m is the number of alternatives.

$Cond_2$: Acceptable Stability in decision making: Alternative $A^{(1)}$ must also be the best when ranked using G_j and/or R_j.

If one of the two conditions is not satisfied, then a set of compromise solution is

a) Alternatives $A^{(1)}$ and $A^{(2)}$ if only $Cond_1$ is satisfied;

b) Alternatives $A^{(1)}, A^{(2)}, \cdots, A^{(M)}$ if only $Cond_2$ is satisfied.

$A^{(M)}$ is determined from the relation $Q\left(A^{(M)}\right) - Q\left(A^{(1)}\right) < \frac{1}{m-1}$.

In order to compare different similarity measure methods when using VIKOR, it is better to use a MAGDM problem with true background, so we choose a numerical example from reference [51], in which a machine tool selection problem for the Pakistan Machine Tool Factory (PMTF) Private Limited has been solved by the early defuzzification, late defuzzification, and non-fuzzy VIKOR methods respectively. Linguistic variables in the form of triangular fuzzy numbers reflect evaluator preferences for the criteria importance weights and the performance ratings. In their application example, there were four evaluators $D_1, \cdot D_2, \cdot D_3, \cdot D_4$, five alternatives $A_1, \cdot A_2, \cdots, \cdot A_5$ and six criteria $C_1, \cdot C_2, \cdots, C_6$. DMs use the linguistic terms shown in Table 3 to evaluate the candidates with respect to each attribute as shown in Table 5 and to assess the importance of the criteria as shown in Table 6.

Table 5: The ratings of candidates given by evaluators under all criteria

Evaluators	Candidates	Criteria					
		C_1	C_2	C_3	C_4	C_5	C_6
D_1	A_1	F	VH	F	P	MP	MH
	A_2	H	VH	MP	CH	MH	H
	A_3	VH	MP	H	F	MP	H
	A_4	H	F	MH	MH	H	H
	A_5	VH	H	VH	MH	MH	H
D_2	A_1	H	F	MP	F	P	H
	A_2	H	H	H	MH	MH	VH
	A_3	VH	MP	MH	F	F	H
	A_4	H	MP	MH	H	MH	MH
	A_5	H	VH	H	VH	H	MH
D_3	A_1	MH	VH	MH	F	F	H
	A_2	VH	H	MP	H	MH	H
	A_3	VH	P	VH	MP	F	H
	A_4	VH	F	MH	MH	F	MH
	A_5	H	VH	VH	MH	F	H
D_4	A_1	MP	H	F	F	MP	H
	A_2	H	VH	F	H	H	MH
	A_3	VH	MP	MH	F	F	MH
	A_4	H	F	MH	H	MH	VH
	A_5	VH	H	VH	H	MH	H

Table 6: The importance weight of the criteria given by evaluators

	C_1	C_2	C_3	C_4	C_5	C_6
D_1	VH	H	M	VH	M	VH
D_2	VH	M	M	VH	H	VH
D_3	VH	H	VH	M	H	H
D_4	H	M	VH	H	H	H

Step 1-2: Using Eq. (31) and Table 3, we can transform the initial fuzzy decision matrix into the normalized fuzzy decision matrix as Table 7.

Table 7: Normalized fuzzy decision matrix

D_i	A_j	Criteria					
		C_1	C_2	C_3	C_4	C_5	C_6
D_1	A_1	(0.4,0.5,0.5,0.6;1)	(0.8,0.9,1.0,1.0;1)	(0.4,0.5,0.5,0.6;1)	(0.1,0.2,0.2,0.3;1)	(0.2,0.3,0.4,0.5;1)	(0.5,0.6,0.7,0.8;1)
	A_2	(0.7,0.8,0.8,0.9;1)	(0.8,0.9,1.0,1.0;1)	(0.2,0.3,0.4,0.5,1)	(0.8,0.9,1.0,1.0;1)	(0.5,0.6,0.7,0.8;1)	(0.7,0.8,0.8,0.9;1)
	A_3	(0.8,0.9,1.0,1.0;1)	(0.2,0.3,0.4,0.5,1)	(0.7,0.8,0.8,0.9;1)	(0.4,0.5,0.5,0.6;1)	(0.2,0.3,0.4,0.5;1)	(0.7,0.8,0.8,0.9;1)
	A_4	(0.7,0.8,0.8,0.9;1)	(0.4,0.5,0.5,0.6;1)	(0.5,0.6,0.7,0.8;1)	(0.5,0.6,0.7,0.8;1)	(0.7,0.8,0.8,0.9;1)	(0.7,0.8,0.8,0.9;1)
	A_5	(0.8,0.9,1.0,1.0;1)	(0.7,0.8,0.8,0.9;1)	(0.8,0.9,1.0,1.0;1)	(0.5,0.6,0.7,0.8;1)	(0.5,0.6,0.7,0.8;1)	(0.7,0.8,0.8,0.9;1)
D_2	A_1	(0.7,0.8,0.8,0.9;1)	(0.4,0.5,0.5,0.6;1)	(0.2,0.3,0.4,0.5,1)	(0.4,0.5,0.5,0.6;1)	(0.1,0.2,0.2,0.3;1)	(0.7,0.8,0.8,0.9;1)
	A_2	(0.7,0.8,0.8,0.9;1)	(0.7,0.8,0.8,0.9;1)	(0.7,0.8,0.8,0.9;1)	(0.5,0.6,0.7,0.8;1)	(0.5,0.6,0.7,0.8;1)	(0.8,0.9,1.0,1.0;1)
	A_3	(0.8,0.9,1.0,1.0;1)	(0.2,0.3,0.4,0.5,1)	(0.5,0.6,0.7,0.8;1)	(0.4,0.5,0.5,0.6;1)	(0.4,0.5,0.5,0.6;1)	(0.7,0.8,0.8,0.9;1)
	A_4	(0.7,0.8,0.8,0.9;1)	(0.2,0.3,0.4,0.5,1)	(0.5,0.6,0.7,0.8;1)	(0.7,0.8,0.8,0.9;1)	(0.5,0.6,0.7,0.8;1)	(0.5,0.6,0.7,0.8;1)
	A_5	(0.7,0.8,0.8,0.9;1)	(0.8,0.9,1.0,1.0;1)	(0.7,0.8,0.8,0.9;1)	(0.8,0.9,1.0,1.0;1)	(0.7,0.8,0.8,0.9;1)	(0.5,0.6,0.7,0.8;1)
D_3	A_1	(0.5,0.6,0.7,0.8;1)	(0.8,0.9,1.0,1.0;1)	(0.5,0.6,0.7,0.8;1)	(0.4,0.5,0.5,0.6;1)	(0.4,0.5,0.5,0.6;1)	(0.7,0.8,0.8,0.9;1)
	A_2	(0.8,0.9,1.0,1.0;1)	(0.7,0.8,0.8,0.9;1)	(0.2,0.3,0.4,0.5,1)	(0.7,0.8,0.8,0.9;1)	(0.5,0.6,0.7,0.8;1)	(0.7,0.8,0.8,0.9;1)
	A_3	(0.8,0.9,1.0,1.0;1)	(0.1,0.2,0.2,0.3;1)	(0.8,0.9,1.0,1.0;1)	(0.2,0.3,0.4,0.5,1)	(0.4,0.5,0.5,0.6;1)	(0.7,0.8,0.8,0.9;1)
	A_4	(0.8,0.9,1.0,1.0;1)	(0.4,0.5,0.5,0.6;1)	(0.5,0.6,0.7,0.8;1)	(0.5,0.6,0.7,0.8;1)	(0.4,0.5,0.5,0.6;1)	(0.5,0.6,0.7,0.8;1)
	A_5	(0.7,0.8,0.8,0.9;1)	(0.8,0.9,1.0,1.0;1)	(0.8,0.9,1.0,1.0;1)	(0.5,0.6,0.7,0.8;1)	(0.4,0.5,0.5,0.6;1)	(0.7,0.8,0.8,0.9;1)
D_4	A_1	(0.2,0.3,0.4,0.5,1)	(0.7,0.8,0.8,0.9;1)	(0.4,0.5,0.5,0.6;1)	(0.4,0.5,0.5,0.6;1)	(0.2,0.3,0.4,0.5;1)	(0.7,0.8,0.8,0.9;1)
	A_2	(0.7,0.8,0.8,0.9;1)	(0.8,0.9,1.0,1.0;1)	(0.4,0.5,0.5,0.6;1)	(0.7,0.8,0.8,0.9;1)	(0.7,0.8,0.8,0.9;1)	(0.5,0.6,0.7,0.8;1)
	A_3	(0.8,0.9,1.0,1.0;1)	(0.2,0.3,0.4,0.5,1)	(0.5,0.6,0.7,0.8;1)	(0.4,0.5,0.5,0.6;1)	(0.4,0.5,0.5,0.6;1)	(0.5,0.6,0.7,0.8;1)
	A_4	(0.7,0.8,0.8,0.9;1)	(0.4,0.5,0.5,0.6;1)	(0.5,0.6,0.7,0.8;1)	(0.7,0.8,0.8,0.9;1)	(0.5,0.6,0.7,0.8;1)	(0.8,0.9,1.0,1.0;1)
	A_5	(0.8,0.9,1.0,1.0;1)	(0.7,0.8,0.8,0.9;1)	(0.8,0.9,1.0,1.0;1)	(0.7,0.8,0.8,0.9;1)	(0.5,0.6,0.7,0.8;1)	(0.7,0.8,0.8,0.9;1)

Step 3: According to Eq. (32), the fuzzy weights of criteria are obtained as Table 8.

Table 8: The importance weight of the criteria given by evaluators

	C_1	C_2	C_3	C_4	C_5	C_6
D_1	(0.70,0.80,0.90,1.00;1)	(0.50,0.60,0.70,0.80;1)	(0.30,0.40,0.50,0.60;1)	(0.70,0.80,0.90,1.00;1)	(0.30,0.40,0.50,0.60;1)	(0.70,0.80,0.90,1.00;1)
D_2	(0.70,0.80,0.90,1.00;1)	(0.30,0.40,0.50,0.60;1)	(0.30,0.40,0.50,0.60;1)	(0.70,0.80,0.90,1.00;1)	(0.50,0.60,0.70,0.80;1)	(0.70,0.80,0.90,1.00;1)
D_3	(0.70,0.80,0.90,1.00;1)	(0.50,0.60,0.70,0.80;1)	(0.70,0.80,0.90,1.00;1)	(0.30,0.40,0.50,0.60;1)	(0.50,0.60,0.70,0.80;1)	(0.50,0.60,0.70,0.80;1)
D_4	(0.50,0.60,0.70,0.80;1)	(0.30,0.40,0.50,0.60;1)	(0.70,0.80,0.90,1.00;1)	(0.50,0.60,0.70,0.80;1)	(0.50,0.60,0.70,0.80;1)	(0.50,0.60,0.70,0.80;1)
ω_k	(0.50,0.75,0.85,1.00;1)	(0.30,0.50,0.60,0.80;1)	(0.30,0.60,0.70,1.00;1)	(0.30,0.65,0.75,1.00;1)	(0.30,0.55,0.65,0.80;1)	(0.50,0.70,0.80,1.00;1)

Step 4: Convert the normalized fuzzy decision matrix into the weighted normalized fuzzy decision matrix by Eq. (33) (as shown in Table 9).

Table 9: The importance weight of the criteria given by evaluators

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	(0.21,0.35,0.42,0.54;1)	(0.19,0.34,0.41,0.54;1)	(0.11,0.26,0.32,0.49;1)	(0.09,0.25,0.28,0.43;1)	(0.07,0.17,0.22,0.33;1)	(0.29,0.43,0.49,0.63;1)
A_2	(0.32,0.49,0.55,0.65;1)	(0.21,0.36,0.44,0.57;1)	(0.11,0.26,0.32,0.50;1)	(0.19,0.42,0.49,0.64;1)	(0.16,0.31,0.39,0.51;1)	(0.30,0.44,0.51,0.64;1)
A_3	(0.34,0.52,0.62,0.68;1)	(0.05,0.13,0.19,0.31;1)	(0.17,0.37,0.45,0.63;1)	(0.10,0.26,0.31,0.46;1)	(0.10,0.23,0.27,0.39;1)	(0.29,0.43,0.49,0.63;1)
A_4	(0.32,0.49,0.55,0.65;1)	(0.10,0.21,0.26,0.39;1)	(0.14,0.31,0.41,0.59;1)	(0.17,0.38,0.45,0.72;1)	(0.15,0.30,0.37,0.49;1)	(0.28,0.42,0.50,0.63;1)
A_5	(0.33,0.50,0.57,0.66;1)	(0.21,0.36,0.44,0.57;1)	(0.21,0.43,0.52,0.67;1)	(0.17,0.40,0.48,0.63;1)	(0.15,0.30,0.37,0.49;1)	(0.29,0.43,0.49,0.63;1)

Step 5: Using Eqs. (34)-(37), the FPIS A^+ and FNIS A^- can be determined as shown in Table 10.

Table 10: The importance weight of the criteria given by evaluators

	C_1	C_2	C_3	C_4	C_5	C_6
A^+	(0.68,0.68,0.68,0.68;1)	(0.57,0.57,0.57,0.57;1)	(0.67,0.67,0.67,0.67;1)	(0.64,0.64,0.64,0.64;1)	(0.51,0.51,0.51,0.51;1)	(0.64,0.64,0.64,0.64;1)
A^-	(0.21,0.21,0.21,0.21;1)	(0.05,0.05,0.05,0.05;1)	(0.11,0.11,0.11,0.11;1)	(0.09,0.09,0.09,0.09;1)	(0.07,0.07,0.07,0.07;1)	(0.28,0.28,0.28,0.28;1)

Step 6: The group utility G_j, individual regret R_j and general VIKOR index Q_j can be computed by Eqs. (38)-(40). To further verify the superiority of the proposed method, the decision results by using different similarity measures mentioned are compared in Section 4.2, as shown in Table 11. In reference [51], A_5 is the best option when using early and late defuzzification VIKOR method, while A_2 is the best option when considering a non-fuzzy information based decision. It implies that different defuzzification methods can affect the final results. In terms of results, both A_2 and A_5 are acceptable solutions.

Table 11: The values of G_j, R_j, Q_j and the ranking order by using different similarity measures.

Similarity measures		A_1	A_2	A_3	A_4	A_5	Ranking order from Q_j (CS)
Khorshidi and Nikfalazar [26]	G_j	6.501	6.683	6.558	6.622	6.720	$A_1 \succ A_4 \succ A_3 \succ A_2 \succ A_5$
	R_j	1.121	1.132	1.145	1.121	1.132	(A_1)
	Q_j	0.000	0.644	0.630	0.279	0.729	
Li and Zeng [29]	G_j	6.573	6.260	6.449	6.408	6.196	$A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
($\varepsilon = 0.01$)	R_j	1.169	1.157	1.169	1.198	1.169	(A_2, A_5)
	Q_j	0.640	0.084	0.476	0.781	0.140	
Li and Zeng [29]($\varepsilon = 0.09$)	G_j	6.549	6.231	6.424	6.381	6.167	$A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
($\varepsilon = 0.01$)	R_j	1.163	1.151	1.163	1.192	1.163	(A_2, A_5)
	Q_j	0.643	0.085	0.479	0.780	0.143	
Xie, et al. [53]	G_j	15.335	14.973	15.236	15.104	14.910	$A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
	R_j	3.101	3.094	3.101	3.116	3.101	(A_2, A_5)
	Q_j	0.672	0.073	0.554	0.727	0.172	
Proposed	G_j	4.856	4.327	4.694	4.515	4.204	$A_2 \succ A_5 \succ A_3 \succ A_1 \succ A_4$
	R_j	0.909	0.894	0.909	0.940	0.909	(A_2, A_5)
	Q_j	0.657	0.095	0.533	0.739	0.157	

According to the results in Table 11, some conclusions can be drawn:

(1) It is obviously that the acceptable alternatives A_2 and A_5 selected by Li and Zeng [29], Xie, et al. [53] and our proposed similarity measure are coincided with the reference [51].

(2) The decision result calculated by the similarity measure of Khorshidi and Nikfalazar [26] is completely different from reference [51] and other methods. This is because this similarity measure is poor in distinguishability, which can be illustrated in Table 2.

(3) Although no matter the value of ε is equal to 0.01 or 0.1, the Q_2 is ranked first by using the similarity measure of Li and Zeng [29]. However, different values of ε lead to different values of G_j, R_j and Q_j, which is unable to promise the robust decision-making results in other numerical examples. With the increase of ε, $sp = \frac{\min(P(A),P(B))+\varepsilon}{\max(P(A),P(B))+\varepsilon}$ becomes less sensitive to the similarity between $P(A)$ and $P(B)$. This makes the overall distinguishability of similarity measure less favorable, which also can be verified in Table 2.

(4) Although the ranking orders computed by the similarity measures of Li and Zeng [29] and Xie, et al. [53] are the same with reference [51] and our proposed method, the values of G_j are larger than 5. It implies that distances between some alternatives and FPIS must be larger than the distance between FNIS and FPIS. However, this phenomenon is unreasonable.

In a word, combining the results of comparative experiment in Section 4.2 and numerical example in this section, it can be seen that our proposed method has advantages in performance and application.

7 Conlusions and future works

In this paper, a novel similarity measure of GTFNs in order to make up for the deficiencies of existing measure methods has been proposed. Then, the superiorities of the proposed method have been proved by the comparison of numerical examples. The results shown that the novel similarity measure has better performance in distinguishability and validity. To solve the MAGDM problem, we have analyzed why and how the novel similarity measure of GTFNs can be used within compromise methods, such as VIKOR. Finally, we have given a numerical example from existing paper to demonstrate the application of the integrated approach proposed by us. Particularly, our paper has two main contributions:

(1) Based on the concept of cosine similarity, the similarity measure between COGs has been improved. Then, it has been combined with the concepts of perimeter, area and geometric distance, which represent the size or shape information of GTFNs. Finally, a novel and more effective similarity measure of GTFNs has been put forward. This similarity measure has higher distinguishability and lower invalidity.

(2) A general approach combined proposed new similarity measure of GTFNs with compromise methods has been developed to deal with MAGDM problem. The comparison results have shown the validity, and the superiority of the novel similarity measure has been proved again.

In the future, the proposed similarity measure of GTFNs is able to be combined with different methods. For example, substituting corresponding steps of Superiority and Inferiority Ranking (SIR) method for the Steps 6-8 of Section 5.2, or replacing the difference between the evaluations of Preference Ranking Organization Method for Enrichment Evaluations (PROMETHEE) with our proposed similarity, and so on. In addition, it is worthy to apply the proposed similarity measure of GTFNs in other problems. For instance, it can be applied to fuzzy risk analysis by measuring the similarity degree between the obtained risk and the given linguistics terms, or to pattern recognition by selecting the maximum degree of similarity between fuzzy numbers to decide the sample belongs to which pattern.

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