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Aggregation of sequence of fuzzy measures

L. Nedović[1] and E. Pap[2]

[1] *Department of Fundamentals Sciences, Faculty of Technical Sciences, University of Novi Sad, Novi Sad, Serbia*
[2] *Department for Postgraduate Studies, Singidunum University, Belgrade, Serbia*

nljubo@uns.ac.rs, epap@singidunum.ac.rs

Abstract

In this paper we present construction of new fuzzy measures by applying extended aggregation function on a sequence of fuzzy measures. According to properties of applied aggregation function and properties of initial fuzzy measures, some properties of constructed fuzzy measure are proved. Additionally, one new extended aggregation function named extended weighted arithmetic mean of distorted arguments is introduced, and its relevant properties are proved. It is shown that this extended aggregation function, with appropriated parameters, can be suitable for construction of new fuzzy measures. Other types of non-additive measures can be constructed in the same way, by applying aggregation function on the initial sequence of non-additive measures.

Keywords: Fuzzy measure, non-additive measure, aggregation function.

1 Introduction

Construction of fuzzy measures using aggregation function is in literature implemented for some particular types of non-additive measures such as *capacities*, *fuzzy measures*, *possibility measures* and *decomposable non-additive measures*, and for some particular types of aggregation functions, such as Choquet integral, OWA aggregation function, t-norms and t-conorms, see [2, 3, 11, 18, 21, 23, 25, 28, 30, 40, 41, 43, 48]. The most frequent topic is aggregation of possibility, necessity and other similar types of fuzzy measures, see [1, 13, 47]. In the majority of these approaches, the underlaying space of mentioned types of fuzzy measures is finite, and applied aggregation function take arguments and values in bounded interval $[0, 1]$.

Aggregation of fuzzy measures is generalized for various types of non-additive measures and other measure functions in the widest sense, defined on arbitrary space equipped with σ-algebra structure, with wide classes and types of aggregation functions. In this paper we present a generalized construction of a new fuzzy measure by applying aggregation function on initial sequence of fuzzy measures. In addition, we present one new aggregation function suitable for the mentioned construction.

Applying aggregation function on a sequence of classical measures we obtain a new measure function only in the case when applied aggregation function is a linear transformation, i.e., when we apply *weighted arithmetic mean* aggregation function. On the other hand, when we apply aggregation function on a sequence of non-additive measures, new non-additive measure of certain type can be obtained, see [4, 6, 9, 16, 26, 48]. There are many definitions and types of *fuzzy measures* and non-additive measures generally, see [7, 12, 15, 19, 28, 31, 34, 35, 36, 39, 44, 45]. Many types of fuzzy integrals can be defined with respect to some fuzzy measure, see [8, 10, 20, 23, 27, 29, 33, 34, 37, 38, 41, 42, 43].

In this paper, we investigate the application of the most common type of aggregation function on the sequence of fuzzy measures of most general type.

In Section 2 we present the definition and relevant properties of aggregation functions, as well as definition and some important possible properties of fuzzy measures. In Section 3 we refer to some known aggregation functions suitable for application in fuzzy measure construction from Section 4. The properties relevant for fuzzy measure construction are presented. In Section 4 we present a new method for the construction of a new fuzzy measure using extended

Corresponding Author: L. Nedović
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aggregation function and sequence of given fuzzy measures. Depending of the properties of initial fuzzy measures and the properties of applied aggregation function, some possible properties of the constructed fuzzy measure are proved. In Section 5 we introduce one new extended aggregation function which is suitable for new fuzzy measure construction. Relevant properties of the introduced extended aggregation function are examined. The proposed extended aggregation function is a generalization of *extended weighted arithmetic mean of powers* function which is introduced in [9], where one application in image segmentation is also presented. Section 6 provides a short conclusion of our results as well as some ideas and guidelines for further investigations.

2 Preliminaries

In this section we present some basic notions related to aggregation functions and fuzzy measures, which are relevant for the results obtained in this article.

Extended aggregation functions represent a wide class of fuzzy operations, see [4, 5, 6, 16, 17, 24, 26], and they have a significant role in the application in many engineering sciences, especially in informatics and other similar areas, see [9, 14, 16, 17, 24, 32, 46]. There are numerous classes of aggregation operators, see [4, 6, 17, 22], where certain types of aggregation functions can represent an appropriated model of information fusion, suitable for some specified application. In literature we can find various definitions of aggregation operators. In this paper we refer to definition with minimal axioms, see for example [5, 16, 24]. In addition, we refer to some more important properties which aggregation operators can have, which are significant for the construction of fuzzy measures presented in Section 4. We start with a basic notion of aggregation of fixed $n \in \mathbb{N}$ input values, following up with a notion of aggregation of various number of input values, see [5, 16]. Input and output values of aggregation operator are real numbers from a finite or infinite interval $I \subseteq \overline{\mathbb{R}}$. Which interval we will use, depends on the nature and meaning of arguments and resulting value, i.e., the choice of interval depends on the subject of application of aggregation operator. Regarding to properties of fuzzy measures considered in Section 4 and possible similar investigations, we will use one of the following intervals I.

[I1] Bounded interval $I_1 = [0, 1]$ (arbitrary bounded interval $[a, b] \subseteq \mathbb{R}$ can be used in some other similar considerations).

[I2] Infinite interval $I_2 = [0, \infty[$.

[I3] Infinite interval $I_3 = [0, \infty]$, where by convention we adopt $\forall a \in [0, \infty]$, $a + \infty = \infty$, $0 \cdot \infty = 0$ and $\forall a \in]0, \infty]$, $a \cdot \infty = \infty$.

Definition 2.1 (Aggregation function)**.** *For each $n \in \mathbb{N}$, an n-*ary aggregation function *is a function $A_{[n]} : I^n \to I$ with the following properties.*

(a01) Boundary conditions *holds, which particulary means that*

$$A_{[n]}(0, \dots, 0) = 0, \tag{1}$$

and, depending on the observed cases for interval I,

[I1] *for $I = I_1 = [0, 1]$,*

$$A_{[n]}(1, \dots, 1) = 1, \tag{2}$$

[I2] *for $I = I_2 = [0, \infty[$,*

$$\lim_{\forall i \in \{1,\dots,n\},\ a_i \to \infty} A_{[n]}(a_1, \dots, a_n) = \infty, \tag{3}$$

[I3] *for $I = I_3 = [0, \infty]$,*

$$A_{[n]}(\infty, \dots, \infty) = \infty. \tag{4}$$

(a02) *A function A is monotonically non-decreasing in each component, i.e., implication*

$$\forall i \in \{1, \dots, n\},\ a_i \le b_i \quad \Rightarrow \quad A_{[n]}(a_1, \dots, a_n) \le A_{[n]}(b_1, \dots, b_n), \tag{5}$$

hold for all $(a_1, \dots, a_n), (b_1, \dots, b_n) \in I^n$.

For $n = 1$, by definition is $A_{[1]}(x) = x$, $x \in I$.

Definition 2.2 (Extended aggregation function)**.** *An* extended aggregation function *is a function* $A : \bigcup_{n=1}^{\infty} I^n \to I$ *such that its restriction* $A_{[n]} : I^n \to I$ *is an* n*-ary aggregation function for every* $n \in \mathbb{N}$.

Further, we list some additional properties which extended aggregation functions can have, see [4, 16], and which are of interest in the various applications and research presented in this paper.

Definition 2.3. *Extended aggregation function* $A : \bigcup_{n=1}^{\infty} I^n \to I$ *can have the following properties.*

(a03) *Function* A *is* continuous, *i.e., every* $A_{[n]} : I^n \to I$, $n \in \mathbb{N}$ *is a continuous aggregation function.*

(a04) *Function* A *is* symmetric in each component, *i.e., for every* $n \in \mathbb{N}$, *each* n*-tuple* $(a_1, \dots, a_n) \in I^n$ *and each permutation* p *of the set* $\{1, \dots, n\}$ *hold* $A_{[n]}(a_1, \dots, a_n) = A_{[n]}(a_{p(1)}, \dots, a_{p(n)})$.

(a05) *Function* A *is* idempotent, *i.e., for every* $n \in \mathbb{N}$ *and each* $(a, \dots, a) \in I^n$ *holds* $A_{[n]}(a, \dots, a) = a$.

(a06) *Function* A *is* additive, *i.e., for every* $n \in \mathbb{N}$ *and all* $(a_1, \dots, a_n) \in I^n$, $(b_1, \dots, b_n) \in I^n$ *that satisfy* $(a_1 + b_1, \dots, a_n + b_n) \in I^n$ *hold* $A_{[n]}(a_1 + b_1, \dots, a_n + b_n) = A_{[n]}(a_1, \dots, a_n) + A_{[n]}(b_1, \dots, b_n)$.

(a07) *Function* A *is* subadditive, *i.e., for every* $n \in \mathbb{N}$ *and all* $(a_1, \dots, a_n) \in I^n$, $(b_1, \dots, b_n) \in I^n$ *that satisfy condition* $(a_1 + b_1, \dots, a_n + b_n) \in I^n$ *holds* $A_{[n]}(a_1 + b_1, \dots, a_n + b_n) \leq A_{[n]}(a_1, \dots, a_n) + A_{[n]}(b_1, \dots, b_n)$.

(a08) *Function* A *is* superadditive, *i.e., for every* $n \in \mathbb{N}$ *and all* $(a_1, \dots, a_n) \in I^n$, $(b_1, \dots, b_n) \in I^n$ *that satisfy condition* $(a_1 + b_1, \dots, a_n + b_n) \in I^n$ *holds* $A_{[n]}(a_1 + b_1, \dots, a_n + b_n) \geq A_{[n]}(a_1, \dots, a_n) + A_{[n]}(b_1, \dots, b_n)$.

(a09) *For every* $n \in \mathbb{N}$, *depending on interval* I, *holds one of following implications.*

- [I1] *For* $I = I_1 = [0, 1]$, $A_{[n]}(a_1, \dots, a_n) < 1 \;\Rightarrow\; \forall i \in \{1, \dots, n\},\ a_i < 1.$
- [I2] *For* $I = I_2 = [0, \infty[$ *and* $(b_1, \dots, b_n) \in [0, \infty]^n$,

$$\lim_{\forall i \in \{1,\dots,n\},\ a_i \to b_i} A_{[n]}(a_1, \dots, a_n) < \infty \;\Rightarrow\; \forall i \in \{1, \dots, n\},\ b_i < \infty.$$

- [I3] *For* $I = I_3 = [0, \infty]$, $A_{[n]}(a_1, \dots, a_n) < \infty \;\Rightarrow\; \forall i \in \{1, \dots, n\},\ a_i < \infty.$

Remark 2.4. (i) *Aggregation function* $A_{[1]}(x)$, $x \in I$ *A obviously satisfies all the properties from Definition* 2.3. *Beside this, function* $A_{[1]}$ *is not of interest in our investigations. Because of that, in considerations and examinations of the properties of extended aggregation functions we will consider restrictions* $A_{[n]}$ *for* $n \geq 2$.

(ii) *Assumption* $(a_1 + b_1, \dots, a_n + b_n) \in I^n$ *in properties* (a06), (a07) *and* (a08) *are necessary only when for* I *we use bounded interval* $I_1 = [0, 1]$. *Instead of demand specified conditions, it can be convenient to consider the extension* $A^* : \bigcup_{n=1}^{\infty} [0, \infty[^n \to [0, 1]$ *of function* A *defined as* $A^*_{[n]}(a_1, \dots, a_n) = \begin{cases} A_{[n]}(a_1, \dots, a_n) & , \quad (a_1, \dots, a_n) \in [0,1]^n \\ 1 & , \quad (a_1, \dots, a_n) \notin [0,1]^n \end{cases}$.

One type of non-additive measures suitable for aggregation presented in Section 4 is a fuzzy measure, see [24, 35, 44, 45]. Fuzzy measures are set functions defined on σ-algebra.

Definition 2.5. *Family of sets* $\mathcal{A} \subseteq \mathcal{P}(X)$ *is* σ-algebra *on the* $X \neq \emptyset$ *if*

(i) $X \in \mathcal{A}$,

(ii) $\forall A, B \in \mathcal{A},\ \ A \setminus B \in \mathcal{A}$,

(iii) $\forall A_i \in \mathcal{A}, i \in \mathbb{N},\ \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

Let $\mathcal{A}$ be a σ-algebra on X. Let I be one of the following intervals: $[0, 1]$, $[0, \infty[$ or $[0, \infty]$.

Definition 2.6. *Function* $m : \mathcal{A} \to I$ *is a* fuzzy measure *on* σ*-algebra* $\mathcal{A}$ *if*

(FM1) $m(\emptyset) = 0$,

(FM2) $\forall A, B \in \mathcal{A}, \; A \subseteq B \; \Rightarrow \; m(A) \le m(B)$.

In the next definition we give some additional properties of fuzzy measures which will be used in this paper, see [35, 44, 45].

Definition 2.7. *Fuzzy measure $m : \mathcal{A} \to I$ can have the following properties.*

(FM3) *For every family of sets $A_i \in \mathcal{A}$, $i \in \mathbb{N}$ nested as $A_1 \subseteq A_2 \subseteq \ldots$ holds*

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} m(A_i) \qquad (m \text{ is continuous from below}). \tag{6}$$

(FM4) *For every family of sets $A_i \in \mathcal{A}$, $i \in \mathbb{N}$ nested as $A_1 \supseteq A_2 \supseteq \ldots$ and satisfying that there exists $n_0 \in \mathbb{N}$ such that $m(A_{n_0}) < \infty$ holds*

$$m\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} m(A_i) \qquad (m \text{ is continuous from above}). \tag{7}$$

(FM5) *For each disjoint pair of sets $A, B \in \mathcal{A}$ holds,*

$$m(A \cup B) \le m(A) + m(B) \qquad (m \text{ is subadditive}). \tag{8}$$

(FM6) *For each disjoint pair of sets $A, B \in \mathcal{A}$ holds*

$$m(A \cup B) \ge m(A) + m(B) \qquad (m \text{ is superadditive}). \tag{9}$$

3 Properties of some aggregation functions

In Examples 3.1 and 3.2, some well known extended aggregation functions are referred and examined, see [4, 6, 9, 16, 17, 32]. As we will see, min and max, also as aggregation function *extended weighted arithmetic mean* from Example 3.2 can be suitable for fuzzy measures aggregation presented in Section 4. In Example 3.5 we present an extended aggregation function called *extended weighted arithmetic mean of powers* which is introduced in [9], and also can be appropriated for the mentioned construction of new fuzzy measures. In [9, 32], properties (a03), (a04), (a05), (a06) and (a07) of all mentioned aggregation functions are analyzed on interval $I_1 = [0, 1]$. In this section, for this functions we analyze properties (a08) and (a09) from Definition 2.3. All the properties from Definition 2.3 of these functions will be considered also on intervals $I_2 = [0, \infty[$ and $I_3 = [0, \infty]$.

Example 3.1 (min - max)**.** *For each $n \in \mathbb{N}$, functions*
$A_{\min} : I^n \to I, \quad A_{\min}(a_1, \ldots, a_n) = \min(a_1, \ldots, a_n)$, *and* $A_{\max} : I^n \to I, \quad A_{\max}(a_1, \ldots, a_n) = \max(a_1, \ldots, a_n)$,
are continuous (a03)*, symmetrical* (a04) *and idempotent* (a05) *aggregation functions, see* [16, 24, 32]*. Properties* (a06) *and* (a07) *are considered in* [9, 32] *on interval $I_1 = [0, 1]$. The same conclusions for the mentioned properties also holds on intervals $I_2 = [0, \infty[$ and $I_3 = [0, \infty]$.*

In [32]*, subadditivity* (a07) *of* max *aggregation function is proved for $I_1 = [0, 1]$, but the same proof obviously holds for intervals $I_2 = [0, \infty[$ and $I_3 = [0, \infty]$ also. Because the same proof of subadditivity of* max *operator also holds on intervals $[-1, 0]$, $]-\infty, 0]$ and $[-\infty, 0]$, superadditivity* (a08) *of* min *operator on all the considered intervals follows:*

$$\begin{aligned}
\min(a_1 + b_1, \ldots, a_n + b_n) &= -\max(-a_1 - b_1, \ldots, -a_n - b_n) \\
&\ge -\left(\max(-a_1, \ldots, -a_n) + \max(-b_1, \ldots, -b_n)\right) \\
&= -\max(-a_1, \ldots, -a_n) - \max(-b_1, \ldots, -b_n) \\
&= \min(a_1, \ldots, a_n) + \min(b_1, \ldots, b_n),
\end{aligned}$$

for all $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in I^n$ (with additional assumption $(a_1 + b_1, \ldots, a_n + b_n) \in I^n$ on interval $I = [0, 1]$). On the other hand, operator max *is not superadditive and operator* min *is not subadditive because, e.g., for $(a_1, a_2) = (0.1, 0.2)$ and $(b_1, b_2) = (0.5, 0.4)$ we obtain*

$$\max(a_1 + b_1, a_2 + b_2) < \max(a_1, a_2) + \max(b_1, b_2),$$
$$\min(a_1 + b_1, a_2 + b_2) > \min(a_1, a_2) + \min(b_1, b_2).$$

Obviously, operator $A_{\max}$ hase property (a09) *for all the considered intervals, and operator $A_{\min}$ does not satisfy* (a09) *on any of them. The considered properties of $A_{\min}$ and $A_{\max}$ are summarized in Table* 1.

With aggregation functions $A_{\min} : I^n \to I$, $n \in \mathbb{N}$ and $A_{\max} : I^n \to I$, $n \in \mathbb{N}$, the corresponding extended aggregation functions are determined.

Table 1: Properties of min and max operators.

	$A_{\min}$	$A_{\max}$
(a03):	YES	YES
(a04):	YES	YES
(a05):	YES	YES
(a06):	no	no
(a07):	no	YES
(a08):	YES	no
(a09):	no	YES

Example 3.2 (WAM)**.** *For arbitrary family*

$$\omega = \left\{ \omega_{n,i} \geq 0 \,\middle|\, n \in \mathbb{N},\ i \in \{1, ..., n\}\, ,\ \sum_{i=1}^{n} \omega_{n,i} = 1 \right\},$$

of real numbers, arbitrary $n \in \mathbb{N}$ and $(a_1, \ldots, a_n) \in I^n$, with the following equation a continuous and idempotent, but not symmetrical extended aggregation function $\mathsf{WAM}^\omega : \bigcup_{n=1}^{\infty} I^n \to I$ is defined,

$$\mathsf{WAM}^\omega_{[n]}(a_1, \ldots, a_n) = \omega_{n,1} a_1 + \cdots + \omega_{n,n} a_n, \tag{10}$$

see [16, 17]*. Operator* WAM^ω *is called* extended weighted arithmetic mean*. For each $n \in \mathbb{N}$, restriction $\mathsf{WAM}^\omega_{[n]} : I^n \to I$ of operator WAM^ω on the set I^n is a linear transformation from I^n to I. This means that extended aggregation function WAM^ω is additive and also subadditive and superadditive. WAM^ω operator generally does not satisfy property* (a09)*. For $n \in \mathbb{N}$, aggregation function $\mathsf{WAM}^\omega_{[n]}$ satisfies* (a09) *only for intervals $I_2 = [0, \infty[$ and $I_3 = [0, \infty]$ and for $\forall i \in \{1, \ldots, n\}$, $\omega_{n,i} > 0$. Namely, in those cases,*

[I2] *for $I = [0, \infty[$ and $(b_1, \ldots, b_n) \in [0, \infty]^n$, equality*

$$\lim_{\forall i \in \{1,\ldots,n\},\ a_i \to b_i} \mathsf{WAM}^\omega_{[n]}(a_1, \ldots, a_n) = \lim_{\forall i \in \{1,\ldots,n\},\ a_i \to b_i} (\omega_{n,1} a_1 + \cdots + \omega_{n,n} a_n) = \infty,$$

can be satisfied only in the case when exists $i \in \{1, \ldots, n\}$ such that $b_i = \infty$,

[I3] *for $I = [0, \infty]$, $\mathsf{WAM}^\omega_{[n]}(a_1, \ldots, a_n) = \omega_{n,1} a_1 + \cdots + \omega_{n,n} a_n = \infty$, can be only when $a_i = \infty$ for some $i \in \{1, \ldots, n\}$.*

All the considered properties are summarized in Table 2.

Table 2: Properties of WAM^ω function.

	WAM^ω
(a03):	YES
(a04):	no
(a05):	YES
(a06):	YES
(a07):	YES
(a08):	YES
(a09):	no

Remark 3.3. *It is known that additivity* (a06) *holds only for extended aggregation function* $A : \bigcup_{n=1}^{\infty} I^n \to I$ *of* extended weighted arithmetic mean *type (see Example* 3.2*), i.e., when every restriction* $A_{[n]} : I^n \to I$*,* $n \in \mathbb{N}$ *is a linear transformation on* I^n*. Namely, homogenity*

$$A_{[n]}(ta_1, \ldots, ta_n) = tA_{[n]}(a_1, \ldots, a_n), \quad t \geq 0, \quad (a_1, \ldots, a_n) \in I^n,$$

(with additional assumption $(ta_1, \ldots, ta_n) \in I^n$ *in the case* $I = [0, 1]$*) of aggregation function* $A_{[n]}$ *follows from their monotonicity* (a02) *and additivity* (a06)*.*

Remark 3.4. *Assumption* $\forall n \in \mathbb{N}$, $\sum_{i=1}^{n} \omega_{n,i} = 1$ *for coefficients* ω *is necessary only when interval* $I = [0, 1]$ *is used, because of boundary condition* (2) *in Definition* 2.1*. In cases* $I = [0, \infty[$ *and* $I = [0, \infty]$*, the mentioned assumption can be replaced with* $\forall n \in \mathbb{N}$, $\exists i \in \{1, \ldots, n\}$, $\omega_{n,i} > 0$*. In that case, operator is generally not idempotent. The same remark holds for coefficients* ω *in the definition of* WAMP *operator in Example* 3.5*.*

In [9], on interval $I = [0, 1]$ one extended aggregation function $\mathsf{WAMP}^{\omega,\lambda}$ called *extended weighted arithmetic mean of powers* is introduced, which is suitable for construction of fuzzy measures presented in Section 4. It is shown in [9] that $\mathsf{WAMP}^{\omega,\lambda}$ operator is also suitable for the construction of distance functions and their application in image segmentation.

Example 3.5 (WAMP)**.** *For arbitrary families of coefficients*

$$\omega = \left\{ \omega_{n,i} \geq 0 \mid n \in \mathbb{N},\ i \in \{1, \ldots, n\},\ \sum_{i=1}^{n} \omega_{n,i} = 1 \right\}, \qquad \lambda = \{\lambda_{n,i} > 0 \mid n \in \mathbb{N},\ i \in \{1, \ldots, n\}\},$$

function $\mathsf{WAMP}^{\omega,\lambda} : \bigcup_{n=1}^{\infty} I^n \to I$ *defined with*

$$\mathsf{WAMP}_{[n]}^{\omega,\lambda}(a_1, \ldots, a_n) = \sum_{i=1}^{n} \omega_{n,i} a_i^{\lambda_{n,i}}, \tag{11}$$

for all $n \in \mathbb{N}$ *and* $(a_1, \ldots, a_n) \in I^n$ *is one extended aggregation function, called* extended weighted arithmetic mean of powers*. It is easy to see that boundary conditions* (a01) *and monotonicity* (a02) *are satisfied in all cases of interval* $I \in \{I_1, I_2, I_3\}$*.*

Properties (a03)…(a07) *of* $\mathsf{WAMP}^{\omega,\lambda}$ *operator on interval* $I = [0, 1]$ *are examined in* [9]*, and it is easy to see that the same conclusions also are valid on intervals* $I = [0, \infty[$ *and* $I = [0, \infty]$*. Operator* $\mathsf{WAMP}^{\omega,\lambda}$ *does not have the property* (a08) *for the same reasons it does not have the property* (a07)*. In general case, operator* $\mathsf{WAMP}^{\omega,\lambda}$ *does not satisfy property* (a09) *for the same reason as operator* WAM^{ω}*. All the considered properties are summarized in Table* 3*.*

Table 3: Properties of $\mathsf{WAMP}^{\omega,\lambda}$ function.

	$\mathsf{WAMP}^{\omega,\lambda}$
(a03):	YES
(a04):	no
(a05):	no
(a06):	no
(a07):	no
(a08):	no
(a09):	no

Proposition 3.6. *For certain specific families of coefficients* ω *and* λ*,* $\mathsf{WAMP}^{\omega,\lambda}$ *operator satisfies some of the properties* (a04)…(a09)*, see* [9]*.*

(i) *Property* (a04) *holds (only) in case* $\omega_{n,i} = \frac{1}{n}$ *and* $\lambda_{n,i} \equiv \lambda_n > 0$ *for all* $n \in \mathbb{N}$ *and* $i \in \{1, \ldots, n\}$.

(ii) *Properties* (a05), (a06), (a07) *and* (a08) *hold for* $\lambda = \{\lambda_{n,i} = 1 \mid n \in \mathbb{N},\ i \in \{1, \ldots, n\}\}$. *In this case actually* $\mathsf{WAMP}^{\omega,\lambda} = \mathsf{WAM}^{\omega}$.

(iii) *Subadditivity* (a07) *holds for* $\lambda = \{\lambda_{n,i} \leq 1 \mid n \in \mathbb{N},\ i \in \{1, \ldots, n\}\}$, *i.e., with concave power functions, see Section* 5.

(iv) *Superadditivity* (a08) *holds for* $\lambda = \{\lambda_{n,i} \geq 1 \mid n \in \mathbb{N},\ i \in \{1, \ldots, n\}\}$, *i.e., with convex power functions, see Section* 5.

(v) *For particular* $n \in \mathbb{N}$ *and aggregation function* $\mathsf{WAMP}^{\omega,\lambda}_{[n]}$, *property* (a08) *holds in the case* $\forall i \in \{1, \ldots, n\}, \omega_{n,i} > 0$, *for the same reasons as for* $\mathsf{WAM}^{\omega}_{[n]}$ *aggregation function.*

(vi) *For each* $n \in \mathbb{N}$, *aggregation function* $\mathsf{WAMP}^{\omega,\lambda}_{[n]}$ *satisfies* (a09) *only for intervals* $I_2 = [0, \infty[$ *and* $I_3 = [0, \infty]$ *and for coefficients* $\forall i \in \{1, \ldots, n\}$, $\omega_{n,i} > 0$, *for the same reasons as for* WAM^{ω} *function.*

In Section 5 we introduce one new extended aggregation function which is a generalization of $\mathsf{WAMP}^{\omega,\lambda}$ operator.

4 Aggregation of non-additive measures

4.1 Previous constructions

Let X be an arbitrary nonempty set, let $I \subset \overline{\mathbb{R}}$, and let $f_i : X \to I$, $i \in \mathbb{N}$ be a sequence of functions which we can interpret as some kind of measure on X in the widest sense. Applying an aggregation function A on functions f_i, $i \in \mathbb{N}$ a new function $f : X \to I$ is constructed, which under some additional assumptions on aggregation function A, is also a measure function of the considered type on the set X. In [9, 32] is presented a construction of bounded distance function $f : X \to [0, 1]$ by applying aggregation function on the sequence of bounded distance functions $f_i : X \to [0, 1]$, $i \in \mathbb{N}$. The properties of the constructed distance function f depend on the properties of the initial distance functions f_i, $i \in \mathbb{N}$, as well as on the properties of the applied aggregation function A. In [9, 32], one application of distance function constructed in this way is presented, where the constructed distance functions are used for image segmentation.

In the literature, construction of new fuzzy measure by applying aggregation function on some given initial fuzzy measures is studied for particular types of initial fuzzy measures, and for particular types of aggregation functions. The most general approach is presented in [47].

In [13], the authors construct a new possibility measure and belief function applying consensus functions, convex combinations, and weighted maximum and minimum aggregation functions. Also, in the same paper, the construction of a new decomposable measure by applying aggregation function generated by t-conorm on the initial decomposable measures is presented.

In [11], aggregation of decomposable measures and bounded possibility and necessity fuzzy measures are presented. In this paper, for fuzzy measures of this type, as appropriated aggregation functions authors consider weighted arithmetic mean, minimum and maximum operators and consensus functions. The authors also present the application of the constructed fuzzy measures in utility theory.

In [2], the author discuss closeness of families of probability and belief measures when some types of aggregation function are applied on them. The considered underlying space is mostly finite. As significant aggregation function for getting closed families of mentioned fuzzy measures, the author study the applications of weighted arithmetic mean, product aggregation function, and the appropriated mixture of them.

Paper [3] is a continuation of the research implemented in [2]. In [3], the considered fuzzy measures, mostly on finite underlying set, are coherent lower probabilities, k-monotone fuzzy measures, belief functions and distorted probabilities. In this research, applied aggregation functions are weighted arithmetic mean, minimum and product-type functions on bounded interval $[0, 1]$. In the same article, the author investigate the construction of aggregation function using sequence of fuzzy measures of the mentioned types.

Paper [47] is devoted to aggregation of bounded fuzzy measures, especially for possibility measures, additive measures, cardinality based measures on finite underlying set, Sugeno measures, quasi-additive measures, and fuzzy measures generated by OWA (*ordered weighted averaging*) aggregation operator. For the aggregation of mentioned and other types of fuzzy measures, the authors consider the following aggregation operators: WAM (*weighted arithmetic mean*), OWA, max, min, *product*, Choquet and Sugeno integral, and *median* operator. In addiction to this special type of fuzzy measures and aggregation functions, in [47] the authors introduce a general construction of new bounded fuzzy measure by applying an arbitrary aggregation function on a sequence of bounded fuzzy measures. In our paper, we expand this method on fuzzy measures with values in unbounded interval. In the same paper, the authors also investigate the construction of new aggregation function using one fuzzy measure as a basic criterion in multi-criteria decision-making process.

4.2 Aggregation of sequence of fuzzy measures

In this subsection, we introduce a construction of a new fuzzy measure by applying arbitrary aggregation function in the sequence of fuzzy measures of any type and with values in bounded or infinite interval.

Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an σ-algebra on $X \neq \emptyset$.

Definition 4.1. *Let $m_i : \mathcal{A} \to I$, $i \in \mathbb{N}$ be a sequence of fuzzy measures on $\mathcal{A}$, where I is one of the interval $I_1 = [0,1]$, $I_2 = [0,\infty[$ or $I_3 = [0,\infty]$. Let $A : \bigcup_{n=1}^{\infty} I^n \to I$ be an arbitrary extended aggregation function. For $n \in \mathbb{N}$, let function $m_{[n]} : \mathcal{A} \to I$ be defined by*

$$m_{[n]}(F) = A_{[n]}(m_1(F), \ldots, m_n(F)), \quad F \in \mathcal{A}. \tag{12}$$

Function $m_{[n]}$ is a set function on $\mathcal{A}$ with values in I as well as functions m_i, $i \in \mathbb{N}$. As we will see, for each $n \in \mathbb{N}$, function $m_{[n]}$ is a fuzzy measure on $\mathcal{A}$, and depending on appropriated properties of aggregation function $A_{[n]}$, fuzzy measure $m_{[n]}$ can inherit some important properties from the initial fuzzy measures m_i, $i \in \mathbb{N}$.

Remark 4.2. (i) *In* [47] *Yager and Alajlan have defined new set function $m_{[n]}$ in the same way as in* (12) *and have proved that $m_{[n]}$ is a fuzzy measure, but only for the bounded fuzzy measures $m_i : \mathcal{A} \to I$, $i \in \mathbb{N}$, i.e., for the case of interval $I = [0,1]$ for fuzzy measures m_i and aggregation function $A_{[n]}$. We extend the definition* (12) *to unbounded fuzzy measures $m_i : \mathcal{A} \to I$, $i \in \mathbb{N}$, i.e., for the cases of $I = [0,\infty[$ and $I = [0,\infty]$. In Theorem* 4.3 *we proved that $m_{[n]}$ is a fuzzy measure in this cases also, and prove some additional properties which $m_{[n]}$ can have in regards to the properties of initial fuzzy measures $m_i : \mathcal{A} \to I$, $i \in \mathbb{N}$ and the properties of aggregation functions $A_{[n]}$.*

(ii) *Value $m_i(F)$ can be interpreted as a measure of set (object) F in certain sense, i.e., the measure of F regarding certain criterion $i \in \{1, \ldots, n\}$. Then, $m_{[n]}(F)$ can be interpreted as a joined measure of F which is obtained using particular measures $m_i(F)$, $i \in \{1, \ldots, n\}$ and joining criterion A. Using suitable aggregation operator A, we can model the total measure of object F in the desired way. So, operator A is a joining criterion for the construction of measure $m_{[n]}(F)$ using particular measures $m_i(F)$, $i \in \{1, \ldots, n\}$.*

Theorem 4.3. *Let $\mathcal{A}$ be a σ-algebra on $X \neq \emptyset$, and let $m_i : \mathcal{A} \to I$, $i \in \mathbb{N}$ be a sequence of fuzzy measures on $\mathcal{A}$. Then, for each $n \in \mathbb{N}$, function $m_{[n]} : \mathcal{A} \to I$ defined by* (12) *is a fuzzy measure on $\mathcal{A}$. Additionally, the following statements hold.*

(a) *Let all fuzzy measures m_i, $i \in \{1, \ldots, n\}$ be continuous from below (property* (FM3)*). If A is a continuous extended aggregation function, then fuzzy measure $m_{[n]}$ is also continuous from below.*

(b) *Let all fuzzy measures m_i, $i \in \{1, \ldots, n\}$ be continuous from above (property* (FM4)*). Let A be a continuous extended aggregation function and, in the case $I = [0,\infty]$, let A additionally have property* (a09)*. Then fuzzy measure $m_{[n]}$ is also continuous from above.*

(c) *Let all fuzzy measures m_i, $i \in \{1, \ldots, n\}$ be subadditive (property* (FM5)*). If A is a subadditive extended aggregation function (property* (a07)*), then fuzzy measure $m_{[n]}$ is also subadditive.*

(d) *Let all fuzzy measures m_i, $i \in \{1, \ldots, n\}$ be superadditive (property* (FM6)*). If A is a superadditive extended aggregation function (property* (a08)*), then fuzzy measure $m_{[n]}$ is also superadditive.*

Proof. For arbitrary $n \in \mathbb{N}$, let $m_i : \mathcal{A} \to I$, $i \in \{1, \ldots, n\}$ be fuzzy measures from Definition 2.6. Because of boundary condition (1) from Definition 2.1 of aggregation function $A_{[n]}$, set function $m_{[n]}$ inherit property (FM1) from fuzzy measures m_i, $i \in \{1, \ldots, n\}$. Also, $m_{[n]}$ inherit property (FM2) from m_i, $i \in \{1, \ldots, n\}$ because of monotonicity condition (5) of aggregation function $A_{[n]}$ from Definition 2.1.

(a) Let $F_k \in \mathcal{A}$, $k \in \mathbb{N}$ be a sequence of sets satisfying $F_1 \subseteq F_2 \subseteq \ldots$. Fuzzy measures m_i, $i \in \{1, \ldots, n\}$ are continuous from below, i.e.,

$$m_i\left(\bigcup_{k=1}^{\infty} F_k\right) = \lim_{k\to\infty} m_i(F_k), \quad i \in \{1, \ldots, n\}.$$

Applying continuity (a03) of aggregation function $A_{[n]}$ we obtain

$$m_{[n]}\left(\bigcup_{k=1}^{\infty} F_k\right) = A_{[n]}\left(m_1\left(\bigcup_{k=1}^{\infty} F_k\right), \dots, m_n\left(\bigcup_{k=1}^{\infty} F_k\right)\right) = A_{[n]}\left(\lim_{k_1\to\infty} m_1(F_{k_1}), \dots, \lim_{k_n\to\infty} m_n(F_{k_n})\right)$$

$$\overset{(a03)}{=} \lim_{\forall i,\, k_i\to\infty} A_{[n]}\left(m_1(F_{k_1}), \dots, m_n(F_{k_n})\right) \overset{(a03)}{=} \lim_{k\to\infty} A_{[n]}\left(m_1(F_k), \dots, m_n(F_k)\right) = \lim_{k\to\infty} m_{[n]}(F_k).$$

(b) Let $F_k \in \mathcal{A}$, $k \in \mathbb{N}$ be a sequence of sets satisfying $F_1 \supseteq F_2 \supseteq \dots$, and, in the case of $I = I_3 = [0,\infty]$ let there exist $k_0 \in \mathbb{N}$ such that $m_{[n]}(F_{k_0}) = A_{[n]}\left(m_1(F_{k_0}), \dots, m_n(F_{k_0})\right) < \infty$. Then for each $i \in \{1, \dots, n\}$ holds $m_i(F_{k_0}) < \infty$, including the case $I = I_3 = [0,\infty]$ because of property (a09) of aggregation function $A_{[n]}$. Using property (FM4) of fuzzy measures m_i, $i \in \{1, \dots, n\}$ we obtain that $m_i\left(\bigcap_{k=1}^{\infty} F_k\right) = \lim_{k\to\infty} m_i(F_k)$ holds for each $i \in \{1, \dots, n\}$, and it follows that

$$m_{[n]}\left(\bigcap_{k=1}^{\infty} F_k\right) = A_{[n]}\left(m_1\left(\bigcap_{k=1}^{\infty} F_k\right), \dots, m_n\left(\bigcap_{k=1}^{\infty} F_k\right)\right) = A_{[n]}\left(\lim_{k_1\to\infty} m_1(F_{k_1}), \dots, \lim_{k_n\to\infty} m_n(F_{k_n})\right).$$

Finally, using continuity (a03) of $A_{[n]}$ it follows that

$$m_{[n]}\left(\bigcap_{k=1}^{\infty} F_k\right) \overset{(a03)}{=} \lim_{\forall i,\, k_i\to\infty} A_{[n]}\left(m_1(F_{k_1}), \dots, m_n(F_{k_n})\right) \overset{(a03)}{=} \lim_{k\to\infty} A_{[n]}\left(m_1(F_k), \dots, m_n(F_k)\right) = \lim_{k\to\infty} m_{[n]}(F_k).$$

Additionally note that $m_{[n]}\left(\bigcap_{k=1}^{\infty} F_k\right)$ has a finite value in the case $I = I_3 = [0,\infty]$ also, because of $m_{[n]}(F_{k_0}) < \infty$.

(c) Let for all fuzzy measures m_i, $i \in \mathbb{N}$ and sets $G, H \in \mathcal{A}$ satisfying $G \cap H = \emptyset$ hold $m_i(G \cup H) \leq m_i(G) + m_i(H)$. Using monotonicity (a02) of aggregation function $A_{[n]}$ we obtain

$$m_{[n]}(G \cup H) = A_{[n]}\left(m_1(G \cup H), \dots, m_n(G \cup H)\right) \leq A_{[n]}\left(m_1(G) + m_1(H), \dots, m_n(G) + m_n(H)\right),$$

and, because of subadditivity (a07) of $A_{[n]}$ it finally follows

$$m_{[n]}(G \cup H) \leq A_{[n]}\left(m_1(G), \dots, m_n(G)\right) + A_{[n]}\left(m_1(H), \dots, m_n(H)\right) \leq m_{[n]}(G) + m_{[n]}(H).$$

(d) Analogously as in proof of statement (c). □

Remark 4.4. (i) *Assumption on* (a09) *property of extended aggregation function A in statement* (b) *of Theorem* 4.3 *is necessary only in the case $I = I_3 = [0,\infty]$, see property* (FM4) *of fuzzy measures in Definition* 2.7*, because in cases $I = I_1 = [0,1]$ and $I = I_2 = [0,\infty[$ fuzzy measure m trivially satisfies the condition that exists $n_0 \in \mathbb{N}$ and $F_{n_0} \in \mathcal{A}$ such that $m(F_{n_0}) < \infty$.*

(ii) *In the proof of Theorem* 4.3 *it can be seen that the same statement and proof hold for fuzzy measures $m : \mathcal{A} \to I$ with values in all considered intervals $I_1 = [0,1]$, $I_2 = [0,\infty[$ and $I_3 = [0,\infty[$. For this reason, the properties of aggregation functions from Examples* 3.1, 3.2 *and* 3.5*, and also the properties of aggregation functions which we introduce in Section* 5*, will be examined for all the mentioned intervals.*

The extended aggregation functions from Examples 3.1, 3.2 and 3.5 can be suitable for the construction of fuzzy measures presented with (12). Table 4 summarizes in which statements of Theorem 4.3 this functions can be applied.

Table 4: Applicability of $A_{\min}$, $A_{\max}$, WAM^{ω} and $\mathsf{WAMP}^{\omega,\lambda}$ operators in Theorem 4.3.

	$A_{\min}$	$A_{\max}$	WAM^{ω}	$\mathsf{WAMP}^{\omega,\lambda}$
(a):	YES	YES	YES	YES
(b):	no	YES	no	no
(c):	no	YES	YES	no
(d):	YES	no	YES	no

Remark 4.5. *With some appropriated parameters ω and λ for WAM^{ω} and $\mathsf{WAMP}^{\omega,\lambda}$, as well as for other extended aggregation functions with values in intervals $I_1 = [0,1]$ and $I_2 = [0,\infty[$, we can state some additional observations about the applicability of the considered extended aggregation functions in Theorem* 4.3.

(i) *Namely, see Example* 3.5, *with powers $\lambda = \{\lambda_{n,i} \le 1 \mid n \in \mathbb{N},\ i \in \{1,\dots,n\}\}$, i.e., with concave power functions, $\mathsf{WAMP}^{\omega,\lambda}$ is applicable in the part* (c) *of Theorem* 4.3.

Analogously, with powers $\lambda = \{\lambda_{n,i} \ge 1 \mid n \in \mathbb{N},\ i \in \{1,\dots,n\}\}$, i.e., with convex power functions, $\mathsf{WAMP}^{\omega,\lambda}$ is applicable in the part (d) *of Theorem* 4.3.

(ii) *We can notice that for $I_1 = [0,1]$ and $I_2 = [0,\infty[$ all aggregation functions satisfy property* (a09), *so that all continuous aggregation functions (as well as the functions presented in Table* 4*) with the values in intervals $I_1 = [0,1]$ and $I_2 = [0,\infty[$ can be applied in statement* (b) *of Theorem* 4.3.

(iii) *On interval $I_3 = [0,\infty]$, with parameters $\omega = \{\omega_{n,i} > 0 \mid n \in \mathbb{N},\ i \in \{1,\dots,n\}\}$, functions WAM^{ω} and $\mathsf{WAMP}^{\omega,\lambda}$ also satisfy* (a09) *and can be applied in statement* (b) *of Theorem* 4.3.

5 Extended weighted arithmetic mean of distorted arguments

In this section we introduce one new extended aggregation function which can be suitable for aggregation of fuzzy measures presented in Section 4. Also, for the application of Theorem 4.3, relevant properties of the introduced extended aggregation function are examined. We call this function *extended weighted arithmetic mean of distorted arguments*, and denote as WAMDA. Function WAMDA is a generalization of WAMP function from Section 3.

Let I be one of the interval $I_1 = [0,1]$, $I_2 = [0,\infty[$ or $I_3 = [0,\infty]$. As parameter of WAMDA function will be the family

$$\mathcal{F} = \{f_{n,i} : I \to I \mid n \in \mathbb{N},\ i \in \{1,...,n\}\} \tag{13}$$

of monotonically nondecreasing functions satisfying boundary conditions

$$f_{n,i}(0) = 0, \tag{14}$$

and

$$f_{n,i}(1) = 1, \tag{15}$$

in the case $I = I_1 = [0,1]$,

$$\lim_{x\to\infty} f_{n,i}(x) = \infty, \tag{16}$$

in the case $I = I_2 = [0,\infty[$, and

$$f_{n,i}(\infty) = \infty, \tag{17}$$

in the case $I = I_3 = [0,\infty]$. For $n = 1$, we take $f_{1,1}(x) = x$, $x \in I$ which satisfies all the mentioned conditions.

Definition 5.1 (WAMDA)**.** *Let $\omega = \left\{\omega_{n,i} \ge 0 \,\middle|\, n \in \mathbb{N},\ i \in \{1,...,n\} \ \wedge\ \forall n \in \mathbb{N},\ \exists i \in \{1,...,n\},\ \omega_{n,i} > 0\right\}$, be an arbitrary family of nonnegative real numbers, satisfying $\sum_{i=1}^{n} \omega_{n,i} = 1$ in the case $I = [0,1]$, and let $f_{n,i} \in \mathcal{F}$, $i \in \{1,\dots,n\}$ be functions* (13) *with the properties described by* (14), (15), (16) *and* (17)*. For each $n \in \mathbb{N}$, let function $\mathsf{WAMDA}^{\omega,\mathcal{F}}_{[n]} :$ $I^n \to I$ is defined by*

$$\mathsf{WAMDA}^{\omega,\mathcal{F}}_{[n]}(a_1,\dots,a_n) = \omega_{n,1} f_{n,1}(a_1) + \cdots + \omega_{n,n} f_{n,n}(a_n), \tag{18}$$

for all $(a_1,\dots,a_n) \in I^n$. Let function $\mathsf{WAMDA}^{\omega,\mathcal{F}} : \bigcup_{n=1}^{\infty} I^n \to I$ is defined with

$$\mathsf{WAMDA}^{\omega,\mathcal{F}}(a_1,\dots,a_n) = \mathsf{WAMDA}^{\omega,\mathcal{F}}_{[n]}(a_1,\dots,a_n), \tag{19}$$

for each $n \in \mathbb{N}$ and all $(a_1,\dots,a_n) \in I^n$. We call functions $\mathsf{WAMDA}^{\omega,\mathcal{F}}_{[n]}$ and $\mathsf{WAMDA}^{\omega,\mathcal{F}}$ weighted arithmetic mean of distorted arguments *and* extended weighted arithmetic mean of distorted arguments*, respectively.*

Remark 5.2. (i) *It is obvious that* $\mathsf{WAMDA}^{\omega,\mathcal{F}}$ *is an extended aggregation function. Namely, boundary conditions* (a01) *follow from the boundary conditions of functions* $f_{n,i}$*,* $n \in \mathbb{N}$*,* $i \in \{1,\ldots,n\}$ *and the properties of coefficients* ω*, and on the other hand, monotonicity* (a02) *follows from the monotonicity of functions* $f_{n,i}$*,* $n \in \mathbb{N}$*,* $i \in \{1,\ldots,n\}$*. Remark* 3.4 *also hold for* WAMDA*.*

(ii) *Extended aggregation function* WAMP *from Example* 3.5 *is a special case of* WAMDA *function from Definition* 5.1*, with power functions* $f_{n,i}(x) = x^{\lambda_{n,i}}$*. Consequently, the properties from Definition* 2.3 *which* WAMP *does not satisfy, in general case are not satisfied by extended aggregation function* WAMDA *either, see Table* 3*. It remains to discus property* (a03)*, and also to analyze for which additional assumptions on parameters,* WAMDA *can have some additional considered properties.*

In formulations of further statements and considerations, we deal with terms of continuous, concave and convex monotonically nondecreasing functions $f : I \to I$. In the case of interval $I = I_3 = [0,\infty]$, for nondecreasing function $f : [0,\infty] \to [0,\infty]$ with boundary conditions $f(0) = 0$ and $f(\infty) = \infty$ it is necessary to define the terms of continuity, convexity and concavity of function $f : [0,\infty] \to [0,\infty]$. For the purpose of our examination, we will adopt the following definitions.

Definition 5.3. (i) *Monotonically nondecreasing function* $f : [0,\infty] \to [0,\infty]$ *satisfying boundary conditions* $f(0) = 0$ *and* $f(\infty) = \infty$ *is continuous on interval* $[0,\infty]$ *if for* $b = \sup\{x \in [0,\infty] \mid f(x) < \infty\}$ *holds* $b > 0$*,* $f : [0,b[\to [0,\infty[$ *is a continuous function, and* $\lim_{x\to b} f(x) = \infty$*. Following monotonicity of function* f*,* $f(b) = \infty$ *must be additionally satisfied.*

(ii) *Monotonically nondecreasing function* $f : [0,\infty] \to [0,\infty]$ *satisfying boundary conditions* $f(0) = 0$ *and* $f(\infty) = \infty$ *is a concave function on interval* $[0,\infty]$ *if its restriction on interval* $[0,\infty[$ *is a concave function* $f : [0,\infty[\to [0,\infty[$ *(with finite values) and* $\lim_{x\to\infty} f(x) = \infty$*. Additionally, we also consider function* $f : [0,\infty] \to [0,\infty]$*,*

$$f(x) = \begin{cases} 0 & , \quad x = 0 \\ \infty & , \quad x > 0, \end{cases}$$

as concave.

(iii) *Monotonically nondecreasing function* $f : [0,\infty] \to [0,\infty]$ *satisfying boundary conditions* $f(0) = 0$ *and* $f(\infty) = \infty$ *is a convex function on interval* $[0,\infty]$ *if its restriction on interval* $[0,\infty[$ *is convex function* $f : [0,\infty[\to [0,\infty[$ *(with finite values).*

In general case, function $\mathsf{WAMDA}^{\omega,\mathcal{F}}$ is not continuous (property (a03)). For example, for $n = 2$, $\omega_{2,1} = \omega_{2,2} = \frac{1}{2}$, and

$$f_{2,1}(x) = f_{2,2}(x) = \begin{cases} 0 & , \quad 0 \le x < 0.5 \\ 0.5 & , \quad x \ge 0.5, \end{cases}$$

aggregation function $\mathsf{WAMDA}^{\omega,\mathcal{F}}_{[2]}$ has discontinuity in $(0.5, 0.5)$. That means, see Remark 5.2, that extended aggregation function $\mathsf{WAMDA}^{\omega,\mathcal{F}}$ generally does not satisfy any of properties from Definition 2.3.

Remark 5.4. *But, for certain specific parameters* ω *and* $\mathcal{F}$*, we can derive some additional conclusions about the properties of function* $\mathsf{WAMDA}^{\omega,\mathcal{F}}$*. That means that for some types of parameters* ω *and* $\mathcal{F}$*, extended aggregation function* $\mathsf{WAMDA}^{\omega,\mathcal{F}}$ *can be suitable for application in Theorem* 4.3*.*

Theorem 5.5. *Let* $f : I \to I$ *be a monotonically nondecreasing and concave function (see Definition* 5.3*) satisfying boundary condition* (14)*, and appropriated boundary condition* (15)*,* (16) *or* (17) *depending on interval* I*. Function* f *is then continuous (see Definition* 5.3*) on* I*, except maybe in* 0*.*

Proof. Let us prove that function f is continuous from the left at any $x_0 > 0$. The proof for continuity from the right in $x_0 > 0$ is analogous. In the case of interval $I = [0,\infty]$ and $x_0 = \infty$, left-continuity follows from the definition of concavity, see Definition 5.3, so we further consider $x_0 < \infty$. Suppose that f has discontinuity in x_0. Function f is monotonically nondecreasing, which implies that there exists $\varepsilon > 0$ such that

$$\forall x < x_0, \quad f(x) < f(x_0) - \varepsilon. \qquad [*]$$

From $\lim_{\alpha\to 1}((1-\alpha)f(0) + \alpha f(x_0)) = f(x_0)$ follows that there exists $\alpha \in [0,1)$ such that

$$(1-\alpha)f(0)+\alpha f(x_0) > f(x_0)-\varepsilon. \qquad [**]$$

On the other hand, for $x=(1-\alpha)\cdot 0+\alpha\cdot x_0 < x_0$ from [*] follows that $f(x) < f(x_0)-\varepsilon$, i.e.,

$$f\left((1-\alpha)\cdot 0+\alpha\cdot x_0\right) < f(x_0)-\varepsilon. \qquad [***]$$

From [**] and [***] follows that $f\left((1-\alpha)\cdot 0+\alpha\cdot x_0\right) < (1-\alpha)f(0)+\alpha f(x_0)$, but the last inequality is in contradiction with the assumption of concavity of function f. □

Example 5.6. *Function f from Theorem 5.5 must not be continuous at 0, which we illustrate with function $f:[0,1]\to[0,1]$ defined by*

$$f(x)=\begin{cases} 0 & , \quad x=0 \\ \frac{1}{2}x+\frac{1}{2} & , \quad x>0, \end{cases}$$

Analogously to Theorem 5.5, the following theorem for convex functions can be proved.

Theorem 5.7. *Let $f:I\to I$ be a monotonically nondecreasing and convex function (see Definition 5.3) satisfying boundary condition (14), and one of the boundary conditions (15), (16) or (17), depending on the choice of interval I. Function f is then continuous (see Definition 5.3) on I, except maybe in 1 in the case of interval $I=[0,1]$, or maybe in ∞ in the case of interval $I=[0,\infty]$.*

Function $f:[0,1]\to[0,1]$, $f(x)=\begin{cases} \frac{1}{2}x^2 & , \quad x<1 \\ 1 & , \quad x=1 \end{cases}$ is monotonically nondecreasing, convex, satisfies boundary conditions $f(0)=0$ and $f(1)=1$, but has discontinuity at the point 1. Also, function $f:[0,\infty]\to[0,\infty]$, $f(x)=\begin{cases} 0 & , \quad x<\infty \\ \infty & , \quad x=\infty \end{cases}$ is monotonically nondecreasing, convex, satisfies boundary conditions $f(0)=0$ and $f(\infty)=\infty$, but has discontinuity at the point ∞.

As a consequence of Theorems 5.5 and 5.7, the following statement is valid.

Proposition 5.8. (i) *If each od the functions $f_{n,i}\in\mathcal{F}$, $i\in\{1,\dots,n\}$ is concave, then aggregation function $\mathsf{WAMDA}_{[n]}^{\omega,\mathcal{F}}: I^n\to I$ can have discontinuity just at the points $(a_1,\dots,a_n)$ with $a_i=0$ for some $i\in\{1,\dots,n\}$.*

(ii) *If each of the functions $f_{n,i}\in\mathcal{F}$, $i\in\{1,\dots,n\}$ is convex, then aggregation function $\mathsf{WAMDA}_{[n]}^{\omega,\mathcal{F}}: I^n\to I$ can have discontinuity just at the points $(a_1,\dots,a_n)$ with*

- [I1] *$a_i=1$ for some $i\in\{1,\dots,n\}$ in the case of interval $I=I_1=[0,1]$,*
- [I3] *$a_i=\infty$ for some $i\in\{1,\dots,n\}$ in the case of interval $I=I_3=[0,\infty]$.*

Remark 5.9. *In general case, extended aggregation function $\mathsf{WAMDA}^{\omega,\mathcal{F}}$ is not continuous. If all functions from $\mathcal{F}$ are continuous, then $\mathsf{WAMDA}^{\omega,\mathcal{F}}$ is also continuous because $\mathsf{WAMDA}_{[n]}^{\omega,\mathcal{F}}$ is a composition of continuous functions for each $n\in\mathbb{N}$. Regarding Theorems 5.5 and 5.7, if all functions from $\mathcal{F}$ are convex, or all of them are concave, and all of them are continuous at borders of interval I (see Definition 5.3), extended aggregation function $\mathsf{WAMDA}^{\omega,\mathcal{F}}$ is then continuous.*

Extended aggregation function $\mathsf{WAMDA}^{\omega,\mathcal{F}}$ is obviously symmetrical (property (a04)) only in the case when $\omega_{n,i}=\frac{1}{n}$, $i\in\{1,\dots,n\}$, $n\in\mathbb{N}$ and $f_{n,i}\equiv f_n$, $i\in\{1,\dots,n\}$, $n\in\mathbb{N}$ for some sequence of functions f_n, $n\in\mathbb{N}$.

It has the property of idempotencity (a05) only in the special case of WAM function with coefficients ω satisfying $\sum_{i=1}^{n}\omega_{n,i}=1$, $n\in\mathbb{N}$.

It has the property of additivity (a06) only in the special case of WAM function. With some additional assumptions about concavity or convexity of functions $\mathcal{F}$, extended aggregation function $\mathsf{WAMDA}^{\omega,\mathcal{F}}$ can be subadditive (a07) or superadditive (a08).

Proposition 5.10. *If each of $f_{n,i}\in\mathcal{F}$, $n\in\mathbb{N}$, $i\in\{1,\dots,n\}$ is a concave function in the sense of Definition 5.3, then $\mathsf{WAMDA}^{\omega,\mathcal{F}}$ is a subadditive extended aggregation function.*

Proof. It is known that from concavity of function $f:I\to I$ and $f(0)=0$ follows their subadditivity. We will prove this statement. Because of concavity of f and $f(0)=0$, for any $t\in[0,1]$ and each $x\in I$, $x\neq\infty$ in the case $I=[0,\infty]$, holds $f(tx)=f\left(tx+(1-t)\cdot 0\right)\geq tf(x)+(1-t)\cdot f(0)=tf(x)$. For $I=[0,\infty]$ and $x=\infty$, there is

- $f(tx) = f(0) = 0 = 0 \cdot \infty = tf(x)$, for $t = 0$, and
- $f(tx) = f(\infty) = \infty = t \cdot \infty = tf(x)$, for $t > 0$.

So, in all the considered cases for interval I we have $f(tx) \geq tf(x)$ for all $t \in [0,1]$ and each $x \in I$. Then for all $x, y \in I$, $x \neq \infty$, $y \neq \infty$ from concavity of f follows subadditivity of f:

$$f(x+y) = \frac{x}{x+y} f(x+y) + \frac{y}{x+y} f(x+y) \leq f\left(\frac{x}{x+y}(x+y)\right) + f\left(\frac{y}{x+y}(x+y)\right) = f(x) + f(y).$$

In the case of $I = [0, \infty]$, when $x = \infty$ or $y = \infty$, consider for example $x = \infty$, we obtain $f(x+y) = f(\infty) = \infty = \infty + f(y) = f(x) + f(y)$. So, in all the considered cases for interval I we have $f(x+y) \leq f(x) + f(y)$ for all $x, y \in I$. We get that for all $f_{n,i} \in \mathcal{F}$, $n \in \mathbb{N}$, $i \in \{1, \ldots, n\}$ holds

$\forall x_i, y_i \in I,\ f_{n,i}(x_i + y_i) \leq f_{n,i}(x_i) + f_{n,i}(y_i)$,

$\forall x_i, y_i \in I,\ \omega_{n,i} f_{n,i}(x_i + y_i) \leq \omega_{n,i} f_{n,i}(x_i) + \omega_{n,i} f_{n,i}(y_i)$,

$\forall x_i, y_i \in I,\ \sum\limits_{i=1}^{n} \omega_{n,i} f_{n,i}(x_i + y_i) \leq \sum\limits_{i=1}^{n} \omega_{n,i} f_{n,i}(x_i) + \sum\limits_{i=1}^{n} \omega_{n,i} f_{n,i}(y_i)$,

i.e., for all $x_i, y_i \in I$,

$$\mathsf{WAMDA}_{[n]}^{\omega,\mathcal{F}}(x_1 + y_1, \ldots, x_n + y_n) \leq \mathsf{WAMDA}_{[n]}^{\omega,\mathcal{F}}(x_1, \ldots, x_n) + \mathsf{WAMDA}_{[n]}^{\omega,\mathcal{F}}(y_1, \ldots, y_n).$$

□

Analogously to Proposition 5.10, the following statement can be proved.

Proposition 5.11. *If each of $f_{n,i} \in \mathcal{F}$, $n \in \mathbb{N}$, $i \in \{1, \ldots, n\}$ is a convex function in the sense of Definition* 5.3*, then* $\mathsf{WAMDA}^{\omega,\mathcal{F}}$ *is a superadditive extended aggregation function.*

Let us examine whether $\mathsf{WAMDA}^{\omega,\mathcal{F}}$ function has property (a09).

[I1] In the case of interval $I = I_1 = [0,1]$, property (a09) is not satisfied. For example, for $n = 2$, $f_{2,1}(x) = f_{2,2}(x) = x$, $x \in I$, $\omega_{2,1} = \omega_{2,2} = 0.5$, and $(a_1, a_2) = (1, 0.6)$ is $\mathsf{WAMDA}_{[2]}^{\omega,\mathcal{F}}(a_1, a_2) = 0.8 < 1$, where $a_1 = 1$.

[I2] In the case of interval $I = I_2 = [0, \infty[$, e.g., for $n = 2$, $f_{2,1}(x) = f_{2,2}(x) = x$, $x \in I$, $\omega_{2,1} = 1$, $\omega_{2,2} = 0$, $(b_1, b_2) = (1, \infty)$, where is $b_2 = \infty$, we have

$$\lim_{\forall i \in \{1,2\},\ a_i \to b_i} \mathsf{WAMDA}_{[2]}^{\omega,\mathcal{F}}(a_1, a_2) = \lim_{\forall i \in \{1,2\},\ a_i \to b_i} \sum_{i=1}^{2} \omega_{2,i} \cdot a_i = 1 < \infty.$$

So, $\mathsf{WAMDA}^{\omega,\mathcal{F}}$ generally does not have property (a09) in this case either. But, with strictly positive coefficients $\omega = \left\{\omega_{n,i} > 0 \,\middle|\, n \in \mathbb{N},\ i \in \{1, ..., n\}\right\}$, it is obvious that for all $n \in \mathbb{N}$, $\lim\limits_{\forall i \in \{1,\ldots,n\},\ a_i \to b_i} \mathsf{WAMDA}_{[n]}^{\omega,\mathcal{F}}(a_1, \ldots, a_n) < \infty$ can be satisfied only when $b_i < \infty$ for all $i \in \{1, \ldots, n\}$, because of $\lim\limits_{x \to \infty} f_{n,i}(x) = \infty$ property of functions $f_{n,i}$, $n \in \mathbb{N}$, $i \in \{1, \ldots, n\}$. Hence, for strictly positive coefficients ω, extended aggregation function $\mathsf{WAMDA}^{\omega,\mathcal{F}}$ has the property (a09).

[I3] In the case of interval $I = I_3 = [0, \infty]$, analogously to the case [I2] we derive that $\mathsf{WAMDA}^{\omega,\mathcal{F}}$ generally does not have property (a09). It is easy to see that $\mathsf{WAMDA}^{\omega,\mathcal{F}}$ satisfies property (a09) only for strictly positive coefficients $\omega = \left\{\omega_{n,i} > 0 \,\middle|\, n \in \mathbb{N},\ i \in \{1, ..., n\}\right\}$ and family $\mathcal{F}$ of functions such that

$$\forall n \in \mathbb{N},\ \forall i \in \{1, \ldots, n\}, \quad f_{n,i}(x) = \infty \ \Rightarrow \ x = \infty.$$

Previous examinations shows that, in general case, $\mathsf{WAMDA}^{\omega,\mathcal{F}}$ does not have any of the considered properties, but with some additional assumptions on functions $\mathcal{F}$, extended aggregation function $\mathsf{WAMDA}^{\omega,\mathcal{F}}$ has the properties suitable for fuzzy measures construction from Section 4, i.e., for application in Theorem 4.3. In that sense, the consequences and conclusions of the previous discussion are summarized in the following Theorem 5.12, where terms of continuity, concavity and convexity of function $f : [0, \infty] \to [0, \infty]$ are used in the sense of Definition 5.3.

Theorem 5.12. *Let* $\mathsf{WAMDA}^{\omega,\mathcal{F}}$ *be an extended aggregation function with parameters* ω *and* $\mathcal{F}$ *from Definition* 5.1*, let* m_i*,* $i \in \mathbb{N}$ *be a family of fuzzy measures, and let for arbitrary* $n \in \mathbb{N}$ *be* $m_{[n]}$ *the fuzzy measure defined by* (12)*.*

(a) *Let each of* $f_{n,i} \in \mathcal{F}$*,* $n \in \mathbb{N}$*,* $i \in \{1, \dots, n\}$ *be a continuous function. In the case of interval* $I = [0, \infty]$*, let* ω *be a family of strictly positive coefficients, and each of the function* $f_{n,i} \in \mathcal{F}$*,* $n \in \mathbb{N}$*,* $i \in \{1, \dots, n\}$ *reach value* $f_{n,i}(x) = \infty$ *only for* $x = \infty$*. Fuzzy measure* $m_{[n]}$ *is then continuous, i.e., it has properties* (FM3) *and* (FM4)*.*

(b) *If every fuzzy measure* m_i*,* $i \in \mathbb{N}$ *is subadditive, and each of the function* $f_{n,i} \in \mathcal{F}$*,* $n \in \mathbb{N}$*,* $i \in \{1, \dots\}$ *is concave, then fuzzy measure* $m_{[n]}$ *is subadditive, i.e., it has the property* (FM5)*.*

(c) *If every fuzzy measure* m_i*,* $i \in \mathbb{N}$ *is superadditive, and each of the function* $f_{n,i} \in \mathcal{F}$*,* $n \in \mathbb{N}$*,* $i \in \{1, \dots\}$ *is convex, then fuzzy measure* $m_{[n]}$ *is superadditive, i.e., it has the property* (FM6)*.*

The following Example presents two $\mathsf{WAMDA}^{\omega,\mathcal{F}}$-type extended aggregation functions which can be applied in the statements of Theorem 4.3.

Example 5.13. *Let* $\omega = \left\{ \omega_{n,i} > 0 \,\middle|\, n \in \mathbb{N},\ i \in \{1, ..., n\} \right\}$ *and* $K = \{k_{n,i} > 0 \mid n \in \mathbb{N},\ i \in \{1, \dots, n\}\}$*, be arbitrary families of positive coefficients, and let functions* $f_{n,i} : [0, \infty[\to [0, \infty[$ *and* $g_{n,i} : [0, \infty[\to [0, \infty[$ *be defined by* $f_{n,i}(x) = \ln(k_i x + 1),\ \ x \in [0, \infty[$ *and* $g_{n,i}(x) = e^{k_i x} - 1,\ \ x \in [0, \infty[$*. For families of functions* $\mathcal{F} = \{f_{n,i} \mid n \in \mathbb{N},\ i \in \{1, \dots, n\}\}$*, and* $\mathcal{G} = \{g_{n,i} \mid n \in \mathbb{N},\ i \in \{1, \dots, n\}\}$*, functions* $\mathsf{WAMDA}^{\omega,\mathcal{F}}$ *and* $\mathsf{WAMDA}^{\omega,\mathcal{G}}$ *are* extended weighted arithmetic mean of distorted arguments *type. All of the functions from* $\mathcal{F}$ *and* $\mathcal{G}$ *are continuous, so that* $\mathsf{WAMDA}^{\omega,\mathcal{F}}$ *and* $\mathsf{WAMDA}^{\omega,\mathcal{G}}$ *are continuous extended aggregation functions. Beside that, they obviously satisfy property* (a09)*.*

(a) *The functions from* $\mathcal{F}$ *are concave too, which by Theorem* 5.12 *implies that applying* $\mathsf{WAMDA}^{\omega,\mathcal{F}}$ *function on the sequence* m_i*,* $i \in \mathbb{N}$ *of continuous and subadditive fuzzy measures (properties* (FM3)*,* (FM4) *and* (FM5)*) we obtain continuous and subadditive fuzzy measures* $m_{[n]}$*,* $n \in \mathbb{N}$*.*

(b) *The functions from* $\mathcal{G}$ *are convex too, which by Theorem* 5.12 *implies that applying* $\mathsf{WAMDA}^{\omega,\mathcal{G}}$ *function on the sequence* m_i*,* $i \in \mathbb{N}$ *of continuous and superadditive fuzzy measures (properties* (FM3)*,* (FM4) *and* (FM6)*) we obtain continuous and superadditive fuzzy measures* $m_{[n]}$*,* $n \in \mathbb{N}$*.*

6 Conclusion

Now we can summarize our obtained results.

1. Applying arbitrary aggregation function on a sequence of fuzzy measures on the some σ-algebra we obtain a new fuzzy measure on same σ-algebra.
2. Applying continuous aggregation function with minimal additional property on a sequence of fuzzy measures we obtain a new continuous fuzzy measure.
3. Applying subadditive aggregation function on a sequence of subadditive fuzzy measures we obtain a new subadditive fuzzy measure.
4. Applying superadditive aggregation function on a sequence of superadditive fuzzy measures we obtain a new superadditive fuzzy measure.

With suitable choice of parameters, extended aggregation function presented in Section 5 can be good for the mentioned construction of new fuzzy measures with desirable properties. Choosing convex and concave functions as the parameters of applied aggregation function, we determine the inheritance of subadditivity or superadditivity property from the initial fuzzy measures on the constructed fuzzy measure. The extended aggregation function presented in Section 5 can be a good model for information fusion whenever subadditivity or superadditivity is important. Also, choosing appropriate family of functions figuring in this aggregation function, we can model the influence of each information in fusion process. Choosing coefficients in the presented aggregation function, we can model a rate of influence of each information in the fusion process.

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