Iranian Journal of Fuzzy Systems

Volume 17, Number 6, (2020), pp. 115-131

University of Sistan and Baluchestan

A modified method on estimating and assessing the process yield with imprecise multiple characteristics

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Abstract

The conventional advanced process yield index S_{pk}^T is widely applied in industry to provide an exact measure of the overall production yield whose quality characteristics are mutually independent and multivariate normally distributed. While one can find numerous studies that consider a crisp estimation of S_{pk}^T to evaluate and test the overall process yield, the recorded measurements of product quality characteristics are not always reported precisely. This paper presents a new fuzzy-based method to assess the overall process yield in the presence of a specified degree of ambiguity for the sample data. After finding a fuzzy estimator of S_{pk}^T based on Buckley's approach, a new fuzzy three-decision testing rule is proposed to evaluate process performance based on critical values and fuzzy p-values. Subsequently, this work extends the application of the proposed method to the class of correlated characteristics by adopting the principal component analysis technique. The introduced fuzzy testing procedure includes the existing customary binary-decision testing rule as a spacial case. In addition, comparative studies are conducted to display the benefits of the proposed rule. Finally, two industrial examples are given for independent and correlated characteristics to guide the practitioners.

Keywords: Process capability indices, process yield, hypothesis testing, p-value, critical value, fuzzy numbers arithmetic.

1 Introduction

Process capability indices (PCIs) are usually applied to measure how much of the process productions meet quality specifications. PCIs are classified into two groups, univariate and multivariate PCIs, as the product quality is determined by one and more than one characteristic, respectively. Some univariate PCIs, which are often applied in capability analysis activities, are C_p, C_{pk}, C_{pm} and S_{pk} [20]. In addition to the PCIs, the process yield, which is the percentage of processed product units passing inspection, is also applied to measure the process performance in the manufacturing industry. The relationship between these two criteria has received widely attention among researchers. Although C_{pk} obtains the upper and lower bounds (an approximate measure) for the process yield [44], and C_{pm} provides a lower bound for the production yield [33], the yield index S_{pk} presented by Boyles [6] has a one-to-one relationship (an exact measure) with the process yield.

Owing to rapid development of new technologies, there exist several quality characteristics that should be monitored simultaneously for an item to describe its quality. To this end, several multivariate PCIs have been developed by researchers. For example, the multivariate capability index MC_{pm} was described by Taam et al. [36] which is the ratio of two volumes of the modified tolerance region and a scaled 99.73% process region. Shahriari and Abdollahzadeh [34] presented a three-component process capability vector to judge the operation of multivariate normal process. The overall capability index C_{pk}^T is a developed form of the univariate index C_{pk} which was introduced by Pearn et al. [31]. It is used to evaluate the performance of a process whose characteristics are mutually independent and normally distributed. Chen et al. [12] introduced an extended version of the process yield index S_{pk} denoted by S_{pk}^T to assess

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Received: December 2019; Revised: April 2020; Accepted: May 2020.

the capability of a process with products needed to measure mutually independent normally characteristics. They also demonstrated, that has an exact one-to-one relation with the total process yield. Then, Pearn et al. [32] developed a generalized index based on the index S_{pk} using the principal component analysis (PCA) technique. Next, Pearn and Cheng [30] investigated the behaviour of sampling distribution of S_{pk}^T based on various combinations of the production parameters and presented a crisp statistical method for estimating the true production yield by the lower confidence bound of S_{pk}^T. They also conducted crisp hypothesis testing on the production yield. To obtain more information about multivariate capability indices, refer [10, 17, 25, 26, 39, 50].

Uncertainty often appears in real applications when measurements of product quality characteristics cannot be precisely recorded or collected. Hence, in such a situation, traditional statistical inference in evaluating process yield may end in unreliable decision results. To remedy this problem, a new attitude of combining fuzziness and randomness has recently developed to assess process capability. Yongting [45] first formed fuzzy index C_p by applying the concept of fuzzy quality. Lee [21] obtained the fuzzy estimation of C_{pk} based on fuzzy numbers when inaccuracy of the measurement is caused by subjective determination of the administrators. Moreover, Chen et al. [11] provided statistical inference to evaluate process performance in a fuzzy environment for processes with one-sided specifications. Tsai and Chen [38] presented a pair of non-linear functions to obtain membership function of fuzzy index C_p. In addition, Parchami et al. [28] and Parchami and Mashinchi [27] studied fuzzy PCIs in the presence of fuzzy specification limits. Abbasi Ganji and Sadeghpour Gildeh [1] applied fuzzy logic to evaluate the univariate process incapability. Recently, Abbasi Ganji and Sadeghpour Gildeh [4] presented two fuzzy capability indices to measure the performance of simple linear profiles as specification limits are vague. One can have a vast knowledge on fuzzy PCIs by studying the review paper written by Kaya and Colak [19].

Though, very few fuzzy decision making methods have been considered by former studies to assess whether the process under investigation satisfies the pre-set requirements or not. Buckley [7], and Buckley and Eslami [9] introduced an approach to find fuzzy estimators of mean and variance that uses a set of confidence intervals to yield a fuzzy triangular number as the estimator. Additionally, Neyman-Pearson lemma for fuzzy hypothesis testing with vague data was presented by Torabi et al. [37]. Later, Parchami et al. [29] studied the problem of fuzzy hypotheses testing with crisp data. They first introduced the notion of fuzzy p-value by using the extension principle and then presented a method to test fuzzy hypotheses by comparing the fuzzy p-value and a fuzzy significance level. Liao and Wu [22] investigated decision rules based on the incapability index C_{pp} in the presence of vague product quality measurements. Recently, Wu and Liao [41] proposed a realistic fuzzy approach based on the index S_{pk} to evaluate process yield for the process with univariate normally distributed quality characteristic. Shu and Wu [35] described the process quality observations by applying bell-shaped fuzzy numbers to derive the fuzzy estimation of index S_{pk} based on these fuzzy numbers. Then, Wu and Liao [42] provided the fuzzy lower confidence bound for S_{pk} to make reliable decisions in a fuzzy environment. Subsequently, Wu and Liao [43] formulated fuzzy numbers to describe the quality characteristic measurements and applied two methods to find the fuzzy estimation of S_{pk} for evaluating and ranking process yields with imprecise data. Afshari and Sadeghpour Gildeh [5] designed a modified decision rule for fuzzy hypotheses testing based on sequential probability ratio test with crisp data. Abbasi Ganji and Sadeghpour Gildeh [2] proposed a fuzzy multivariate capability vector $\widetilde{NMPCV}$ and fuzzy capability index $\widetilde{MC_{pm}}$ to assess the performance of manufacturing process when the target and specification limits of every characteristic are imprecise. Later, Abbasi Ganji and Sadeghpour Gildeh [3] discussed about the $\widetilde{NMPCV}$ as the collected data from the process are considered to be ambiguous.

According to the literature, the existing imprecise-capability-index-based approaches for evaluating process yield widely focus on univariate quality characteristics. Although the existing methods are suitable to such a situation, they are not helpful when a manufactured product described by multiple characteristics. Methods for measuring and testing yield for processes with imprecise multiple characteristics have been comparatively neglected. Considering the above discussion, the appealing properties of index S_{pk}^T caused the authors to apply index S_{pk}^T to make a fuzzy criterion to assess the operation of manufacturing process in a fuzzy environment when its characteristics are mutually independent and normally distributed. Therefore, in this paper, we extend the classic results of Pearn and Cheng [30] to provide an imprecise procedure to evaluate and test the process yield based on an extended version of Buckley's approach [7] when the underlying sample data have a certain degree of imprecision. We also develop the proposed rule to a larger class of the correlated characteristics by using PCA technique.

This paper is organized as follows. Next section discusses the crisp estimation of S_{pk}^T, its generalization based on PCA, and presents the critical value and p-value based on the approximated normal distribution of estimated S_{pk}^T. Some related definitions and properties of fuzzy sets, as well as Buckleys approach, are considered in Section 3. The λ-cut intervals of fuzzy estimator for S_{pk}^T are also obtained in Section 3. After finding the fuzzy triangular shaped estimator of S_{pk}^T, we propose a fuzzy-three-decision testing rule to assess the process performance via critical value and fuzzy p-value in Section 4. In addition, two industrial examples for both independent and correlated observations are given in Section 5. To show the advantages of the proposed manner, the comparative studies are also conducted in this section. Finally, some concluding remarks are presented in the last section.

2 The advanced process yield index S_{pk}^T and its crisp estimation

Numerous capability indices have been proposed to assess the performance of processes with multiple characteristics. This section deals with the index S_{pk}^T as well as brief discussion on the statistical properties of its crisp-based estimator.

2.1 The advanced process index S_{pk}^T

Let $\mathbf{X} = (X_1, X_2, \cdots, X_v)'$ be a random vector of v interested quality characteristics of the product so that it has a multivariate normal distribution, $N_v(\boldsymbol{\mu}, \sum)$, where $\boldsymbol{\mu} = (\mu_1, \mu_2, \cdots, \mu_v)'$ is the mean vector and $\sum = [\sigma_{ij}]_{i,j=1,2,\cdots,v}$ displays the variance-covariance matrix such that σ_{ij} is the covariance between the ith and jth variables. Considering the mentioned assumptions, the overall production yield, which is the percentage of processed product units passing the inspection successfully, can be obtained as

$$\text{Yield} = P(\text{LSL}_1 \le X_1 \le \text{USL}_1, \text{LSL}_2 \le X_2 \le \text{USL}_2, \cdots, \text{LSL}_v \le X_v \le \text{USL}_v),$$

where USL_j and LSL_j are the upper and lower specification limits for jth component of $\mathbf{X}$, respectively. Chen et al. [12] provided the advanced yield index as

$$S_{pk}^T = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\left[\prod_{j=1}^{v}(2\Phi(3S_{pkj}) - 1) + 1\right]\right\}, \tag{1}$$

where $\Phi(.)$ is the cumulative distribution function of the standard normal distribution, $\Phi^{-1}(.)$ represents the inverse function of $\Phi(.)$, and S_{pkj} denotes the individual yield index of the jth characteristic for $j = 1, 2, \ldots, v$ that was proposed by Boyles [6] for processes with a single normal characteristic,

$$S_{pkj} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\left[\Phi\left(\frac{\text{USL}_j - \mu_j}{\sigma_j}\right) + \Phi\left(\frac{\mu_j - \text{LSL}_j}{\sigma_j}\right)\right]\right\}, \tag{2}$$

so that its one-to-one relationship with individual process yield and singular process nonconformities p_j measured in parts per million (ppm) can be displayed as $\text{yield}_j = 2\Phi(3S_{pkj}) - 1$ and $p_j = (2 - 2\Phi(3S_{pkj})) \times 10^6$ for $j = 1, 2, \ldots, v$, respectively. Similar to what we mentioned about S_{pkj} above, the overall index S_{pk}^T also has a one-to-one relation with the overall production yield and the overall process nonconformities p as follows (see [32])

$$\text{Yield} = \prod_{j=1}^{v}\text{yield}_j = 2\Phi(3S_{pk}^T) - 1, \quad p = \prod_{j=1}^{v} p_j = (2 - 2\Phi(3S_{pk}^T)) \times 10^6. \tag{3}$$

2.2 The crisp estimator of S_{pk}^T

The natural estimator $\widehat{S}_{pkj}$ $(j = 1, 2, \ldots, v)$, can be obtained by (4) when one applies the sample mean $\overline{X}_j = \frac{1}{n}\sum_{i=1}^{n} X_{ji}$ and the sample standard deviation $S_j = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_{ji} - \overline{X}_j)^2}$ as estimators for μ_j and σ_j, respectively.

$$\widehat{S}_{pkj} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\left[\Phi\left(\frac{\text{USL}_j - \overline{X}_j}{S_j}\right) + \Phi\left(\frac{\overline{X}_j - \text{LSL}_j}{S_j}\right)\right]\right\}. \tag{4}$$

As a result, the natural estimator of S_{pk}^T is determined by replacing S_{pkj} with the estimators of v individual indices $\widehat{S}_{pkj}$ in (1), then we have

$$\widehat{S}_{pk}^T = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\left[\prod_{j=1}^{v}(2\Phi(3\widehat{S}_{pkj}) - 1) + 1\right]\right\}. \tag{5}$$

Pearn and Cheng [30] showed that the approximate sampling distribution of $\widehat{S}_{pk}^{T}$ is a normal distribution as

$$\widehat{S}_{pk}^{T} \sim N\left(S_{pk}^{T}, \frac{(S_{pk}^{T})^2}{2n}\right). \tag{6}$$

As mentioned earlier, two assumptions normality and independence of characteristics are essential to design and apply S_{pk}^{T}. Yet, these considered hypotheses may be invalid during practical problems. Box-Cox powerful transformations can be helpful to transform the distribution of a non-normal data set into a normal one. In addition, a recommended way to solve the correlation restriction is to apply the PCA for converting a number of correlated variables into a number of uncorrelated linear variables [30]. Briefly, assume the variance-covariance matrix $\sum$ related to the random vector $\mathbf{X}$ has the eigenvalue-eigenvector couples $(e_1, \mathbf{u}_1), (e_2, \mathbf{u}_2), \ldots, (e_v, \mathbf{u}_v)$. Also, let $\mathbf{LSL} = (\mathrm{LSL}_1, \mathrm{LSL}_2, \cdots, \mathrm{LSL}_v)'$ and $\mathbf{USL} = (\mathrm{USL}_1, \mathrm{USL}_2, \cdots, \mathrm{USL}_v)'$ be the lower and upper specification limit vectors, respectively, related to each quality characteristic. The linear combinations of the original variables $\mathbf{X} = (X_1, X_2, \ldots, X_v)'$ which are defined by $\mathrm{PC}_j = \mathbf{e}_j'\mathbf{X} = e_{j1}X_1 + e_{j2}X_2 + \cdots + e_{jv}X_v$, for $j = 1, 2, \ldots, v$, are called the principal components of $\mathbf{X}$. For the transformed vector $\mathbf{PC} = (\mathrm{PC}_1, \mathrm{PC}_2, \ldots, \mathrm{PC}_v)'$, we have $\sum_{j=1}^{v} Var(X_j) = \sum_{j=1}^{v} Var(\mathrm{PC}_j) = \sum_{j=1}^{v} e_j$ (see [18]). The ratio $\frac{\sum_{j=1}^{k} e_j}{\sum_{j=1}^{v} e_j}$ is the portion of explained variation due to the first k principal components of the total process variation, which can be applied to reduce the dimension by determining the significant components without losing remarkable information. The components of $\mathbf{PC}$ can be treated as new individual quality characteristics, which are mutually independent and normally distributed. Pearn et al. [32] developed the S_{pk}^{T} and proposed index $TS_{pk,PC}$ to measure the performance of a process based on principal components as

$$TS_{pk,PC} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\left[\prod_{j=1}^{v}\left(2\Phi(3S_{pkj,PC_j}) - 1\right) + 1\right]\right\}, \tag{7}$$

where S_{pkj,PC_j} denotes the univariate measure of process yield index for jth principal component PC_j. We note that the natural estimator $\widehat{TS}_{pk,PC}$ can be obtained by inserting $\widehat{S}_{pkj,PC_j}$ instead of S_{pkj,PC_j} in (7), where

$$\widehat{S}_{pkj,PC_j} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\left[\Phi\left(\frac{\mathrm{USL}_{PC_j} - \overline{X}_{PC_j}}{S_{PC_j}}\right) + \Phi\left(\frac{\overline{X}_{PC_j} - \mathrm{LSL}_{PC_j}}{S_{PC_j}}\right)\right]\right\}, \tag{8}$$

in which $\overline{X}_{PC_j}$, S_{PC_j}, USL_{PC_j} and LSL_{PC_j} are the mean, standard deviation, upper and lower specification limits of PC_j, respectively, so that $\overline{X}_{\mathrm{PC}_j} = \mathbf{u}_j'\overline{\mathbf{X}}$, $S_{\mathrm{PC}_j}^2 = e_j$, $\mathrm{LSL}_{\mathrm{PC}_j} = \mathbf{u}_j'\mathbf{LSL}$ and $\mathrm{USL}_{\mathrm{PC}_j} = \mathbf{u}_j'\mathbf{USL}$ for $j = 1, 2, \cdots, v$.

Remark 2.1. *If $\boldsymbol{T} = (T_1, T_2, \ldots, T_v)'$ is the process target vector whose jth component related to the target value of jth interested quality characteristic, then the target value of PC_j is defined as $\mathrm{T}_{\mathrm{PC}_j} = \boldsymbol{u}_j'\mathbf{T}$.*

We use this method in a real world example to develop our proposed approach practically to a larger class of quality characteristics in Section 5.

2.3 Critical value and p-value to test S_{pk}^{T} with crisp data

To determine whether a production process is capable or not, one considers the statistical testing with null hypothesis $\mathrm{H}_\circ$: Yield $\leq c$ (process is incapable) versus alternative hypothesis H_1 : Yield $> c$ (process is capable), where c is a designed constant. Since there exists a one-to-one relationship (see (3)) between the production yield and the yield index S_{pk}^{T}, testing the above hypotheses is equivalent to testing $\mathrm{H}_\circ : S_{pk}^{T} \leq s$ versus $\mathrm{H}_1 : S_{pk}^{T} > s$, where $s = \frac{1}{3}\Phi^{-1}(\frac{c+1}{2})$. For crisp sample data, one can apply two decision rules based on critical value or p-value to assess the process performance. Suppose the probability that an incapable process is mistakenly considered capable, is α (the significant level of statistical testing). Then, the hypothesis $\mathrm{H}_\circ$ is rejected if $\widehat{S}_{pk}^{T} > s_\circ$. The $s_\circ$, which is just as a function of given-fixed s, is the critical value that can be expressed as (see [30])

$$s_\circ = s + \frac{z_\alpha s}{\sqrt{2n}}, \tag{9}$$

where z_α is the upper 100α percentile of the standard normal distribution.

Moreover, one can compare the p-value, the probability that $\widehat{S}_{pk}^T$ is greater than the observed index value s^* , with the test size α. If the calculated p-value is less than α, then we reject the null hypothesis and declare that the process is capable. The p-value can be calculated as

$$\text{p-value} = P(\widehat{S}_{pk}^T > s^* \mid S_{pk}^T = s) = P\Big(Z > \frac{(s^* - s)\sqrt{2n}}{s}\Big) = 1 - \Phi\Big(\frac{(s^* - s)\sqrt{2n}}{s}\Big). \quad (10)$$

3 Fuzzy estimation of S_{pk}^T with uncertain measurements

Quality inspectors often need to examine two or more quality characteristics of manufacturing process in order to evaluate the process performance, simultaneously. Although in some cases, the recorded characteristics are crisp, they cannot be precisely monitored or collected in many real world situations. Hence, the conventional crisp-based approach cannot be used in situations involving insufficiently precise measurements of multiple product quality. Therefore, in this section we propose the fuzzy estimator of S_{pk}^T by an extended version of the approach of Buckley [7] for a process with mutually independent and multivariate normally distributed characteristics. First, some related notions and properties of fuzzy sets are reminded. For more details about fuzzy sets, refer to [8, 14, 24, 40].

Definition 3.1. *Let R be the set of real numbers. A fuzzy subset $\tilde{A}$ of R is determined by its membership function $\eta_{\tilde{A}} : R \to [0,1]$, which assigns to each element $x \in R$ a real number $\eta_{\tilde{A}}(x)$ in the interval $[0,1]$, where the value $\eta_{\tilde{A}}(x)$ is termed the grade of membership of x in $\tilde{A}$. $\tilde{A}$ is named a normal fuzzy subset if there is an x such that $\eta_{\tilde{A}}(x) = 1$, and $\tilde{A}$ is called a convex fuzzy subset if $\eta_{\tilde{A}}(\gamma x_1 + (1-\gamma)x_2) \geq min(\eta_{\tilde{A}}(x_1), \eta_{\tilde{A}}(x_2))$ for all x_1, x_2 in R and all $\gamma \in [0,1]$.*

Definition 3.2. *A λ-cut interval of a fuzzy subset $\tilde{A}$ is a non-fuzzy set of elements belonging to $\tilde{A}$, at least to the degree of membership λ, which is denoted by $\tilde{A}[\lambda] = \{x \in R \mid \eta_{\tilde{A}}(x) \geq \lambda\}$, for $\lambda \in [0,1]$.*

Definition 3.3. *A fuzzy subset $\tilde{A}$ is called a fuzzy number if (a) $\tilde{A}$ is a normal and convex fuzzy subset, (b) $\eta_{\tilde{A}}$ is upper semi-continuous (the function $\eta_{\tilde{A}}$ is said to be upper semi-continuous at the point $x_\circ \in R$ if $\eta_{\tilde{A}}(x_\circ) \geq limsup_{x \to x_\circ} \eta_{\tilde{A}}(x)$), (c) The λ-cut of $\tilde{A}$ is bounded for each $\lambda \in [0,1]$.*

From Zadeh [46], if $\tilde{A}$ is a fuzzy number, then $\tilde{A}[\lambda]$ is a closed interval that is denoted by $\tilde{A}[\lambda] = [L_{\tilde{A}}(\lambda), R_{\tilde{A}}(\lambda)]$, where $L_{\tilde{A}}(\lambda)$ and $R_{\tilde{A}}(\lambda)$ are the lower and upper bounds of the closed interval $\tilde{A}[\lambda]$, respectively, and we have $L_{\tilde{A}}(\lambda) = inf\{x \in R, \eta_{\tilde{A}}(x) \geq \lambda\}$ and $R_{\tilde{A}}(\lambda) = sup\{x \in R, \eta_{\tilde{A}}(x) \geq \lambda\}$.

Definition 3.4. *A trapezoidal fuzzy quantity $\tilde{A}$ is a fuzzy set denoted $T_r = (a_1, a_2, a_3, a_4)$ if its membership function is as follows:*

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & , \quad x < a_1, \\ \frac{x-a_1}{a_2-a_1} & , \quad a_1 \leq x < a_2, \\ 1 & , \quad a_2 \leq x \leq a_3, \\ \frac{a_4-x}{a_4-a_3} & , \quad a_3 < x \leq a_4, \\ 0 & , \quad a_4 < x. \end{cases}$$

Trapezoidal fuzzy quantity with $a_2 = a_3$ is called triangular fuzzy number.

Definition 3.5. *The ranking function is one of the tools that is applied to order fuzzy quantities. If $\tilde{M}$ and $\tilde{N}$ are two fuzzy quantities, then $\tilde{M} \geq_R \tilde{N}$ if and only if $R(\tilde{M}) \geq R(\tilde{N})$, so that for each fuzzy quantity $\tilde{W}$ with the lower and upper bounds of λ-cut interval, $L_{\tilde{W}}(\lambda)$ and $R_{\tilde{W}}(\lambda)$, $R(\tilde{W})$ is defined as $R(\tilde{W}) = \frac{1}{2}\int_0^1 L_{\tilde{W}}(\lambda) + R_{\tilde{W}}(\lambda) d\lambda$ (see [16]).*

Proposition 3.6. *Let $\tilde{Z}_1, \tilde{Z}_2, \ldots, \tilde{Z}_v$ be v fuzzy numbers with their corresponding membership functions $\eta_{\tilde{Z}_1}(z_1)$, $\eta_{\tilde{Z}_2}(z_2)$, $\ldots$, $\eta_{\tilde{Z}_v}(z_v)$, respectively. We can combine these fuzzy numbers into a v-dimensional fuzzy vector $\tilde{\boldsymbol{Z}} = (\tilde{Z}_1, \tilde{Z}_2, \ldots, \tilde{Z}_v)$ which is determined by a membership function vector $\eta_{\tilde{\boldsymbol{Z}}}(z_1, z_2, \cdots, z_v)$. The function $\eta_{\tilde{\boldsymbol{Z}}} : R^v \to [0,1]$ has the following properties (see [15])*
(a) $0 \leq \eta_{\tilde{\boldsymbol{Z}}}(\boldsymbol{z}) \leq 1$ for $\boldsymbol{z} = (z_1, z_2, \cdots, z_v) \in R^v$,
(b) $sup_{\boldsymbol{z} \in R^v} \eta_{\tilde{\boldsymbol{Z}}}(\boldsymbol{z}) = 1$,
(c) for all $\lambda \in (0,1]$, the λ-cuts of $\tilde{\boldsymbol{Z}}$, $\tilde{\boldsymbol{Z}}[\lambda] = [L_{\tilde{\boldsymbol{Z}}}(\lambda), R_{\tilde{\boldsymbol{Z}}}(\lambda)]$, is closed compact and convex subset of R^v.

From Filzmoser and Viertl [15], one way to combine the membership functions $\eta_{\tilde{Z}_1}(z_1)$, $\eta_{\tilde{Z}_2}(z_2)$, $\ldots, \eta_{\tilde{Z}_v}(z_v)$ into the membership function vector $\eta_{\tilde{\boldsymbol{Z}}}(\boldsymbol{z})$ is the minimum combination rule:

$$\eta_{\tilde{\boldsymbol{Z}}}(\boldsymbol{z}) = min\{\eta_{\tilde{Z}_1}(z_1), \eta_{\tilde{Z}_2}(z_2), \ldots, \eta_{\tilde{Z}_v}(z_v)\}, \quad \forall \boldsymbol{z} \in R^v.$$

Moreover, the λ-cuts $\tilde{\boldsymbol{Z}}[\lambda]$ are Cartesian products of the λ-cuts of the v fuzzy numbers $\tilde{Z}_1, \tilde{Z}_2, \ldots, \tilde{Z}_v$, i.e.

$$\tilde{\boldsymbol{Z}}[\lambda] = \tilde{Z}_1[\lambda] \times \tilde{Z}_2[\lambda] \times \ldots \times \tilde{Z}_v[\lambda], \quad \forall \lambda \in (0,1].$$

Proposition 3.7. *Suppose that F is the set of all fuzzy numbers. Let $g : R^v \to R$ be a real-valued function, and $\tilde{Z}_1, \tilde{Z}_2, \ldots, \tilde{Z}_v$ be v fuzzy numbers. Assume $g : F^v \to F$ is a fuzzy-valued function induced by $g(z_1, z_2, \ldots, z_v)$. Subsequently, $\tilde{w} = g(\tilde{Z}_1, \tilde{Z}_2, \ldots, \tilde{Z}_v)$ is again a fuzzy number so that under the extension principle (see [47, 48, 49]) its membership function, $\forall w \in R$, is as follows:*

$$\eta_{\tilde{w}}(w) = \begin{cases} sup\Big\{\eta_{\tilde{\boldsymbol{Z}}}(z_1, z_2, \ldots, z_v) : g(z_1, z_2, \ldots, z_v) = w\Big\} & if \quad g^{-1}(w) \neq 0, \\ 0 & if \quad g^{-1}(w) = 0. \end{cases}$$

In addition, the λ-cut of $\tilde{w}$ is given as $\tilde{w}[\lambda] = [L_{\tilde{w}}(\lambda), R_{\tilde{w}}(\lambda)]$, where $L_{\tilde{w}}(\lambda) = inf_{w \in \tilde{w}[\lambda]}(w)$ and $R_{\tilde{w}}(\lambda) = sup_{w \in \tilde{w}[\lambda]}(w)$ are the lower and upper bounds of $\tilde{w}[\lambda]$ (see [13]).

Remark 3.8. *If g is a continuous and monotonic function in each variable, then in order to determine the lower and upper bounds of $\tilde{w}[\lambda]$, one needs only to find the value of function g at the endpoints of the λ-cuts $\tilde{Z}_1[\lambda]$, $\tilde{Z}_2[\lambda]$, ..., $\tilde{Z}_v[\lambda]$.*

3.1 Buckley's approach to produce triangular shaped fuzzy numbers

Here, we review the approach, which was applied by Buckley [7], to obtain the triangular shaped fuzzy numbers from a set of confidence intervals. Let Y be a random variable with probability density function (PDF) $f(y, \theta)$ for individual parameter θ. Let θ be unknown and it should be estimated through a random sample Y_1, Y_2, ..., Y_n. We can obtain a point estimate θ^* for θ by using the given values of these random variables $Y_i = y_i$, for $i = 1, 2, \ldots, n$ based on an appropriate statistic such as $H = h(Y_1, Y_2, \ldots, Y_n)$. But we would never expect this point estimate exactly equals θ, so we often compute a $(1-\beta)\%$ confidence interval for θ as $[L(\beta), R(\beta)]$, $\forall \beta \in (0,1)$. The 0% confidence interval for θ is obtained as $[L(100\%), R(100\%)] = [\hat{\theta}, \hat{\theta}]$, and also $[L(0\%), R(0\%)] = \Theta$ is the 100% confidence interval so that Θ is the whole parameter space. Buckley [7] put these confidence intervals, one on top of the other, to construct a triangular shaped fuzzy number $\tilde{\hat{\theta}}$ whose λ-cut sets are the confidence intervals. In this way, one uses more information in $\tilde{\hat{\theta}}$ than just a point estimator, or just a single interval estimate.

3.2 λ-cut of fuzzy estimator for S_{pk}^T

Assume that interested multiple quality characteristics follow a multivariate normal distribution such that each characteristic has two-sided specification limits. Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be the collected observations. According to (1), it can be seen that the advanced index S_{pk}^T is a function of v individual indices S_{pk_1}, S_{pk_2}, ..., S_{pk_v} such as $S_{pk}^T = g(S_{pk_1}, S_{pk_2}, \ldots, S_{pk_v})$. From (5), the estimate of S_{pk}^T is obtained by replacing S_{pk_j} with their corresponding estimation $\hat{S}_{pk_j}$ $(j = 1, 2, \ldots, v)$. On the other hand, equation (4) displays that each individual estimation $\hat{S}_{pk_j}$ is a real-valued function of observations such as $w_j = f(x_{j1}, x_{j2}, \ldots, x_{jn})$. However, the collected observations for estimating S_{pk_j} may be vague in most real applications. In this case, the most logical method to estimate S_{pk_j} is to treat the original observations as imprecise data and describe them by the fuzzy numbers $\tilde{x}_{j1}, \tilde{x}_{j2}, \ldots, \tilde{x}_{jn}$. Hence, in order to estimate S_{pk_j}, the fuzzy estimations $\tilde{\overline{X}}_j = \frac{1}{n}\sum_{i=1}^n \tilde{X}_{ji}$ and $\tilde{S}_j = \sqrt{\frac{1}{n-1}\sum_{i=1}^n (\tilde{X}_{ji} - \tilde{\overline{X}}_j)^2}$ are used instead of $\overline{X}_j$ and S_j, respectively, in (4). By the concept of Buckley [7], Wu and Liao [41] produced the fuzzy estimation of S_{pk_j} ($\tilde{\hat{S}}_{pk_j}$) whose λ-cut set is as follows

$$\begin{aligned} \tilde{\hat{S}}_{pk_j}[\lambda] &= \Big[L_{\tilde{\hat{S}}_{pk_j}}(\lambda), R_{\tilde{\hat{S}}_{pk_j}}(\lambda)\Big], \\ &= \Bigg[\frac{1}{3}\Phi^{-1}\Bigg\{\frac{1}{2}\Phi\Bigg(\frac{\mathrm{USL}_j - \overline{X}_j - t_{\frac{\lambda}{2},n-1}\frac{S_j}{\sqrt{n}}}{\sqrt{(n-1)S_j^2/\chi^2_{1-\frac{\lambda}{2},n-1}}}\Bigg) + \frac{1}{2}\Phi\Bigg(\frac{\overline{X}_j + t_{\frac{\lambda}{2},n-1}\frac{S_j}{\sqrt{n}} - \mathrm{LSL}_j}{\sqrt{(n-1)S_j^2/\chi^2_{1-\frac{\lambda}{2},n-1}}}\Bigg)\Bigg\}, \\ &\quad \frac{1}{3}\Phi^{-1}\Bigg\{\frac{1}{2}\Phi\Bigg(\frac{\mathrm{USL}_j - \overline{X}_j + t_{\frac{\lambda}{2},n-1}\frac{S_j}{\sqrt{n}}}{\sqrt{(n-1)S_j^2/\chi^2_{\frac{\lambda}{2},n-1}}}\Bigg) + \frac{1}{2}\Phi\Bigg(\frac{\overline{X}_j - t_{\frac{\lambda}{2},n-1}\frac{S_j}{\sqrt{n}} - \mathrm{LSL}_j}{\sqrt{(n-1)S_j^2/\chi^2_{\frac{\lambda}{2},n-1}}}\Bigg)\Bigg\}\Bigg], \end{aligned} \tag{11}$$

for $\overline{X}_j \geq T_j$ (T_j is the target value of jth quality characteristic), and

$$\begin{aligned}\tilde{\hat{S}}_{pk_j}[\lambda] &= \left[L_{\tilde{\hat{S}}_{pk_j}}(\lambda), R_{\tilde{\hat{S}}_{pk_j}}(\lambda)\right],\\ &= \left[\frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{\mathrm{USL}_j - \overline{X}_j + t_{\frac{\lambda}{2},n-1}\frac{S_j}{\sqrt{n}}}{\sqrt{(n-1)S_j^2/\chi^2_{1-\frac{\lambda}{2},n-1}}}\right) + \frac{1}{2}\Phi\left(\frac{\overline{X}_j - t_{\frac{\lambda}{2},n-1}\frac{S_j}{\sqrt{n}} - \mathrm{LSL}_j}{\sqrt{(n-1)S_j^2/\chi^2_{1-\frac{\lambda}{2},n-1}}}\right)\right\},\right.\\ &\quad \left.\frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{\mathrm{USL}_j - \overline{X}_j - t_{\frac{\lambda}{2},n-1}\frac{S_j}{\sqrt{n}}}{\sqrt{(n-1)S_j^2/\chi^2_{\frac{\lambda}{2},n-1}}}\right) + \frac{1}{2}\Phi\left(\frac{\overline{X}_j + t_{\frac{\lambda}{2},n-1}\frac{S_j}{\sqrt{n}} - \mathrm{LSL}_j}{\sqrt{(n-1)S_j^2/\chi^2_{\frac{\lambda}{2},n-1}}}\right)\right\}\right],\end{aligned} \tag{12}$$

for $\overline{X}_j < T_j$, where $t_{\frac{\lambda}{2},n-1}$ is the upper $\frac{\lambda}{2}$ percentile of the t distribution with $n-1$ degrees of freedom, $\chi^2_{\frac{\lambda}{2},n-1}$ and $\chi^2_{1-\frac{\lambda}{2},n-1}$ are the upper $\frac{\lambda}{2}$ and $1-\frac{\lambda}{2}$ percentile of the Chi-Square distribution with $n-1$ degrees of freedom, respectively.

Now, we focus on finding λ-cut intervals of imprecise estimate of S^T_{pk} ($\tilde{\hat{S}}^T_{pk}$) based on the Buckley's approach. Let $\tilde{\hat{S}}_{pk_1}$, $\tilde{\hat{S}}_{pk_2}$, $\ldots$, $\tilde{\hat{S}}_{pk_v}$ be fuzzy estimators of v individual indices S_{pk_1}, S_{pk_2}, $\ldots$, S_{pk_v}, respectively, which are obtained by Buckley's approach. Then, we combine these fuzzy numbers into a v-dimensional fuzzy number vector as $(\tilde{\hat{S}}_{pk_1}, \tilde{\hat{S}}_{pk_2}, \ldots, \tilde{\hat{S}}_{pk_v})$. According to Proposition 3.6, the λ-cut interval $(\tilde{\hat{S}}_{pk_1}, \tilde{\hat{S}}_{pk_2}, \ldots, \tilde{\hat{S}}_{pk_v})[\lambda]$ is obtained by the Cartesian products of the λ-cut intervals of the v fuzzy numbers $\tilde{\hat{S}}_{pk_j}$ $(j = 1, 2, \ldots, v)$, i.e., $\forall\lambda \in (0, 1]$ we have

$$(\tilde{\hat{S}}_{pk_1}, \tilde{\hat{S}}_{pk_2}, \ldots, \tilde{\hat{S}}_{pk_v})[\lambda] = \tilde{\hat{S}}_{pk_1}[\lambda] \times \tilde{\hat{S}}_{pk_2}[\lambda] \times \ldots \times \tilde{\hat{S}}_{pk_v}[\lambda].$$

To obtain the λ-cut sets of $\tilde{\hat{S}}^T_{pk}$, assume that $a_1 \in \tilde{\hat{S}}_{pk_1}[\lambda]$, $a_2 \in \tilde{\hat{S}}_{pk_2}[\lambda]$, $\ldots$, $a_v \in \tilde{\hat{S}}_{pk_v}[\lambda]$, then from (5) we have

$$\widehat{S}^T_{pk}(a_1, a_2, \ldots, a_v) = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\left[\prod_{j=1}^{v}(2\Phi(3a_j) - 1) + 1\right]\right\}, \tag{13}$$

where a_j $(j = 1, 2, \ldots, v)$ is ranged through its interval. From (13), one can conclude that $\widehat{S}^T_{pk}(a_1, a_2, \ldots, a_v)$ is always an increasing function of a_j $(j = 1, 2, \ldots, v)$. Therefore, according to Remark 3.8 it results

$$\begin{aligned}\tilde{\hat{S}}^T_{pk}[\lambda] &= \left[L_{\tilde{\hat{S}}^T_{pk}}(\lambda), R_{\tilde{\hat{S}}^T_{pk}}(\lambda)\right],\\ &= \left[\frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\left[\prod_{j=1}^{v}(2\Phi(3L_{\tilde{\hat{S}}_{pk_j}}(\lambda)) - 1) + 1\right]\right\},\ \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\left[\prod_{j=1}^{v}(2\Phi(3R_{\tilde{\hat{S}}_{pk_j}}(\lambda)) - 1) + 1\right]\right\}\right],\end{aligned} \tag{14}$$

in which $L_{\tilde{\hat{S}}_{pk_j}}(\lambda)$ and $R_{\tilde{\hat{S}}_{pk_j}}(\lambda)$ are the lower and upper bounds of $\tilde{\hat{S}}_{pk_j}[\lambda]$ obtained by (11) and (12).

4 Assessing the process performance based on fuzzy estimator

In this section, we construct decision rules based on both critical value and p-value, separately, to test the process performance.

4.1 Critical-value-based decision rule and testing method

For the first time, Neyman and Pearson [23] represented a three-decision testing problem, that is (a) accept $\mathrm{H}_\circ$ and reject H_1, (b) reject $\mathrm{H}_\circ$ and accept H_1, (c) neither accept nor reject both $\mathrm{H}_\circ$ and H_1. As mentioned earlier, we want to test $\mathrm{H}_\circ : S^T_{pk} \leq s$ versus $\mathrm{H}_1 : S^T_{pk} > s$ in the presence of imprecise measurements. In fact, this hypothesis testing would be done with fuzzy test statistic $\tilde{\hat{S}}^T_{pk}$ whose λ-cut interval is obtained by (14). To judge the process performance based on critical value, it needs to obtain the critical value $s_\circ$ using (9). The ultimate decision thus depends on the relation between the fuzzy test statistic $\tilde{\hat{S}}^T_{pk}$ and $s_\circ$. To recognize the relationship between $\tilde{\hat{S}}^T_{pk}$ and $s_\circ$, we compare the position of $s_\circ$ to the λ-cuts of $\tilde{\hat{S}}^T_{pk}$. According to (14), since $\forall\lambda \in (0,1]$ all λ-cuts $\tilde{\hat{S}}^T_{pk}[\lambda]$ are closed, the finite interval $\left[L_{\tilde{\hat{S}}^T_{pk}}(\lambda), R_{\tilde{\hat{S}}^T_{pk}}(\lambda)\right]$ can be tested with the critical value $s_\circ$. As a result, a new fuzzy three-decision testing is proposed as follows

(a) If $L_{\tilde{\hat{S}}^T_{pk}}(\lambda) > s_\circ$, then reject $H_\circ$ and accept H_1.

(b) If $R_{\tilde{\hat{S}}^T_{pk}}(\lambda) < s_\circ$, then accept $H_\circ$ and reject H_1.

(c) If $L_{\tilde{\hat{S}}^T_{pk}}(\lambda) \le s_\circ \le R_{\tilde{\hat{S}}^T_{pk}}(\lambda)$, then compute the value of degree d, which is defined as $d = \frac{R_{\tilde{\hat{S}}^T_{pk}}(\lambda) - \mathrm{Max}\left\{L_{\tilde{\hat{S}}^T_{pk}}(\lambda), s_\circ\right\}}{R_{\tilde{\hat{S}}^T_{pk}}(\lambda) - L_{\tilde{\hat{S}}^T_{pk}}(\lambda)}$. If d (or $1-d$) is close to one, then reject (or accept) $H_\circ$ with degree d (or $1-d$) at the level λ. Otherwise, neither accept nor reject both $H_\circ$ and H_1. Here, "close" defines that d (or $1-d$) is not smaller than 0.9.

Remark 4.1. *If $L_{\tilde{\hat{S}}^T_{pk}}(\lambda) = R_{\tilde{\hat{S}}^T_{pk}}(\lambda)$, then the proposed three-decision testing rule with imprecise measurements is converted to crisp case with a binary testing rule.*

In the following, we give a brief statement of the operating procedure of the proposed testing rule step by step to assess whether the process satisfies the pre-set capability requirements or not.

Step 1. Set the capability requirement s (or determine the definition of "capable"), and α risk (the chance of incorrectly concluding an incapable process as capable, which is usually set to 0.01, 0.025, and 0.05).

Step 2. Compute the critical value $s_\circ$ using (9) based on the given values s, α, and sample size n.

Step 3. Specify the user-approved degree of ambiguity λ of the sample data.

Step 4. For a specified λ in Step 3, calculate $\tilde{\hat{S}}^T_{pk}[\lambda] = \left[L_{\tilde{\hat{S}}^T_{pk}}(\lambda), R_{\tilde{\hat{S}}^T_{pk}}(\lambda)\right]$ applying (14).

Step 5. Make a decision about the process performance. Conclude that the process is capable if $L_{\tilde{\hat{S}}^T_{pk}}(\lambda) > s_\circ$. Derive that the process is incapable if $R_{\tilde{\hat{S}}^T_{pk}}(\lambda) < s_\circ$. Otherwise, one is not able to decide surely about the capability or incapability of the process, so that the process is considered capable with d degree or incapable with $1-d$ degree at level λ if d (or $1-d$) is close to one.

4.2 Fuzzy p-value-based decision rule and testing method

The p-value, which displays the real risk of incorrect judging an incapable process as a capable one, is also widely applied to make decisions in testing process performance. Similar to what we did find the λ-cut interval of $\tilde{\hat{S}}^T_{pk}$, one can obtain the λ-cut set of fuzzy p-value ($\tilde{p}$) which is shown by symbol $\tilde{p}[\lambda]$. By knowing that $\forall\lambda \in (0,1]$ the λ-cut intervals $\tilde{\hat{S}}^T_{pk}[\lambda] = \left[L_{\tilde{\hat{S}}^T_{pk}}(\lambda), R_{\tilde{\hat{S}}^T_{pk}}(\lambda)\right]$ are closed and finite intervals, they can be used to determine the corresponding λ-cut intervals of $\tilde{p}$. So, we have

$$\begin{aligned}
\tilde{p}[\lambda] &= \left[L_{\tilde{p}}(\lambda), R_{\tilde{p}}(\lambda)\right], \\
&= \left[P\left(\hat{S}_{pk} > R_{\tilde{\hat{S}}^T_{pk}}(\lambda) \mid S^T_{pk} = s\right), P\left(\hat{S}_{pk} > L_{\tilde{\hat{S}}^T_{pk}}(\lambda) \mid S^T_{pk} = s\right)\right], \\
&= \left[1 - \Phi\left(\frac{\left(R_{\tilde{\hat{S}}^T_{pk}}(\lambda) - s\right)\sqrt{2n}}{s}\right), 1 - \Phi\left(\frac{\left(L_{\tilde{\hat{S}}^T_{pk}}(\lambda) - s\right)\sqrt{2n}}{s}\right)\right]. \qquad (15)
\end{aligned}$$

It is clear that $\forall\lambda \in (0,1]$, $L_{\tilde{p}}(\lambda) > 0$ and $R_{\tilde{p}}(\lambda) \le 1$. Hence, $\tilde{p}[\lambda]$ can be explained as probabilities and compared with α risk of the test. Consequently, the fuzzy p-value-based decision rule can be constructed by recognizing the position of α according to the λ-cuts $\left[L_{\tilde{p}}(\lambda), R_{\tilde{p}}(\lambda)\right]$ as follows

(a) If $R_{\tilde{p}}(\lambda) < \alpha$, then reject $H_\circ$ and accept H_1.

(b) If $L_{\tilde{p}}(\lambda) > \alpha$, then accept $H_\circ$ and reject H_1.

(c) If $L_{\tilde{p}}(\lambda) \leq \alpha \leq R_{\tilde{p}}(\lambda)$, then calculate the value of degree $d^{'}$, which is defined as $d^{'} = \frac{\text{Min}\left\{R_{\tilde{p}}(\lambda),\alpha\right\} - L_{\tilde{p}}(\lambda)}{R_{\tilde{p}}(\lambda) - L_{\tilde{p}}(\lambda)}$. If $d^{'}$ (or $1-d^{'}$) is close to one, then reject (or accept) $\text{H}_\circ$ with degree $d^{'}$ (or $1-d^{'}$) at level λ. Otherwise, neither accept nor reject both $\text{H}_\circ$ and H_1. Here, "close" defines that $d^{'}$ (or $1-d^{'}$) is not smaller than 0.9.

Remark 4.2. *If $L_{\tilde{p}}(\lambda) = R_{\tilde{p}}(\lambda)$, then the proposed fuzzy p-value-based decision testing rule with imprecise measurements is turned to crisp case with a binary testing rule.*

Finally, the summary of the operating procedure of the proposed p-value-based testing rule with ambiguous measurements is given step-by-step as follows

Step 1. Define the capability requirement s, and α risk.

Step 2. Determine the user-approved degree of ambiguity λ of the sample data.

Step 3. For the specified λ in Step 2, calculate $\tilde{\hat{S}}^T_{pk}[\lambda] = \left[L_{\tilde{\hat{S}}^T_{pk}}(\lambda), R_{\tilde{\hat{S}}^T_{pk}}(\lambda)\right]$ using (14).

Step 4. Calculate $\tilde{p}[\lambda] = \left[L_{\tilde{p}}(\lambda), R_{\tilde{p}}(\lambda)\right]$ using (15) based on the given values s and λ.

Step 5. Form a judgement about the process performance. Conclude that the process is capable if $R_{\tilde{p}}(\lambda) < \alpha$. Decide that the process is incapable if $L_{\tilde{p}}(\lambda) > \alpha$. Otherwise, one is not able to judge surely about the capability or incapability of the process, so that the process is considered capable with $d^{'}$ degree or incapable with $1-d^{'}$ degree at level λ if $d^{'}$ (or $1-d^{'}$) is close to one.

5 Comparative study and application in industry

In this section, two practical examples are given to clarify the operating of the proposed method to determine whether the manufacturing process satisfies the pre-set capability requirement or not. Some comparative studies are also conducted.

Example 5.1 (Independent quality characteristics)**.** *In this example, we want to test the performance of three processes A, B, and C whose quality characteristics X_1, X_2 are bivariate normally distributed and independent. The advanced yield index S^T_{pk} is used to test the process performance. Let the process target and the engineering specification limits of product characteristics be $\boldsymbol{T} = (177, 53)^{'}$, $\mathbf{LSL} = (112.7, 32.7)^{'}$ and $\mathbf{USL} = (241.3, 73.3)^{'}$, respectively. The capability requirement is described $s = 1$ based on the written agreement between producer and consumer. In fact, we aim to measure that the mentioned processes meet the capability requirement $S^T_{pk} > 1$, which is equivalent to having overall nonconformities not exceeding 2699 ppm. The sample means and sample variances are computed for each process when $n = 25$ as follows*

process A: $\overline{\mathbf{X}}_{(A)} = (176.50, 53.04)^{'}$, $\mathbf{S^2}_{(A)} = (350.00, 40.00)^{'}$,

process B: $\overline{\mathbf{X}}_{(B)} = (180.00, 54.00)^{'}$, $\mathbf{S^2}_{(B)} = (295.00, 27.01)^{'}$,

process C: $\overline{\mathbf{X}}_{(C)} = (178.00, 53.99)^{'}$, $\mathbf{S^2}_{(C)} = (180.00, 20.00)^{'}$.

To judge the yield performance, $\text{H}_\circ : S^T_{pk} \leq 1$ versus $\text{H}_1 : S^T_{pk} > 1$, we assume that the nominal size α is equal to 0.05. Then, from (9) the critical value is obtained as $s_\circ = 1.2326$. According to equations (11) and (12) and using sample information, λ-cut intervals of individual $\tilde{\hat{S}}_{pk_j}$ $(j = 1, 2)$ for process A are obtained as follows

$$\begin{aligned}
\tilde{\hat{S}}_{pk_1(A)}[\lambda] &= \left[L_{\tilde{\hat{S}}_{pk_1(A)}}(\lambda), R_{\tilde{\hat{S}}_{pk_1(A)}}(\lambda)\right], \\
&= \Bigg[\frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{64.5 + 3.741651 t_{\frac{\lambda}{2},24}}{\sqrt{8400/\chi^2_{1-\frac{\lambda}{2},24}}}\right) + \frac{1}{2}\Phi\left(\frac{63.8 - 3.741651 t_{\frac{\lambda}{2},24}}{\sqrt{8400/\chi^2_{1-\frac{\lambda}{2},24}}}\right)\right\}, \\
&\quad \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{64.5 - 3.741651 t_{\frac{\lambda}{2},24}}{\sqrt{8400/\chi^2_{\frac{\lambda}{2},24}}}\right) + \frac{1}{2}\Phi\left(\frac{63.8 + 3.741651 t_{\frac{\lambda}{2},24}}{\sqrt{8400/\chi^2_{\frac{\lambda}{2},24}}}\right)\right\}\Bigg],
\end{aligned} \tag{16}$$

$$\begin{aligned}
\tilde{\hat{S}}_{pk_2(A)}[\lambda] &= \left[L_{\tilde{\hat{S}}_{pk_2(A)}}(\lambda), R_{\tilde{\hat{S}}_{pk_2(A)}}(\lambda)\right], \\
&= \Bigg[\frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{20.26 - 1.264911 t_{\frac{\lambda}{2},24}}{\sqrt{960/\chi^2_{1-\frac{\lambda}{2},24}}}\right) + \frac{1}{2}\Phi\left(\frac{20.34 + 1.264911 t_{\frac{\lambda}{2},24}}{\sqrt{960/\chi^2_{1-\frac{\lambda}{2},24}}}\right)\right\}, \\
&\quad \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{20.26 + 1.264911 t_{\frac{\lambda}{2},24}}{\sqrt{960/\chi^2_{\frac{\lambda}{2},24}}}\right) + \frac{1}{2}\Phi\left(\frac{20.34 - 1.264911 t_{\frac{\lambda}{2},24}}{\sqrt{960/\chi^2_{\frac{\lambda}{2},24}}}\right)\right\}\Bigg].
\end{aligned} \tag{17}$$

Putting equations (16) and (17) in equation (14), we compute the lower and upper bounds of λ-cut intervals of fuzzy estimator of overall $\tilde{\hat{S}}^T_{pk(A)}$ as follows

$$L_{\tilde{\hat{S}}^T_{pk(A)}}(\lambda) = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\left(\left(2\Phi(3L_{\tilde{\hat{S}}_{pk_1(A)}}(\lambda)) - 1\right)\left(2\Phi(3L_{\tilde{\hat{S}}_{pk_2(A)}}(\lambda)) - 1\right) + 1\right)\right\}, \tag{18}$$

$$R_{\tilde{\hat{S}}^T_{pk(A)}}(\lambda) = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\left(\left(2\Phi(3R_{\tilde{\hat{S}}_{pk_1(A)}}(\lambda)) - 1\right)\left(2\Phi(3R_{\tilde{\hat{S}}_{pk_2(A)}}(\lambda)) - 1\right) + 1\right)\right\}. \tag{19}$$

For instance, if $\lambda = 0.6$, then according to equations (16) and (17) we have

$$\tilde{\hat{S}}_{pk_1(A)}[0.6] = [1.0368, 1.2130],\ \tilde{\hat{S}}_{pk_2(A)}[0.6] = [0.9702, 1.1305].$$

Therefore the lower and upper bound of 0.6-cut set of fuzzy estimator $\tilde{\hat{S}}^T_{pk(A)}$ is obtained as

$$L_{\tilde{\hat{S}}^T_{pk(A)}}(0.6) = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\left(\left(2\Phi(3\times 1.0368) - 1\right)\left(2\Phi(3\times 0.9702) - 1\right) + 1\right)\right\} = 0.9260,$$

$$R_{\tilde{\hat{S}}^T_{pk(A)}}(0.6) = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\left(\left(2\Phi(3\times 1.2130) - 1\right)\left(2\Phi(3\times 1.1305) - 1\right) + 1\right)\right\} = 1.0998,$$

i.e. $\tilde{\hat{S}}^T_{pk(A)}[0.6] = [0.9260, 1.0998]$. *Some λ-cut intervals of $\tilde{\hat{S}}^T_{pk(A)}$ are computed for several values $\lambda \in [0,1]$ and reported in Table 1. Now, putting λ-cut sets, one on the top up the others, the membership function of $\tilde{\hat{S}}^T_{pk(A)}$ is made for process A as demonstrated in Figure 1. Similar to what explained for process A, we compute and set λ-cut intervals of $\tilde{\hat{S}}^T_{pk}$ for processes B and C. The obtained λ-cut intervals for processes B and C are also reported in Table 1. In addition, their corresponding membership functions are illustrated in Figure 1.*

	Process A	Process B	Process C
λ	$\hat{S}^T_{pk(A)}[\lambda]$	$\hat{S}^T_{pk(B)}[\lambda]$	$\hat{S}^T_{pk(C)}[\lambda]$
1	1.0181	1.1749	1.4474
0.95	[1.0075, 1.0286]	[1.1609, 1.1888]	[1.4311, 1.4637]
0.9	[0.9967, 1.0390]	[1.1469, 1.2028]	[1.4148, 1.4801]
0.85	[0.9857, 1.0492]	[1.1329, 1.2169]	[1.3984, 1.4966]
0.8	[0.9744, 1.0594]	[**1.1186**, **1.2311**]	[1.3818, 1.5133]
0.75	[0.9629, 1.0695]	[1.1042, 1.2455]	[1.3650, 1.5303]
0.7	[0.9510, 1.0796]	[1.0895, 1.2601]	[1.3479, 1.5475]
0.65	[0.9388, 1.0897]	[1.0745, 1.2750]	[1.3304, 1.5652]
0.6	[0.9260, 1.0998]	[1.0590, 1.2903]	[1.3125, 1.5833]
0.55	[0.9127, 1.1100]	[1.0431, 1.3059]	[1.2939, 1.6019]
0.5	[0.8988, 1.1202]	[1.0264, 1.3221]	[1.2746, 1.6212]
0.45	[0.8841, 1.1305]	[1.0090, 1.3387]	[1.2544, 1.6412]
0.4	[0.8684, 1.1410]	[0.9906, 1.3561]	[**1.2329**, **1.6621**]
0.35	[0.8514, 1.1517]	[0.9708, 1.3743]	[1.2100, 1.6841]
0.3	[0.8329, 1.1627]	[0.9494, 1.3934]	[1.1851, 1.7074]
0.25	[0.8122, 1.1742]	[0.9256, 1.4137]	[1.1574, 1.7324]
0.2	[0.7885, 1.1863]	[0.8986, 1.4355]	[1.1259, 1.7595]
0.15	[0.7602, 1.1995]	[0.8664, 1.4595]	[1.0884, 1.7896]
0.1	[**0.7237**, **1.2146**]	[0.8254, 1.4869]	[1.0405, 1.8250]
0.05	[0.6690, 1.2338]	[0.7642, 1.5218]	[0.9688, 1.8723]
0.025	[0.6214, 1.2475]	[0.7113, 1.5473]	[0.9065, 1.9094]

Table 1: λ-cut sets of fuzzy estimator of overall $\tilde{\hat{S}}^T_{pk}$ in Example 5.1

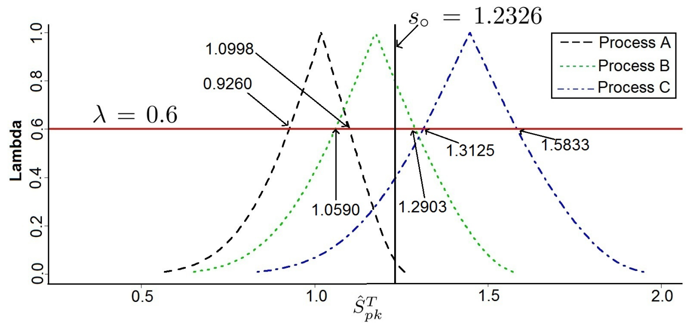

Figure 1: The membership functions of fuzzy estimators $\tilde{\hat{S}}^T_{pk(A)}$, $\tilde{\hat{S}}^T_{pk(B)}$, and $\tilde{\hat{S}}^T_{pk(C)}$ with line 0.6 level

Let the imprecision degree of sample data is specified $\lambda = 0.6$ based on the knowledge of production experts. So, from Table 1 we have

$$\tilde{\hat{S}}^T_{pk(A)}[0.6] = [0.9260, 1.0998],\ \tilde{\hat{S}}^T_{pk(B)}[0.6] = [1.0590, 1.2903],\ \tilde{\hat{S}}^T_{pk(C)}[0.6] = [1.3125, 1.5833],$$

as clarified in Figure 1. Now, we want to assess the performance of the underlying processes at $\lambda = 0.6$. Moreover, the reject boundary of null hypothesis ($s_\circ = 1.2326$) is represented in Figure 1 by vertical line. From Figure 1 and considering the proposed critical-value-based testing rule, it results: Process A is incapable, since $R_{\tilde{\hat{S}}^T_{pk(A)}}(0.6) = 1.0998 < s_\circ = 1.2326$. Also, process C is capable because of $L_{\tilde{\hat{S}}^T_{pk(C)}}(0.6) = 1.3125 > s_\circ = 1.2326$. Finally, we cannot decide surely about the capability or incapability of the process B, because $L_{\tilde{\hat{S}}^T_{pk(B)}}(0.6) = 1.0590 < s_\circ < R_{\tilde{\hat{S}}^T_{pk(B)}}(0.6) = 1.2903$. In this case, according to the proposed testing rule we should compute the value of degree $d = \frac{1.2903-1.2326}{1.2903-1.0590} = 0.24$, so it results $1 - d = 0.76$. Since none of them (d or $1 - d$) is close to one, we neither accept nor reject the hypothesis $H_\circ$ at level $\lambda = 0.6$. It means that we need to take another sample to judge the performance of process B at the specified imprecision level.

Besides, it can be found that the amount of data ambiguity (λ) affects our decision, as presented in Table 1. For $\lambda \geq 0.1$ and $\lambda \geq 0.8$, one derives that the processes A and B are incapable, respectively, but for $\lambda < 0.1$ and $\lambda < 0.8$, we can not make a decision about the capability or incapability of processes A and B, respectively. Similarly, process C is capable for $\lambda \geq 0.4$, while its efficiency is not clearly determined for $\lambda < 0.4$. It means that we should devote a degree of confidence ($0 < d < 1$) to our decision when the critical value ($s_\circ$) is located between the lower and upper bounds of fuzzy estimator. If the value assigned to our decision (d or $1-d$) is close to one, then we make our final decision about capability or incapability of the process based on the corresponding degree, otherwise, additional information is required by taking another sample to reach a definitive final decision. For instance, if $\lambda = 0.75$, then the final decision about the performance of process B is as follows: according to Table 1, we have $L_{\tilde{\hat{S}}^T_{pk(B)}}(0.75) = 1.1042 < s_\circ = 1.2326 < R_{\tilde{\hat{S}}^T_{pk(B)}}(0.75) = 1.2455$, so it results $d = \frac{1.2455-1.2326}{1.2455-1.1042} = 0.0912$ and $1 - d = 1 - \frac{1.2455-1.2326}{1.2455-1.1042} = 0.9088 > 0.9$. Therefore, process B is incapable with degree 0.9088 at level 0.75. It means that process B does not meet the capability requirement with a probability of %90.88 when the sample data are recorded with an imprecision degree of 0.75. In other words, the overall nonconformities exceed 2699 ppm with a probability of %90.88 when the observations are collected with the mentioned uncertainty degree.

Similarly, the yield performances of three mentioned processes can be measured based on the fuzzy p-value when $\lambda = 0.6$. For this purpose, we should find λ-cut intervals of fuzzy p-value for three processes A, B and C. Based on the sample information and using equation (15), these intervals are obtained. The method used to plot membership functions of fuzzy p-values is similar to the method we explained above for fuzzy overall S^T_{pk}. So to save the capacity of paper, we ignore more details. Figure 2 plots the membership functions of the fuzzy p-values $\tilde{p}_{(A)}$, $\tilde{p}_{(B)}$ and $\tilde{p}_{(C)}$ as well as the reject border of the null hypothesis ($\alpha = 0.05$ shown by vertical line). The 0.6-cut intervals of fuzzy p-values for processes A, B and C are recorded as follows

$$\tilde{p}_{(A)}[0.6] = [0.24001, 0.69943],\ \tilde{p}_{(B)}[0.6] = [0.02004, 0.33806],\ \tilde{p}_{(C)}[0.6] = [0.00002, 0.01356]$$

as shown in Figure 2.

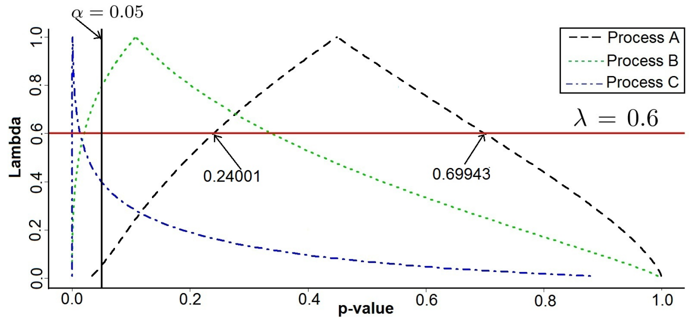

Figure 2: The membership functions of fuzzy p-values $\tilde{p}_{(A)}$, $\tilde{p}_{(B)}$, $\tilde{p}_{(C)}$ with reject border $\alpha = 0.05$

By analysing Figure 2 and considering the proposed p-value-based-decision testing rule, similar conclusions are obtained about the performance of the processes A, B and C. From Figure 2, the performance of the processes A and C is distinguished certainty so that it results process A is incapable since $L_{\tilde{p}_{(A)}}(0.6) = 0.24001 > \alpha = 0.05$, *while process C is capable because* $R_{\tilde{p}_{(C)}}(0.6) = 0.01356 < \alpha = 0.05$. *At last, the performance of process B cannot be identified certainty due to* $L_{\tilde{p}_{(B)}}(0.6) = 0.02004 < \alpha = 0.05 < R_{\tilde{p}_{(B)}}(0.6) = 0.33806$. *But according to the introduced testing rule it is incapable with* $1 - d' = 0.906$ *degree at level* $\lambda = 0.6$.

Now, assume that one wants to test the classical hypothesis on precise overall S^T_{pk}. *To test the null hypothesis* $\mathrm{H}_\circ : S^T_{pk} \leq 1$ *at level* $\alpha = 0.05$, *we implement according to the existing classic multivariate way proposed by Pearn and Cheng [30]. So, after estimating* S^T_{pk} *based on the random sample, it is compared with the critical value* $s_\circ = 1.2326$. *Based on the Pearn and Cheng's manner [30], the hypothesis* $H_\circ$ *will be rejected at level* α *if* $\hat{S}^T_{pk}$ *is greater than* $s_\circ$. *Based on the sample information and applying equations (4) and (5),* S^T_{pk} *is computed as* $\hat{S}^T_{pk(A)} = 1.0181$, $\hat{S}^T_{pk(B)} = 1.1749$ *and* $\hat{S}^T_{pk(C)} = 1.4474$ *for processes A, B and C, respectively. Since* $\hat{S}^T_{pk(A)}$ *and* $\hat{S}^T_{pk(B)}$ *are both less than* $s_\circ = 1.2326$, *it results that processes A and B are incapable. In addition, one can conclude that process C is capable, since* $\hat{S}^T_{pk(C)}$ *is larger than* $s_\circ$. *Importantly, these results can be seen in both Table 1 and Figure 1 when the value of imprecision degree of data is* $\lambda = 1$ *or when fuzzy random variables reduce to random ones. On the other hand, judgements can be formed based on the p-value. By using (10), the crisp p-values for processes A, B, and C are obtained as* $p_{(A)} = 0.4484$, $p_{(B)} = 0.1080$ *and* $p_{(C)} = 0.0007$, *respectively. Knowing that the process is considered capable if the obtained p-value is less than* $\alpha = 0.05$, *decision making about the performance of above three processes based on the p-value would be the same to critical-value based decision. It is worth to mention that the obtained results can be clearly derived from Figure 2 as* $\lambda = 1$ *or when vague random variables reduce to random ones. Hence, the existing traditional manner is a spacial case of the introduced fuzzy method.*

The above discussion displays that in the traditional case, the processes A and B are incapable under the operation of Pearn and Cheng's suggested method while in a fuzzy environment processes A and B are incapable as amount of data ambiguity is at least 0.1 and 0.8, respectively. It means that for membership grade less than 0.1 and 0.8, we can not decide definitely about processes A and B, respectively, and sometimes more information is required. Likewise, the process C is capable in the customary plan while it is capable in the fuzzy way whenever the data imprecision value is reported 0.4 or more. As a result, the proposed technique is affected by uncertainty degree and is more informative than the existing traditional one, so that the presented method leads to the classical method when there is no ambiguity in the recorded data or when fuzzy random variables reduce to random ones.

Example 5.2 (correlated quality characteristics). *In the manufacturing process of a product (studied by Shahriari and Abdollahzadeh [34]) two correlated and normally distributed quality characteristics are measured, including brinell hardness* (X_1), *and tensile strength* (X_2). *The process target and specification limits* $(\mathrm{LSL}, \mathrm{USL})'$ *for characteristics* X_1, *and* X_2 *are specified to* $(177, 53)'$, $(112.7, 241.3)'$, *and* $(32.7, 73.3)'$, *respectively. Also suppose that the pre-described capability requirement is fixed at* $s = 1$ *according to the buyer-supplier purchasing agreement. To determine whether the underlying process meets the pre-set capability requirement, we test the null hypothesis* $\mathrm{H}_\circ : S^T_{pk} \leq 1$ *versus* $\mathrm{H}_1 : S^T_{pk} > 1$ *with* $\alpha = 0.05$. *To apply the proposed testing rule, after taking a sample of size 25 from the considered normal dataset (these data are accessible in [34]), the sample mean vector, sample variance-covariance matrix and correlation matrix are computed as*

$$\overline{\mathbf{X}} = \begin{bmatrix} 177.2 \\ 52.32 \end{bmatrix}, \quad \boldsymbol{S} = \begin{bmatrix} 337.8 & 85.3308 \\ 85.3308 & 33.6247 \end{bmatrix}, \quad \boldsymbol{r} = \begin{bmatrix} 1.0000 & 0.8 \\ 0.8 & 1.0000 \end{bmatrix}.$$

According to the obtained sample correlation matrix $\mathbf{r}$*, a significant positive correlation exists between two original variables. By performing the PCA, the eigenvalues* e_1*,* e_2*, eigenvectors* $\boldsymbol{u}_1$*,* $\boldsymbol{u}_2$ *of the sample covariance matrix, and the percentage of explained variance* $(e_j/\sum_{j=1}^{2} e_j) \times 100$ *are reported in Table 2.*

	Eigenvector	
Characteristic	$\mathbf{u}_1$	$\mathbf{u}_2$
X_1	0.9674	-0.2528
X_2	0.2528	0.9674
Eigenvalues	360.1027	11.3219
%Explained of total variability	0.9695	0.0304

Table 2: Eigenvectors and eigenvalues of the sample covariance

From Table 2, the most of the total variability of the process (about %96.95) is described by the first PC. In other words, the first PC summarizes very well the sample variation. Hence, ignoring the last PC is logical to reduce the dimension problem. As a result, the two high correlated original variables X_1*, and* X_2 *can be substituted by the first principal component, which is normally distributed. The transformed sample mean, upper and lower specification limits, target values and the variance of the principal component* PC_1 *are equal to* $\overline{X}_{PC_1} = 184.6712$*,* $\mathrm{USL}_{PC_1} = 251.9932$*,* $\mathrm{LSL}_{PC_1} = 117.3061$*,* $\mathrm{T}_{PC_1} = 184.6496$*, and* $S^2_{PC_1} = 360.1027$*, respectively. According to (9), the critical value for hypothesis testing is* $s_\circ = 1.2326$*. Now, suppose that there exists a degree of imprecision of the sample data. Let it is set to* $\lambda = 0.8$ *based on the knowledge of production engineers. Then by replacing original variables* (X_j) *with the principal components* (PC_j) *in (11), (12) and (14), the 0.8-cut interval of fuzzy estimator can be found as* $\widetilde{\widetilde{TS}}_{pk,PC}[0.8] = \left[L_{\widetilde{\widetilde{TS}}_{pk,PC}}(0.8), R_{\widetilde{\widetilde{TS}}_{pk,PC}}(0.8)\right] = [1.1222, 1.2083]$*. Since* $R_{\widetilde{\widetilde{TS}}_{pk,PC}}(0.8) = 1.2083 < s_\circ = 1.2326$*, it results that the process is incapable using the proposed critical-value-based testing rule at level 0.8.*

Some λ*-cut intervals of* $\widetilde{\widetilde{TS}}_{pk,PC}$ *are reported in Table 3. From Table 3, it appears that the manufacturing process is incapable for ambiguity degree equivalent to 0.7 or more. Otherwise, one can not judge surly the process capability. Figure 3 displays the membership function of* $\widetilde{\widetilde{TS}}_{pk,PC}$ *to assist the reader to obtain a better viewpoint of the interpretation of the proposed fuzzy index.*

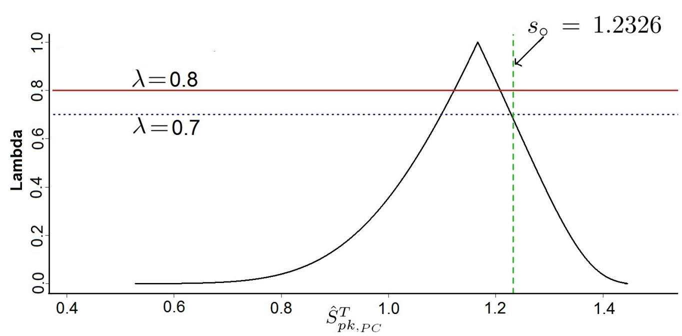

Figure 3: The membership function of fuzzy estimator $\tilde{\hat{S}}^T_{pk,PC}$ in Example 5.2

It is worth to mention that the same decision can be made by taking the proposed p-value-based testing rule. By inserting $\widetilde{\widetilde{TS}}_{pk,PC}$ *instead of* $\tilde{\hat{S}}^T_{pk}$ *in (15), the* λ*-cut of p-value is* $\tilde{p}[0.8] = [0.0730, 0.1937]$*, so that the value of lower bound 0.0730 is more than* $\alpha = 0.05$*. Therefore, the null hypothesis is accepted and concluded that the process is incapable at level 0.8.*

In the following, we make a decision regarding to the process performance under the present methods, such as ones proposed by Abbasi Ganji and Sadeghpour Gildeh [2]. Two fuzzy multivariate process capability indices, $\widetilde{MC}_{pm}$ *and* $\widetilde{NMC}_{pm}$ *were introduced by them based on the classic-original work of Taam et al. [36] and Shahriari and Abdollahzadeh [34], respectively, for multivariate normally distributed observations of correlated characteristics with uncertain process specification limits and process target. They apply ranking function to asses the process efficiency in their imple-*

mentation. The $\widetilde{MC}_{pm}$ is the proportion of two elliptical regions, modified tolerance volume (R1) and 99.73%-covered process volume (R2). Based on $\widetilde{MC}_{pm}$, the process is considered capable if its ranking function is at least one i.e. $R(\widetilde{MC}_{pm}) \geq 1$. Similar to the $\widetilde{MC}_{pm}$, the index $\widetilde{NMC}_{pm}$ compares two above elliptical regions, except that it employs the correlation structure of observed samples to recognize R1. The index $\widetilde{NMC}_{pm}$ is the first component of a three-component vector $\widetilde{NMPCV} = [\widetilde{NMC}_{pm}, \widetilde{PV}, LI]$ in which $\widetilde{PV}$ is a tool for testing the process mean equality with the target. And LI is a measure, taking values 0 or 1, compares the position of R2 with the tolerance region. Under $\widetilde{NMPCV}$, a process is named capable if the three below conditions are satisfied simultaneously: $R(\widetilde{NMC}_{pm}) \geq 1$, $LI = 1$ and $\widetilde{PV}$ is approximately greater than 0.05. Complete information is detailed in Abbasi Ganji and Sadeghpour Gildeh [2]. They discussed about the performance of the above-mentioned manufacturing process based on the above-given sample information with trapezoidal fuzzy quantity specification limits $\widetilde{\mathrm{USL}}_1 = \mathrm{T}_r(240, 240.3, 242.3, 242.6)$, $\widetilde{\mathrm{LSL}}_1 = \mathrm{T}_r(111, 111.7, 113.7, 114)$ and target $\widetilde{\mathrm{T}}_1 = \mathrm{T}_r(176, 176.5, 177.5, 178)$ for the brinell hardness (X_1), and $\widetilde{\mathrm{USL}}_2 = \mathrm{T}_r(72, 72.3, 74.3, 74.6)$, $\widetilde{\mathrm{LSL}}_2 = \mathrm{T}_r(31, 31.7, 33.7, 34)$ with target $\widetilde{\mathrm{T}}_2 = \mathrm{T}_r(52, 52.5, 53.5, 54)$ for the tensile strength (X_2). The calculated values of $\widetilde{MC}_{pm}$ and $\widetilde{NMPCV}$ are given in Table 3 (the reported values in the first and second columns of this table were depicted by Abbasi Ganji and Sadeghpour Gildeh [2]). According to the reported values in Table 3, the process is considered capable under the operation of $\widetilde{MC}_{pm}$, whereas it would be incapable under the operation of $\widetilde{NMPCV}$. By using graphical illustrations, Abbasi Ganji and Sadeghpour Gildeh [2] showed that $\widetilde{MC}_{pm}$ overestimates the process capability.

$\widetilde{MC}_{pm}$	$\widetilde{NMPCV}$	$\widetilde{\widetilde{TS}}_{pk,PC}$
approx. between 1.4867 and 1.8847,	[approx. between 0.9935 and 1.0409,	$\widetilde{\widetilde{TS}}_{pk,PC}[1] = 1.1664$
$R(\widetilde{MC}_{pm}) = 1.67915$	approx. between 0.9530 and 0.9943, 0]	$\widetilde{\widetilde{TS}}_{pk,PC}[0.9] = [1.1447, 1.1875]$
	$R(\widetilde{NMC}_{pm}) = 1.0172$	$\widetilde{\widetilde{TS}}_{pk,PC}[0.8] = [1.1222, 1.2083]$
		$\widetilde{\widetilde{TS}}_{pk,PC}[0.7] = [1.0985, 1.2288]$
		$\widetilde{\widetilde{TS}}_{pk,PC}[0.65] = [1.0861, 1.2390]$

Table 3: Comparison between the proposed method with the existing methods

As above discussion, the obtained results from Table 3 are outlined below. (i) The introduced plan performs better than $\widetilde{MC}_{pm}$ since $\widetilde{MC}_{pm}$ overestimates the process capability due to ignoring the correlation structure of the gathered samples. (ii) The presented technique is more informative than the $\widetilde{NMPCV}$ because of being sensitive to ambiguity degree of recorded data. The process is incapable under $\widetilde{NMPCV}$, while it would be incapable under the proposed method as imprecision degree of observations is at least 0.7. It means that we can not make a decision surly for uncertainty degree less than 0.7, and sometimes it needs to take more information to judge.

6 Conclusion

Process yield is a numerical benchmark that is routinely used in manufacturing industry to evaluate performance of the process. In today's modern world, quality controllers mostly need to inspect two or more quality characteristics to access the process performance, simultaneously. Although, there exist several multivariate capability indices to determine the process performance, some of them can be used to compute the production yield for processes with uncertain-measured multiple characteristics. In this study, a fuzzy advanced S_{pk}^T-based estimator was presented to evaluate and test the overall production yield of process whose quality characteristics are mutually independent and multivariate normally distributed. In our methodology, we considered a certain degree of imprecision on the gathered samples and combined the values of fuzzy individual yield indices to generate the membership function of fuzzy advanced S_{pk}^T-based estimator under Buckley's approach. Moreover, a fuzzy three-decision testing rule was made to judge the process yield based on the critical values and fuzzy p-values. In order to develop the proposed method to a larger class of quality characteristics, the PCA technique was used when one deals with the correlated characteristics. Based on the comparative studies, the results are described as follows. (i) A main benefit of the proposed method is that the critical value is independent of variable dimensions, and hence it can be employed to assess multidimensional process with different dimensions. (ii) The introduced fuzzy three-decision testing rule leads to the existing classical binary-decision testing method presented by Pearn and Cheng [30] when fuzzy random variables reduce to random ones or when underlying observations are monitored precisely. (iii) The presented method accomplishes better than the existing $\widetilde{MC_{pm}}$ because $\widetilde{MC_{pm}}$ overestimates the process efficiency because of disregarding the correlation between the observed data during its implementation. (iv) The suggested rule is more informative than the present $\widetilde{NMPCV}$ due to being delicate to imprecision degree of reported data. At last, two real data examples with independent and correlated characteristics were discussed to demonstrate the application of the proposed rule in real world problems. As future study, one can extend the proposed fuzzy testing rule to assess the multivariate profile performance in a fuzzy environment.

Acknowledgement

The authors would like to thank the reviewers for their excellent and useful suggestions to improve this paper. This research was supported by a grant from Ferdowsi University of Mashhad, Iran; No. 2/52113.

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