Iranian Journal of Fuzzy Systems
Volume 18, Number 1, (2021), pp. 17-33
Original paper

The cross-migrativity equation with respect to semi-t-operators

Y. Y. Zhao[1] and F. Qin[2]

[1] *School of Mathematics, Shandong University, 250100 Jinan, PR China*
[2] *College of Mathematics and Information Science, Jiangxi Normal University, 330022 Nanchang, PR China*

783992484@qq.com, qinfeng923@163.com

Abstract

The cross-migrativity has been investigated for families of certain aggregation operators, such as t-norms, t-subnorms and uninorms. In this paper, we aim to study the cross-migrativity property for semi-t-operators, which are generalizations of t-operators by omitting commutativity. Specifically, we give all solutions of the cross-migrativity equation for all possible combinations of semi-t-operators. Moreover, it is shown that if a semi-t-operator F is α-cross-migrative over another semi-t-operator G, then G must be a semi-nullnorm except one case. Finally, it is pointed out that the cross-migrativity property between two semi-t-operators is always determined by their underlying operators except a few cases.

Keywords: Fuzzy connectives, cross-migrativity, semi-t-operators, semi-nullnorms.

1 Introduction

The aggregation function is an essential tool in many fields like mathematics, computer sciences, economics and social sciences [2, 4, 10, 11]. And many researchers focus on analysis of interesting properties of aggregation functions [1], like idempotency, modularity, Frank and Alsina equations [3], migrativity [9, 16, 19, 22] and distributivity. Thereinto, migrativity is an important property of binary operations defined on the unit interval because it plays a key role in decision making processes and image processing. Hence, Fodor and Rudas [7] considered the migrative funcitional equation

$$T(\alpha x, y) = T(x, \alpha y), \quad \text{for all } (x, y) \in [0, 1]^2 \tag{1}$$

and fully characterized all continuous Archimedean t-norms satisfying Eq.(1). And then they [8] extended Eq.(1) into

$$T(T_0(\alpha, x), y) = T(x, T_0(\alpha, y)), \quad \text{for all } (x, y) \in [0, 1]^2 \tag{2}$$

where T_0 is a t-norm. Moreover, when Fodor et al. [6] studied the computing of aggregation operators, they found the classical commuting equation over the t-norms

$$T(T_0(x, y), T_0(u, v)) = T_0(T(x, u), T(y, v)) \tag{3}$$

has only solution $T = T_0$. By fixing $u = 1$ and writing $y = \alpha,\ v = y$ in Eq.(3), we can obtain a weaker functional equation

$$T(T_0(\alpha, x), y) = T_0(x, T(\alpha, y)), \quad \text{for all } (x, y) \in [0, 1]^2. \tag{4}$$

The t-norm T satisfying Eq.(4) is said to be *α-cross-migrative* with respect to T_0 (or shortly T is (α, T_0)-cross-migrative) [6]. The cross-migrative property has been studied for t-norms [6, 12], t-subnorms [18] and uninorms [20, 21]. The paper will focus on studying the cross-migrativity with respect to *semi-t-operators* and give the sufficient and necessary conditions of the cross-migrativity equations for all combinations of semi-t-operators.

Corresponding Author: F. Qin

Received: September 2019; Revised: June 2020; Accepted: August 2020.

The remainder of this paper is organized as follows. In Section 2, we will recall some results and structures related to basic fuzzy logic connectives used in this paper and define the cross-migrativity with respect to semi-t-operators. In Sections 3, 4, 5, 6, we will respectively characterize all solutions of the cross-migrativity equations for four kinds of possible combinations of semi-t-operators. In Section 7, we give our conclusions and suggestions for further research.

2 Preliminaries

In this section, to make this work self-contained, we recall some definitions and results employed in the paper.

Definition 2.1. [15]

(i) A binary operator $T : [0,1]^2 \to [0,1]$ is called a semi-t-norm *if it is increasing, associative and it has a neutral element* 1*,*

(ii) A binary operator $S : [0,1]^2 \to [0,1]$ is called a semi-t-conorm *if it is increasing, associative and it has a neutral element* 0*.*

Clearly, a t-norm is a commutative semi-t-norm while a t-conorm is a commutative semi-t-conorm.

Definition 2.2. [13] *The operation $F : [0,1]^2 \to [0,1]$ is called a* t-operator *if it is commutative, associative, increasing and such that $F(0,0) = 0$, $F(1,1) = 1$ and the functions F_0 and F_1 are continuous, where $F_0(x) = F(0,x)$ and $F_1(x) = F(1,x)$.*

In fact, T. Calvo [3] introduced the notion of nullnorms in order to generalize the functional equations of Frank and Alsina into uninorms and nullnorms. In [14], M. Mas has proven that t-operators and nullnorms are equivalent. But it is interesting that their own generalizations semi-nullnorms and semi-t-operators are not equivalent. Now, let us introduce definitions of semi-nullnorms.

Definition 2.3. [5] *The operation $F : [0,1]^2 \to [0,1]$ is called a* semi-nullnorm *if it is increasing, associative, has an absorbing element $k \in [0,1]$ and satisfies*

(i) $F(0,x) = F(x,0) = x$, for all $x \le k$.

(ii) $F(1,x) = F(x,1) = x$, for all $x \ge k$.

Note that definitions of semi-nullnorms in this paper is slightly different with ones in [5], since we require the associativity. Thus the set of all semi-nullnorms is a proper subset for the set of all semi-t-operators.

Theorem 2.4. [5] *A binary operation F is a semi-nullnorm with an absorbing element $k \in (0,1)$ if and only if there exists a semi-t-norm T and a semi-t-conorm S such that F is give by*

$$F(x,y) = \begin{cases} kS(\frac{x}{k}, \frac{y}{k}) & \text{if } (x,y) \in [0,k]^2, \\ k + (1-k)T(\frac{x-k}{1-k}, \frac{y-k}{1-k}) & \text{if } (x,y) \in [k,1]^2, \\ k & \text{otherwise.} \end{cases} \tag{5}$$

Definition 2.5. [5] *A binary operation $F : [0,1]^2 \to [0,1]$ is called a* semi-t-operator *if it is associative, non-decreasing, fulfills $F(0,0) = 0, F(1,1) = 1$ and such that the functions F_0, F_1, F^0, F^1 are continuous, where $F_0(x) = F(0,x), F_1(x) = F(1,x), F^0(x) = F(x,0)$ and $F^1(x) = F(x,1)$.*

Let $\mathcal{F}_{a,b}$ denote the family of all semi-t-operators such that $F(0,1) = a$ and $F(1,0) = b$. Then we obtain the following theorem.

Theorem 2.6. [17] *Let $F : [0,1]^2 \to [0,1]$, write $F(0,1) = a$, $F(1,0) = b$. Operation $F \in \mathcal{F}_{a,b}$ if and only if there exists a semi-t-norm T_F and a semi-t-conorm S_F such that*

$$F(x,y) = \begin{cases} aS_F(\frac{x}{a}, \frac{y}{a}) & \text{if } (x,y) \in [0,a]^2, \\ b + (1-b)T_F(\frac{x-b}{1-b}, \frac{y-b}{1-b}) & \text{if } (x,y) \in [b,1]^2, \\ a & \text{if } (x,y) \in [0,a] \times [a,1], \\ b & \text{if } (x,y) \in [b,1] \times [0,b], \\ x & \text{otherwise,} \end{cases} \tag{6}$$

for $a \leq b$ *and*

$$F(x,y) = \begin{cases} bS_F(\frac{x}{b}, \frac{y}{b}) & \text{if } (x,y) \in [0,b]^2, \\ a + (1-a)T_F(\frac{x-a}{1-a}, \frac{y-a}{1-a}) & \text{if } (x,y) \in [a,1]^2, \\ a & \text{if } (x,y) \in [0,a] \times [a,1], \\ b & \text{if } (x,y) \in [b,1] \times [0,b], \\ y & \text{otherwise}, \end{cases} \tag{7}$$

for $a \geq b$.

Now, let us define the cross-migrativity of semi-t-operators.

Definition 2.7. *Let* $\alpha \in [0,1]$ *and* $F,\ G : [0,1]^2 \to [0,1]$ *two semi-t-operators.* F *is said to be* α-cross-migrative with respect to G *or* F *is* (α, G)-cross-migrative *or* (F,G) is α-cross-migrative *if*

$$F(G(\alpha,x),y) = G(x, F(\alpha,y)) \quad \text{for all } x,y \in [0,1]. \tag{8}$$

According to the order relationship between a,b and c,d, we need to consider the following four cases:
(1) the cross-migrativity of $F \in \mathcal{F}_{a,b}$ with $a \geq b$ over $G \in \mathcal{F}_{c,d}$ with $c \geq d$;
(2) the cross-migrativity of $F \in \mathcal{F}_{a,b}$ with $a \geq b$ over $G \in \mathcal{F}_{c,d}$ with $c \leq d$;
(3) the cross-migrativity of $F \in \mathcal{F}_{a,b}$ with $a \leq b$ over $G \in \mathcal{F}_{c,d}$ with $c \geq d$;
(4) the cross-migrativity of $F \in \mathcal{F}_{a,b}$ with $a \leq b$ over $G \in \mathcal{F}_{c,d}$ with $c \leq d$.
Next, let us study them in turn.

3 Cross-migrativity of $F \in \mathcal{F}_{a,b}$ with $a \geq b$ over $G \in \mathcal{F}_{c,d}$ with $c \geq d$

In this section, depending on the order relationship among α, a, b, c, d, we need consider the following three different cases: (1) $\alpha \leq \min(b,d)$, (2) $\alpha \geq \max(a,c)$, (3) $\min(b,d) < \alpha < \max(a,c)$. Now, let us consider the first subcase.

3.1 $\alpha \leq \min(b,d)$

Lemma 3.1. *Let* $\alpha \leq \min(b,d)$, $F \in \mathcal{F}_{a,b}$ *with* $a \geq b$ *and* $G \in \mathcal{F}_{c,d}$ *with* $c \geq d$. *If* F *is* α*-cross-migrative over* G, *then* $a = c \geq b = d \geq \alpha$.

Proof. Assume that $a > c$. Taking $x = 0, y = 1$, then we have that $a = F(\alpha,1) = F(G(\alpha,0),1) = G(0,F(\alpha,1)) = G(0,a) = c$. It contradicts with $a > c$. Hence $a \leq c$.

Taking $x = 1, y = 0$, then we have that $b = F(c,0) = F(G(\alpha,1),0) = G(1,F(\alpha,0)) = G(1,\alpha) = d$.

Taking $x = 1, y = 1$, then we have that $c = F(c,1) = F(G(\alpha,1),1) = G(1,F(\alpha,1)) = G(1,a) = a$.

Therefore, $\alpha \leq b = d \leq a = c$. □

Theorem 3.2. *Let* $\alpha \leq \min(b,d)$, $F \in \mathcal{F}_{a,b}$ *with* $a \geq b$ *and* $G \in \mathcal{F}_{c,d}$ *with* $c \geq d$. *Then* F *is* α*-cross-migrative over* G *if and only if*

(i) $\alpha \leq b = d \leq a = c$,

(ii) S_F *is* $\frac{\alpha}{b}$*-cross-migrative over* S_G,

where S_F *and* S_G *are the underlying semi-t-conorms of* F *and* G, *respectively.*

Proof. (⇒) Firstly, we can directly get from Lemma 3.1 that (i). To prove (ii), consider $x, y \in [0,b]$ in Eq.(8), we have that the left-hand side of Eq.(8) is

$$F(G(\alpha,x),y) = F(bS_G(\frac{\alpha}{b}, \frac{x}{b}), y) = bS_F(S_G(\frac{\alpha}{b}, \frac{x}{b}), \frac{y}{b}), \tag{9}$$

while the right-hand side of Eq.(8) is

$$G(x, F(\alpha,y)) = G(x, bS_F(\frac{\alpha}{b}, \frac{y}{b})) = bS_G(\frac{x}{b}, S_F(\frac{\alpha}{b}, \frac{y}{b})). \tag{10}$$

Note that F is α-cross-migrative over G, then $S_F(S_G(\frac{\alpha}{b}, \frac{x}{b}), \frac{y}{b}) = S_G(\frac{x}{b}, S_F(\frac{\alpha}{b}, \frac{y}{b}))$. Thus we get that S_F is $\frac{\alpha}{b}$-cross-migrative over S_G because of $\frac{x}{b}, \frac{y}{b} \in [0,1]$. Hence (ii) is proved.

($\Leftarrow$) Suppose that $x, y \in [0,1]$, then we have the following cases to be checked.

Assume that $y \geq a$, from structures of F, G and $\alpha \leq b = d \leq a = c$, then we have that $G(\alpha, x) \leq G(\alpha, 1) = a$, $G(x, a) = a$, and then $F(G(\alpha, x), y) = a = G(x, a) = G(x, F(\alpha, y))$.

Assume that $b \leq y \leq a$, from structures of F, G and $\alpha \leq b = d \leq a = c$, then we have that $F(G(\alpha, x), y) = y = G(x, y) = G(x, F(\alpha, y))$.

Assume that $y \leq b$, from structures of F, G, $\alpha \leq b = d \leq a = c$ and S_F is $\frac{\alpha}{b}$-cross-migrative over S_G, then we have that $bS_F(\frac{\alpha}{b}, \frac{y}{b}) \leq b$. For $x \in [0,1]$, there exist the following subcases to consider.

- If $x \geq a$, then $F(G(\alpha, x), y) = F(a, y) = b = G(x, bS_F(\frac{\alpha}{b}, \frac{y}{b})) = G(x, F(\alpha, y))$.
- If $x \in [b, a]$, then $F(G(\alpha, x), y) = F(x, y) = b = G(x, bS_F(\frac{\alpha}{b}, \frac{y}{b})) = G(x, F(\alpha, y))$.
- If $x \leq b$, then $F(G(\alpha, x), y) = F(bS_G(\frac{\alpha}{b}, \frac{x}{b}), y) = bS_F(S_G(\frac{\alpha}{b}, \frac{x}{b}), \frac{y}{b}) = bS_G(\frac{x}{b}, S_F(\frac{\alpha}{b}, \frac{y}{b})) = G(x, bS_F(\frac{\alpha}{b}, \frac{y}{b})) = G(x, F(\alpha, y))$. □

3.2 $\alpha \geq \max(a, c)$

Lemma 3.3. *Let $\alpha \geq \max(a, c)$, $F \in \mathcal{F}_{a,b}$ with $a \geq b$ and $G \in \mathcal{F}_{c,d}$ with $c \geq d$. If F is α-cross-migrative over G, then $\alpha \geq a = c \geq b = d$.*

Proof. Assume that $b < d$. Taking $x = 1, y = 0$, then we have that $b = F(\alpha, 0) = F(G(\alpha, 1), 0) = G(1, F(\alpha, 0)) = G(1, b) = d$. It contradicts $b < d$. Thus $b \geq d$.

Taking $x = 0, y = 1$, then we have that $a = F(d, 1) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 1)) = G(0, \alpha) = c$.

Taking $x = d, y = 0$, then we have that $d = F(d, 0) = F(G(\alpha, d), 0) = G(d, F(\alpha, 0)) = G(d, b) = b$.

Therefore, $b = d \leq a = c \leq \alpha$. □

Theorem 3.4. *Let $\alpha \geq \max(a, c)$, $F \in \mathcal{F}_{a,b}$ with $a \geq b$ and $G \in \mathcal{F}_{c,d}$ with $c \geq d$. Then F is α-cross-migrative over G if and only if*

(i) $b = d \leq a = c \leq \alpha$.

(ii) T_F *is* $\frac{\alpha - a}{1 - a}$*-cross-migrative over* T_G.

Proof. ($\Rightarrow$) We can directly obtain from Lemma 3.3 that (i). To prove (ii), similarly, consider $x, y \in [a, 1]$ in Eq.(8), according to $b = d \leq a = c \leq \alpha$, we have that the left-hand side of Eq.(8) is

$$F(G(\alpha, x), y) = F(a + (1-a)T_G(\frac{\alpha - a}{1 - a}, \frac{x - a}{1 - a}), y) = a + (1-a)T_F(T_G(\frac{\alpha - a}{1 - a}, \frac{x - a}{1 - a}), \frac{y - a}{1 - a}), \tag{11}$$

while the right-hand side of Eq.(8) is

$$G(x, F(\alpha, y)) = G(x, a + (1-a)T_F(\frac{\alpha - a}{1 - a}, \frac{y - a}{1 - a})) = a + (1-a)T_G(\frac{x - a}{1 - a}, T_F(\frac{\alpha - a}{1 - a}, \frac{y - a}{1 - a})). \tag{12}$$

Note that F is α-cross-migrative over G, then $T_F(T_G(\frac{\alpha - a}{1 - a}, \frac{x - a}{1 - a}), \frac{y - a}{1 - a}) = T_G(\frac{x - a}{1 - a}, T_F(\frac{\alpha - a}{1 - a}, \frac{y - a}{1 - a}))$. Thus we get that T_F is $\frac{\alpha - a}{1 - a}$-cross-migrative over T_G because of $\frac{x - a}{1 - a}, \frac{y - a}{1 - a} \in [0,1]$. Hence (ii) is proved.

($\Leftarrow$) Suppose that $x, y \in [0,1]$, then we have the following cases to be checked.

Assume that $y \leq b$, from the structure of T, G and $b = d \leq a = c \leq \alpha$, we have that $G(\alpha, x) \geq G(\alpha, 0) = b$, $F(\alpha, y) = b$, $G(x, b) = b$, then $F(G(\alpha, x), y) = b = G(x, b) = G(x, F(\alpha, y))$.

Assume that $y \in [b, a]$, then we have $G(x, F(\alpha, y)) = G(x, y) = y = F(G(\alpha, x), y)$.

Assume that $y \geq a$, then we need to consider two subcases: $x \leq a$ and $x \geq a$.

- If $x \leq a$, then $G(x, F(\alpha, y)) = G(x, a + (1-a)T_F(\frac{\alpha - a}{1 - a}, \frac{y - a}{1 - a})) = a$ because of $a + (1-a)T_F(\frac{\alpha - a}{1 - a}, \frac{y - a}{1 - a}) \geq a$ and $F(G(\alpha, x), y) = a$ because of $G(\alpha, x) \leq G(\alpha, a) = a$. Thus $G(x, F(\alpha, y)) = a = F(G(\alpha, x), y)$.
- If $x \geq a$, then $F(G(\alpha, x), y) = F(a + (1-a)T_G(\frac{\alpha - a}{1 - a}, \frac{x - a}{1 - a}), y) = a + (1-a)T_F(T_G(\frac{\alpha - a}{1 - a}, \frac{x - a}{1 - a}), \frac{y - a}{1 - a}) = a + (1-a)T_G(\frac{x - a}{1 - a}, T_F(\frac{\alpha - a}{1 - a}, \frac{y - a}{1 - a})) = G(x, a + (1-a)T_F(\frac{\alpha - a}{1 - a}, \frac{y - a}{1 - a})) = G(x, F(\alpha, y))$ because of T_F is $\frac{\alpha - a}{1 - a}$-cross-migrative over T_G. □

3.3 $\min(b, d) < \alpha < \max(a, c)$

For this case, there are four different subcases to be considered: (1) $a \leq c$ and $b \leq d$, (2) $a \leq c$ and $b \geq d$, (3) $a \geq c$ and $b \leq d$, (4) $a \geq c$ and $b \geq d$. But it is interesting that the above four cases have the same results. Now we start Subcase (1).

Lemma 3.5. *Let $b = \min(b, d) < \alpha < \max(a, c) = c$, $F \in \mathcal{F}_{a,b}$ with $a \geq b$ and $G \in \mathcal{F}_{c,d}$ with $c \geq d$. If F is α-cross-migrative over G, then $a = c > \alpha > b = d$.*

Proof. Taking $x = 1, y = 0$, we have that $b = F(c, 0) = F(G(\alpha, 1), 0) = G(1, F(\alpha, 0)) = G(1, b) = d$.

Assume that $b \leq a \leq \alpha < c$. Taking $x = 1$ and $y = 1$, then we have that $c = F(c, 1) = F(G(\alpha, 1), 1) = G(1, F(\alpha, 1)) = G(1, \alpha) = \alpha$, which contradicts $\alpha < c$. Thus $b < \alpha < a \leq c$.

Taking $x = 1, y = 1$, then we have that $c = F(c, 1) = F(G(\alpha, 1), 1) = G(1, F(\alpha, 1)) = G(1, a) = a$.

Therefore it follows that $b = d < \alpha < a = c$. □

For Subcase (2), we have the following lemma.

Lemma 3.6. *Let $d = \min(b, d) < \alpha < \max(a, c) = c$, $F \in \mathcal{F}_{a,b}$ with $a \geq b$ and $G \in \mathcal{F}_{c,d}$ with $c \geq d$. If F is α-cross-migrative over G, then $b = d < \alpha < a = c$.*

Proof. Assume that $a < \alpha < c$. Taking $x = 0$ and $y = 1$, then we have that $a = F(d, 1) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 1)) = G(0, \alpha) = \alpha$, which contradicts with $\alpha > a$. Thus $d < \alpha \leq a \leq c$.

Assume that $d < \alpha < b$. Taking $x = 1, y = 0$, then we have that $b = F(c, 0) = F(G(\alpha, 1), 0) = G(1, F(\alpha, 0)) = G(1, \alpha) = \alpha$, which contradicts $d < \alpha < b$. Thus $b \leq \alpha \leq a \leq c$.

Taking $x = 1, y = 1$, then we have that $c = F(c, 1) = F(G(\alpha, 1), 1) = G(1, F(\alpha, 1)) = G(1, a) = a$.

Taking $x = 0, y = 0$, then we have that $d = F(d, 0) = F(G(\alpha, 0), 0) = G(0, F(\alpha, 0)) = G(0, b) = b$.

Therefore it follows that $a = c > \alpha > b = d$. □

For Subcase (3), see the following lemma.

Lemma 3.7. *Let $b = \min(b, d) < \alpha < \max(a, c) = a$, $F \in \mathcal{F}_{a,b}$ with $a \geq b$ and $G \in \mathcal{F}_{c,d}$ with $c \geq d$. If F is α-cross-migrative over G, then $b = d < \alpha < a = c$.*

Proof. Taking $x = 0, y = 1$, then we have that $G(\alpha, 0) \leq d \leq c \leq a$, and then it holds that $a = F(G(\alpha, 0), 1) = G(0, F(\alpha, 1)) = G(0, a) = c$.

Taking $x = 1, y = 0$, then we have that $G(\alpha, 1) = c \geq d \geq b$, and then it holds that $b = F(G(\alpha, 1), 0) = G(1, F(\alpha, 0)) = G(1, b) = d$.

Therefore it follows that $a = c > \alpha > b = d$. □

For Subcase (4), we have the following lemma.

Lemma 3.8. *Let $d = \min(b, d) < \alpha < \max(a, c) = a$, $F \in \mathcal{F}_{a,b}$ with $a \geq b$ and $G \in \mathcal{F}_{c,d}$ with $c \geq d$. If F is α-cross-migrative over G, then $b = d < \alpha < a = c$.*

Proof. Taking $x = 0, y = 1$, then we have that $a = F(d, 1) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 1)) = G(0, a) = c$.

Assume that $d < \alpha < b \leq a = c$. Taking $x = 1, y = 0$, then we have that $b = F(c, 0) = F(G(\alpha, 1), 0) = G(1, F(\alpha, 0)) = G(1, \alpha) = \alpha$, which contradicts $\alpha < b$. Thus $b \leq \alpha < c$.

Taking $x = d, y = 0$, then we have that $d = F(d, 0) = F(G(\alpha, d), 0) = G(d, F(\alpha, 0)) = G(d, b) = b$.

Therefore it follows that $a = c > \alpha > b = d$. □

Theorem 3.9. *Let $\min(b, d) < \alpha < \max(a, c)$, $F \in \mathcal{F}_{a,b}$ with $a \geq b$ and $G \in \mathcal{F}_{c,d}$ with $c \geq d$, then F is α-cross-migrative over G if and only if $b = d < \alpha < a = c$.*

Proof. ($\Rightarrow$) From Lemmas 3.5, 3.6, 3.7 and 3.8, we can directly get the result.

($\Leftarrow$) Suppose that $x, y \in [0, 1]$, we consider the following cases.

Assume that $y \leq b$, there exist the following subcases to be checked.

- If $x \geq a$, then $F(G(\alpha, x), y) = F(a, y) = b = G(x, b) = G(x, F(\alpha, y))$.
- If $x \in [b, a]$, then $F(G(\alpha, x), y) = F(x, y) = b = G(x, b) = G(x, F(\alpha, y))$.
- If $x \leq b$, then $F(G(\alpha, x), y) = F(b, y) = b = G(x, b) = G(x, F(\alpha, y))$.

Assume that $y \in [b, a]$, we have that $F(G(\alpha, x), y) = y = G(x, y) = G(x, F(\alpha, y))$.

Assume that $y \geq a$, we have the following subcases to be checked.

- If $x \geq a$, then $F(G(\alpha, x), y) = F(a, y) = a = G(x, a) = G(x, F(\alpha, y))$.
- If $x \in [b, a]$, then $F(G(\alpha, x), y) = F(x, y) = a = G(x, a) = G(x, F(\alpha, y))$.
- If $x \leq b$, then $F(G(\alpha, x), y) = F(b, y) = a = G(x, a) = G(x, F(\alpha, y))$. □

Remark 3.10. *By means of Theorems 3.2, 3.4, 3.9, we know that $F \in \mathcal{F}_{a,b}$ with $a \geq b$ is not α-cross-migrativity over $G \in \mathcal{F}_{c,d}$ with $c \geq d$ except $F, G \in \mathcal{F}_{a,b}$. Moveover, if $F, G \in \mathcal{F}_{a,b}$, we find the fact as follow:*

(i) For $\alpha \leq b$, (α, G)-cross-migrativity of F is completely determined by the underlying semi-t-conorms S_F and S_G, which indicates that the underlying semi-t-conorms S_F and S_G have corresponding cross-migrativity, that is, the cross-migrative property of two semi-t-operators is completely ascribed to the cross-migrative property of their corresponding underlying semi-t-conorms.

(ii) For $\alpha \geq a$, (α, G)-cross-migrativity of F is completely determined by the underlying semi-t-norms T_F and T_G, which indicates that the underlying semi-t-norms T_F and T_G have corresponding cross-migrativity, that is, the cross-migrative property of two semi-t-operators is completely described to the cross-migrative property of their corresponding underlying semi-t-norms.

(iii) For $b < \alpha < a$, F is always (α, G)-cross-migrative.

Example 3.11. *Let F, $G \in \mathcal{F}_{0.6,0.4}$ be two semi-t-operators as follows:*

$$F(x,y) = \begin{cases} \max(x,y) & \textit{if } (x,y) \in [0,0.4]\times[0.2,0.4] \cup [0.2,0,4]\times[0,0.4], \\ x+y-5xy & \textit{if } (x,y) \in [0,0.2]^2, \\ 2.5xy-1.5x-1.5y+1.5 & \textit{if } (x,y) \in [0.6,1]^2, \\ 0.6 & \textit{if } (x,y) \in [0,0.6]\times[0.6,1], \\ 0.4 & \textit{if } (x,y) \in [0.4,1]\times[0,0.4], \\ y & \textit{otherwise}, \end{cases} \tag{13}$$

$$G(x,y) = \begin{cases} \max(x,y) & \textit{if } (x,y) \in [0,0.4]^2, \\ 2.5xy-1.5x-1.5y+1.5 & \textit{if } (x,y) \in [0.6,1]^2, \\ 0.6 & \textit{if } (x,y) \in [0,0.6]\times[0.6,1], \\ 0.4 & \textit{if } (x,y) \in [0.4,1]\times[0,0.4], \\ y & \textit{otherwise}, \end{cases} \tag{14}$$

(i) If $\alpha \leq 0.4$, then (S_F, S_G) is 0.5-cross-migrative, and then (F, G) is 0.2-cross-migrative.

(ii) If $\alpha \in (0.4, 0.6)$, then (F, G) is 0.5-cross-migrative.

(iii) If $\alpha \geq 0.6$, then (T_F, T_G) is 0.6-cross-migrative, and then (F, G) is 0.84-cross-migrative.

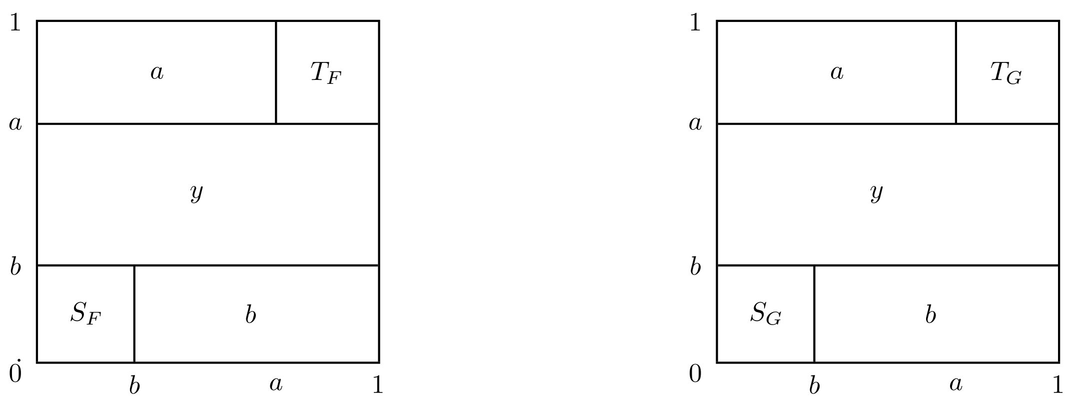

Fig.1. Structures of F (left) and G (right) in Section 3 when $a \geq b$.

4 Cross-migrativity of $F \in \mathcal{F}_{a,b}$ with $a \geq b$ ove $G \in \mathcal{F}_{c,d}$ with $c \leq d$

In this section, depending on the order relationship among α, a, b, c, d, we need to consider three different cases: (1) $\alpha \leq \min(b, c)$, (2) $\alpha \geq \max(a, d)$, (3) $\min(b, c) < \alpha < \max(a, d)$. Now, let us consider the first case.

4.1 $\alpha \leq \min(b,c)$

Lemma 4.1. *Let $\alpha \leq \min(b,c)$, $F \in \mathcal{F}_{a,b}$ with $a \geq b$ and $G \in \mathcal{F}_{c,d}$ with $c \leq d$. If F is α-cross-migrative over G, then $\alpha \leq a = b = c = d$.*

Proof. Assume that $a > c$. Taking $x = 0, y = 1$ in Eq.(8), then we have that $a = F(\alpha,1) = F(G(\alpha,0),1) = G(0,F(\alpha,1)) = G(0,a) = c$, which contradicts with $a > c$. Thus $a \leq c$, i.e. $\alpha \leq b \leq a \leq c \leq d$.

Next, taking $x = 1, y = 0$ in Eq.(8), then we get that $b = F(c,0) = F(G(\alpha,1),0) = G(1,F(\alpha,0)) = G(1,\alpha) = d$.

Because of $b \leq a \leq c \leq d$ and $b = d$, we obtain that $a = b = c = d$. □

Theorem 4.2. *Let $\alpha \leq \min(b,c)$, $F \in \mathcal{F}_{a,b}$ with $a \geq b$ and $G \in \mathcal{F}_{c,d}$ with $c \leq d$. Then F is α-cross-migrative over G if and only if*

(i) $\alpha \leq a = b = c = d$.

(ii) S_F is $\frac{\alpha}{a}$-cross-migrative over S_G, where S_F and S_G are the underlying semi-t-conorms of F and G, respectively.

Proof. ($\Rightarrow$) Clearly, we can directly get (i) from Lemma 4.1. To prove (ii), consider $x,y \in [0,a]$ in Eq.(8), according to $\alpha \leq b = d = a = c$, we know that the left-hand side of Eq.(8) is

$$F(G(\alpha,x),y) = F(aS_G(\frac{\alpha}{a},\frac{x}{a}),y) = aS_F(S_G(\frac{\alpha}{a},\frac{x}{a}),\frac{y}{a}), \tag{15}$$

while the right-hand side of Eq.(8) is

$$G(x,F(\alpha,y)) = G(x,aS_F(\frac{\alpha}{a},\frac{y}{a})) = aS_G(\frac{x}{a},S_F(\frac{\alpha}{a},\frac{y}{a})). \tag{16}$$

Hence it follows that $S_F(S_G(\frac{\alpha}{a},\frac{x}{a}),\frac{y}{a}) = S_G(\frac{x}{a},S_F(\frac{\alpha}{a},\frac{y}{a}))$. Note that $\frac{x}{a},\frac{y}{a} \in [0,1]$, then we get that S_F is $\frac{\alpha}{b}$-cross-migrative over S_G. Thus we have proven (ii).

($\Leftarrow$) Suppose that $x,y \in [0,1]$, we consider the following cases:

Assume that $x,y \in [a,1]$, from structures of F,G and $\alpha \leq b = d = a = c$, we have that $F(G(\alpha,x),y) = F(a,y) = a = G(x,a) = G(x,F(\alpha,y))$.

Assume that $x,y \in [0,a]$, from structures of T,G, $\alpha \leq b = d = a = c$, S_F is $\frac{\alpha}{b}$-cross-migrative over S_G, that is, $S_F(S_G(\frac{\alpha}{a},\frac{x}{a}),\frac{y}{a}) = S_G(\frac{x}{a},S_F(\frac{\alpha}{a},\frac{y}{a}))$, then it follows that $F(G(\alpha,x),y) = F(aS_G(\frac{\alpha}{a},\frac{x}{a}),y) = aS_F(S_G(\frac{\alpha}{a},\frac{x}{a}),\frac{y}{a}) = aS_G(\frac{x}{a},S_F(\frac{\alpha}{a},\frac{y}{a})) = G(x,aS_F(\frac{\alpha}{a},\frac{y}{a})) = G(x,F(\alpha,y))$.

Assume that $(x,y) \in [0,a] \times [a,1] \cup [a,1] \times [0,a]$. Without loss of generality, we further assume that $x \geq y$, then it holds that $F(G(\alpha,x),y) = F(a,y) = a = G(x,aS_F(\frac{\alpha}{a},\frac{y}{a})) = G(x,F(\alpha,y))$. □

4.2 $\alpha \geq \max(a,d)$

Lemma 4.3. *Let $\alpha \geq \max(a,d)$, $F \in \mathcal{F}_{a,b}$ with $a \geq b$ and $G \in \mathcal{F}_{c,d}$ with $c \leq d$. If F is α-cross-migrative over G, then $\alpha \geq a = c = b = d$.*

Proof. Assume that $b < d$. Taking $x = 1, y = 0$, then we have that $b = F(\alpha,0) = F(G(\alpha,1),0) = G(1,F(\alpha,0)) = G(1,b) = d$, which contradicts $b < d$. Thus $b \geq d$, that is, $\alpha \geq a \geq b \geq d \geq c$.

Next, taking $x = 0, y = 1$, then we have that $a = F(d,1) = F(G(\alpha,0),1) = G(0,F(\alpha,1)) = G(0,\alpha) = c$.

Because of $a \geq b \geq d \geq c$ and $a = c$, we obtain that $a = b = c = d$.

Therefore it follows that $\alpha \geq b = d = a = c$. □

Theorem 4.4. *Let $\alpha \geq \max(a,d)$, $F \in \mathcal{F}_{a,b}$ with $a \geq b$ and $G \in \mathcal{F}_{c,d}$ with $c \leq d$. Then F is α-cross-migrative over G if and only if*

(i) $\alpha \geq b = d = a = c$.

(ii) T_F is $\frac{\alpha-a}{1-a}$-cross-migrative over T_G.

Proof. ($\Rightarrow$) (i) is clearly established from Lemma 4.3. To prove (ii), consider $x,y \in [a,1]$ in Eq.(8), we have from $b = d = a = c \leq \alpha$ that the left-hand side of Eq.(8) is

$$F(G(\alpha,x),y) = F(a + (1-a)T_G(\frac{\alpha-a}{1-a},\frac{x-a}{1-a}),y) = a + (1-a)T_F(T_G(\frac{\alpha-a}{1-a},\frac{x-a}{1-a}),\frac{y-a}{1-a}), \tag{17}$$

while the right-hand side of Eq.(8) is

$$G(x,F(\alpha,y)) = G(x,a+(1-a)T_F(\frac{\alpha-a}{1-a},\frac{y-a}{1-a})) = a+(1-a)T_G(\frac{x-a}{1-a},T_F(\frac{\alpha-a}{1-a},\frac{y-a}{1-a})). \tag{18}$$

Thus it follows that $T_F(T_G(\frac{\alpha-a}{1-a},\frac{x-a}{1-a}),\frac{y-a}{1-a}) = T_G(\frac{x-a}{1-a},T_F(\frac{\alpha-a}{1-a},\frac{y-a}{1-a}))$. Note that $\frac{x-a}{1-a},\frac{y-a}{1-a}\in[0,1]$, we get that T_F is $\frac{\alpha-a}{1-a}$-cross-migrative over T_G.

($\Leftarrow$) Suppose that $x,y\in[0,1]$, we consider the following cases:

Assume that $x,y\in[0,a]$, from structures of F,G and $\alpha\geq b=d=a=c$, we have that $F(G(\alpha,x),y)=F(a,y)=a=G(x,a)=G(x,F(\alpha,y))$.

Assume that $x,y\in[a,1]$, from structures of F and G, $\alpha\geq b=d=a=c$, T_F is $\frac{\alpha-a}{1-a}$-cross-migrative over T_G, namely, $T_F(T_G(\frac{\alpha-a}{1-a},\frac{x-a}{1-a}),\frac{y-a}{1-a}) = T_G(\frac{x-a}{1-a},T_F(\frac{\alpha-a}{1-a},\frac{y-a}{1-a}))$, then we have that $F(G(\alpha,x),y)=F(a+(1-a)T_G(\frac{\alpha-a}{1-a},\frac{x-a}{1-a}),y) = a+(1-a)T_F(T_G(\frac{\alpha-a}{1-a},\frac{x-a}{1-a}),\frac{y-a}{1-a}) = a+(1-a)T_G(\frac{x-a}{1-a},T_F(\frac{\alpha-a}{1-a},\frac{y-a}{1-a})) = G(x,a+(1-a)T_F(\frac{\alpha-a}{1-a},\frac{y-a}{1-a})) = G(x,F(\alpha,y))$.

Assume that $(x,y)\in[0,a]\times[a,1]\cup[a,1]\times[0,a]$. Without loss of generality, we further assume that $x\geq y$, then it holds that $F(G(\alpha,x),y)=F(a+(1-a)T_G(\frac{\alpha-a}{1-a},\frac{x-a}{1-a}),y)=a=G(x,a)=G(x,F(\alpha,y))$. □

Remark 4.5. *For cases $\alpha\leq\min(b,c)$ and $\alpha\geq\max(a,d)$, if F is the (α,G)-cross-migrativity, then both F and G degenerate into semi-nullnorms shown in the Fig.2. Furthermore, from Theorems 4.2 and 4.4, the cross-migrativity between semi-t-operators is completely determined by their own underlying operators.*

Example 4.6.

$$F(x,y)=\begin{cases}\min(x+y,0,4) & \textit{if } (x,y)\in[0,0.4]^2,\\ \max(x,y) & \textit{if } (x,y)\in[0.4,0.5]\times[0,0.5]\cup[0,0.4]\times[0.4,0.5],\\ 5xy+4-4x-4y & \textit{if } (x,y)\in[0.8,1]^2,\\ \min(x,y) & \textit{if } (x,y)\in[0.5,0.8]\times[0.8,1]\cup[0.8,1]\times[0.5,0.8],\\ 0.5 & \textit{otherwise.}\end{cases} \tag{19}$$

$$G(x,y)=\begin{cases}\max(x,y) & \textit{if } (x,y)\in[0,0.5]^2,\\ \min(x,y) & \textit{if } (x,y)\in[0.5,1]^2,\\ 0.5 & \textit{otherwise.}\end{cases} \tag{20}$$

(i) If $\alpha\leq0.5$, then (S_F,S_G) is 0.8-cross-migrative, and then (F,G) is 0.4-cross-migrative.

(ii) If $\alpha\geq0.5$, then (T_F,T_G) is 0.6-cross-migrative, and then (F,G) is 0.8-cross-migrative.

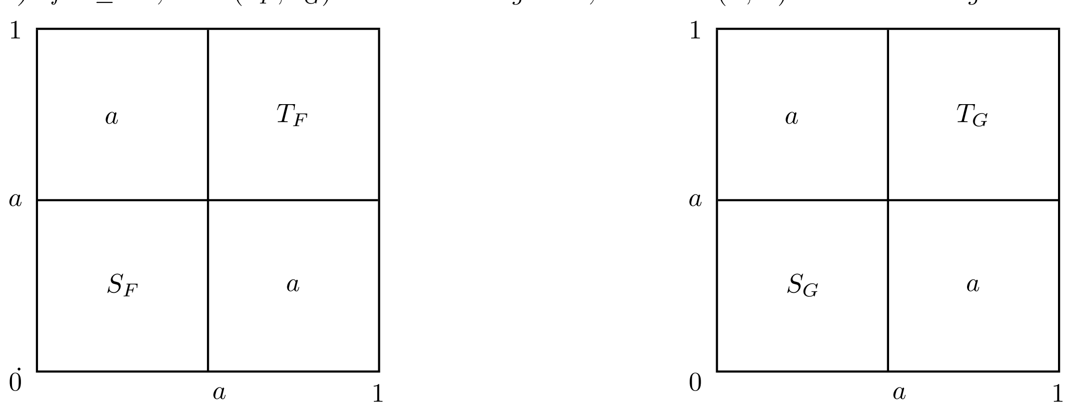

Fig.2. Structure of F (left) and G (right) in Subsection 4.1 (4.2)

4.3 $\min(b,c)<\alpha<\max(a,d)$

For this case, there are four different subcases to be considered: (1) $a\geq d,b\geq c$, (2) $a\geq d,b\leq c$, (3) $a\leq d,b\geq c$, (4) $a\leq d,b\leq c$. But it is interesting that the above four cases have the same result that F is not α-cross-migrative over G. Now we start Subcase (1).

Lemma 4.7. *Let $c = \min(b,c) < \alpha < \max(a,d) = a$, $F \in \mathcal{F}_{a,b}$ with $a \geq b$ and $G \in \mathcal{F}_{c,d}$ with $c \leq d$. Then F is not α-cross-migrative over G.*

Proof. Taking $x = 0$ and $y = 1$, from $G(\alpha,0) \leq G(\alpha,1) \leq G(a,1) = a$, then we have that $a = F(G(\alpha,0),1) = G(0,F(\alpha,1)) = G(0,a) = c$, which contradicts with $c < \alpha < a$. □

For Subcase (2), there is the following lemma.

Lemma 4.8. *Let $b = \min(b,c) < \alpha < \max(a,d) = a$, $F \in \mathcal{F}_{a,b}$ with $a \geq b$ and $G \in \mathcal{F}_{c,d}$ with $c \leq d$. Then F is not α-cross-migrative over G.*

Proof. Taking $x = 0, y = 1$, then we have that $a = F(G(\alpha,0),1) = G(0,F(\alpha,1)) = G(0,a) = c$. Note that $a \geq d \geq c$ and $a = c$, we get that $a = c = d$.

Next, taking $x = 1, y = 0$, we have that $b = F(a,0) = F(G(\alpha,1),0) = G(1,F(\alpha,0)) = G(1,b) = a$. Then we can get that $a = b = c = d$. But it contradicts $b < \alpha < a$. □

For Subcase (3), we have the following lemma.

Lemma 4.9. *Let $c = \min(b,c) < \alpha < \max(a,d) = d$, $F \in \mathcal{F}_{a,b}$ with $a \geq b$ and $G \in \mathcal{F}_{c,d}$ with $c \leq d$. Then F is not α-cross-migrative over G.*

Proof. Assume that $\alpha < b$, taking $x = 1, y = 0$, then we have that $\alpha = F(\alpha,0) = F(G(\alpha,1),0) = G(1,F(\alpha,0)) = G(1,\alpha) = d$. But it contradicts $\alpha < d$. Thus it holds that $b \leq \alpha < d$.

Taking $x = 1, y = 0$ again, then we have that $b = F(\alpha,0) = F(G(\alpha,1),0) = G(1,F(\alpha,0)) = G(1,b) = d$. But it contradicts $b \leq \alpha < d$. □

For Subcase (4), we have the following lemma.

Lemma 4.10. *Let $b = \min(b,c) < \alpha < \max(a,d) = d$, $F \in \mathcal{F}_{a,b}$ with $a \geq b$ and $G \in \mathcal{F}_{c,d}$ with $c \leq d$. Then F is not α-cross-migrative over G.*

Proof. Taking $x = 1, y = 0$, then from $G(\alpha,1) \geq c \geq b$, it follows that $b = F(G(\alpha,1),0) = G(1,F(\alpha,0)) = G(1,b) = d$. But it contradicts $b < \alpha < d$. □

Remark 4.11. *From Theorems 4.2, 4.4, Lemmas 4.7, 4.8, 4.9, 4.10, if $F \in \mathcal{F}_{a,b}$ with $a \geq b$ over $G \in \mathcal{F}_{c,d}$ with $c \leq d$ is α-cross-migrative, then we can boldly assert these two semi-t-operators must be semi-nullnorm.*

5 Cross-migrativity of $F \in \mathcal{F}_{a,b}$ with $a \leq b$ over $G \in \mathcal{F}_{c,d}$ with $c \leq d$

In this section, depending on the order relationship among α, a, b, c, d, we will consider three different cases: (1) $\alpha \leq \min(a,c)$, (2) $\alpha \geq \max(b,d)$, (3) $\min(a,c) < \alpha < \max(b,d)$. Next, let us consider the first case.

5.1 $\alpha \leq \min(a,c)$

Lemma 5.1. *Let $\alpha \leq \min(a,c)$, $F \in \mathcal{F}_{a,b}$ with $a \leq b$ and $G \in \mathcal{F}_{c,d}$ with $c \leq d$. If F is α-cross-migrative over G, then $\alpha \leq a \leq c = d \leq b$.*

Proof. Assume that $a > c$, taking $x = 0, y = 1$, then we have that $a = F(\alpha,1) = F(G(\alpha,0),1) = G(0,F(\alpha,1)) = G(0,a) = c$, which contradicts $a > c$. Thus $a \leq c$.

Next, taking $x = 1, y = 1$, then we have that $c = F(c,1) = F(G(\alpha,1),1) = G(1,F(\alpha,1)) = G(1,a) = d$.

Assume that $d > b$. Taking $x = 1, y = 0$, then we obtain that $b = F(c,0) = F(G(\alpha,1),0) = G(1,F(\alpha,0)) = G(1,\alpha) = d$, which contradicts $d > b$. Therefore $\alpha \leq a \leq c = d \leq b$. □

Theorem 5.2. *Let $\alpha \leq \min(a,c)$, $F \in \mathcal{F}_{a,b}$ with $a \leq b$ and $G \in \mathcal{F}_{c,d}$ with $c \leq d$. Then F is α-cross-migrative over G if and only if one of the following two cases holds.*

(i) If $\alpha \leq a = c = d \leq b$, then S_F is $\frac{\alpha}{a}$-cross-migrative over S_G.

(ii) If $\alpha \leq a < c = d \leq b$, and let $x_0 = \sup\{x \in [0,1] | G(\alpha,x) \leq a\}$, then one of the following two subcases holds.

(a) If $G(\alpha,x_0)=a$*, then* $G(x,a)=\begin{cases}G(\alpha,x)=G(x,\alpha) & \text{if } x>x_0,\\ a & \text{if } x\le x_0,\end{cases}$ *and* $F(G(\alpha,x),y)=G(x,F(\alpha,y))$ *when* $x\le x_0, y\le a$.

(b) If $G(\alpha,x_0)<a$*, then* $G(x,a)=\begin{cases}G(\alpha,x)=G(x,\alpha) & \text{if } x\ge x_0,\\ a & \text{if } x<x_0,\end{cases}$ *and* $F(G(\alpha,x),y)=G(x,F(\alpha,y))$ *when* $x<x_0, y\le a$.

Proof. For (i), clearly, it follows from Lemma 5.1 that $\alpha\le a\le c=d\le b$. Further, if $\alpha\le a=c=d\le b$ and $x\le a, y\le a$, then, from structures of F,G, it follows that $aS_F(\frac{c}{a}S_G(\frac{\alpha}{c},\frac{x}{c}),\frac{y}{a})=aS_F(S_G(\frac{\alpha}{a},\frac{x}{a}),\frac{y}{a})=cS_G(\frac{x}{c},\frac{a}{c}S_F(\frac{\alpha}{a},\frac{y}{a}))=aS_G(\frac{x}{a},S_F(\frac{\alpha}{a},\frac{y}{a}))$. Note that $\frac{\alpha}{a},\frac{x}{a},\frac{y}{a}\in[0,1]$, we can obtain that $S_F(S_G(\frac{\alpha}{a},x),y)=S_G(x,S_F(\frac{\alpha}{a},y))$ for all $x,y\in[0,1]$, i.e. S_F is $\frac{\alpha}{a}$-cross-migrative over S_G.

Conversely, suppose that $x,y\in[0,1]$, we consider the following cases:

Assume that $x,y\in[a,1]$, from structures of F,G and $\alpha\le d=a=c\le b$, we have that $F(G(\alpha,x),y)=F(a,y)=a=G(x,a)=G(x,F(\alpha,y))$.

Assume that $x,y\in[0,a]$, from structures of F,G and $\alpha\le d=a=c\le b$ and S_F is $\frac{\alpha}{b}$-cross-migrative over S_G, we have that $S_F(S_G(\frac{\alpha}{a},\frac{x}{a}),\frac{y}{a})=S_G(\frac{x}{a},S_F(\frac{\alpha}{a},\frac{y}{a}))$, then $F(G(\alpha,x),y)=F(aS_G(\frac{\alpha}{a},\frac{x}{a}),y)=aS_F(S_G(\frac{\alpha}{a},\frac{x}{a}),\frac{y}{a})=aS_G(\frac{x}{a},S_F(\frac{\alpha}{a},\frac{y}{a}))=G(x,aS_F(\frac{\alpha}{a},\frac{y}{a}))=G(x,F(\alpha,y))$.

Assume that $(x,y)\in[0,a]\times[a,1]\cup[a,1]\times[0,a]$, without loss of generality, we further assume that $x\ge y$, then $F(G(\alpha,x),y)=F(a,y)=a=G(x,aS_F(\frac{\alpha}{a},\frac{y}{a}))=G(x,F(\alpha,y))$.

For (ii), we only prove (ii)(a) because (ii)(b) is similar. By Lemma 5.1, we can assume that $\alpha\le a<c=d\le b$. Specially, we know from Eq.(8) that $F(G(\alpha,x),y)=G(x,F(\alpha,y))$ when $x\le x_0, y\le a$. Further, it follows from structure of G and definition of x_0 that $G(x,a)=a$ when $x\le x_0$. Next, we prove the last result in (ii)(a), that is, $G(x,a)=G(\alpha,x)=G(x,\alpha)$ holds when $x>x_0$. Note that $a<G(\alpha,x)\le c$ when $x>x_0$, then we have from structure of F and Eq.(8) that

$$G(\alpha,x)=F(G(\alpha,x),y)=G(x,F(\alpha,y)). \tag{21}$$

Specially, take $y=0$ and $y=1$ in Eq.(21) respectively, then we can obtain the required result.

Conversely, suppose that $x,y\in[0,1]$, we consider the following cases:

Assume that $x\le x_0, y\le a$, then it holds that Eq.(8) since the assumption $F(G(\alpha,x),y)=G(x,F(\alpha,y))$ when $x\le x_0, y\le a$.

Assume that $x\le x_0, y\ge a$, then it follows from structures of F,G and assumption $G(x,a)=a$ when $x\le x_0$ that $F(G(\alpha,x),y)=a=G(x,a)=G(x,F(\alpha,y))$.

Assume that $x>x_0$, then it follows from structures of F,G that $G(x,\alpha)\le G(x,F(\alpha,y))\le G(x,a)$. Further, using assumption $G(x,a)=G(\alpha,x)=G(x,\alpha)$ when $x>x_0$, we can obtain that $G(x,F(\alpha,y))=G(\alpha,x)$. Therefore, it holds that $F(G(\alpha,x),y)=G(\alpha,x)=G(x,F(\alpha,y))$. □

Remark 5.3. *(i) For Case (i) in Theorem 5.2, when F is α-cross-migrative over G, then G must be a semi-nullnorm and the cross-migrativity is completely determined by their underlying semi-t-conorms S_F and S_G.*

(ii) For Case (ii) in Theorem 5.2, when F is α-cross-migrative over G, then G must be a semi-nullnorm and the cross-migrativity is completely determined by the cross-migrativity of F,G on the domain $x\le x_0, y\le a$ or $x<x_0, y\le a$, and by the values of $G(x,a)$. These results are different from the above other obtained results.

5.2 $\alpha\ge\max(b,d)$

Lemma 5.4. *Let $\alpha\ge\max(b,d)$, $F\in\mathcal{F}_{a,b}$ with $a\le b$ and $G\in\mathcal{F}_{c,d}$ with $c\le d$. If F is α-cross-migrative over G, then $a\le c=d\le b\le\alpha$.*

Proof. Assume that $d<a$. Taking $x=0, y=1$, then we can get that $a=F(d,1)=F(G(\alpha,0),1)=G(0,F(\alpha,1))=G(0,\alpha)=c$, then $a=c>d$, which contradicts $c\le d$. Thus $d\ge a$.

Assume that $b<d$. Taking $x=1, y=0$, then we get that $b=F(\alpha,0)=F(G(\alpha,1),0)=G(1,F(\alpha,0))=G(1,b)=d$, which contradicts $b<d$. Thus $b\ge d$.

Next, taking $x=0, y=0$, then we have that $d=F(d,0)=F(G(\alpha,0),0)=G(0,F(\alpha,0))=G(0,b)=c$.

Thus it follows that $\alpha\ge b\ge c=d\ge a$. □

Theorem 5.5. *Let $\alpha\ge\max(b,d)$, $F\in\mathcal{F}_{a,b}$ with $a\le b$ and $G\in\mathcal{F}_{c,d}$ with $c\le d$. Then F is α-cross-migrative over G if and only if one of the following results is true,* *(i) If $a \leq c = d = b \leq \alpha$, then T_F is $\frac{\alpha-b}{1-b}$-cross-migrative over T_G.*

(ii) If $a \leq c = d < b \leq \alpha$, and denote $x_0 = \inf\{x \in [0,1] | G(\alpha, x) \geq b\}$, then one of the following two subcases holds.

(a) If $G(\alpha, x_0) = b$, then $G(x, b) = \begin{cases} G(\alpha, x) = G(x, \alpha) & \text{if } x < x_0, \\ b & \text{if } x \geq x_0, \end{cases}$ and $F(G(\alpha, x), y) = G(x, F(\alpha, y))$ when $x \geq x_0, y \geq b$.

(b) If $G(\alpha, x_0) < b$, then $G(x, b) = \begin{cases} G(\alpha, x) = G(x, \alpha) & \text{if } x \leq x_0, \\ b & \text{if } x > x_0, \end{cases}$ and $F(G(\alpha, x), y) = G(x, F(\alpha, y))$ when $x > x_0, y \geq b$.

Proof. For (i), clearly, it follows from Lemma 5.4 that $\alpha \leq a \leq c = d \leq b \leq \alpha$. Further, if $\alpha \leq a = c = d \leq b \leq \alpha$ and $x, y \in [a, 1]$ in Eq.(8), since F is α-cross-migrative over G, then we have that $b+(1-b)T_F(T_G(\frac{\alpha-b}{1-b}, \frac{x-b}{1-b}), \frac{y-b}{1-b}) = F(b+(1-b)T_G(\frac{\alpha-b}{1-b}, \frac{x-b}{1-b}), y) = F(G(\alpha, x), y) = G(x, F(\alpha, y)) = G(x, b+(1-b)T_F(\frac{\alpha-b}{1-b}, \frac{y-b}{1-b})) = b+(1-b)T_G(\frac{x-b}{1-b}, T_F(\frac{\alpha-b}{1-b}, \frac{y-b}{1-b}))$, and then $T_F(T_G(\frac{\alpha-b}{1-b}, \frac{x-b}{1-b}), \frac{y-b}{1-b}) = T_G(\frac{x-b}{1-b}, T_F(\frac{\alpha-b}{1-b}, \frac{y-b}{1-b}))$. Note that $\frac{x-b}{1-b}, \frac{y-b}{1-b} \in [0,1]$, thus we get that T_F is $\frac{\alpha-b}{1-b}$-cross-migrative over T_G.

Conversely, suppose that $x, y \in [0,1]$, we consider the following cases:

Assume that $x, y \in [0, b]$, from structure of F, G and $a \leq c = d = b \leq \alpha$, we have that $F(G(\alpha, x), y) = F(d, y) = d = G(x, d) = G(x, F(\alpha, y))$.

Assume that $x, y \in [b, 1]$, from structure of F, G, $a \leq c = d = b \leq \alpha$ and T_F is $\frac{\alpha-b}{1-b}$-cross-migrative over T_G, that is, $T_F(T_G(\frac{\alpha-b}{1-b}, \frac{x-b}{1-b}), \frac{y-b}{1-b}) = T_G(\frac{x-b}{1-b}, T_F(\frac{\alpha-b}{1-b}, \frac{y-b}{1-b}))$, then we have that $F(G(\alpha, x), y) = F(b + (1-b)T_G(\frac{\alpha-b}{1-b}, \frac{x-b}{1-b}), y) = b + (1-b)T_F(T_G(\frac{\alpha-b}{1-b}, \frac{x-b}{1-b}), \frac{y-a}{1-a}) = b + (1-b)T_G(\frac{x-b}{1-b}, T_F(\frac{\alpha-b}{1-b}, \frac{y-b}{1-b})) = G(x, b + (1-b)T_F(\frac{\alpha-b}{1-b}, \frac{y-b}{1-b})) = G(x, F(\alpha, y))$.

Assume that $(x, y) \in [0, b] \times [b, 1] \cup [b, 1] \times [0, b]$, without loss of generality, we further assume that $x > y$, then it follows that $F(G(\alpha, x), y) = F(b + (1-b)T_G(\frac{\alpha-b}{1-b}, \frac{x-b}{1-b}), y) = b = G(x, b) = G(x, F(\alpha, y))$.

For (ii), we only prove (ii)(a) because (ii)(b) is similar. By Lemma 5.4, we can assume that $a \leq c = d \leq b \leq \alpha$. Specially, we know from Eq.(8) that $F(G(\alpha, x), y) = G(x, F(\alpha, y))$ when $x \geq x_0, y \geq b$. Further, it follows from structure of G and definition of x_0 that $G(x, b) = b$ when $x \geq x_0$. Next, we prove the remaining result in (ii)(a), that is, $G(x, b) = G(\alpha, x) = G(x, \alpha)$ holds when $x < x_0$. Note that $c \leq G(\alpha, x) < b$ when $x < x_0$, then we have from structure of F and Eq.(8) that

$$G(\alpha, x) = G(x, F(\alpha, y)). \tag{22}$$

Specially, take $y = 0$ and $y = 1$ in Eq.(21) respectively, then we can obtain the required result.

Conversely, suppose that $x, y \in [0,1]$, we consider the following cases:

Assume that $x \geq x_0, y \geq b$, then it holds that Eq.(8) since the assumption $F(G(\alpha, x), y) = G(x, F(\alpha, y))$ when $x \geq x_0, y \geq b$.

Assume that $x \geq x_0, y \leq b$, then it follows from structures of F, G and assumption $G(x, b) = b$ when $x \geq x_0$ that $F(G(\alpha, x), y) = b = G(x, b) = G(x, F(\alpha, y))$.

Assume that $x < x_0$, then it follows from structures of F, G that $G(x, b) \leq G(x, F(\alpha, y)) \leq G(x, \alpha)$. Further, using assumption $G(x, b) = G(\alpha, x) = G(x, \alpha)$ when $x < x_0$, we can obtain that $G(x, F(\alpha, y)) = G(\alpha, x)$. Therefore, it holds that $F(G(\alpha, x), y) = G(\alpha, x) = G(x, F(\alpha, y))$. □

Remark 5.6. *(i) For Case (i) in Theorem 5.5, when F is α-cross-migrative over G, then G must be a semi-nullnorm and the cross-migrativity is completely determined by their underlying semi-t-conorms T_F and T_G.*

(ii) For Case (ii) in Theorem 5.5, when F is α-cross-migrative over G, then G must be a semi-nullnorm and the cross-migrativity is completely determined by the cross-migrativity of F, G on the domain $x \geq x_0, y \geq b$ or $x > x_0, y \geq b$, and by the values of $G(x, b)$. These results are different from the above other obtained results.

5.3 $\min(b, d) < \alpha < \max(a, c)$

For this case, there are four different subcases to be considered: (1) $a \leq c, b \leq d$, (2) $a \leq c, b \geq d$, (3) $a \geq c, b \leq d$, (4) $a \geq c, b \geq d$. Now we start Subcase (1).

Lemma 5.7. *Let $a = \min(a, c) < \alpha < \max(b, d) = d$, $F \in \mathcal{F}_{a,b}$ with $a \leq b$ and $G \in \mathcal{F}_{c,d}$ with $c \leq d$. If F is α-cross-migrative over G, then $a < \alpha < b = c = d$.*

Proof. Assume that $c < \alpha < d$, taking $x = 0, y = 1$, then we get that $F(G(\alpha,0),1) = F(\alpha,1) = \alpha = G(0,F(\alpha,1)) = G(0,\alpha) = c$, and then $\alpha = c$ contradicts $c < \alpha < d$, thus $a < \alpha \le c$.

Taking $x = 1, y = 1$, then we can obtain that $F(G(\alpha,1),1) = F(c,1) = c = G(1,F(\alpha,1)) = G(1,\alpha) = d$, then $a < \alpha < c = d$ and $a \le b \le d = c$.

Taking $x = 1, y = 0$, then we have that $F(G(\alpha,1),0) = F(c,0) = b = G(1,F(\alpha,0)) = d$, and then $a < \alpha < b = c = d$. □

Theorem 5.8. *Let $a = \min(a,c) < \alpha < \max(b,d) = d$, $F \in \mathcal{F}_{a,b}$ with $a \le b$ and $G \in \mathcal{F}_{c,d}$ with $c \le d$. Then F is α-cross-migrative over G if and only if the following two results are true.*

(i) $a < \alpha < b = c = d$.

(ii) $G(x,\alpha) = G(\alpha,x)$, *for all* $x \in [0,c]$.

Proof. ($\Rightarrow$) (i) is clearly established from Lemma 5.7. To prove (ii), consider $x, y \in [0,c]$ in Eq.(8), since $a < \alpha \le cS_G(\frac{\alpha}{c},\frac{x}{c}) \le c = b$ and F is α-cross-migrative over G, we have that $G(\alpha,x) = F(G(\alpha,x),y) = G(x,F(\alpha,y)) = G(x,\alpha)$ for all $x \in [0,c]$. Thus the result is proved.

($\Leftarrow$) Suppose that $x, y \in [0,1]$, we consider the following cases:

If $x \ge c$, then, from $a < \alpha < b = c = d$ and the structures of F and G, we have that $F(G(\alpha,x),y) = F(c,y) = c = G(x,\alpha) = G(x,F(\alpha,y))$ for all $y \in [0,1]$.

If $x \le c$, then, since $a < \alpha < b = c = d$, $G(x,\alpha) = G(\alpha,x)$ for all $x \in [0,c]$ and the structures of F and G, we obtain that $F(G(\alpha,x),y) = F(cS_G(\frac{\alpha}{c},\frac{x}{c}),y) = cS_G(\frac{\alpha}{c},\frac{x}{c}) = G(\alpha,x) = G(x,\alpha) = G(x,F(\alpha,y))$ for all $y \in [0,1]$. □

Remark 5.9. *For Subcase $a = \min(a,c) < \alpha < \max(b,d) = d$, the (α,G)-cross-migrativity of F is completely determined by a portion of α-section of G and has nothing to do with the remaining of $[0,1]^2$.*

For Subcase (2), there is the following lemma.

Lemma 5.10. *Let $a = \min(a,c) < \alpha < \max(b,d) = b$, $F \in \mathcal{F}_{a,b}$ with $a \le b$ and $G \in \mathcal{F}_{c,d}$ with $c \le d$. If F is α-cross-migrative over G, then $a < \alpha \le c = d \le b$ or $a \le c = d \le \alpha < b$.*

Proof. Assume that $c < \alpha < d$, then, taking $x = 1, y = 0$, we can get that $\alpha = F(\alpha,0) = F(G(\alpha,1),0) = G(1,F(\alpha,0)) = G(1,\alpha) = d$, which contradicts $\alpha < d$. Taking Thus it follows that $a < \alpha \le c$ or $d \le \alpha < b$.

If $a < \alpha \le c$, then, taking $x = 1, y = 0$ in Eq. 8, we can get that $c = F(c,0) = F(G(\alpha,1),0) = G(1,F(\alpha,0)) = G(1,\alpha) = d$. So $a < \alpha \le c = d \le b$.

If $d \le \alpha < b$, then, taking $x = 0, y = 1$ in Eq. 8, we can get that $d = F(d,1) = F(G(\alpha,0),1) = G(0,F(\alpha,1)) = G(0,\alpha) = c$. So $a \le c = d \le \alpha < b$.

In a word, we have always that $a < \alpha \le c = d \le b$, or $a \le c = d \le \alpha < b$.

□

Theorem 5.11. *Let $a = \min(a,c) < \alpha < \max(b,d) = b$, $F \in \mathcal{F}_{a,b}$ with $a \le b$ and $G \in \mathcal{F}_{c,d}$ with $c \le d$. Then F is α-cross-migrative over G if and only if one of the following conditions is true*

(i) $a < \alpha \le c = d \le b$ *and* $G(x,\alpha) = G(\alpha,x)$, *for all* $x \in [0,c]$.

(ii) $a \le c = d \le \alpha < b$, *and* $G(x,\alpha) = G(\alpha,x)$, *for all* $x \in [c,1]$.

Proof. ($\Rightarrow$) We can directly get from Lemma 5.10 that $a < \alpha \le c = d \le b$ or $a \le c = d \le \alpha < b$.

On the one hand, for the case $a < \alpha \le c = d \le b$, consider $x, y \in [0,c]$ in Eq.(8), from the structures of F and G and $a < \alpha \le cS_G(\frac{\alpha}{c},\frac{x}{c}) \le c \le b$, we have that $G(\alpha,x) = F(cS_G(\frac{\alpha}{c},\frac{x}{c}),y) = F(G(\alpha,x),y) = G(x,F(\alpha,y)) = G(x,\alpha)$ for all $x \in [0,c]$. Thus (i) is proved.

On the other hand, for the case $a \le c = d \le \alpha < b$, consider $x, y \in [c,1]$ in Eq.(8), the same procedure may be easily adapted to obtaining $G(x,\alpha) = G(\alpha,x)$ for all $x \in [c,1]$. Thus (ii) is proved.

($\Leftarrow$) For Case (i), we need to consider the following cases:

If $x \ge c, y \in [0,1]$, from (i) and structures of F and G, then it follows that $F(G(\alpha,x),y) = F(c,y) = c = G(x,\alpha) = G(x,F(\alpha,y))$.

If $x \le c, y \in [0,1]$, from (i) and structures of F and G, then it holds that $F(G(\alpha,x),y) = F(cS_G(\frac{\alpha}{c},\frac{x}{c}),y) = cS_G(\frac{\alpha}{c},\frac{x}{c}) = G(\alpha,x) = G(x,\alpha) = G(x,F(\alpha,y))$.

For Case (ii), we can prove that the result is true by using the same methods. □

Remark 5.12. *For the case $a = \min(a, c) < \alpha < \max(b, d) = b$, the (α, G)-cross-migrativity of F is completely determined by a portion of α-section of G, and has nothing to do with the remaining of $[0, 1]^2$.*

For Subcase (3), there is the following lemma.

Lemma 5.13. *Let $c = \min(a, c) < \alpha < \max(b, d) = d$, $F \in \mathcal{F}_{a,b}$ with $a \le b$ and $G \in \mathcal{F}_{c,d}$ with $c \le d$, then F is not α-cross-migrative over G.*

Proof. Assume that $a \le \alpha \le b$, then, taking $x = 0,\ y = 1$ in Eq.(8), we have that $\alpha = F(\alpha, 1) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 1)) = c$, which contradicts $\alpha > c$. Assume that $c < \alpha < a$, then, taking $x = 0,\ y = 1$ in Eq.(8), we have that $a = F(\alpha, 1) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 1)) = c$, which contradicts $a > \alpha > c$. Assume that $b < \alpha < d$, then, taking $x = 0,\ y = 1$ in Eq.(8), we have that $\alpha = F(\alpha, 1) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 1)) = c$, which contradicts $\alpha > c$.

Therefore, F is not α-cross-migrative over G. □

For Subcase (4), there is the following lemma.

Lemma 5.14. *Let $c = \min(a, c) < \alpha < \max(b, d) = b$, $F \in \mathcal{F}_{a,b}$ with $a \le b$ and $G \in \mathcal{F}_{c,d}$ with $c \le d$. If F is α-cross-migrative over G, then $a = c = d < \alpha < b$.*

Proof. Assume that $c < \alpha < d$, then, taking $x = 1,\ y = 0$ in Eq.(8), we can get that $\alpha = F(\alpha, 0) = F(G(\alpha, 1), 0) = G(1, F(\alpha, 0)) = G(1, \alpha) = d$, which contradicts $c < \alpha < d$. So $d \le \alpha < b$. On the one hand, taking $x = 0,\ y = 0$ in Eq.(8), we can get that $d = F(d, 0) = F(G(\alpha, 0), 0) = G(0, F(\alpha, 0)) = G(0, \alpha) = c$. On the other hand, taking $x = 0,\ y = 1$ in Eq.(8), we can get that $a = F(c, 1) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 1)) = c$.

Sum up, it follows that $a = c = d < \alpha < b$. □

Theorem 5.15. *Let $c = \min(a, c) < \alpha < \max(b, d) = b$, $F \in \mathcal{F}_{a,b}$ with $a \le b$ and $G \in \mathcal{F}_{c,d}$ with $c \le d$. Then F is α-cross-migrative over G if and only if*

(i) $a = c = d < \alpha < b$.

(ii) $G(x, \alpha) = G(\alpha, x)$, *for all* $x \in [a, 1]$.

Proof. ($\Rightarrow$) We can directly get from Lemma 5.14 that (i) is true. To prove (ii), from (i) and the structures of F and G and $a \le a + (1-a)T_G(\frac{\alpha-a}{1-a}, \frac{x-a}{1-a}) \le \alpha < b$, we have that $G(\alpha, x) = F(G(\alpha, x), y) = G(x, F(\alpha, y)) = G(x, \alpha)$ for all $x \in [a, 1]$. Thus (ii) is proved.

($\Leftarrow$) Suppose that $x, y \in [0, 1]$. If $x \le a, y \in [0, 1]$, from (i) and the structures of F and G, then $F(G(\alpha, x), y) = F(a, y) = a = G(x, \alpha) = G(x, F(\alpha, y))$. If $x \ge a,\ y \in [0, 1]$, from (i), (ii), $a \le a + (1-a)T_G(\frac{\alpha-a}{1-a}, \frac{x-a}{1-a}) \le \alpha < b$ and the structures of F and G, then $F(G(\alpha, x), y) = F(a + (1-a)T_G(\frac{\alpha-a}{1-a}, \frac{x-a}{1-a}), y) = a + (1-a)T_G(\frac{\alpha-a}{1-a}, \frac{x-a}{1-a}) = G(\alpha, x) = G(x, \alpha) = G(x, F(\alpha, y))$.

□

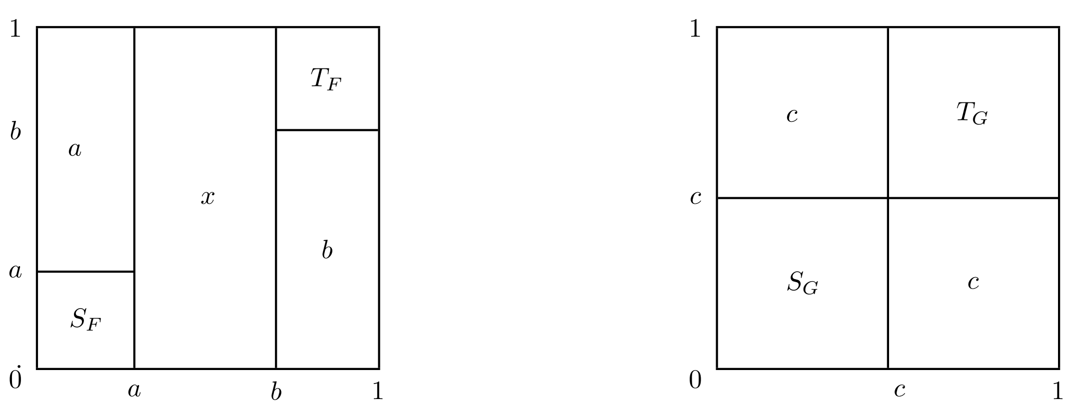

Fig.3. Structures of F (left) and G (right) in Section 5

Remark 5.16. *For the case $c = \min(a, c) < \alpha < \max(b, d) = b$, the (α, G)-cross-migrativity of F is completely determined by a portion of α-section of G, and has nothing to do with the remaining of $[0, 1]^2$.*

6 Cross-migrativity of $F \in \mathcal{F}_{a,b}$ with $a \leq b$ over $G \in \mathcal{F}_{c,d}$ with $c \geq d$

In this section, depending on the order relationship among α, a, b, c, d, we will consider three different cases: (1) $\alpha \leq \min(a, d)$, (2) $\alpha \geq \max(b, c)$, (3) $\min(a, d) < \alpha < \max(b, c)$. It is very interesting that the three cases have the same results with Section 5. Next, let us consider the first case.

6.1 $\alpha \leq \min(a, d)$

Lemma 6.1. *Let $\alpha \leq \min(a, d)$, $F \in \mathcal{F}_{a,b}$ with $a \leq b$ and $G \in \mathcal{F}_{c,d}$ with $c \geq d$. If F is α-cross-migrative over G, then $\alpha \leq a \leq c = d \leq b$.*

Proof. Assume that $a > c$, then, taking $x = 0, y = 1$ in Eq.(8), we have that $a = F(\alpha, 1) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 1)) = G(0, a) = c$, which contradicts $a > c$. Therefore $a \leq c$.

Assume that $d < a$, then, taking firstly $x = 1, y = 1$ in Eq.(8), we obtain that $c = F(c, 1) = F(G(\alpha, 1), 1) = G(1, F(\alpha, 1)) = G(1, a) = a$. So it follows that $d < a = c$. Further, taking $x = 1, y = 0$ in Eq.(8), then we get that $a = c = F(c, 0) = F(G(\alpha, 1), 0) = G(1, F(\alpha, 0)) = G(1, \alpha) = d$. Therefore it holds that $a = d$, but it contradicts $d < a$.

Thus we have $\alpha \leq a \leq c = d \leq b$. □

Theorem 6.2. *Let $\alpha \leq \min(a, d)$, $F \in \mathcal{F}_{a,b}$ with $a \leq b$ and $G \in \mathcal{F}_{c,d}$ with $c \geq d$. Then F is α-cross-migrative over G if and only if one of the following two cases holds:*

(i) If $\alpha \leq a = c = d \leq b$, then S_F is $\frac{\alpha}{a}$-cross-migrative over S_G.

(ii) If $\alpha \leq a < c = d \leq b$, and let $x_0 = \sup\{x \in [0, 1] | G(\alpha, x) \leq a\}$, then one of the following two subcases holds.

(a) If $G(\alpha, x_0) = a$, then $G(x, a) = \begin{cases} G(\alpha, x) = G(x, \alpha) & \text{if } x > x_0, \\ a & \text{if } x \leq x_0, \end{cases}$ and $F(G(\alpha, x), y) = G(x, F(\alpha, y))$ when $x \leq x_0, y \leq a$.

(b) If $G(\alpha, x_0) < a$, then $G(x, a) = \begin{cases} G(\alpha, x) = G(x, \alpha) & \text{if } x \geq x_0, \\ a & \text{if } x < x_0, \end{cases}$ and $F(G(\alpha, x), y) = G(x, F(\alpha, y))$ when $x < x_0, y \leq a$.

Proof. The proof is omitted because it is similar to that of Theorem 5.2. □

6.2 $\alpha \geq \max(b, c)$

Lemma 6.3. *Let $\alpha \geq \max(b, c)$, $F \in \mathcal{F}_{a,b}$ with $a \leq b$ and $G \in \mathcal{F}_{c,d}$ with $c \geq d$. If F is α-cross-migrative over G, then $a \leq c = d \leq b \leq \alpha$.*

Proof. Assume that $d < a$, then, taking $x = 0, y = 1$ in Eq.(8), we can get that $a = F(d, 1) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 0)) = G(0, \alpha) = c$. Thus it follows that $d < a = c \leq b$. Further, taking $x = 0, y = 0$ in Eq.(8), then we can get that $d = F(d, 0) = F(G(\alpha, 0), 0) = G(0, F(\alpha, 0)) = c$, then $d = c = a$, which contradicts the assumption $d < a$. Therefore it follows that $d \geq a$.

Next, taking $x = 0, y = 1$ in Eq.(8), then we can get that $d = F(d, 1) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 0)) = G(0, \alpha) = c$.

Finally, assume that $b < c$, then taking $x = 1, y = 0$, we can get that $b = F(\alpha, 0) = F(G(\alpha, 1), 0) = G(1, F(\alpha, 0)) = G(1, b) = c$, which contradicts the assumption $b < c$. Therefore it follows that $b \geq c$. Thus we prove that $b \geq d = c \geq a$. □

Theorem 6.4. *Let $\alpha \geq \max(b, c)$, $F \in \mathcal{F}_{a,b}$ with $a \leq b$ and $G \in \mathcal{F}_{c,d}$ with $c \geq d$. Then F is α-cross-migrative over G if and only if one of the following results is true*

(i) If $a \leq c = d = b \leq \alpha$, then T_F is $\frac{\alpha-b}{1-b}$-cross-migrative over T_G.

(ii) If $a \leq c = d < b \leq \alpha$, and write $x_0 = \inf\{x \in [0, 1] | G(\alpha, x) \geq b\}$, then one of the following two subcases holds.

(a) If $G(\alpha, x_0) = b$, then $G(x, b) = \begin{cases} G(\alpha, x) = G(x, \alpha) & \text{if } x < x_0, \\ b & \text{if } x \geq x_0, \end{cases}$ and $F(G(\alpha, x), y) = G(x, F(\alpha, y))$ when $x \geq x_0, y \geq b$.

(b) If $G(\alpha, x_0) < b$, then $G(x,b) = \begin{cases} G(\alpha,x) = G(x,\alpha) & \text{if } x \leq x_0, \\ b & \text{if } x > x_0, \end{cases}$ and $F(G(\alpha,x),y) = G(x,F(\alpha,y))$ when $x > x_0, y \geq b$.

Proof. The proof is omitted because it is similar to that of Theorem 5.5. □

6.3 $\min(a,d) < \alpha < \max(b,c)$

For this case, there are four different subcases to be considered: (1) $a \leq d, b \leq c$, (2) $a \leq d, b \geq c$, (3) $a \geq d, b \leq c$, (4) $a \geq d, b \geq c$. Now we start Subcase (1).

Lemma 6.5. *Let $a = \min(a,d) < \alpha < \max(b,c) = c$, $F \in \mathcal{F}_{a,b}$ with $a \leq b$ and $G \in \mathcal{F}_{c,d}$ with $c \geq d$, if F is α-cross-migrative over G, then $a < \alpha < b = c = d$.*

Proof. Assume that $d \leq \alpha < c$, then, taking $x = 1, y = 1$ in Eq.(8), we get that $c = F(c,1) = F(G(\alpha,1),1) = G(1,F(\alpha,1)) = G(1,\alpha) = \alpha$, which contradicts $d \leq \alpha < c$, thus $a < \alpha < d$.

Taking $x = 1, y = 1$ in Eq.(8), then we get that $c = F(c,1) = F(G(\alpha,1),1) = G(1,F(\alpha,1)) = G(1,\alpha) = d$, so $d = c$. Further, taking $x = 1, y = 0$ in Eq.(8), then we have that $b = F(c,0) = F(G(\alpha,1),0) = G(1,F(\alpha,0)) = d$.

Sum up, we have proven that $a < \alpha < b = c = d$. □

Theorem 6.6. *Let $a = \min(a,d) < \alpha < \max(b,c) = c$, $F \in \mathcal{F}_{a,b}$ with $a \leq b$ and $G \in \mathcal{F}_{c,d}$ with $c \geq d$. Then F is α-cross-migrative over G if and only if*

(i) $a < \alpha < b = c = d$.

(ii) $G(x,\alpha) = G(\alpha,x)$, for all $x \in [0,c]$.

Proof. The proof is omitted because it is similar to that of Theorem 5.8. □

For Subcase (2), we have the following lemma.

Lemma 6.7. *Let $a = \min(a,d) < \alpha < \max(b,c) = b$, $F \in \mathcal{F}_{a,b}$ with $a \leq b$ and $G \in \mathcal{F}_{c,d}$ with $c \geq d$. If F is α-cross-migrative over G, then $a < \alpha \leq d = c \leq b$ or $a \leq d = c \leq \alpha < b$.*

Proof. Assume that $d < \alpha < c$, then, taking $x = 0, y = 1$ in Eq.(8), we can get that $d = F(d,0) = F(G(\alpha,0),1) = G(0,F(\alpha,1)) = G(0,\alpha) = \alpha$, which contradicts the assumption $d < \alpha$. Thus it follows that $a < \alpha \leq d$ or $c \leq \alpha < b$.

If $a < \alpha \leq d$, then, taking $x = 1, y = 0$ in Eq.(8), we can get that $c = F(c,0) = F(G(\alpha,1),0) = G(1,F(\alpha,0)) = G(1,\alpha) = d$. Therefore $a < \alpha \leq d = c \leq b$.

If $c \leq \alpha < b$, then, taking $x = 0, y = 0$ in Eq.(8), we can get that $d = F(d,1) = F(G(\alpha,0),1) = G(0,F(\alpha,1)) = G(0,\alpha) = c$. Hence $a \leq c = d \leq \alpha < b$. □

Theorem 6.8. *Let $a = \min(a,c) < \alpha < \max(b,d) = b$, $F \in \mathcal{F}_{a,b}$ with $a \leq b$ and $G \in \mathcal{F}_{c,d}$ with $c \leq d$. Then F is α-cross-migrative over G if and only if one of the following conditions is true*

(i) $a < \alpha \leq c = d \leq b$, and $G(x,\alpha) = G(\alpha,x)$, for all $x \in [0,c]$.

(ii) $a \leq c = d \leq \alpha < b$, and $G(x,\alpha) = G(\alpha,x)$, for all $x \in [c,1]$.

Proof. The proof is omitted because it is similar to that of Theorem 5.11. □

For Subcase (3), we have the following lemma.

Lemma 6.9. *Let $d = \min(a,d) < \alpha < \max(b,c) = c$, $F \in \mathcal{F}_{a,b}$ with $a \leq b$ and $G \in \mathcal{F}_{c,d}$ with $c \geq d$, then F is not α-cross-migrative over G.*

Proof. Assume that $\alpha \leq b$, then taking $x = 0, y = 0$ in Eq.(8), we can get that $d = F(d,0) = F(G(\alpha,0),0) = G(0,F(\alpha,0)) = G(0,\alpha) = \alpha$, which contradicts $d < \alpha$. Thus it holds that $\alpha > b$. Again taking $x = 0, y = 0$ in Eq.(8), we can get that $d = F(d,0) = F(G(\alpha,0),0) = G(0,F(\alpha,0)) = G(0,b) = b$. Note that $d = \min(a,d) \leq a \leq b$ and $b = d$, then it gets that $\alpha > a = d = b$. Finally, taking $x = 0, y = 1$ in Eq.(8), we can get that $d = F(d,1) = F(G(\alpha,0),1) = G(0,F(\alpha,1)) = G(0,\alpha) = \alpha$, which contradicts $\alpha > a = d = b$.

Therefore F is not α-cross-migrative over G. □

For Subcase (4), we have the following lemma.

Lemma 6.10. *Let $d = \min(a, d) < \alpha < \max(b, c) = b$, $F \in \mathcal{F}_{a,b}$ with $a \le b$ and $G \in \mathcal{F}_{c,d}$ with $c \ge d$. If F is α-cross-migrative over G, then $a = c = d < \alpha < b$.*

Proof. Assume that $\alpha \le c$, then taking $x = 0, y = 0$ in Eq.(8), we can get that $d = F(d, 0) = F(G(\alpha, 0), 0) = G(0, F(\alpha, 0)) = G(0, \alpha) = \alpha$, which contradicts $d < \alpha$. Thus it follows that $\alpha > c$. Again taking $x = 0, y = 0$ in Eq.(8), then we can get that $d = F(d, 0) = F(G(\alpha, 0), 0) = G(0, F(\alpha, 0)) = G(0, \alpha) = c$. Note that $d = \min(a, d) \le a$, then it gets that $\alpha > d = c$. Finally, taking $x = 0, y = 1$ in Eq.(8), then we can get that $a = F(d, a) = F(G(\alpha, 0), 1) = G(0, F(\alpha, 1)) = G(0, \alpha) = d$.

Hence it holds that $a = c = d < \alpha < b$. □

Theorem 6.11. *Let $\min(a, c) < \alpha < \max(b, d)$, $F \in \mathcal{F}_{a,b}$ with $a \le b$ and $G \in \mathcal{F}_{c,d}$ with $c \le d$. Then F is α-cross-migrative over G if and only if*

(i) $a = c = d < \alpha < b$.

(ii) $G(x, \alpha) = G(\alpha, x)$, *for all* $x \in [a, 1]$.

Proof. The proof is omitted because it is similar to that of Theorem 5.15. □

Remark 6.12. *By comparing results of Section 5, note that the cross-migrativity of a semi-t-operator $F \in \mathcal{F}_{a,b}$ over another semi-t-operator $G \in \mathcal{F}_{c,d}$ with $a \le b$ and $c \ge d$ is the same with the case $a \le b$, $c \le b$.*

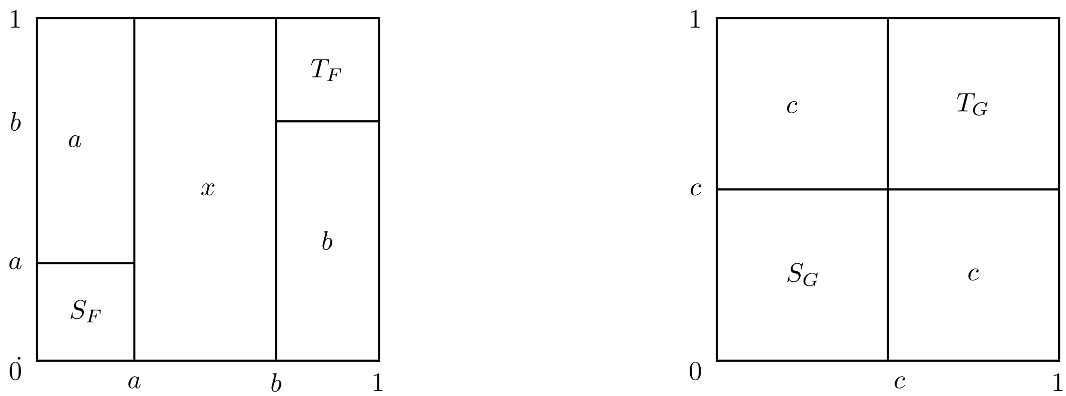

Fig.4. Structures of F (left) and G (right) in Section 6

7 Conclusions

In the paper, depending on the order relationship of α, a, b, c, d, we studied the cross-migrative property between semi-t-operators and gave all solutions of the cross-migrativity equation of a semi-t-operator $F \in F_{a,b}$ over another semi-t-operator $G \in F_{c,d}$. We have found that G degrades into a semi-nullnorm except the case $a \ge b, c \ge d$ if F is α-cross-migrative over G. But for the case $a \ge b, c \ge d$, there must be $a = c, b = d$, which doesn't mean $F = G$. Specially, the cross-migrative property between two semi-t-operators is always determined by their underlying operators. In the future, we will intensively study the cross-migrative property between semi-t-conorms and (or) semi-t-norms. Meanwhile, according to summarizing all of results in this paper, it is easy to find that the semi-nullnorms play the important role.

Acknowledgement

This research was supported by the National Natural Science Foundation of China (No. 61563020), the Key Program of Jiangxi Natural Science Foundation (No. 20171ACB20010) and the Jiangxi Natural Science Foundation (No. 20151BAB201019).

The authors wish to express their appreciation for several excellent suggestions for improvements in this paper made by the referees.

References

[1] J. Aczel, *Lectures on functional equations and their applications*, New York, Academy, Press, 1966.

[2] G. Beliakov, A. Pradera, T. Calvo, *Aggregation functions: A guide for practitioners*, Heidelberg, Springer, 2007.

[3] T. Calvo, B. De Baets, J. Fodor, *The functional equations of Frank and Alsina for uninorms and nullnorms*, Fuzzy Sets and Systems, **120** (2001), 385-394.

[4] T. Calvo, G. Mayor, R. Mesiar, *Aggregation operators: New trends and applications*, Heidelberg, Springer, 2002.

[5] P. Drygaś, *Distributivity between semi-t-operators and semi-nullnorms*, Fuzzy Sets and Systems, **264** (2015), 100-109.

[6] J. Fordor, E. P. Klement, R. Mesiar, *Cross-migrative triangular norms*, International Journal of Intelligent Systems, **27** (2012), 411-428.

[7] J. Fordor, I. J. Rudas, *On the continuous triangular norms that are migrative*, Fuzzy Sets and Systems, **158** (2007), 1692-1697.

[8] J. Fordor, I. J. Rudas, *An extension of the migrative property for triangular norms*, Fuzzy Sets and Systems, **168** (2011), 70-80.

[9] J. Fordor, I. J. Rudas, *Migrative t-norms with respect to continuous ordinal sums*, Information Science, **181** (2011), 4860-4866.

[10] M. Grabisch, J. L. Marichal, R. Mesiar, E. Pap, *Aggregation functions*, New York, Cambridge University Press, 2009.

[11] E. P. Klement, R. Mesiar, E. Pap, *Triangular norms*, Dordrecht, Kluwer Academic Publishers, 2000.

[12] S. Li, F. Qin, J. Fodor, *On the cross-migrativity with respect to continuous t-norms*, International Journal of Intelligent Systems, **30** (2014), 225-235.

[13] M. Mas, G. Mayor, J. Torrens, *T-operators*, International Journal Uncertain Fuzziness Knowledge Based Systems, **7** (1999), 31-50.

[14] M. Mas, G. Mayor, J. Torrens, *The modularity condition for uninorms and t-operators*, Fuzzy Sets and Systems, **126** (2002), 207-218.

[15] R. Mesiar, H. Bustince, J. Fernandez, *On the α-migrativity of semecopulas, quasi-copulas, and copulas*, Information Science, **180** (2010), 1967-1976.

[16] Y. Su, H. W. Liu, J. V. Riera, D. R. Aguilera, J. Torrens, *The migrativity equation for uninorms revisited*, Fuzzy Sets and Systems, **323** (2017), 56-78.

[17] Y. Su, W. W. Zong, H. W. Liu, P. J. Xue, *Migrative property for uninorms and semi-t-operators*, Information Science, **325** (2015), 455-465.

[18] H. D. Wang, F. Qin, *On the cross-migrativity of triangular subnorms*, IEEE International Conference on Fuzzy Systems, (2014), 1139-1142.

[19] L. Wu, Y. Ouyang, *On the migrativity of triangular subnorms*, Fuzzy Sets and Systems, **226** (2013), 89-98.

[20] H. Zhan, H. W. Liu, *The cross-migrative property for uninorms*, Aequationes Mathematics, **90** (2016), 1219-1239.

[21] H. Zhan, H. W. Liu, *Cross-migrative property uninorms with different neutral element*, Journal of Intelligent Fuzzy Systems, **32** (2017), 1877-1889.

[22] W. W. Zong, Y. Su, H. W. Liu, *Migrative property for nullnorms*, International Journal Uncertainty Fuzziness Knowledge-Based Systems, **22** (2014), 749-759.