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Original paper

On the existence of solutions to interval-valued differential equations with length constraints

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Abstract

In this paper, we consider the existence of solutions to first-order interval-valued differential equations with length constraints under gH-differentiability. By using the fixed point theory of compact maps on metric spaces, we provide some sufficient conditions for the existence of solutions. At the end of this paper, we show some examples to illustrate our results.

Keywords: Interval-valued differential equations, gH-differentiability, length constraints, switching points.

1 Introduction

Uncertain differential equations were introduced to discuss problems with fuzzy, interval or set-valued uncertainties that appear in physics, biology, engineering and other fields. Some researchers considered uncertain differential equations as Lipschitz continuous uncertain processes with stationary and independent increments [4, 12, 14, 10], some others applied differential inclusions method to study uncertain differential equations [2, 5, 1, 9]. We can also find some works in the literature using Hukuhara derivative and generalized Hukuhara derivative of set-valued mappings to discuss fuzzy and interval-valued differential equations, see, for instance, [13, 3, 11, 7, 6].

In [13], the authors pointed out the difficulty that arises in the solvability of uncertain differential equations caused by the fact that fuzzy or interval-valued differential equations may have only solutions with increasing length of their support under Hukuhara's differentiability. By using the concepts of gH-derivative and gH-differentiability, interval-valued differential equations may have solutions with nonmonotonic length and some switching points. From then on, these ideas were applied to study initial value problems for interval-valued differential equations, see [3, 11, 7, 6].

In the literature, the works about interval-valued differential equations under gH-differentiability noticed the length of the solutions as an interesting notion whose behavior is a consequence of the particular type of gH-differentiability considered. However, none of them considered the existence of solutions with length constraints. Consequently, in this paper, we consider the existence of solutions to interval-valued differential equations subject to initial conditions

$$\begin{cases} x'(t) = f(t, x(t)), \\ x(0) = x_0, \end{cases} \tag{1}$$

where $x_0 \in K_C$ and $f : [0, T] \times K_C \to K_C$ is continuous, and we discuss the existence of solutions to (1) with length constraints, for instance, in the form $x^+(t) - x^-(t) \leq M$, where M is a fixed positive real number. Moreover, only natural switching points, which satisfy $x^+(t) - x^-(t) = 0$ or $x^+(t) - x^-(t) = M$, are permitted in our approach.

The results can be applied to discuss some interval-valued dynamical systems with length constraints. The study of solutions subject to some bounds or restrictions is natural in many different control processes, such as the study of

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automatic pressure or temperature control system. In these examples, one is interested in obtaining solutions predicting the evolution of pressure or temperature whose values are kept below or/and over some specific fixed values all along the process.

On the other hand, in many different applications related to models involving uncertainty, it is interesting to study the existence of solutions subject to a restriction on the length. In the literature, it is suggested that the combination of different types of differentiability through the introduction of switching points is an interesting procedure in order to adapt the properties of the solutions to the particularities of the dynamical processes under consideration. Moreover, the use of gH-differentiability and the introduction of switching points can also be important to get solutions, corresponding to processes subject to uncertainty, where the degree of imprecision must be necessarily controlled in order to produce significant responses. This way, we are mainly interested in solutions whose degree of uncertainty does not exceed the value $M > 0$ fixed. We could make a similar study to seek solutions whose degree of imprecision lies between the fixed numbers m and M, with $0 \leq m < M$. However, due to the interpretation of the control of degree of uncertainty, the study of solutions whose length is below a certain number $M > 0$ will be our main objective.

The paper is organized as follows. In Section 2, we recall some basic definitions and facts about interval operations, interval differential equations, and generalized Hukuhara differentiability. In Section 3, some important lemmas and sufficient conditions for the existence of solutions to interval-valued differential equations with length constraints and natural switching points are proved. In Section 4, we show some examples to illustrate our theorems.

2 Preliminaries

Let K_C be the family of all non-empty compact convex subsets of $\mathbb{R}$, that is, $K_C = \{[a,b] \mid a,b \in \mathbb{R},\ a \leq b\}$. For every $u,v \in K_C$, $u+v = \{a+b \mid a \in u,\ b \in v\}$ and $ku = \{ka \mid a \in u\}$, $k \in \mathbb{R}$. Moreover,

$$u - v = u + (-1)v = \{a - b \mid a \in u,\ b \in v\}.$$

We see that $u - u \neq \{0\}$ unless $u = \{a\}$ is a singleton. Thus, the concept of Hukuhara difference was introduced in [8].

Definition 2.1. [8] *Let $u,v,w \in K_C$, $w = u \ominus v$ is called the Hukuhara difference of u and v if $u = v + w$.*

However, $u \ominus v$ exists if and only if $len(u) \geq len(v)$, where $len(u) = u^+ - u^-$, for $u = [u^-, u^+]$. To overcome this situation, Luciano Stefanini and Barnabás Bede introduced the definition of generalized Hukuhara difference as we recall below.

Definition 2.2. [13] *Let $u,v \in K_C$. The generalized Hukuhara difference (gH-difference, for short) of u and v is defined as follows*

$$u \ominus_g v = \begin{cases} u \ominus v, & \text{if } u \ominus v \text{ exists}, \\ (-1)(v \ominus u), & \text{if } v \ominus u \text{ exists}. \end{cases}$$

Let $H : K_C \times K_C \to [0, +\infty)$, $H(u,v) = \max\{|u^- - v^-|, |u^+ - v^+|\}$, where $u = [u^-, u^+]$ and $v = [v^-, v^+]$. It is well-known that H is the Pompeiu-Hausdorff metric on K_C and (K_C, H) is a complete and locally compact metric space.

Lemma 2.3. *Let $u,v,w \in K_C$. The following properties are valid:*

(i) $len(u+v) = len(u-v) = len(u) + len(v)$.

(ii) *If $u \ominus v$ exists, then* $len(u \ominus v) = len(u) - len(v)$.

(iii) *If $u \subseteq v$, then* $H(u,\{0\}) \leq H(v,\{0\})$.

(iv) $H(u+v,\{0\}) \leq H(u,\{0\}) + H(v,\{0\})$.

(v) $H(ku,kv) = |k| H(u,v)$, $k \in \mathbb{R}$.

(vi) $H(u+w,v+w) = H(u,v)$.

Proof. Properties (v-vi) are from [13], here we just prove (i)-(iv). Indeed:

(i) $len(u+v) = u^+ + v^+ - u^- - v^- = len(u) + len(v)$. On the other hand,

$$len(u-v) = len[u^- - v^+, u^+ - v^-] = u^+ - u^- + v^+ - v^- = len(u) + len(v).$$

(ii) $len(u \ominus v) = len[u^- - v^-, u^+ - v^+] = u^+ - u^- - v^+ + v^- = len(u) - len(v)$.

(iii) The definition of Pompeiu-Hausdorff distance provides that $H(u,\{0\}) = \sup_{a\in u}\{|a|\}$, thus $u \subseteq v$ implies that $H(u,\{0\}) \le H(v,\{0\})$.

(iv) $H(u+v,\{0\}) = \max\{|u^-+v^-|, |u^++v^+|\} \le \max\{|u^-|+|v^-|, |u^+|+|v^+|\}$, $H(u,\{0\})+H(v,\{0\}) = \max\{|u^-|,|u^+|\}+\max\{|v^-|,|v^+|\}$, thus we have $H(u+v,\{0\}) \le H(u,\{0\}) + H(v,\{0\})$.

□

Definition 2.4. *Let $f : [a,b] \to K_C$. The function f is said to be continuous at $t_0 \in [a,b]$ if $\forall\varepsilon > 0$, $\exists\delta > 0$ such that $H(f(t), f(t_0)) < \varepsilon$ for $t \in [a,b]$ with $|t - t_0| < \delta$. Moreover, f is said to be continuous on $[a,b]$ if f is continuous at each point in $[a,b]$.*

Let $C([a,b],K_C)$ be the set of continuous functions on $[a,b]$, and define $D : C([a,b],K_C) \times C([a,b],K_C) \to [0,+\infty)$, given by $D(f,g) = \max_{t\in[a,b]} H(f(t), g(t))$. It is known that $(C([a,b],K_C), D)$ is a complete metric space.

In addition, let $f : [a,b] \to K_C$ and $f(t) = [f^-(t), f^+(t)]$. The function f is integrable on $[a,b]$ if f^-, f^+ are integrable functions on $[a,b]$. If $f \in C([a,b],K_C)$, then f is integrable on $[a,b]$.

Lemma 2.5. [2, 13] *Let $f : [a,b] \to K_C$ be an interval-valued function such that $f(t) = [f^-(t), f^+(t)]$ and let $t_0 \in (a,b)$, then*

$$\lim_{t\to t_0} f(t) = \left[\lim_{t\to t_0} f^-(t), \lim_{t\to t_0} f^+(t)\right] \text{ and } \lim_{t\to t_0} f(t) = f(t_0) \Leftrightarrow \lim_{t\to t_0} (f(t) \ominus_g f(t_0)) = \{0\}.$$

The limits above are taken in the metric H. If $t_0 = a$ or $t_0 = b$, then the limit is understood as a one-sided limit.

Now, we recall the definition of gH-derivative introduced by Luciano Stefanini and Barnabás Bede (see [13]).

Definition 2.6. [13] *Let $t_0 \in (a,b)$. The gH-derivative of a function $f : (a,b) \to K_C$ at t_0 is defined as*

$$f'(t_0) = \lim_{h\to 0} \frac{1}{h}[f(t_0+h) \ominus_g f(t_0)].$$

Lemma 2.7. [3] *Let $f : [a,b] \to K_C$ be an interval-valued function such that $f(t) = [f^-(t), f^+(t)]$. If f is gH-differentiable at $t_0 \in (a,b)$, then f is continuous at t_0 and one of the following cases holds:*

(i) f^- and f^+ are differentiable at t_0 and

$$f'(t_0) = [\min\{(f^-)'(t_0), (f^+)'(t_0)\}, \max\{(f^-)'(t_0), (f^+)'(t_0)\}].$$

(ii) f^- and f^+ are not differentiable at t_0, but $(f^-)'_-(t_0)$, $(f^-)'_+(t_0)$, $(f^+)'_-(t_0)$ and $(f^+)'_+(t_0)$ exist and satisfy $(f^-)'_-(t_0) = (f^+)'_+(t_0)$, $(f^-)'_+(t_0) = (f^+)'_-(t_0)$. Moreover,

$$f'(t_0) = [\min\{(f^-)'_-(t_0), (f^+)'_-(t_0)\}, \max\{(f^-)'_-(t_0), (f^+)'_-(t_0)\}].$$

Definition 2.8. [13] *Let $f : [a,b] \to K_C$ and $f(t) = [f^-(t), f^+(t)]$. Then:*
(i) f is said to be (I)-gH-differentiable at $t_0 \in (a,b)$ if f^- and f^+ are both differentiable at t_0 and $f'(t_0) = [(f^-)'(t_0), (f^+)'(t_0)]$.
(ii) f is said to be (II)-gH-differentiable at $t_0 \in (a,b)$ if f^- and f^+ are both differentiable at t_0 and $f'(t_0) = [(f^+)'(t_0), (f^-)'(t_0)]$.

Definition 2.9. [13] *Let $f : [a,b] \to K_C$ be gH-differentiable, and $t_0 \in (a,b)$. Then:*
(i) $t_0 \in (a,b)$ is said to be a type-I switching point for the differentiability of f, if in any neighborhood of t_0 there exist $t_1 < t_0 < t_2$ such that f is (I)-gH-differentiable at t_1 while it is not (II)-gH-differentiable at that point, and f is (II)-gH-differentiable at t_2 while it is not (I)-gH-differentiable at that point.
(ii) $t_0 \in (a,b)$ is said to be a type-II switching point for the differentiability of f, if in any neighborhood of t_0 there exist $t_1 < t_0 < t_2$ such that f is (II)-gH-differentiable at t_1 while it is not (I)-gH-differentiable at that point, and f is (I)-gH-differentiable at t_2 while it is not (II)-gH-differentiable at that point.

Definition 2.10. *Let $f : [a,b] \to K_C$ be gH-differentiable and $M > 0$. The point $t_0 \in (a,b)$ is said to be a natural switching point for the length constraint $len(f(t)) \le M$, if t_0 is a type-I switching point of f and satisfies $len(f(t_0)) = M$, or t_0 is a type-II switching point of f and satisfies $len(f(t_0)) = 0$.*

Lemma 2.11. [13] *The interval differential equation (without switching point)*

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{2}$$

is equivalent to the integral equation

$$x(t) \ominus_g x_0 = \int_{t_0}^{t} f(s, x(s))\, ds \tag{3}$$

on some interval $[t_0, t_0 + \delta]$, with $\delta > 0$.

Lemma 2.12. [4] *Suppose that $f : [a,b] \to K_C$ is integrable, then $len(\int_a^b f(t)\,dt) = \int_a^b len(f(t))\,dt$.*

3 Interval-valued differential equations with length constraints

In this section, we consider the interval-valued differential equation (1) with length constraint $len(x(t)) \le M$ for all $t \in [0,T]$, where $M > 0$ and the initial condition satisfies that $len(x_0) < M$.

Firstly, we define two maps A and B on $C([0,T], K_C)$. For every $x \in C([0,T], K_C)$, if

$$len(x_0) + len\left(\int_0^T f(t, x(t))\, dt\right) \le M, \tag{4}$$

we define $(Ax)(t) = x_0 + \int_0^t f(s, x(s))\, ds$, for $t \in [0,T]$. If

$$nM < len(x_0) + len\left(\int_0^T f(t, x(t))\, dt\right) \le (n+1)M, \ n \ge 1, \tag{5}$$

let

$$(Ax)(t) = \begin{cases} x_0 + \int_0^t f(s, x(s))\, ds, \quad t \in [0, \alpha(x,1)], \\ (Ax)(\alpha(x,j)) \ominus (-1) \int_{\alpha(x,j)}^t f(s, x(s))\, ds, \\ \qquad \text{for } t \in (\alpha(x,j), \alpha(x,j+1)] \text{ and } j \text{ odd}, \\ (Ax)(\alpha(x,j)) + \int_{\alpha(x,j)}^t f(s, x(s))\, ds, \\ \qquad \text{for } t \in (\alpha(x,j), \alpha(x,j+1)] \text{ and } j \text{ even}, \end{cases}$$

where

$$\alpha(x,j) = \min\left\{ t \in [0,T] \,:\, len\left(\int_0^t f(s, x(s))\, ds\right) = jM - len(x_0) \right\},$$

for $j = 1, 2, ..., n$ and $\alpha(x, n+1) = T$. By Lemma 2.12, we see that $0 < \alpha(x,1) < \alpha(x,2) < \cdots < \alpha(x, n+1)$.

On the other hand, if $len(x_0) \ge len\left(\int_0^T f(t, x(t))\, dt\right)$, then we define

$$(Bx)(t) = x_0 \ominus (-1) \int_0^t f(s, x(s))\, ds, \quad t \in [0,T].$$

If $(n-1)M < len\left(\int_0^T f(t,x(t))\,dt\right) - len(x_0) \le nM$, $n \ge 1$, then

$$(Bx)(t) = \begin{cases} x_0 \ominus (-1)\int_0^t f(s,x(s))\,ds, & t \in [0,\beta(x,1)], \\ (Bx)(\beta(x,j)) + \int_{\beta(x,j)}^t f(s,x(s))\,ds, & \\ \quad \text{for } t \in (\beta(x,j),\beta(x,j+1)] \text{ and } j \text{ odd}, & \\ (Bx)(\beta(x,j)) \ominus (-1)\int_{\beta(x,j)}^t f(s,x(s))\,ds, & \\ \quad \text{for } t \in (\beta(x,j),\beta(x,j+1)] \text{ and } j \text{ even}, & \end{cases}$$

where

$$\beta(x,j) = \min\left\{t \in [0,T] \,:\, len\left(\int_0^t f(s,x(s))\,ds\right) = (j-1)M + len(x_0)\right\},$$

for $j = 1,2,...,n$ and $\beta(x,n+1) = T$. By Lemma 2.12, we also have $0 < \beta(x,1) < \beta(x,2) < \cdots < \beta(x,n+1)$.

We see that A, B are well-defined self-maps on $C([0,T],K_C)$ and satisfy $len((Ax)(t)) \le M$, $len((Bx)(t)) \le M$ for all $x \in C([0,T],K_C)$, $t \in [0,T]$. If there exists $x \in C([0,T],K_C)$ such that $Ax = x$ or $Bx = x$, then x is a gH-differentiable solution to (1) with the length constraint $len(x(t)) \le M$, for all $t \in [0,T]$.

Let $f(t,x) = [f^-(t,x), f^+(t,x)]$, the continuity of f guarantees that f^-, $f^+ : [0,T] \times K_C \to \mathbb{R}$ are also continuous. For every $x \in C([0,T],K_C)$ and $t \in [0,T]$, let $(Ax)(t) = [(A_-x)(t),(A_+x)(t)]$, $(Bx)(t) = [(B_-x)(t),(B_+x)(t)]$.

Lemma 3.1. *Suppose that $f : [0,T] \times K_C \to K_C$ is continuous, and that $\{x_k\} \subset C([0,T],K_C)$ satisfies $\lim_{k\to+\infty} x_k = x^*$. Moreover, assume that there exists $n \ge 1$ such that*

$$nM < len(x_0) + len\left(\int_0^T f(t,x^*(t))\,dt\right) < (n+1)M. \tag{6}$$

(i) Let $\alpha(x^*,0) = 0$. Then there exists $N > 0$ such that, for every $k > N$, $\alpha(x^*,j-1) < \alpha(x_k,j) < \alpha(x^*,j+1)$ for all $j = 1,2,...,n$.

(ii) For every $j = 1,2,...,n$,

$$\lim_{k\to+\infty} len\left(\int_{\alpha(x^*,j)}^{\alpha(x_k,j)} f(s,x^*(s))\,ds\right) = 0. \tag{7}$$

Proof. (i) For a particular $j \in \{1,\dots,n\}$, we assume that there exists $\{x_{k_i}\}$ an infinite subsequence of $\{x_k\}$ satisfying $\alpha(x_{k_i},j) \le \alpha(x^*,j-1)$. By $\{\alpha(x_{k_i},j)\} \subset [0,T]$, we see that $\{\alpha(x_{k_i},j)\}$ has convergent subsequences. For convenience, we suppose that the convergent subsequence is $\{\alpha(x_{k_i},j)\}$ and $\alpha(x_{k_i},j) \to \tilde{\alpha}_j$, then $\tilde{\alpha}_j \le \alpha(x^*,j-1)$.

Based on the definition of $\alpha(x_{k_i},j)$, and by the uniform continuity of f^- and f^+ on compact sets, we have

$$len\left(\int_0^{\tilde{\alpha}_j} f(s,x^*(s))\,ds\right) = \lim_{k_i\to+\infty} len\left(\int_0^{\alpha(x_{k_i},j)} f(s,x_{k_i}(s))\,ds\right) = jM - len(x_0).$$

This implies that $len\left(\int_0^{\tilde{\alpha}_j} f(s,x^*(s))\,ds\right) > len\left(\int_0^{\alpha(x^*,j-1)} f(s,x^*(s))\,ds\right)$, which contradicts $\tilde{\alpha}_j \le \alpha(x^*,j-1)$. Hence, there exist at most a finite number of elements in $\{x_k\}$ such that $\alpha(x_k,j) \le \alpha(x^*,j-1)$. Similarly, we can prove that there exist at most a finite number of elements in $\{x_k\}$ such that $\alpha(x_k,j) \ge \alpha(x^*,j+1)$.

Summing it all up, there exists $N_j > 0$ such that $\alpha(x^*,j-1) < \alpha(x_k,j) < \alpha(x^*,j+1)$ for all $k > N_j$. Taking $N = \max\{N_1,N_2,...,N_n\}$, we can complete the proof of (i).

(ii) If $\lim_{k\to+\infty} \alpha(x_k,j) = \alpha(x^*,j)$ for $j = 1,2,...,n$, then (7) holds. If there exists j such that $\alpha(x_k,j) \nrightarrow \alpha(x^*,j)$, we prove that (7) still holds.

Firstly, let γ_1, γ_2 be the minimum and maximum cluster points of $\{\alpha(x_k,j)\}$, respectively. We prove that $\gamma_1 \ge \alpha(x^*,j)$ and $len\left(\int_{\alpha(x^*,j)}^{\gamma_2} f(t,x^*(t))\,dt\right) = 0$.

Suppose that γ is one of the cluster points of $\{\alpha(x_k, j)\}$, that is, there exists $\{x_{k_i}\} \subseteq \{x_k\}$ such that $\alpha(x_{k_i}, j) \to \gamma$. Based on the definition of $\alpha(x_{k_i}, j)$ and the properties of f^-, f^+, we have

$$len\left(\int_0^{\alpha(x_{k_i},j)} f(s, x_{k_i}(s))\, ds\right) = jM - len(x_0). \tag{8}$$

Let $k_i \to +\infty$, we get that $len\left(\int_0^{\gamma} f(s, x^*(s))\, ds\right) = jM - len(x_0)$. Thus $\gamma \geq \alpha(x^*, j)$. By Lemma 2.3(i), we have

$$len\left(\int_{\alpha(x^*,j)}^{\gamma} f(s, x^*(s))\, ds\right) = len\left(\int_0^{\gamma} f(s, x^*(s))\, ds\right) - len\left(\int_0^{\alpha(x^*,j)} f(s, x^*(s))\, ds\right) = 0.$$

Hence, we have $\gamma_1 \geq \alpha(x^*, j)$ and $len\left(\int_{\alpha(x^*,j)}^{\gamma_2} f(t, x^*(t))\, dt\right) = 0$. Lemma 2.12 implies that

$$\int_{\alpha(x^*,j)}^{\gamma_2} len(f(t, x^*(t)))\, dt = len\left(\int_{\alpha(x^*,j)}^{\gamma_2} f(t, x^*(t))\, dt\right) = 0.$$

Then, by the nonnegativity of $len(f(t, x^*(t)))$, we see that $len(f(t, x^*(t))) \equiv 0$ for $t \in [\alpha(x^*, j), \gamma_2]$.

Secondly, the continuity of f and x^* guarantees that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for $|t - \alpha(x^*, j)| < \delta$,

$$|len(f(t, x^*(t)))| = |len(f(t, x^*(t))) - len(f(\alpha(x^*, j), x^*(\alpha(x^*, j))))| < \varepsilon. \tag{9}$$

For $|t - \gamma_2| < \delta$,

$$|len(f(t, x^*(t)))| = |len(f(t, x^*(t))) - len(f(\gamma_2, x^*(\gamma_2)))| < \varepsilon. \tag{10}$$

Let δ be a smaller positive number if it is necessary, then we have $[\alpha(x^*, j) - \delta, \gamma_2 + \delta] \subset [0, T]$. The definitions of γ_1, γ_2 and $\gamma_1 \geq \alpha(x^*, j)$ imply that there exists $N > 0$ such that, for every $k > N$, $\alpha(x_k, j) \in [\alpha(x^*, j) - \delta, \gamma_2 + \delta]$. By Lemma 2.3(i) and (9), (10), we have

$$\begin{aligned} len\left(\int_{\alpha(x^*,j)}^{\alpha(x_k,j)} f(s, x^*(s))\, ds\right) &\leq len\left(\int_{\alpha(x^*,j)-\delta}^{\gamma_2+\delta} f(s, x^*(s))\, ds\right) \\ &= len\left(\int_{\alpha(x^*,j)-\delta}^{\alpha(x^*,j)} f(s, x^*(s))\, ds\right) + len\left(\int_{\gamma_2}^{\gamma_2+\delta} f(s, x^*(s))\, ds\right) \leq 2\delta\varepsilon. \end{aligned}$$

That is, (7) holds even for $\alpha(x_k, j) \nrightarrow \alpha(x^*, j)$. □

Lemma 3.2. *Suppose that $f : [0, T] \times K_C \to K_C$ is continuous, then A and B are both continuous on $C([0, T], K_C)$.*

Proof. Here, we just prove that $A_+ : C([0, T], K_C) \to C([0, T], \mathbb{R})$ is continuous.

For every $x \in C([0, T], K_C)$, if (4) holds, then we have

$$(A_+x)(t) = x_0^+ + \int_0^t f^+(s, x(s))\, ds, t \in [0, T]. \tag{11}$$

If (5) is satisfied, then

$$(A_+x)(t) = \begin{cases} x_0^+ + \int_0^t f^+(s, x(s))\, ds, \quad t \in [0, \alpha(x, 1)], \\ (A_+x)(\alpha(x, j)) + \int_{\alpha(x,j)}^t f^-(s, x(s))\, ds, \\ \quad \text{for } t \in (\alpha(x, j), \alpha(x, j + 1)] \text{ and } j \text{ odd}, \\ (A_+x)(\alpha(x, j)) + \int_{\alpha(x,j)}^t f^+(s, x(s))\, ds, \\ \quad \text{for } t \in (\alpha(x, j), \alpha(x, j + 1)] \text{ and } j \text{ even}, \end{cases} \tag{12}$$

where $j = 1, 2, ..., n$.

Let $\{x_k\} \subset C([0,T], K_C)$ satisfy that $\lim_{k\to+\infty} x_k = x^*$, then, for every $\varepsilon > 0$, there exists $N_0 > 0$ such that, for every $t \in [0,T]$ and $k > N_0$,

$$\int_0^T \left(|f^+(s, x_k(s)) - f^+(s, x^*(s))| + |f^-(s, x_k(s)) - f^-(s, x^*(s))|\right) ds < \varepsilon. \tag{13}$$

Now, we prove that $\lim_{k\to+\infty} A_+x_k = A_+x^*$ in three steps.

Step 1. If x^* satisfies that $len(x_0) + len\left(\int_0^T f(t, x^*(t))\,dt\right) < M$, then the continuity of f guarantees that there exists $N > N_0$ such that A_+x_k can be written as (11) for all $k > N$. In addition, for every $t \in [0,T]$,

$$|(A_+x_k)(t) - (A_+x^*)(t)| = \left|\int_0^t \left(f^+(s, x_k(s)) - f^+(s, x^*(s))\right) ds\right| \leq \int_0^T \left|f^+(s, x_k(s)) - f^+(s, x^*(s))\right| ds < \varepsilon.$$

Step 2. Suppose that x^* satisfies (6). By the continuity of f and Lemma 3.1(i), there exists $N_1 > N_0$ such that x_k satisfies (6) and $\max\{\alpha(x_k, j), \alpha(x^*, j)\} \leq \min\{\alpha(x_k, j+1), \alpha(x^*, j+1)\}$ for all $k > N_1$ and $j = 1, 2, \cdots, n$.

Let $\check{t}_{k,j} := \min\{\alpha(x_k, j), \alpha(x^*, j)\}$, $\hat{t}_{k,j} := \max\{\alpha(x_k, j), \alpha(x^*, j)\}$. For every $k > N_1$ and $t \in [0, \check{t}_{k,1}]$,

$$(A_+x_k)(t) - (A_+x^*)(t) = \int_0^t (f^+(s, x_k(s)) - f^+(s, x^*(s)))\,ds.$$

For every $t \in [\check{t}_{k,j}, \hat{t}_{k,j}]$, $j = 1, 2, ..., n$, we have

$$(A_+x_k)(t) - (A_+x^*)(t) = \begin{cases} (A_+x_k)(\alpha(x_k, j)) - (A_+x^*)(\alpha(x_k, j)) \\ \quad + \int_{\alpha(x_k,j)}^t (f^-(s, x_k(s)) - f^+(s, x^*(s)))\,ds, \\ \quad \text{for } j \text{ odd and } \alpha(x_k, j) < \alpha(x^*, j); \\ (A_+x_k)(\alpha(x_k, j)) - (A_+x^*)(\alpha(x_k, j)) \\ \quad + \int_{\alpha(x_k,j)}^t (f^+(s, x_k(s)) - f^-(s, x^*(s)))\,ds, \\ \quad \text{for } j \text{ even and } \alpha(x_k, j) < \alpha(x^*, j); \\ (A_+x_k)(\alpha(x^*, j)) - (A_+x^*)(\alpha(x^*, j)) \\ \quad + \int_{\alpha(x^*,j)}^t (f^+(s, x_k(s)) - f^-(s, x^*(s)))\,ds, \\ \quad \text{for } j \text{ odd and } \alpha(x_k, j) \geq \alpha(x^*, j); \\ (A_+x_k)(\alpha(x^*, j)) - (A_+x^*)(\alpha(x^*, j)) \\ \quad + \int_{\alpha(x^*,j)}^t (f^-(s, x_k(s)) - f^+(s, x^*(s)))\,ds, \\ \quad \text{for } j \text{ even and } \alpha(x_k, j) \geq \alpha(x^*, j). \end{cases}$$

For every $t \in [\hat{t}_{k,j}, \check{t}_{k,j+1}]$, $j = 1, 2, ..., n$, we have

$$(A_+x_k)(t) - (A_+x^*)(t) = \begin{cases} (A_+x_k)(\hat{t}_{k,j}) - (A_+x^*)(\hat{t}_{k,j}) \\ \quad + \int_{\hat{t}_{k,j}}^t (f^-(s, x_k(s)) - f^-(s, x^*(s)))\,ds, \text{ for } j \text{ odd}, \\ (A_+x_k)(\hat{t}_{k,j}) - (A_+x^*)(\hat{t}_{k,j}) \\ \quad + \int_{\hat{t}_{k,j}}^t (f^+(s, x_k(s)) - f^+(s, x^*(s)))\,ds, \text{ for } j \text{ even}. \end{cases}$$

Consequently, for all $k > N_1$ and $j = 1, 2, ..., n$,

$$|(A_+x_k)(t) - (A_+x^*)(t)| \le \begin{cases} \int_0^T |f^+(s, x_k(s)) - f^+(s, x^*(s))|\, ds,\ t \in [0, \check{t}_{k,1}], \\ |(A_+x_k)(\check{t}_{k,j}) - (A_+x^*)(\check{t}_{k,j})| \\ \quad + \int_{\check{t}_{k,j}}^{\hat{t}_{k,j}} \big(|f^-(s, x_k(s)) - f^-(s, x^*(s))| \\ \qquad + |f^+(s, x_k(s)) - f^+(s, x^*(s))| \\ \qquad + 2\, |f^+(s, x^*(s)) - f^-(s, x^*(s))| \big)\, ds,\ t \in [\check{t}_{k,j}, \hat{t}_{k,j}], \\ |(A_+x_k)(\hat{t}_{k,j}) - (A_+x^*)(\hat{t}_{k,j})| \\ \quad + \int_{\hat{t}_{k,j}}^{\check{t}_{k,j+1}} \big(|f^-(s, x_k(s)) - f^-(s, x^*(s))| \\ \qquad + |f^+(s, x_k(s)) - f^+(s, x^*(s))| \big)\, ds,\ t \in [\hat{t}_{k,j}, \check{t}_{k,j+1}]. \end{cases}$$

By (13), we have $|(A_+x_k)(t) - (A_+x^*)(t)| < \varepsilon$, for $t \in [0, \check{t}_{k,1}]$. Moreover, Lemma 3.1(ii) implies that there exists $N > N_1$ such that for $k > N$ and $j = 1, 2, ..., n$,

$$\int_{\alpha(x^*, j)}^{\alpha(x_k, j)} \big| f^+(s, x^*(s)) - f^-(s, x^*(s)) \big|\, ds < \varepsilon. \tag{14}$$

Thus, we have

$$\begin{aligned} &|(A_+x_k)(t) - (A_+x^*)(t)| \le 4j\varepsilon,\ \text{ for } t \in [\check{t}_{k,j}, \hat{t}_{k,j}], \\ &|(A_+x_k)(t) - (A_+x^*)(t)| \le (4j+1)\varepsilon,\ \text{ for } t \in [\hat{t}_{k,j}, \check{t}_{k,j+1}], \end{aligned}$$

where $j = 1, 2, ..., n$.

Summing it all up, there exists $N > N_1$ such that, for $k > N$ and $t \in [0, T]$, $|(A_+x_k)(t) - (A_+x^*)(t)| \le (4n+1)\varepsilon$.

Step 3. Suppose that x^* satisfies

$$len(x_0) + len\left(\int_0^T f(t, x^*(t))\, dt \right) = nM. \tag{15}$$

The continuity of f provides that there exists $N > N_0$ such that

$$(n-1)M < len(x_0) + len\left(\int_0^T f(t, x_k(t))\, dt \right) < (n+1)M,\ \ k > N.$$

If $(n-1)M < len(x_0) + len\left(\int_0^T f(t, x_k(t))\, dt \right) \le nM$ for all $k > N$, then we can prove that $A_+x_k \to A_+x^*$ with the same method in Step 1 for $n = 1$ and Step 2 for $n \ge 2$. Otherwise, assume that there exists $\{x_{k_i}\} \subseteq \{x_k\}$ satisfying

$$nM < len(x_0) + len\left(\int_0^T f(t, x_{k_i}(t))\, dt \right) < (n+1)M.$$

If $n = 1$, for every $k_i > N$,

$$(A_+x_{k_i})(t) - (A_+x^*)(t) = \begin{cases} \int_0^t \left(f^+(s, x_{k_i}(s)) - f^+(s, x^*(s)) \right) ds, \quad t \in [0, \alpha(x_{k_i}, 1)], \\ \int_0^{\alpha(x_{k_i}, 1)} f^+(s, x_{k_i}(s))\, ds + \int_{\alpha(x_{k_i}, 1)}^t f^-(s, x_{k_i}(s))\, ds \\ \quad - \int_0^t f^+(s, x^*(s))\, ds, \quad t \in (\alpha(x_{k_i}, 1), T]. \end{cases}$$

Thus, for every $t \in [0, T]$, and $k_i > N$, we have

$$|(A_+x_{k_i})(t) - (A_+x^*)(t)| \le \int_0^{\alpha(x_{k_i},1)} \left|f^+(s, x_{k_i}(s)) - f^+(s, x^*(s))\right| ds + \int_{\alpha(x_{k_i},1)}^T (\left|f^-(s, x_{k_i}(s)) - f^-(s, x^*(s))\right| + \left|f^-(s, x^*(s)) - f^+(s, x^*(s))\right|) ds. \tag{16}$$

By (13) and (14), $|(A_+x_{k_i})(t) - (A_+x^*)(t)| < 3\varepsilon$ holds for all k_i large enough and $t \in [0, T]$.

If $n \ge 2$, similarly to Step 2, we can prove that there exists $N > N_0$ such that, for $k > N$ and $t \in [0, \check{t}_{k,n}]$,

$$|(A_+x_k)(t) - (A_+x^*)(t)| < (4(n-1)+1)\varepsilon. \tag{17}$$

For $t \in [\check{t}_{k,n}, \hat{t}_{k,n}] = [\alpha(x_k, n), T]$,

$$\begin{aligned} |(A_+x_k)(t) - (A_+x^*)(t)| &\le |(A_+x_k)(\alpha(x_k, n)) - (A_+x^*)(\alpha(x_k, n))| \\ &\quad + \int_{\alpha(x_k,n)}^T (\left|f^-(s, x_k(s)) - f^+(s, x^*(s))\right| \\ &\qquad + \left|f^+(s, x_k(s)) - f^-(s, x^*(s))\right|) ds \\ &\le |(A_+x_k)(\alpha(x_k, n)) - (A_+x^*)(\alpha(x_k, n))| \\ &\quad + \int_{\alpha(x_k,n)}^T (\left|f^-(s, x_k(s)) - f^-(s, x^*(s))\right| \\ &\qquad + \left|f^+(s, x_k(s)) - f^+(s, x^*(s))\right| \\ &\qquad + 2\left|f^-(s, x^*(s)) - f^+(s, x^*(s))\right|) ds. \end{aligned}$$

By (13), (14) and (17), we have $|(A_+x_k)(t) - (A_+x^*)(t)| < 4n\varepsilon$ for k large enough and $t \in [0, T]$.

Similarly, we can also prove that A_-, B_+ and B_- are all continuous, that is, A and B are continuous on $C([0, T], K_C)$. □

Lemma 3.3. *Suppose that $f : [0, T] \times K_C \to K_C$ is continuous. For every $t \in [0, T]$ and $x \in C([0, T], K_C)$, $(Ax)(t), (Bx)(t) \subseteq x_0 + \int_0^t f(s, x(s))\, ds$.*

Proof. By (11) and (12), we see that $(A_+x)(t) \le x_0^+ + \int_0^t f^+(s, x(s))\, ds$, $t \in [0, T]$.

We can also check that $(B_+x)(t) \le x_0^+ + \int_0^t f^+(s, x(s))\, ds$ and $(A_-x)(t), (B_-x)(t) \ge x_0^- + \int_0^t f^-(s, x(s))\, ds$ for all $t \in [0, T]$. Thus, $(Ax)(t), (Bx)(t) \subseteq x_0 + \int_0^t f(s, x(s))\, ds$ for every $t \in [0, T]$ and $x \in C([0, T], K_C)$. □

Theorem 3.4. *Let $x_0 \in K_C$ with $0 < len(x_0) \le M$, $f : [0, T] \times K_C \to K_C$ be continuous and nontrivial, that is, $len(x) > 0$ gives $len(f(t, x)) > 0$ for every $t \in [0, T]$. If there exists $\sigma \in (0, 1)$ such that*

$$\lim_{H(x,\{0\})\to+\infty} \frac{H(f(t, x), \{0\})}{H(x, \{0\})} < \frac{\sigma}{T} \tag{18}$$

holds uniformly for $t \in [0, T]$, then problem (1) has at least two gH-differentiable solutions that satisfy the length constraint $len(x(t)) \le M$ and have only natural switching points.

Proof. By (18), there exists $r_0 > H(x_0, \{0\})$ such that, for every $t \in [0, T]$ and $x \in K_C$ satisfying $H(x, \{0\}) > r_0$, we have $H(f(t, x), \{0\}) \le \frac{\sigma}{T} H(x, \{0\})$.

Let $r > r_0$ and $E_r = \{x \in C([0, T], K_C) \mid D(x, \{0\}) \le r\}$, $B_{r_0} = \{x \in K_C \mid H(x, \{0\}) < r_0\}$. For every $x \in E_r$, let $U_x = \{t \in [0, T] \mid x(t) \in \overline{B_{r_0}}\}$ and $V_x = [0, T] \setminus U_x$. By Lemma 3.3 and Lemma 2.3(iii,iv), we have, for $x \in E_r$,

$$\begin{aligned} H((Ax)(t), \{0\}) &\le H(x_0 + \int_0^t f(s, x(s))\, ds, \{0\}) \\ &\le H(x_0, \{0\}) + H(\int_0^t f(s, x(s))\, ds, \{0\}) \\ &\le H(x_0, \{0\}) + \int_0^T H(f(s, x(s)), \{0\})\, ds \end{aligned}$$

$$
\begin{aligned}
&\leq H(x_0,\{0\}) + \max_{(t,x)\in[0,T]\times\overline{B_{r_0}}} H(f(t,x),\{0\}) \int_{U_x} dt \\
&\quad + \int_{V_x} \frac{\sigma}{T} H(x(t),\{0\})\, dt \\
&\leq H(x_0,\{0\}) + T\cdot \max_{[0,T]\times\overline{B_{r_0}}} H(f(t,x),\{0\}) + \sigma r.
\end{aligned}
$$

Let r be a bigger positive number if it is necessary, then we have $H((Ax)(t),\{0\}) \leq r$, that is, $AE_r \subseteq E_r$.

On the other hand, for every $x \in E_r$ and $0 \leq t_1 \leq t_2 \leq T$,

$$
\begin{aligned}
H((Ax)(t_1),(Ax)(t_2)) &= \max\{|(A_-x)(t_1) - (A_-x)(t_2)|, |(A_+x)(t_1) - (A_+x)(t_2)|\} \\
&\leq \int_{t_1}^{t_2} \left(|f^-(s,x(s))| + |f^+(s,x(s))|\right) ds \\
&\leq \max_{(t,x)\in[0,T]\times\overline{B_r}} 2H(f(t,x),\{0\})\cdot(t_2 - t_1).
\end{aligned}
$$

We see that A_-E_r and A_+E_r are both uniformly bounded and equicontinuous, thus A_-E_r and A_+E_r are sequentially compact, which provide that AE_r is also sequentially compact. Therefore, A has at least one fixed point in E_r. Similarly, we can prove that B has at least one fixed point in E_r. The nontrivial property of f and $len(x_0) > 0$ guarantee that the fixed points of A and B are different.

As a result, problem (1) has at least two gH-differentiable solutions, which satisfy the length constraint $len(x(t)) \leq M$ and have only natural switching points. □

Remark 3.5. *If the conditions in Theorem 3.4 are all fulfilled, then the map $F : C([0,T],K_C) \to C([0,T],K_C)$, which is defined as*

$$
(Fx)(t) = x_0 + \int_0^t f(s,x(s))\, ds, \quad t \in [0,T],
$$

has also at least one fixed point. That is, problem (1) has at least one gH-differentiable solution without switching point and length constraints.

Consequently, let $M_2 > M_1 > 0$, the gH-differentiable solutions of (1) with the different length constraints $len(x(t)) \leq M_1$ and $len(x(t)) \leq M_2$ might be exactly as the fixed point of F, which is (I)-gH-differentiable on $[0,T]$ and does not have switching point.

However, if x is a gH-differentiable solution to (1) satisfying the length constraint $len(x(t)) \leq M_1$ and has at least one type-I switching point on $(0,T)$, then x is not a solution to (1) satisfying $len(x(t)) \leq M_2$ with only natural switching points. In fact, x satisfies $len(x(t)) = M_1$ at the type-I switching point, which would not be a type-I natural switching point to the solutions under length constraint $len(x(t)) \leq M_2$.

Remark 3.6. *Besides, similarly to the establishment of an upper bound constraint of length, we can also consider a lower bound constraint of length $len(x(t)) \geq m$, or both upper and lower constraints of length $m \leq len(x(t)) \leq M$. Modifying the maps A and B according to the new length constraints, then we can prove that the new maps are also continuous on $C([0,T],K_C)$. Similarly to Theorem 3.4, there exists at least two gH-differentiable solutions to problem (1) satisfying the corresponding restrictions.*

However, with the aim of controlling the degree of imprecision, the study of solutions whose length is below a certain number $M > 0$ has been our main objective.

4 Examples

In this section, we provide some examples to show that the hypotheses in Theorem 3.4 are feasible.

Example 4.1. *Consider the interval-valued differential equation*

$$
\begin{cases} x'(t) = 2t\cdot[1,2], \ t \in [\frac{1}{2},2], \\ x(\frac{1}{2}) = [\frac{1}{4},\frac{1}{2}]. \end{cases} \tag{19}
$$

Let the length constraint be $len(x(t)) \leq 1$. Obviously, condition (18) is satisfied. Theorem 3.4 implies that there exist at least two gH-differentiable solutions to problem (19) with $len(x(t)) \leq 1$ and only natural switching points.

The solutions of $x'(t) = 2t \cdot [1, 2]$, $x(t_0) = x_0$, *are*

$$x(t) = x_0 + [t^2 - t_0^2, 2t^2 - 2t_0^2], \text{ or } x(t) = x_0 + [2t^2 - 2t_0^2, t^2 - t_0^2].$$

Thus, the two solutions to (19) with the length constraint $len(x(t)) \le 1$ *and only natural switching points (see Fig. 1) can be written as*

$$x_1(t) = \begin{cases} [t^2, 2t^2],\ t \in \left[\frac{1}{2}, 1\right], \\ \left[2t^2 - 1, t^2 + 1\right],\ t \in \left[1, \sqrt{2}\right], \\ \left[t^2 + 1, 2t^2 - 1\right],\ t \in \left[\sqrt{2}, \sqrt{3}\right], \\ \left[2t^2 - 2, t^2 + 2\right],\ t \in \left[\sqrt{3}, 2\right], \end{cases}$$

and

$$x_2(t) = \begin{cases} [2t^2 - \frac{1}{4}, t^2 + \frac{1}{4}], t \in \left[\frac{1}{2}, \sqrt{\frac{1}{2}}\right], \\ \left[t^2 + \frac{k}{4}, 2t^2 - \frac{k}{4}\right], t \in \left[\sqrt{\frac{k}{2}}, \sqrt{\frac{k+2}{2}}\right], k = 1,\ 5; \\ \left[2t^2 - \frac{5}{4}, t^2 + \frac{5}{4}\right], t \in \left[\sqrt{\frac{3}{2}}, \sqrt{\frac{5}{2}}\right]; \\ \left[2t^2 - \frac{9}{4}, t^2 + \frac{9}{4}\right], t \in \left[\sqrt{\frac{7}{2}}, 2\right]. \end{cases}$$

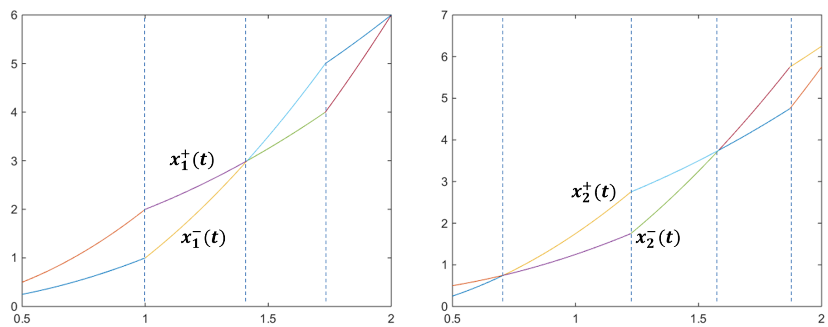

Figure 1: The solutions to problem (19) with length constraint $len(x(t)) \le 1$ and only natural switching points.

Example 4.2. *Consider the interval-valued differential equation*

$$\begin{cases} x'(t) = \frac{1}{4}x(t),\ \ t \in [0, \pi], \\ x(0) = [0, e^{-\frac{1}{4}}]. \end{cases} \tag{20}$$

Let the length constraint be $len(x(t)) \le 1$.

Here $f(t, x) = \frac{1}{4}x$. *Let* $1 > \sigma > \frac{\pi}{4}$, *we can check that condition (18) holds. By Theorem 3.4, problem (20) has at least two gH-differentiable solutions with* $len(x(t)) \le 1$ *and only natural switching points. By a direct calculation, the two solutions can be written as*

$$x_1(t) = \begin{cases} e^{\frac{t}{4}}[0, e^{-\frac{1}{4}}],\ t \in [0, 1], \\ \frac{1}{2}[e^{\frac{1}{4}(t-1)} - e^{-\frac{1}{4}(t-1)}, e^{\frac{1}{4}(t-1)} + e^{-\frac{1}{4}(t-1)}],\ t \in [1, \pi], \end{cases}$$

$$x_2(t) = \frac{1}{2}[e^{\frac{t-1}{4}} - e^{-\frac{t+1}{4}}, e^{\frac{t-1}{4}} + e^{-\frac{t+1}{4}}],\ t \in [0, \pi].$$

We can check that x_1 *and* x_2 *satisfy the length constraint* $len(x(t)) \le 1$. *In fact,*

$$len(x_1(t)) = \begin{cases} e^{\frac{t-1}{4}},\ t \in [0, 1], \\ e^{-\frac{1}{4}(t-1)},\ t \in [1, \pi], \end{cases}$$

and $len(x_2(t)) = e^{-\frac{t+1}{4}}$, $t \in [0, \pi]$ *(see Fig. 2).*

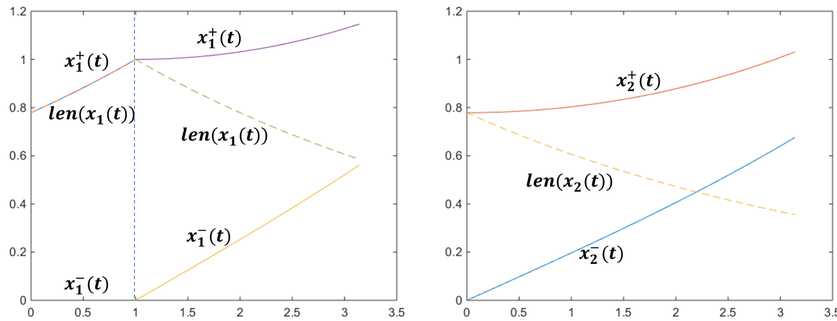

Figure 2: The solutions to problem (20) with length constraint $len(x(t)) \leq 1$ and only natural switching points.

Moreover, let the length constraint be $len(x(t)) \leq e^{\frac{1}{2}}$, *then* x_2 *is still a solution to (20) satisfying the new length constraint, but* x_1 *is no longer a solution to (20) with the new length constraint and only natural switching points. The new solution is*

$$\tilde{x}_1(t) = \begin{cases} e^{\frac{t}{4}}[0, e^{-\frac{1}{4}}],\ t \in [0, 3], \\ \frac{1}{2}[e^{\frac{1}{4}(t-1)} - e^{-\frac{1}{4}(t-5)}, e^{\frac{1}{4}(t-1)} + e^{-\frac{1}{4}(t-5)}],\ t \in [3, \pi]. \end{cases}$$

5 Conclusions

In this paper, we consider the existence of gH-differentiable solutions with length constraint to first order interval-valued differential equation. The switching points are not preset, but determined naturally by the length of the solutions. By constructing two continuous self-maps on $C([0, T], K_C)$, we provide some sufficient conditions for the existence of gH-differentiable solutions to problem (1) with the length constraint $len(x(t)) \leq M$, where $M > 0$. The results can be applied to study the solutions to some interval-valued dynamical systems for which the degree of uncertainty is controlled by a fixed value.

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