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Original paper

On Goursat problem for fuzzy random partial differential equations under generalized Lipschitz conditions

N. T. Kim Son[1], H. V. Long[2] and N. P. Dong[3]

[1] *Faculty of Natural Science, Hanoi Metropolitan University, Hanoi, Vietnam*

[2] *Division of Computational Mathematics and Engineering, Institute for Computational Science, Ton Duc Thang University, Ho Chi Minh City, Vietnam*

[2] *Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam*

[3] *Department of Mathematics, Hanoi Pedagogical University 2, Hanoi, Vietnam*

ntkson@daihocthudo.edu.vn, hoangvietlong@tdtu.edu.vn, nguyenphuongdong@hpu2.edu.vn

Abstract

Fuzzy random partial differential equations (PDEs) present a connection between random dynamical systems with nonstatistical inexactness data. These blended models could be efficiently used in modeling dynamical systems working in vagueness and ambiguity environments such as fuzzy random adaptive control, fuzzy random financial prediction, fuzzy random biological modeling, etc. In this article, we study Goursat problem for fuzzy random wave equations in the framework of generalized complete metric spaces in the sense of Luxemburg. We consider equations under generalized Hukuhara differentiability. The force functions are constrained by generalized Lipschitz conditions, that makes the range of PDEs types wider than using unbounded and locally Lipschitz conditions. The existence, uniqueness and boundedness of fuzzy solutions are investigated by employing Picard successive approximation method and Luxemburg fixed point theorem. Some illustrated examples are given to demonstrate for theoretical results.

Keywords: Fuzzy random partial differential equations, generalized metric spaces, generalized Lipschitz conditions.

1 Introduction

Random dynamical systems in fact are disturbed by indeterminable, inadequate or uncertain elements. Parameters of the dynamical systems, input data, initial conditions, external forces are often not observed with the exactness. The uncertainty in parameters result from ambiguity of data obtained by measurements or due to lack of human knowledge. It is arising one more additional source of uncertainty in random dynamical models. In these situations, the quantities may not be taken single value, there may be taken value in a set of intervals or a descriptive set of linguistic terms in connection with random evens input. Thus, the deterministic or non-fuzzy random models are often impossible to represent all characteristics of the considered systems. So that, we need to incorporate the value of random variables of dynamical systems by fuzzy values that follows some interesting types of differential equations (DEs) and partial equations (PDEs), namely fuzzy random DEs, fuzzy random PDEs and so on.

The first effort including the indeterministic into random experiment was proposed by Kwakernaak [10], called fuzzy random variable. Not all properties of random variables were carried out, however, Kwakernaak gave enough theoretical foundation for consideration the simple application of many problems related to nonnegative random variables consisting impreciseness. In a slightly different manner, Puri and Ralescu [21] introduced the concept of fuzzy random variables as a tool for presentation of the connection between random experiment with nonstatistical impreciseness data. Guo and Guo [7] analyzed the nature of error contributions caused by coupling translation between the Grey differential equation and the coupled regression according to fuzzy random variables.

Malinowski [16] established some successive approximation schemes to prove the results of existence and uniqueness of solutions for fuzzy random differential equations and stochastic integral equations and then, some properties of

Corresponding Author: H. V. Long

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mild fuzzy random solution are investigated in [17]. Fuzzy random DEs based on Itô integral was firstly presented in [15]. Next, the concept of stochastic differential equation was extended into fractional order [18]. This interesting subject has been further developed to fuzzy random fractional PDEs in [11]. In the fuzzy random differential equation theory, the conditions placed on the force function or the coefficients of equations always play an essential role. In [15, 17], the fuzzy and set-valued stochastic DEs have been studied with assumptions of global Lipschitz coefficients, in which Malinowski used Picard or Maruyama successive approximation schemes to prove the existence and uniqueness of solutions. The paper [19] developed them by considering local Lipschitz condition and linear growth conditions. The paper [20] accomplished the existence and uniqueness of solutions to set-valued and fuzzy stochastic DEs under conditions which are considered as the weakest conditions in the class of admissible coefficients.

The study of fuzzy theory in partial differential equations was first performed by Buckley and Feuring [4] for some simple linear equations. Bertone et al. [3] defined fuzzy solutions for some classical models of PDEs with fuzzy parameters. Zadeh extension principle was used to obtain the stability properties with respect to the initial boundary data. Long et al. [12] studied the existence and uniqueness of fuzzy solutions for some classes of hyperbolic PDEs under generalized Hukuhara derivatives and then, they expanded research for fractional order PDEs [13]. In recent research, Gouyandeh et.al. [6] applied fuzzy Fourier transform to study a class of fuzzy heat equation under generalized Hukuhara differentiability. One important class of hyperbolic partial differential equation is the Goursat problem, that is a second-order hyperbolic differential equations in two independent variables with given values on two characteristic curves emanating from the same point. The study of Goursat problems in real-world environments that contain uncertain and vagueness leads to an interesting concepts of partial differential equations, namely the fuzzy Goursat problems. One of pioneer work in this area, fuzzy Goursat problem for PDEs with the classical Lipschitz conditions, were studied in [25]. Khastan and Lopez [9] established some sufficient and necessary conditions for the existence of the uniqueness solutions of fuzzy Goursat problem in a closed rectangle by the use of Banach fixed point theorem. Beside, the stochastic Goursat problem for PDEs was well-studied in [22]. However, there has been no study of the random fuzzy Goursat problem. In this paper, we study the fuzzy random Goursat problem for partial differential equations

$$\mathcal{D}^{\sigma}_{ts} w(t,s,\omega) \overset{D\mathbb{P}.1}{=} h_{\omega}(t,s,w(t,s,\omega),w_t(t,s,\omega),w_s(t,s,\omega)),$$

with local conditions $w(t,0,\omega) \overset{[0,c]\mathbb{P}.1}{=} \varphi(t,\omega)$ and $w(0,s,\omega) \overset{[0,b]\mathbb{P}.1}{=} \psi(s,\omega)$.

Fuzzy random fractional partial integro-differential equations under Caputo gH-differentiability have been studied with the Lipschitz assumptions and Picard approximation method in [11]. In this paper, we impose some generalized Lipschitz conditions **(A3)** or **(A4)**. Here, the generalized Lipschitz coefficients may depend on independent variables and it may not be bounded. This paper's approach is based on the characteristics of external forces that requires us to build suitable generalized metrics in solutions spaces. Furthermore, we exploit the approximation method and Luxemburg fixed point theorem to establish the existence and uniqueness as well as the boundedness of solutions.

This paper is organized as follows: Some useful notions on fuzzy calculus and random analysis are given in Section 2. In addition, Luxemburg fixed point theorem in generalized metric spaces are also recalled in this section. Goursat problem for fuzzy random partial differential equation is stated in Section 3 with the introductions of fuzzy stochastic processes and sequence of approximate mild fuzzy random solutions. In Section 4, Theorem 4.3 and Theorem 4.6 focus on the solvability of the problem in type 1. The existence and uniqueness of mild fuzzy random solution in type 2 are presented in Theorem 5.2 of Section 5. The boundedness of solutions is obtained in Theorem 6.1 of Section 6. Finally, some conclusions and future works are discussed in Section 7.

2 Preliminaries

2.1 A brief review of fuzzy calculus and fuzzy random variables

In this section, we will briefly recall some preliminaries on fuzzy calculus, for a more details on the background of fuzzy analysis, the readers are kindly requested to go through in [1, 5]. By $\mathcal{K}_C$, we denote the set of all nonempty compact, convex subsets of $\mathbb{R}$ and $||.||$ is the Euclidean norm in $\mathbb{R}$. The distance between A and B in $\mathcal{K}_C$ is defined by Hausdorff metric

$$\rho_H(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}||a-b||, \sup_{b\in B}\inf_{a\in A}||a-b||\right\}.$$

The Hukuhara difference $A\ominus B$ of A and $B\in\mathcal{K}_C$ is an element $C\in\mathcal{K}_C$ such that $A\ominus B = C$. The generalized Hukuhara (gH) difference of $A,B\in\mathcal{K}_C$ is defined by $A\ominus_{gH} B = C \Leftrightarrow A = B + C$ or $B = A + (-1)C$.

We denote by E the space of all fuzzy numbers on $\mathbb{R}$, where each fuzzy number can be seen as a nonempty subset $\{(x,u(x))\}\subset\mathbb{R}\times[0,1]$ of a function u: $\mathbb{R}\to[0,1]$ being normal, fuzzy convex, semi-continuous and compact support.

Denote by $[v]^\theta = \{t \in \mathbb{R} : v(t) \geq \theta\}$ if $0 < \theta \leq 1$ and $[v]^0 = \mathrm{cl}\{t \in \mathbb{R} : v(t) > 0\}$. Then, the fuzzy number v has the parametric form $[v]^\theta = [v_{l\theta}, v_{r\theta}] \in \mathcal{K}_C$ for all $\theta \in [0,1]$. It is well-known that (E, ρ) is a complete metric space with supremum metric $\rho(w,v) = \sup_{0\leq\theta\leq 1} \rho_H\left([w]^\theta, [v]^\theta\right)$. The addition, multiplication rules and gH-difference in the space E are correspondingly defined level-wise set through their parametric forms in $\mathcal{K}_C$.

A fuzzy-valued function $w : I \subset \mathbb{R}^2 \to E$ is said to be partial gH-differentiable with respect to (w.r.t.) variable t at a point $(t_0, s_0) \in I$ if there exist an element $w_t(t_0, s_0) \in E$ such that the limit in following right-hand side exists

$$w_t(t_0, s_0) = \lim_{h\to 0} \frac{1}{h}(w\,(t_0 + h, s_0) \ominus_{gH} w\,(t_0, s_0)).$$

Similarly, we can define the partial gH-derivatives of the function w w.r.t. variable s and higher order derivatives.

The gH-differentiability function w is called partial (i)-gH differentiable w.r.t. t at $(t_0, s_0) \in I$ if

$$\left[\frac{\partial w}{\partial t}(t_0, s_0)\right]^\theta = \Big[(w_{l\theta})_t(t_0, s_0), (w_{r\theta})_t(t_0, s_0)\Big],$$

and the gH-differentiability function w is called partial (ii)-gH differentiable w.r.t. t at $(t_0, s_0) \in I$ if

$$\Big[w_t\,(t_0, s_0)\Big]^\theta = \Big[(w_{r\theta})_t(t_0, s_0), (w_{l\theta})_t(t_0, s_0)\Big].$$

If w (w.r.t. variable t) and w_t (w.r.t. variable s) have the same type of gH-differentiability at $(t_0, s_0) \in I$ then w is called of type (i) of gH-differentiable of order 2 at (t_0, s_0). Then, the gH-derivative of order 2 of w is denoted by $\mathcal{D}^1_{ts} w(t_0, s_0) = w_{ts}(t_0, s_0)$, whose parametric form is $\Big[\mathcal{D}^1_{ts} w(t_0, s_0)\Big]^\theta = \Big[(w_{l\theta})_{ts}(t_0, s_0), (w_{r\theta})_{ts}(t_0, s_0)\Big]$.
If w (w.r.t variable t) and w_t (w.r.t. variable s) have different types of gH-differentiability at $(t_0, s_0) \in I$ then w is called of type (ii) of gH-differentiable of order 2 at (t_0, s_0). Then, the gH-derivative in order 2 of w is denoted by $\mathcal{D}^2_{ts} w(t_0, s_0) = w_{ts}(t_0, s_0)$, whose parametric form is $\Big[w_{ts}(t_0, s_0)\Big]^\theta = \Big[(w_{r\theta})_{ts}(t_0, s_0), (w_{l\theta})_{ts}(t_0, s_0)\Big]$.

Lemma 2.1. *Let $v_n : I \subset \mathbb{R}^2 \to E$ be a sequence of fuzzy-valued functions which are partial gH-differentiable w.r.t. variable t and its partial gH-derivative w.r.t variable t, denoted by $(v_n)_t(t,s)$, is continuous on I. Assume that*

(i) *The function v_n is partial (i)-gH differentiable w.r.t. variable t for all $n \in \mathbb{N}$,*

(ii) *There exists a function $\Phi : I \subset \mathbb{R}^2 \to E$ such that $\{v_n(t_0, s_0)\}$ converges to $\Phi(t_0, s_0)$ for each $(t_0, s_0) \in I$,*

(iii) *There exists a function $g : I \subset \mathbb{R}^2 \to E$ such that the sequence $\{(v_n)_t\}$ uniformly converges to g on I.*

Then, the sequence $\{v_n\}$ uniformly converges to a function v defined by $v(t,s) := \Phi(t_0, s_0) + \int_{t_0}^t g(\nu, s)d\nu$. Moreover, the function $v(t,s)$ is partial (i)-gH differentiable w.r.t. variable t on I and $v_t(t,s) = g(t,s)$ for all $(t,s) \in I$.

Proof. Since $(v_n)_t \overset{I}{\rightrightarrows} g$ and $(v_n)_t(t,s)$ is continuous for each $(t,s) \in I$, we imply that $g(t,s)$ is also continuous on I. Thus, for each $(t,s) \in I$, we denote $v(t,s) = \Phi(t_0, s_0) + \int_{t_0}^t g(\nu, s)d\nu$. Since v_n is partial (i)-gH differentiable w.r.t. t, we have $v_t(t,s) = g(t,s)$. Moreover, by fundamental theorem of integral, we also have

$$v_n(t,s) = v_n(t_0, s_0) + \int_{t_0}^t (v_n)_\nu(\nu, s)d\nu, \quad \text{for each } s \text{ is fixed.}$$

On the other hand, since $(v_n)_t(t,s) \overset{I}{\rightrightarrows} g$, we directly obtain $\int_{t_0}^t (v_n)_\nu(\nu, s)d\nu \rightrightarrows \int_{t_0}^t g(\nu, s)d\nu$. Finally, by using the hypothesis (ii), we can conclude that $\{v_n\}$ uniformly converges to v on I. □

Remark 2.1. *In Lemma 2.1, if we replace the condition (i) by the condition "(i') The function v_n is (ii)-gH differentiable w.r.t. t for all $n \in \mathbb{N}^*$ " then the sequence $\{v_n\}$ uniformly converges to a function v, defined by*

$$v(t,s) := \Phi(t_0, s_0) \ominus (-1) \int_{t_0}^t g(\nu, s)d\nu.$$

Moreover, $v(t,s)$ is partial (ii)-gH differentiable w.r.t. variable t and $v_t(t,s) = g(t,s)$ for all $(t,s) \in I$.

Proof. By doing the similar arguments as in Lemma 2.1, we achieved the proof. □

In the following, we will recall from [11] some necessary notions in the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$

(*i*) $u : \Omega \to E$ is called a fuzzy random variable if the mapping $[u]^\theta : \Omega \to \mathcal{K}_C$ is measurable for all $\theta \in [0, 1]$.

(*ii*) $u : I \times \Omega \to E$ is called a fuzzy stochastic process if $u(\cdot, \cdot, \omega)$ is a fuzzy-valued function for each $\omega \in \Omega$ and the mapping $u(x, y, \cdot)$ is a random fuzzy variable for each $(x, y) \in I$.

(*iii*) A fuzzy stochastic process $u : I \times \Omega \to E$ is called continuous if for almost every $\omega \in \Omega$, the trajectory $u(\cdot, \cdot, \omega)$ is continuous on I w.r.t. the metric ρ.

(*iv*) Let η, μ be fuzzy random variables. The formula $P(\{\omega \in \Omega : \eta(\omega) = \mu(\omega)\}) = 1$ will be written in the compact form $\eta(\omega) \overset{\mathbb{P}.1}{=} \mu(\omega)$. The notions for inequalities are defined similarly.

(*v*) Let u, v be fuzzy stochastic processes. The formula $P(\{\omega \in \Omega : u(x, y, \omega) = v(x, y, \omega),\ \forall x, y \in I\}) = 1$ can be rewritten by $u(x, y, \omega) \overset{I\mathbb{P}.1}{=} v(x, y, \omega)$ for short, and similarly for the other relations.

2.2 Luxemburg fixed point theorem

Let X be a non-empty set and $\kappa : X \times X \to [0, \infty]$ be a non-negative function. Then, the space (X, κ) is said to be a generalized complete metric space in the Luxemburg sense [14] if two following conditions are fulfilled:

(i) The function κ satisfies all the axioms of a metric.

(ii) If $\{x_n\}_{n=1}^\infty \subset X$ satisfies $\lim\limits_{m,n\to\infty} \kappa(x_n, x_m) = 0$ then there exists $x \in X$ such that $\lim_{n\to\infty} \kappa(x_n, x) = 0$.

Remark 2.2. *Since the fact that NOT every pair $x, y \in X$ necessarily has a finite distance $\kappa(x, y)$, the generalized complete metric space in sense of Luxemburg differs far from the usually complete metric spaces.*

Theorem 2.2 (Luxemburg Theorem, [14])**.** *Let (X, κ) be a generalized complete metric space in Luxemburg sense and an operator $\mathscr{F} :\ X \to X$ satisfies following conditions:*

1^0) *There exists $\lambda \in (0, 1)$ such that $\kappa(\mathscr{F}(t), \mathscr{F}(s)) \le \lambda\kappa(t, s)$ holds for all $t, s \in X$ satisfying $\kappa(t, s) < \infty$,*

2^0) *For $t_0 \in X$ and $t_n = \mathscr{F}(t_{n-1})$, there exists $N = N(t_0) \in \mathbb{N}$ such that $\kappa(t_N, t_{N+k}) < \infty$ for all $k \in \mathbb{N}$.*

3^0) *If $\mathscr{F}(x) = x$ and $\mathscr{F}(y) = y$ then $\kappa(x, y) < \infty$ for all $x, y \in X$.*

Then, $\mathscr{F}$ has a unique fixed point in X. Moreover, $\{t_n\}_{n=1}^\infty$ converges to this fixed point for any initial value $t_0 \in X$.

3 Problem statement

3.1 Fuzzy random Goursat problems

Let $C^1(D\times\Omega, E)$ be the space of all continuous fuzzy stochastic processes $u : D\times\Omega \to E$ whose partial gH-derivatives $u_t(t, s, \omega)$ and $u_s(t, s, \omega)$ exist. Firstly, we introduce some notations that will be used throughout this paper

$$D = [0, c] \times [0, b] \qquad \hat{D} = (0, c] \times (0, b] \qquad W = D \times [C^1(D \times \Omega, E)]^3 \qquad W_1 = \hat{D} \times [C^1(\hat{D} \times \Omega, E)]^3.$$

This paper's aim is to investigate the following Goursat problem

$$\mathcal{D}^\sigma_{ts} w(t, s, \omega) \overset{D\mathbb{P}.1}{=} h_\omega(t, s, w(t, s, \omega), w_t(t, s, \omega), w_s(t, s, \omega)), \tag{1}$$

subject to the local conditions

$$w(t, 0, \omega) \overset{[0,c]\mathbb{P}.1}{=} \varphi(t, \omega), \quad w(0, s, \omega) \overset{[0,b]\mathbb{P}.1}{=} \psi(s, \omega) \tag{2}$$

where the notion $\mathcal{D}^\sigma_{ts}$ stands for the gH-derivative operator in type $\sigma \in \{1, 2\}$, $h_\omega : W_1 \to E$ is a given fuzzy-valued function, $\varphi : [0, c] \times \Omega \to E$ and $\psi : [0, b] \times \Omega \to E$ are continuously gH-differentiable fuzzy stochastic processes satisfying $\varphi(0, \omega) \overset{\mathbb{R}.1}{=} \psi(0, \omega)$. The differences $\psi(s, \omega) \ominus \varphi(0, \omega)$ and $\varphi(t, \omega) \ominus \varphi(0, \omega)$ are assumed to exist. In addition, for simplicity in representation, some following notations are used

(N1) For each $(t,s,\omega)\in D\times\Omega$, denote $z(t,s,\omega):=(w(t,s,\omega),w_t(t,s,\omega),w_s(t,s,\omega))^{\mathrm{t}}$.

(N2) For each $(t,s,\omega)\in D\times\Omega$, denote $g(t,s,\omega)\overset{D\mathbb{P}.1}{=}\varphi(t,\omega)+\psi(s,\omega)\ominus\varphi(0,\omega)$.

(N3) Denote $S(D\times\Omega)$ by the set of all functions $w\in C^1(D\times\Omega,E)$ satisfying the condition (2).

3.2 Some basic lemmas

Hypothesis (A1): The mapping $h_\omega:W_1\to E$, defined by $(t,s,\omega)\mapsto h_\omega(t,s,z(t,s,\omega))$, is a fuzzy stochastic process and $h_\omega(\cdot,\cdot,z(\cdot,\cdot,\omega)):D\to E$ is a continuous fuzzy-valued function for each $\omega\in\Omega$.

Lemma 3.1. *Let $z=(w,w_t,w_s)\in(C^1(D\times\Omega,E))^3$. If the hypothesis (**A1**) holds then for each $(t,s)\in D$, the mappings $v_1(t,s,\omega)=\int_0^t\int_0^s h_\omega(\nu,\tau,z(\nu,\tau,\omega))d\nu d\tau$, $v_2(t,s,\omega)=\int_0^t h_\omega(\nu,s,z(\nu,s,\omega))d\nu$ and $v_3(t,s,\omega)=\int_0^s h_\omega(t,\tau,z(t,\tau,\omega))d\tau$ are fuzzy stochastic processes.*

Proof. For each $\theta\in[0,1]$, the $\theta-$level sets of $v_1(t,s,\omega)$ are given by $[v_1(t,s,\omega)]^\theta:=\int_0^t\int_0^s[h_\omega(\nu,\tau,z(\nu,\tau,\omega))]^\theta d\nu d\tau$. By the hypothesis (**A1**), we have $h_\omega(\cdot,\cdot,z(\cdot,\cdot,\omega)):D\to E$ is continuous with $\mathbb{P}.1$. Next, by applying Proposition 5.1 in [8], we immediately obtain $[h_\omega(\nu,\tau,w,w_\nu,w_\tau)]^\theta=h_\omega\left(\nu,\tau,[w]^\theta,[w_\nu]^\theta,[w_\tau]^\theta\right)$.

Moreover, it is well-known that the multi-valued function $h_\omega(\nu,\tau,[w]^\theta,[w_\nu]^\theta,[w_\tau]^\theta)$is continuous with $\mathbb{P}.1$ and measurable w.r.t. $\omega\in\Omega$, i.e., it is a Caratheodory mappings on $D\times[C^1(D\times\Omega,\mathcal{K}_C)]^3$ with $\mathbb{P}.1$. Hence, by Caratheodory parametric theorem, the multi-valued function $h_\omega(\nu,\tau,[w]^\theta,[w_\nu]^\theta,[w_\tau]^\theta)$is measurable in $D\times[C^1(D\times\Omega,\mathcal{K}_C)]^3$. This implies that $h_\omega(t,s,w(t,s,\omega),w_t(t,s,\omega),w_s(t,s,\omega))$ is integrable in D and the following integral

$$[v_1(t,s,\omega)]^\theta=\int_0^t\int_0^s h_\omega(\nu,\tau,[w(\nu,\tau,\omega)]^\theta,[w_\nu(\nu,\tau,\omega)]^\theta,[w_\tau(\nu,\tau,\omega)]^\theta)d\nu d\tau,$$

is measurable in $D\times\Omega$ or equivalently, the function $v_1(t,s,\omega)$ is a fuzzy stochastic process. By doing similar arguments, we also prove that the functions $v_2(t,s,\omega)$ and $v_3(t,s,\omega)$ are also fuzzy stochastic processes. □

Lemma 3.2. *Let $z=(w,w_t,w_s)\in(C^1(D\times\Omega,E))^3$ and the function $g(t,s,\omega)$ be defined in **(N2)**. If the hypothesis (**A1**) holds then the fuzzy-valued mappings S_i are fuzzy stochastic processes for all $i=\overline{1,3}$ and $(t,s)\in D$, where*

$$\mathrm{S}_1(t,s,\omega)=g(t,s,\omega)+\int_0^t\int_0^s h_\omega(\nu,\tau,z(\nu,\tau,\omega))d\nu d\tau,$$
$$\mathrm{S}_2(t,s,\omega)=g_t(t,s,\omega)+\int_0^s h_\omega(\nu,s,z(\nu,s,\omega))d\nu,$$
$$\mathrm{S}_3(t,s,\omega)=g_s(t,s,\omega)+\int_0^t h_\omega(t,\tau,z(t,\tau,\omega))d\tau.$$

Proof. By definitions of $\varphi(t,\omega)$, $\psi(s,\omega)$ and $g(t,s,\omega)$, it follows that $g(t,s,\omega)$ is a fuzzy stochastic process and hence, the multi-valued function $[g(t,s,\cdot)]^\theta:\Omega\to\mathcal{K}_C$ is measurable for each $\theta\in[0,1]$. In addition, since the hypothesis (**A1**) holds, we deduce that the multi-valued function $\omega\mapsto[h_\omega(t,s,z(t,s,\omega))]^\theta$ is also measurable. Next, we have

$$\left[g(t,s,\omega)+\int_0^t\int_0^s h_\omega(\nu,\tau,z(\nu,\tau,\omega))d\nu d\tau\right]^\theta=[g(t,s,\omega)]^\theta+\left[\int_0^t\int_0^s h_\omega(\nu,\tau,z(\nu,\tau,\omega))d\nu d\tau\right]^\theta.$$

According to Lemma 3.1, it is well-known that $\left[\int_0^t\int_0^s h_\omega(\nu,\tau,z(\nu,\tau,\omega))d\nu d\tau\right]^\theta$ is measurable. Hence, the multi-valued mapping $\omega\mapsto[g(t,s,\omega)]^\theta+\left[\int_0^t\int_0^s h_\omega(\nu,\tau,z(\nu,\tau,\omega))d\nu d\tau\right]^\theta$ is also measurable for each $\theta\in[0,1]$, which follows that $\mathrm{S}_1(t,s,\omega)$ is a fuzzy stochastic process. By similar arguments, the proof is completed. □

Remark 3.1. *If all hypotheses of Lemma 3.2 are satisfied then the following fuzzy-valued mappings*

$$\hat{\mathrm{S}}_1(t,s,\omega)=g(t,s,\omega)\ominus(-1)\int_0^t\int_0^s h_\omega(\nu,\tau,z(\nu,\tau,\omega))d\nu d\tau,$$

$$\hat{\mathrm{S}}_2(t,s,\omega) = g_t(t,s,\omega) \ominus (-1)\int_0^s h_\omega(\nu,s,z(\nu,s,\omega))d\nu,$$
$$\hat{\mathrm{S}}_3(t,s,\omega) = g_s(t,s,\omega) \ominus (-1)\int_0^t h_\omega(t,\tau,z(t,\tau,\omega))d\tau.$$

are fuzzy stochastic processes provided that the H-differences are well-defined. Indeed, since the H-difference $g(t,s,\omega)\ominus(-1)\int_0^t\int_0^s h_\omega(\nu,\tau,z(\nu,\tau,\omega))d\nu d\tau$ *exists with* $\mathbb{P}.1$ *for all* $(t,s)\in D$ *then for each* $\theta\in[0,1]$*, we have*

$$\left[g(t,s,\omega)\ominus(-1)\int_0^t\int_0^s h_\omega(\nu,\tau,z(\nu,\tau,\omega))d\nu d\tau\right]^\theta = [g(t,s,\omega)]^\theta \ominus (-1)\left[\int_0^t\int_0^s h_\omega(\nu,\tau,z(\nu,\tau,\omega))d\nu d\tau\right]^\theta.$$

which means that the H-difference $[g(t,s,\omega)]^\theta \ominus \left[(-1)\int_0^t\int_0^s h_\omega(\nu,\tau,z(\nu,\tau,\omega))d\nu d\tau\right]^\theta$ *exists with* $\mathbb{P}.1$*. Thus, the multi-valued mapping* $\omega\mapsto[g(t,s,\omega)]^\theta\ominus(-1)\left[\int_0^t\int_0^s h_\omega(\nu,\tau,z(\nu,\tau,\omega))d\nu d\tau\right]^\theta$ *is measurable for each* $\theta\in[0,1]$*, that means* $\hat{\mathrm{S}}_1(t,s,\omega)$ *is a fuzzy stochastic process. Similarly, we also have* $\hat{\mathrm{S}}_2(t,s,\omega)$*,* $\hat{\mathrm{S}}_3(t,s,\omega)$ *are fuzzy stochastic processes.*

3.3 Mild fuzzy random solutions

By sketching to Lemma 4.1 in [12], we assume that the function $w\in S(D\times\Omega)$ satisfies the Goursat problem $(1)-(2)$ and the hypothesis (**A1**) is fulfilled. Then, the function w satisfies one of following integro-differential equations:

$$w(t,s,\omega) \overset{D\mathbb{P}.1}{=} g(t,s,\omega) + \int_0^t\int_0^s h_\omega(\nu,\tau,z(\nu,\tau,\omega))d\nu d\tau, \tag{3}$$
$$w(t,s,\omega) \overset{D\mathbb{P}.1}{=} g(t,s,\omega) \ominus (-1)\int_0^t\int_0^s h_\omega(\nu,\tau,z(\nu,\tau,\omega))d\nu d\tau. \tag{4}$$

Definition 3.3. *A fuzzy stochastic process* $w: D\times\Omega\to E$ *is said to be*

(i) **A mild fuzzy random solution in type 1** *of Goursat problem (1) − (2) if it is continuous and satisfies random integro-differential equation (3).*

(ii) **A mild fuzzy random solution in type 2** *of Goursat problem (1) − (2) if it is continuous and satisfies random integro-differential equation (4).*

Remark 3.2. *Note that the concept of fuzzy random solution proposed in Definition 3.3 is of mild sense, i.e., there exists a fuzzy stochastic process* w^* *which is a fuzzy solution of the integro-differential equation (3) [or (4)], but it may not be the fuzzy random solution of the Goursat problem (1) − (2).*

Example 3.4. *Assume that* $\Omega=[0,1]$*,* $\mathcal{F}$ *is a Borel* $\sigma-$ *field on* Ω *and* $\mathbb{P}$ *is Lebesgue measure on* $(\Omega,\mathcal{F})$*. Let* $D=[0,4]\times[0,1]$ *and* $f_\omega: D\times\left[C^1(D\times\Omega,E)\right]^3\to E$ *be a fuzzy-valued function having triangular values as outputs*

$$f(t,s,z(t,s,\omega)) = e^{s+\omega}\left(t^2, t^2+1, 2t^2\right).$$

The $\theta-$*level sets* $[f(t,s,z(t,s,\omega))]^\theta = \left[e^{s+\omega}(t^2+\theta), e^{s+\omega}(2t^2-\theta(t^2-1))\right]$ *for each* $\theta\in[0,1]$*. Now, we consider a Goursat problem for following fuzzy random partial differential equation*

$$\mathcal{D}^\sigma_{ts}w(t,s,\omega) \overset{D\mathbb{P}.1}{=} f(t,s,z(t,s,\omega)) \quad (t,s)\in(0,4]\times(0,1], \tag{5}$$

subject to the conditions

$$\begin{cases} w(t,0,\omega) \overset{D\mathbb{P}.1}{=} \tilde{a}\omega & t\in[0,4], \\ w(0,s,\omega) \overset{D\mathbb{P}.1}{=} \tilde{a}\omega e^s & s\in[0,1], \end{cases} \tag{6}$$

where $z := (w, w_t, w_s) \in \left[C^1(D \times \Omega, E)\right]^3$ *and* $\tilde{a} = (1, 2, 3) \in E$*. Here,* $g(t, s, \omega) \overset{D\mathbb{P}.1}{=} \tilde{a}\omega e^s$ *is a continuously (i)-gH-differentiable fuzzy stochastic process on* $D \times \Omega$*. According to the formulas (3) and (4), the mild fuzzy random solution of Goursat problem (5) - (6) is given by*

$$w(t, s, \omega) \ominus_{gH} g(t, s, \omega) \overset{D\mathbb{P}.1}{=} \int_0^t \int_0^s f\left(\nu, \tau, z(\nu, \tau, \omega)\right) d\nu d\tau, \tag{7}$$

which means that for each $\theta \in [0, 1]$*, the following expression holds*

$$[w(t, s, \omega) \ominus_{gH} g(t, s, \omega)]^\theta = \left[\left(e^{s+\omega} - e^\omega\right)\left(\frac{t^3}{3} + \theta t\right), \left(e^{s+\omega} - e^\omega\right)\left((2-\theta)\frac{t^3}{3} + \theta t\right)\right].$$

Denote by $\mathrm{F}(t, s, \omega)$ *the fuzzy-valued function whose* $\theta-$*level sets is given in right-hand side. Then, by using Negoita-Ralescu characterization theorem [1], we obtain*

$$w(t, s, \omega) \ominus_{gH} g(t, s, \omega) \overset{D\mathbb{P}.1}{=} \mathrm{F}(t, s, \omega) \Leftrightarrow w(t, s, \omega) \overset{D\mathbb{P}.1}{=} \left[\begin{array}{l} \tilde{a}\omega e^s + \mathrm{F}(t, s, \omega) \\ \tilde{a}\omega e^s \ominus (-1)\mathrm{F}(t, s, \omega) \end{array}\right.$$

where the function $\mathrm{F}(t, s, \omega) \overset{D\mathbb{P}.1}{=} \left(e^{s+\omega} - e^\omega\right)\left(\frac{t^3}{3}, \frac{t^3}{3} + t, \frac{2t^3}{3}\right)$*. However, it should be noted that the function* $\mathrm{F}(t, s, \omega)$ *is not partial gH-differentiable w.r.t. variable* t *on* $[0, 4] \times [0, 1]$*. Indeed, for each* $\theta \in [0, 1]$*, we have*

$$\begin{cases} \left(\mathrm{F}_\theta^-\right)'(t, s, \omega) \leq \left(\mathrm{F}_\theta^+\right)'(t, s, \omega) \text{ for all } (t, s) \in D \text{ and } \theta \in [0, 1], \\ \left(\mathrm{F}_\theta^-\right)'(t, s, \omega) = \left(e^{s+\omega} - e^\omega\right)\left(t^2 + \theta\right) \text{ is increasing w.r.t } \theta \text{ for all } (t, s) \in D, \\ \left(\mathrm{F}_\theta^+\right)'(t, s, \omega) = \left(e^{s+\omega} - e^\omega\right)\left((2-\theta)t^2 + \theta\right) \text{ isn't always decreasing w.r.t } \theta \text{ for all } (t, s) \in D. \end{cases}$$

Since the third assertion doesn't hold then according to Proposition 21 in [2], it implies that $\mathrm{F}(t, s, \omega)$ *is not partial gH-differentiable in both two types w.r.t variable* t *on* D*. Thus, the function* w *is a solution of the integro-differential equation (7), but it doesn't satisfy the fuzzy random partial differential equation (5).*

3.4 Sequence of approximate mild fuzzy random solutions

For each $n \in \mathbb{N}$, we define a sequence of fuzzy-valued function $w_n : D \times \Omega \to E$ by

$$w_n(t, s, \omega) \overset{D\mathbb{P}.1}{=} g(t, s, \omega) + \int_0^t \int_0^s h_\omega(\nu, \tau, z_{n-1}(\nu, \tau, \omega)) d\nu d\tau, \quad n \geq 1, \tag{8}$$

where $w_0 \in S(D \times \Omega)$ is the first term and $z_n := (w_n, (w_n)_t, (w_n)_s)$. It should be noted that the proposed sequence $\{w_n\}$ will approximate the mild fuzzy random solution of type 1 of Goursat problem $(1) - (2)$. Next, the approximate sequences for the partial gH-derivatives of type (i) (or type (ii)) are given by

$$(w_n)_t(t, s, \omega) \overset{D\mathbb{P}.1}{=} g_t(t, s, \omega) + \int_0^s h_\omega(t, \tau, z_{n-1}(t, \tau, \omega)) d\tau,$$
$$(w_n)_s(t, s, \omega) \overset{D\mathbb{P}.1}{=} g_s(t, s, \omega) + \int_0^t h_\omega(\nu, s, z_{n-1}(\nu, s, \omega)) d\nu.$$

Similarly, let $w_0 \in S(D \times \Omega)$ be the first term. Then, we also determine a sequence $w_n : D \times \Omega \to E$, that approximates the mild fuzzy random solution of type 2 of Goursat problem $(1) - (2)$, by

$$w_n(t, s, \omega) \overset{D\mathbb{P}.1}{=} g(t, s, \omega) \ominus (-1) \int_0^t \int_0^s h_\omega(\nu, \tau, z_{n-1}(\nu, \tau, \omega)) d\nu d\tau, \quad n \geq 1. \tag{9}$$

Additionally, the approximate sequences of the partial gH-derivatives $(w_n)_t$ and $(w_n)_s$ are as follows

$$(w_n)_t(t, s, \omega) \overset{D\mathbb{P}.1}{=} g_t(t, s, \omega) \ominus (-1) \int_0^s h_\omega(t, \tau, z_{n-1}(t, \tau, \omega)) d\tau,$$
$$(w_n)_s(t, s, \omega) \overset{D\mathbb{P}.1}{=} g_s(t, s, \omega) \ominus (-1) \int_0^t h_\omega(\nu, s, z_{n-1}(\nu, s, \omega)) d\nu.$$

Hypothesis (A2): There exists a function $M : \Omega \to [0, \infty)$ such that for all $z := (w, w_t, w_s) \in [C^1(D \times \Omega, E)]^3$,

$$\rho(h_\omega(t, s, z(t, s, \omega)), \hat{0}) \overset{D\mathbb{P}.1}{\leq} M(\omega).$$

Lemma 3.5. *Under assumptions that the hypotheses* (**A1**) *and* (**A2**) *are fulfilled, the fuzzy-valued function* $w_n(\cdot,\cdot,\omega): D\to E$ *defined by* (8) *are continuous on* D *with* $\mathbb{P}.1$ *for each* $\omega\in\Omega$ *and* $n\in\mathbb{N}$.

Proof. Let $(t,s)\in D$ be fixed and consider a sequence $\{(t_m,s_m)\}_{m\geq 1}\subset\mathbb{R}^2$ converging to (t,s). For each $n\in\mathbb{N}$, it suffices to prove that $\rho(w_n(t_m,s_m,\omega),w_n(t,s,\omega))\to 0$ as $m\to\infty$. Indeed, for each $\omega\in\Omega$, we have

$$\begin{aligned}\rho(w_n(t_m,s_m,\omega),w_n(t,s,\omega)) &\overset{D\mathbb{P}.1}{\leq} \rho(g(t_m,s_m,\omega),g(t,s,\omega))\\ &+\rho\left(\int_0^{t_m}\int_0^{s_m} h_\omega\left(\nu,\tau,w_n,(w_n)_\nu,(w_n)_\tau\right)d\tau d\nu,\int_0^{t}\int_0^{s} h_\omega\left(\nu,\tau,w_n,(w_n)_\nu,(w_n)_\tau\right)d\tau d\nu\right)\\ &\overset{D\mathbb{P}.1}{=}\rho(g(t_m,s_m,\omega),g(t,s,\omega))+\mathcal{I}_m.\end{aligned}$$

Now, we consider four following cases:
Case 1: If $t<t_m$ and $s<s_m$, we have

$$\begin{aligned}\mathcal{I}_m &\overset{D\mathbb{P}.1}{\leq} \rho\left(\int_0^{t}\int_0^{s} h_\omega\left(\nu,\tau,w_n,(w_n)_\nu,(w_n)_\tau\right)d\tau d\nu,\int_0^{t}\int_0^{s} h_\omega\left(\nu,\tau,w_n,(w_n)_\nu,(w_n)_\tau\right)d\tau d\nu\right)\\ &+\rho\left(\int_t^{t_m}\int_0^{s} h_\omega(\nu,\tau,w_n,(w_n)_\nu,(w_n)_\tau)d\nu d\tau,\hat{0}\right)+\rho\left(\int_0^{t}\int_s^{s_m} h_\omega(\nu,\tau,w_n,(w_n)_\nu,(w_n)_\tau)d\nu d\tau,\hat{0}\right)\\ &+\rho\left(\int_t^{t_m}\int_s^{s_m} h_\omega(\nu,\tau,w_n,(w_n)_\nu,(w_n)_\tau)d\nu d\tau,\hat{0}\right)\\ &\overset{D\mathbb{P}.1}{\leq} M(\omega)\left[\int_t^{t_m}\int_0^{s}d\tau d\nu+\int_0^{t}\int_s^{s_m}d\tau d\nu+\int_t^{t_m}\int_s^{s_m}d\tau d\nu\right]\overset{D\mathbb{P}.1}{\leq} M(\omega)(t_m s_m-ts).\end{aligned}$$

By **(N2)**, it follows that $g(t,s,\omega)$ is a continuous fuzzy stochastic process and hence, $\rho(g(t_m,s_m,\omega),g(t,s,\omega))\to 0$ as $m\to\infty$. Thus, we directly obtain $\rho(w_n(t_m,s_m,\omega),w_n(t,s,\omega))\to 0$ as $m\to\infty$.
Case 2: If $t>t_m, s>s_m$ then by similar arguments, we also get $w_n(\cdot,\cdot,\omega)$ is continuous on D with $\mathbb{P}.1$ for each $\omega\in\Omega$.
Case 3: If $t<t_m, s>s_m$ then one gets

$$\begin{aligned}\mathcal{I}_m &\overset{D\mathbb{P}.1}{\leq} \rho\left(\int_0^{t}\int_0^{s_m} h_\omega(\nu,\tau,w_n,(w_n)_\nu,(w_n)_\tau)d\tau d\nu,\int_0^{t}\int_0^{s_m} h_\omega(\nu,\tau,w_n,(w_n)_\nu,(w_n)_\tau)d\tau d\nu\right)\\ &+\rho\left(\int_t^{t_m}\int_0^{s_m} h_\omega(\nu,\tau,w_n,(w_n)_\nu,(w_n)_\tau)d\tau d\nu,\int_0^{t}\int_{s_m}^{s} h_\omega(\nu,\tau,w_n,(w_n)_\nu,(w_n)_\tau)d\tau d\nu\right)\\ &\overset{D\mathbb{P}.1}{\leq} M(\omega)\left[\int_t^{t_m}\int_0^{s_m}d\tau d\nu+\int_0^{t}\int_{s_m}^{s}d\nu d\tau\right]\overset{D\mathbb{P}.1}{\leq} M(\omega)(t_m s_m-ts),\end{aligned}$$

which follows that $\rho(w_n(t_m,s_m,\omega),w_n(t,s,\omega))\to 0$ as $m\to\infty$. Thus, w_n is continuous with $\mathbb{P}.1$.
Case 4: The last case when $t>t_m$ and $s<s_m$ can be proved similarly. □

4 Solvability under generalized Lipschitz conditions

4.1 Generalized Lipschitz condition in Type 1

In this section, we consider the Goursat problem (1) − (2) under following generalized Lipschitz condition
Hypothesis (**A3**)**:** There exist $k>0$, $C>0$, $0<\sigma<1$, $\beta<\sigma$ such that $9k(1-\sigma)^2<(1-\beta)^2$ and the inequalities

$$\rho(h_\omega(t,s,z_1),h_\omega(t,s,z_2))\overset{D\mathbb{P}.1}{\leq}\frac{k}{ts}\left[\rho\left(\varphi_1,\varphi_2\right)+\frac{t}{\sqrt{k}}\rho(\psi_1,\psi_2)+\frac{s}{\sqrt{k}}\rho(\eta_1,\eta_2)\right], \tag{10}$$

$$\rho(h_\omega(t,s,z_1),h_\omega(t,s,z_2))\overset{D\mathbb{P}.1}{\leq}\frac{C}{t^\beta s^\beta}\left[\rho^\sigma(\varphi_1,\varphi_2)+t^\sigma\rho^\sigma(\psi_1,\psi_2)+s^\sigma\rho^\sigma(\eta_1,\eta_2)\right] \tag{11}$$

hold for all $(t,s)\in\hat{D}$, where $z_i(t,s,\omega)=(\varphi_i(t,s,\omega),\psi_i(t,s,\omega),\eta_i(t,s,\omega))\in[C^1(D\times\Omega,E)]^3$, $i=1,2$.

Definition 4.1. *Define a new generalized metric on* $S(D\times\Omega)$ *compatible with the generalized Lipschitz conditions*

$$d(w_1,w_2)=\sup_{\hat{D}\times\Omega}\frac{\rho(w_1,w_2)+\frac{t}{\sqrt{k}}\rho((w_1)_t,(w_2)_t)+\frac{s}{\sqrt{k}}\rho((w_1)_s,(w_2)_s)}{(ts)^{r\sqrt{k}}}, \tag{12}$$

where k *is defined in the hypothesis* ***(A3)*** *and* r *is the biggest number such that* $r^2k(1-\sigma)^2<(1-\beta)^2$.

Remark 4.1. *It should be noted that $d : S(D \times \Omega) \times S(D \times \Omega) \to [0, \infty]$ could get the infinite value and hence, NOT every pair of elements $w_1, w_2 \in S(D \times \Omega)$ necessarily has a finite distance.*

Lemma 4.2. $(S(D \times \Omega), d)$ *is a generalized complete metric space in sense of Luxemburg.*

Proof. Inheriting properties of ρ, the metric d satisfies all axioms of a metric. Now, let $\{w_n\}$ be a Cauchy sequence in $S(D \times \Omega)$. Then, for all $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $m, n > N$

$$d(w_m, w_n) = \sup_{\hat{D}\times\Omega} \frac{\rho(w_m, w_n) + \frac{t}{\sqrt{k}}\rho((w_m)_t, (w_n)_t) + \frac{s}{\sqrt{k}}\rho((w_m)_s, (w_n)_s)}{(ts)^{r\sqrt{k}}} < \varepsilon,$$

for all $(t, s, \omega) \in D \times \Omega$, that means the sequences $\{w_n\}_{n\geq 1}$, $\{(w_n)_t\}_{n\geq 1}$, $\{(w_n)_s\}_{n\geq 1}$ are Cauchy sequences on (E, ρ). Thus, there exist fuzzy-valued functions $v_1,\ v_2,\ v_3 \in S(D \times \Omega)$ such that $\{w_n\}_{n\geq 1}$, $\{(w_n)_t\}_{n\geq 1}$ and $\{(w_n)_s\}_{n\geq 1}$ uniformly converge to these sequences, respectively. In addition, by Lemma 2.1, it yields $w_t = v_2$ and $w_s = v_3$. Thus, we immediately get that $\lim_{n\to\infty} d(w_n, v_1) = 0$, which completes the proof. $\square$

Theorem 4.3. *If the hypotheses* (**A1**), (**A2**) *and* (**A3**) *are fulfilled then the Goursat problem (1) − (2) has a unique mild fuzzy random solution in type 1. Moreover, the approximate solution sequence, defined by* (8)*, uniformly converges to this unique solution on W.*

Proof. Our aim is to prove the existence and uniqueness of the type 1 - mild fuzzy random solution w^*. For this aim, we will build an operator $\mathscr{F}$ that is known as solution operator corresponding to the Goursat problem (1) − (2) and show that the operational equation $\mathscr{F}[x] = x$ has a unique solution $w^* \in S(D \times \Omega)$ by using Theorem 2.2. Indeed, by Lemma 3.2 and Lemma 3.5, it implies that the mappings $w_n(t, s, \omega)$, $(w_n)_t(t, s, \omega)$ and $(w_n)_s(t, s, \omega)$ are continuous fuzzy stochastic processes for all $(t, s, \omega) \in D \times \Omega$ and $n \in \mathbb{N}$. Define an operator $\mathscr{F} : S(D \times \Omega) \to S(D \times \Omega)$ by

$$\mathscr{F}[w](t, s, \omega) \overset{D\mathbb{P}.1}{=} g(t, s, \omega) + \int_0^t \int_0^s h_\omega(\nu, \tau, w(\nu, \tau, \omega), w_\nu(\nu, \tau, \omega), w_\tau(\nu, \tau, \omega))d\nu d\tau.$$

Step 1. The operator $\mathscr{F}$ maps $S(D \times \Omega)$ into itself. Indeed, for each $w \in S(D \times \Omega)$, since h_ω is continuous on W, $\int_0^t \int_0^s h_\omega(\nu, \tau, z(\nu, \tau, \omega))d\nu d\tau \in C^1(D \times \Omega)$ and $g \in C^1(D \times \Omega)$, we deduce that $\mathscr{F}[w] \in C^1(D \times \Omega)$. In addition,

$$\begin{cases} \mathscr{F}[w](t, 0, \omega) \overset{[0,c]\mathbb{P}.1}{=} g(t, 0, \omega) \overset{[0,c]\mathbb{P}.1}{=} \varphi(t, \omega) \\ \mathscr{F}[w](0, s, \omega) \overset{[0,b]\mathbb{P}.1}{=} g(0, s, \omega) \overset{[0,b]\mathbb{P}.1}{=} \psi(s, \omega), \end{cases}$$

which proves that $\mathscr{F}[w] \in S(D \times \Omega)$.

Step 2. We rewrite Picard approximate sequence (8) as $w_n \overset{D\mathbb{P}.1}{=} \mathscr{F}[w_{n-1}]$ and the approximate sequences for partial derivative of w_n as $(w_n)_t \overset{D\mathbb{P}.1}{=} (\mathscr{F}[w_{n-1}])_t$ and $(w_n)_s \overset{D\mathbb{P}.1}{=} (\mathscr{F}[w_{n-1}])_s$, respectively. Now, we will apply Theorem 2.2 to show the unique existence of fixed point of $\mathscr{F}$:

Condition (1^0)**:** Let $w_1, w_2 \in S(D \times \Omega)$ such that $d(w_1, w_2) < \infty$. Then, by the inequality (10), we have

$$\begin{aligned} \rho(\mathscr{F}[w_1](t, s, \omega), \mathscr{F}[w_2](t, s, \omega)) &\overset{D\mathbb{P}.1}{\leq} \int_0^t \int_0^s \rho(h_\omega(\nu, \tau, z_1(\nu, \tau, \omega)), h_\omega(\nu, \tau, z_2(\nu, \tau, \omega)))d\nu d\tau \\ &\overset{D\mathbb{P}.1}{\leq} k \int_0^t \int_0^s \frac{\rho(w_1, w_2) + \frac{\nu}{\sqrt{k}}\rho((w_1)_\nu, (w_2)_\nu) + \frac{\tau}{\sqrt{k}}\rho((w_1)_\tau, (w_2)_\tau)}{(\nu\tau)^{r\sqrt{k}}} (\nu\tau)^{r\sqrt{k}-1} d\nu d\tau. \end{aligned}$$

Using Definition 4.1, it yields

$$\rho(\mathscr{F}[w_1](t, s, \omega), \mathscr{F}[w_2](t, s, \omega)) \overset{D\mathbb{P}.1}{\leq} \left(k \int_0^t \int_0^s (\nu\tau)^{r\sqrt{k}-1} d\nu d\tau\right) d(w_1, w_2) \overset{D\mathbb{P}.1}{\leq} \frac{(ts)^{r\sqrt{k}}}{r} d(w_1, w_2).$$

Similarly, we also obtain

$$\begin{aligned}
&\frac{t}{\sqrt{k}}\rho\left(\frac{\partial}{\partial t}\mathscr{F}[w_1](t,s,\omega),\frac{\partial}{\partial t}\mathscr{F}[w_2](t,s,\omega)\right)\\
&\quad\overset{D\mathbb{P}.1}{\leq} t\sqrt{k}\int_0^s\frac{\rho(w_1,w_2)+\frac{t}{\sqrt{k}}\rho((w_1)_t,(w_2)_t)+\frac{\tau}{\sqrt{k}}\rho((w_1)_\tau,(w_2)_\tau)}{(t\tau)^{r\sqrt{k}}}(t\tau)^{r\sqrt{k}-1}d\tau \overset{D\mathbb{P}.1}{\leq} \frac{(ts)^{r\sqrt{k}}}{r}d(w_1,w_2),\\
&\frac{s}{\sqrt{k}}\rho\left(\frac{\partial}{\partial s}\mathscr{F}[w_1](t,s,\omega),\frac{\partial}{\partial s}\mathscr{F}[w_2](t,s,\omega)\right)\\
&\quad\overset{D\mathbb{P}.1}{\leq} s\sqrt{k}\int_0^t\frac{\rho(w_1,w_2)+\frac{\nu}{\sqrt{k}}\rho((w_1)_\nu,(w_2)_\nu)+\frac{s}{\sqrt{k}}\rho((w_1)_s,(w_2)_s)}{(\nu s)^{r\sqrt{k}}}(\nu s)^{r\sqrt{k}-1}d\nu \overset{D\mathbb{P}.1}{\leq} \frac{(ts)^{r\sqrt{k}}}{r}d(w_1,w_2).
\end{aligned}$$

Next, applying the formula (12) for $\mathscr{F}[w_1](t,s,\omega)$ and $\mathscr{F}[w_2](t,s,\omega)$, we directly get

$$d(\mathscr{F}[w_1](t,s,\omega),\mathscr{F}[w_2](t,s,\omega)) \overset{D\mathbb{P}.1}{\leq} \frac{d(w_1,w_2)}{r}+\frac{d(w_1,w_2)}{r}+\frac{d(w_1,w_2)}{r} \overset{D\mathbb{P}.1}{\leq} \frac{3}{r}d(w_1,w_2).$$

By assumption (**A3**), it implies that $3<r<\frac{1}{\sqrt{k}}\frac{1-\beta}{1-\sigma}$, or equivalently, $0<\frac{3}{r}<1$. Hence, the condition 1^0) holds.

Condition (2^0)**:** For each $(t,s,\omega)\in D\times\Omega$, two first terms of the sequence $\{w_n\}_{n\geq 1}$ satisfy

$$\begin{aligned}
\rho(w_2,w_1) &\overset{D\mathbb{P}.1}{\leq} \int_0^t\int_0^s \rho(h_\omega(\nu,\tau,w_1,(w_1)_n u,(w_1)_\tau,h_\omega(\nu,\tau,w_0,(w_0)_\nu,(w_0)_\tau))d\nu d\tau\\
&\overset{D\mathbb{P}.1}{\leq} \int_0^t\int_0^s 2M(\omega)d\nu d\tau \overset{D\mathbb{P}.1}{\leq} 2M(\omega)st.\\
\rho((w_2)_t,(w_1)_t) &\overset{D\mathbb{P}.1}{\leq} \int_0^s \rho(h_\omega(t,\tau,w_1,(w_1)_t,(w_1)_\tau,h_\omega(t,\tau,w_0,(w_0)_t,(w_0)_\tau))d\tau\\
&\overset{D\mathbb{P}.1}{\leq} \int_0^s 2M(\omega)d\tau \overset{D\mathbb{P}.1}{\leq} 2M(\omega)s.
\end{aligned}$$

By doing similar arguments, we also get $\rho((w_2)_s,(w_1)_s) \overset{D\mathbb{P}.1}{\leq} 2M(\omega)t$. Next, by using formula (11), we receive

$$\begin{aligned}
\rho(w_3,w_2) &\overset{D\mathbb{P}.1}{\leq} \int_0^t\int_0^s \frac{\lambda}{\nu^\beta\tau^\beta}[\rho^\sigma(w_2,w_1)+s^\sigma\rho^\sigma((w_2)_\nu,(w_1)_\nu)+t^\sigma\rho^\sigma((w_2)_\tau,(w_1)_\tau)]d\nu d\tau\\
&\overset{D\mathbb{P}.1}{\leq} \lambda\int_0^t\int_0^s\frac{(2M(\omega)\nu\tau)^\sigma+\tau^\sigma(2M(\omega)\nu)^\sigma+\nu^\sigma(2M(\omega)\tau)^\sigma}{(\nu\tau)^\beta}d\nu d\tau\\
&\overset{D\mathbb{P}.1}{\leq} 3\lambda\int_0^t\int_0^s\frac{(2M(\omega))^\sigma(\nu\tau)^\sigma}{(\nu\tau)^\beta}dsdt \overset{D\mathbb{P}.1}{\leq} 3\lambda(2M(\omega))^\sigma(ts)^{\sigma-\beta}(ts).
\end{aligned}$$

Additionally, we also get $\rho((w_3)_t,(w_2)_t) \overset{D\mathbb{P}.1}{\leq} 3\lambda(2M(\omega))^\sigma(ts)^{\sigma-\beta}s$ and $\rho((w_3)_s,(w_2)_s) \overset{D\mathbb{P}.1}{\leq} 3\lambda(2M(\omega))^\sigma(ts)^{\sigma-\beta}t$. Then, by using induction principle, we receive

$$\begin{aligned}
\rho(w_{n+3},w_{n+2}) &\overset{D\mathbb{P}.1}{\leq} (3\lambda)^{1+\sigma+\ldots+\sigma^n}(2M(\omega))^{\sigma^{n+1}}(ts)^{(\sigma-\beta)(1+\sigma+\ldots+\sigma^n)}(ts),\\
\rho\left((w_{n+3})_t,(w_{n+2})_t\right) &\overset{D\mathbb{P}.1}{\leq} (3\lambda)^{1+\sigma+\ldots+\sigma^n}(2M(\omega))^{\sigma^{n+1}}(ts)^{(\sigma-\beta)(1+\sigma+\ldots+\sigma^n)}s,\\
\rho\left((w_{n+3})_s,(w_{n+2})_s\right) &\overset{D\mathbb{P}.1}{\leq} (3\lambda)^{1+\sigma+\ldots+\sigma^n}(2M(\omega))^{\sigma^{n+1}}(ts)^{(\sigma-\beta)(1+\sigma+\ldots+\sigma^n)}t.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\rho(w_{n+3},w_{n+2})+\frac{t}{\sqrt{k}}\rho\left((w_{n+3})_t,(w_{n+2})_t\right)+\frac{s}{\sqrt{k}}\rho\left((w_{n+3})_s,(w_{n+2})_s\right)\\
&\qquad\overset{D\mathbb{P}.1}{\leq} (3\lambda)^{1+\sigma+\ldots+\sigma^n}(2M(\omega))^{\sigma^{n+1}}(ts)^{(\sigma-\beta)(1+\sigma+\ldots+\sigma^n)+1}\left(1+\frac{2}{\sqrt{k}}\right).
\end{aligned}$$

On the other hand, note that $(\sigma-\beta)(1+\sigma+\ldots+\sigma^n)+1=\frac{1-\beta}{1-\sigma}(1-\sigma^{n+1})+\sigma^{n+1}$ and $r^2k(1-\sigma)^2<(1-\beta)^2$. Hence, there exists $N'(k)\in\mathbb{N}$ such that for all $n\geq N'(k)$, we have $(\sigma-\beta)(1+\sigma+\ldots+\sigma^n)+1>r\sqrt{k}$, that means $d(w_{n+3},w_{n+2})<\infty$, or equivalently, $d(w_{n+1},w_n)<\infty$ for all $n\geq N(k)=N'(k)+2$. Thus, we can conclude that

$$d(w_N,w_{N+p})\leq d(w_N,w_{N+1})+d(w_{N+1},w_{N+2})+\ldots+d(w_{N+p-1},w_{N+p})\leq\infty \quad \text{for all } p\in\mathbb{N},$$

which means the condition 2^0) is satisfied.

Condition (3^0)**:** Assume that $w,\overline{w}\in S(D\times\Omega)$ are two fixed points of the operator $\mathscr{F}$, i.e. $\mathscr{F}[w]\overset{D\mathbb{P}.1}{=}w$ and $\mathscr{F}[\overline{w}]\overset{D\mathbb{P}.1}{=}\overline{w}$. Then, by similar method as in the proof of condition 2^0), we receive

$$\rho(w,\overline{w})+\frac{t}{\sqrt{k}}\rho(w_t,\overline{w}_t)+\frac{s}{\sqrt{k}}\rho(w_s,\overline{w}_s)\overset{D\mathbb{P}.1}{\leq}(3\lambda)^{\frac{1-\sigma^{n+1}}{1-\sigma}}(2M(\omega)ts)^{\sigma^{n+1}}(ts)^{\frac{1-\beta}{1-\sigma}(1-\sigma^{n+1})}\left(1+\frac{2}{\sqrt{k}}\right),$$

which follows that $d(w,v)<\infty$ and condition 3^0) holds. Finally, Theorem 2.2 can be applied to guarantee the existence and uniqueness of mild fuzzy random solution in type 1 of the Goursat problem. $\square$

In following, we give an example to illustrate the effectiveness of Theorem 4.3.

Example 4.4. *Assume that $\Omega=[0,1]$, $\mathcal{F}$ is a Borel $\sigma-$ field on Ω and $\mathbb{P}$ is Lebesgue measure on $(\Omega,\mathcal{F})$. Denote the functional space $W=D\times[C^1(D\times\Omega,E)]^3$ with $D=[0,a]\times[1,b]$. We consider the Goursat problem for following fuzzy random partial differential equation*

$$\mathcal{D}^{\sigma}_{ts}w(t,s,\omega)\overset{D\mathbb{P}.1}{=}\frac{t}{3\sqrt{2Ra}\sqrt[4]{ts}}w_t(t,s,\omega), \tag{13}$$

where $\mathcal{D}^{\sigma}_{ts}w$ is gH-derivatives of w in type $\sigma\in\{1,2\}$ and $t\in[0,a], s\in[1,b]$, $a>0, b>1$.

For each $(t,s,\omega)\in D\times\Omega$, let $h_\omega(t,s,w,w_t,w_s)=\frac{t}{3\sqrt{2Ra}\sqrt[4]{ts}}w_t(t,s,\omega)$. It is easy to see that $h_\omega:W\to E$ is a fuzzy stochastic process for all $(t,s)\in D$ and $h_\omega(\cdot,\cdot,z(\cdot,\cdot,\omega))$ is a continuous fuzzy-valued mapping with $\mathbb{P}.1$ for each $\omega\in\Omega$. Moreover, for all $w\in C^1_r(D,E):=\{w\in C^1(D\times\Omega,E):\rho(w_t,\hat{0})\leq r\}$, we have

$$\rho\left(h_\omega(t,s,w,w_t,w_s),\hat{0}\right)\overset{D\mathbb{P}.1}{\leq}\frac{t}{3\sqrt{2ra}\sqrt[4]{ts}}\rho(w_t,\hat{0})\overset{D\mathbb{P}.1}{\leq}\frac{t^{\frac{3}{4}}r}{3\sqrt{2ra}\sqrt[4]{s}}\overset{D\mathbb{P}.1}{\leq}\frac{r}{3\sqrt{2ra}}=M(\omega).$$

Thus, we have

$$\begin{aligned}\rho(h_\omega(t,s,u,u_t,u_s),h_\omega(t,s,w,w_t,w_s))&\overset{D\mathbb{P}.1}{\leq}\frac{t}{3\sqrt{2Ra}\sqrt[4]{ts}}\rho(u_t,w_t)\overset{D\mathbb{P}.1}{=}\frac{1}{3\sqrt{2Ra}\sqrt[4]{ts}}\sqrt{t\rho(u_t,w_t)}\sqrt{t\rho(u_t,w_t)}\\ \overset{D\mathbb{P}.1}{\leq}\frac{\sqrt{2Ra}}{3\sqrt{2Ra}\sqrt[4]{ts}}\sqrt{t\rho(u_t,w_t)}\overset{D\mathbb{P}.1}{\leq}\frac{1}{3\sqrt[4]{ts}}\sqrt{t\rho(u_t,w_t)}&\overset{D\mathbb{P}.1}{\leq}\frac{1}{3\sqrt[4]{ts}}\left(\sqrt{\rho(u,w)}+\sqrt{t\rho(u_t,w_t)}+\sqrt{s\rho(u_s,w_s)}\right).\end{aligned}$$

Therefore, we can see that hypotheses $(A1)-(A3)$ are satisfied. Thus, the Goursat problem for fuzzy random partial differential equation (13) has a unique integral solution in D.

4.2 Generalized Lipschitz condition of Type 2

By using another generalized Lipschitz conditions, we receive the existence and uniqueness of mild fuzzy random solutions of Goursat problem in a different generalized metric space of Luxemburg sense.

Hypothesis (A4): The fuzzy stochastic process $h_\omega(t,s,z(t,s,\omega))$ is continuous on W_1 and satisfies

$$\begin{aligned}&\rho(h_\omega(t,s,z(t,s,\omega)),\hat{0})\overset{D\mathbb{P}.1}{\leq}A(\omega)(ts)^p,\\ &\rho(h_\omega(t,s,z_1(t,s,\omega)),h_\omega(t,s,z_2(t,s,\omega)))\overset{D\mathbb{P}.1}{\leq}\frac{C(\omega)}{(ts)^r}(\rho^q(\varphi_1(t,s,\omega),\varphi_2(t,s,\omega))\\ &\qquad\qquad+t^q\rho^q(\psi_1(t,s,\omega),\psi_2(t,s,\omega)))+s^q\rho^q(\eta_1(t,s,\omega),\eta_2(t,s,\omega)),\end{aligned}$$

where (t,s,z), (t,s,z_1), $(t,s,z_2)\in W_1$, $p\geq 0$, $A(\omega)\overset{\mathbb{P}.1}{>}0$, $q\geq 1$, $c>0$ and $q(1+p)-r=p$.

Denote by $S^*(\hat{D}\times\Omega)=\Big\{w\in C^1(\hat{D}\times\Omega,E): w \text{ satisfies the condition } (2)\Big\}$ with the metric

$$\hat{d}(w,v)=\sup_{\hat{D}\times\Omega}\Big\{(ts)^{-p-1}e^{-\lambda(t+s)}\big(\rho(w,v)+t\rho(w_t,v_t)+s\rho(w_s,v_s)\big)\Big\}.$$

Then, the space $S^*(\hat{D}\times\Omega)$ endowed with $\hat{d})$ is a metric space. Moreover, we have following result:

Lemma 4.5. *The metric space $(S^*(\hat{D}\times\Omega),\hat{d})$ is a generalized metric space in sense of Luxemburg.*

Proof. It is inferred from definition of the metric $\hat{d}(\cdot,\cdot)$ and Lemma 4.2. □

Remark 4.2. *Note that the space $\mathscr{F}(S(\hat{D}\times\Omega))$ is a complete metric space endowed with the following metric*

$$\hat{\rho}(u,v)=\sup_{D\times\Omega}\{\rho(u,v)+\rho(u_t,v_t)+\rho(u_s,v_s)\}.$$

It is claimed that the space $S^*(\hat{D}\times\Omega)$ is a subset of the space $S(D\times\Omega)$. Indeed, for each $u\in S^*(\hat{D}\times\Omega)$, we have $u=\lim_{n\to\infty}\mathscr{F}[v_n]=\lim_{n\to\infty}w_n$ w.r.t. metric $\hat{\rho}$, where $\{v_n\}\subset S(D\times\Omega)$, $w_n=\mathscr{F}[v_n]$. Additionally, since $(S(D\times\Omega),\hat{\rho})$ is a complete metric space, we deduce that $u\in S(D\times\Omega)$. For all $u\in S^*(\hat{D}\times\Omega)$, we have $\mathscr{F}[u]\in\mathscr{F}(S(D\times\Omega))\subset S^*(\hat{D}\times\Omega)$. Hence, it yields $\mathscr{F}(S^*(\hat{D}\times\Omega))\subset S^*(\hat{D}\times\Omega)$.

Theorem 4.6. *If $(\mathbf{A1}),(\mathbf{A2})$ and $(\mathbf{A4})$ are fulfilled, there exists a unique mild fuzzy random solution of type 1 of the problem (1) − (2). Moreover, the successive approximate sequence (8) uniformly converges to this unique solution.*

Proof. Let $w_1=\mathscr{F}[w]$, $v_1=\mathscr{F}[v]\in S^*(\hat{D}\times\Omega)$ such that $\hat{d}(w_1,v_1)<\infty$. For each $(t,s,\omega)\in\hat{D}\times\Omega$, we have

$$\begin{aligned}\rho(w_1,v_1)&\overset{\hat{D}\mathbb{P}.1}{\le}\int_0^t\int_0^s[\rho(h_\omega(\tau,\nu,w,w_\nu,w_\tau),\hat{0})+\rho(h_\omega(\tau,\nu,v,v_\nu,v_\tau),\hat{0})]d\nu d\tau\\&\overset{\hat{D}\mathbb{P}.1}{\le}\int_0^t\int_0^s 2A(\omega)(\nu\tau)^p d\nu d\tau\overset{\hat{D}\mathbb{P}.1}{\le}\frac{2A(\omega)}{(p+1)^2}(ts)^{p+1}.\end{aligned}$$

Similarly, we also receive $t\rho((w_1)_t,(v_1)_t)\overset{\hat{D}\mathbb{P}.1}{\le}\frac{2A(\omega)}{(p+1)^2}(ts)^{p+1}$ and $s\rho((w_1)_s,(v_1)_s)\overset{\hat{D}\mathbb{P}.1}{\le}\frac{2A(\omega)}{(p+1)^2}(ts)^{p+1}$. Thus, for each $(t,s,\omega)\in\hat{D}\times\Omega$, we have

$$\begin{aligned}\rho\left(\mathscr{F}[w_1],\mathscr{F}[v_1]\right)&\overset{\hat{D}\mathbb{P}.1}{\le}C(\omega)\int_0^t\int_0^s\frac{\rho^q(w_1,v_1)+\tau^q\rho^q((w_1)_\tau,(v_1)_\tau)+\nu^q\rho^q((w_1)_\nu,(v_1)_\nu)}{(\nu\tau)^r}d\nu d\tau\\&\overset{\hat{D}\mathbb{P}.1}{\le}C(\omega)\int_0^t\int_0^s\frac{\rho^{q-1}(w_1,v_1)+\tau^{q-1}\rho^{q-1}((w_1)_\tau,(v_1)_\tau)+\nu^{q-1}\rho^{q-1}((w_1)_\nu,(v_1)_\nu)}{(\nu\tau)^{r-(p+1)}e^{-\lambda(\nu+\tau)}}\\&\qquad\times\frac{\rho(w_1,v_1)+\tau\rho((w_1)_\tau,(v_1)_\tau)+\nu\rho((w_1)_\nu,(v_1)_\nu)}{(\nu\tau)^{p+1}e^{\lambda(\nu+\tau)}}d\nu d\tau\\&\overset{\hat{D}\mathbb{P}.1}{\le}3C(\omega)\left(\frac{2A(\omega)}{(p+1)^2}\right)^{q-1}\hat{d}(w_1,v_1)\int_0^t\int_0^s(\nu\tau)^pe^{\lambda(\nu+\tau)}d\nu d\tau.\end{aligned}$$

Denote $N_1(\lambda)=\inf_{\hat{D}}(r+1+t\lambda)$ and $N_2(\lambda)=\inf_{\hat{D}}(r+1+s\lambda)$. Then, by using $\frac{\partial^2}{\partial t\partial s}[(ts)^{p+1}e^{\lambda(t+s)}]=(ts)^pe^{\lambda(t+s)}(p+1+t\lambda)(p+1+s\lambda)$, one gets

$$\int_0^t\int_0^s(\nu\tau)^pe^{\lambda(\nu+\tau)}d\nu d\tau=\int_0^t\int_0^s\frac{\frac{\partial^2}{\partial\tau\partial\nu}[(\tau\nu)^{p+1}e^{\lambda(\tau+\nu)}]}{(p+1+\nu\lambda)(p+1+\tau\lambda)}d\nu d\tau\le\frac{1}{N_1(\lambda)N_2(\lambda)}(ts)^{p+1}e^{\lambda(t+s)}.$$

Thus, we obtain $\rho\left(\mathscr{F}[w_1], \mathscr{F}[v_1]\right) \overset{\hat{D}\mathbb{P}.1}{\leq} \dfrac{3C(\omega)}{N_1(\lambda)N_2(\lambda)}\left(\dfrac{2A(\omega)}{(p+1)^2}\right)^{q-1}\hat{d}(w_1,v_1)(ts)^{p+1}e^{\lambda(t+s)}$. In addition, we also have

$$\begin{aligned}
t\rho\left((\mathscr{F}[w_1])_t, (\mathscr{F}[v_1])_t\right) &\overset{\hat{D}\mathbb{P}.1}{\leq} tC(\omega)\int_0^s \frac{\rho^{q-1}(w_1,v_1) + t^{q-1}\rho^{q-1}((w_1)_t,(v_1)_t) + \tau^{q-1}\rho^{q-1}((w_1)_\tau,(v_1)_\tau)}{(t\tau)^{r-(p+1)}e^{-\lambda(\tau+t)}} \\
&\qquad\qquad \times \frac{\rho(w_1,v_1) + t\rho((w_1)_t,(v_1)_t) + \tau\rho((w_1)_\tau,(v_1)_\tau)}{(t\tau)^{p+1}e^{\lambda(\nu+t)}}d\tau \\
&\overset{\hat{D}\mathbb{P}.1}{\leq} 3tC(\omega)\left(\frac{2A(\omega)}{p+1}\right)^{q-1}\hat{d}(w_1,v_1)\int_0^s (t\tau)^{(p+1)(q-1)-r+(p+1)}e^{\lambda(\tau+t)}d\tau \\
&\overset{\hat{D}\mathbb{P}.1}{\leq} 3tC(\omega)\left(\frac{2A(\omega)}{p+1}\right)^{q-1}\hat{d}(w_1,v_1)\int_0^s (t\tau)^{p}e^{\lambda(\tau+t)}d\tau. \qquad (14)
\end{aligned}$$

Note that $\displaystyle\int_0^s \tau^p e^{\lambda\tau}d\tau = \int_0^s \frac{1}{p+1+\tau\lambda}(p+1+\tau\lambda)\tau^p e^{\lambda\tau}d\tau \leq \frac{1}{N_2(\lambda)}s^{p+1}e^{\lambda s}$. Hence, the inequality (14) becomes

$$t\rho\left((\mathscr{F}[w_1])_t, (\mathscr{F}[v_1])_t\right) \overset{\hat{D}\mathbb{P}.1}{\leq} 3C(\omega)\left(\frac{2A(\omega)}{p+1}\right)^{q-1}\hat{d}(w_1,v_1)\frac{1}{N_2(\lambda)}(ts)^{p+1}e^{\lambda(t+s)}.$$

Similarly, we have $s\rho\left((\mathscr{F}[w_1])_s, (\mathscr{F}[v_1])_s\right) \overset{\hat{D}\mathbb{P}.1}{\leq} \dfrac{3C(\omega)(ts)^{p+1}e^{\lambda(t+s)}}{N_1(\lambda)}\left(\dfrac{2A(\omega)}{p+1}\right)^{q-1}\hat{d}(w_1,v_1)$. Then, we immediately get $\hat{d}\left(\mathscr{F}[w_1], \mathscr{F}[v_1]\right) \overset{\hat{D}\mathbb{P}.1}{\leq} \dfrac{9C(\omega)[2A(\omega)]^{q-1}}{(p+1)^{q-1}}\dfrac{1}{N(\lambda)}\hat{d}(w_1,v_1)$, where $N(\lambda) = \min\{N_1(\lambda), N_2(\lambda), N_1(\lambda)N_2(\lambda)\}$. Let $\lambda > 0$ be such that $\frac{9C(\omega)[2A(\omega)]^{q-1}}{(p+1)^{q-1}N(\lambda)} < 1$, that follows the condition 1^0) holds. The conditions 2^0) and 3^0) are implied from

$$\hat{d}(w_n, w_{n+1}) = \sup_{\hat{D}\times\Omega}\frac{\rho(w_n,w_{n+1}) + t\rho((w_n)_t,(w_{n+1})_t) + s\rho((w_n)_s,(w_{n+1})_s)}{(ts)^{p+1}e^{\lambda(t+s)}} \overset{\hat{D}\mathbb{P}.1}{\leq} \frac{6A(\omega)}{(p+1)e^{\lambda(t+s)}} < \infty. \qquad (15)$$

Therefore, the proof is completed. □

Example 4.7. *Assume that $\Omega = [0,1]$, $\mathcal{F}$ is a Borel $\sigma-$ field on Ω and $\mathbb{P}$ is Lebesgue measure on $(\Omega, \mathcal{F})$. Consider the Goursat problem for following fuzzy random partial differential equation*

$$\begin{cases}
\mathcal{D}^\sigma_{ts}w(t,s,\omega) \overset{D\mathbb{P}.1}{=} w(t,s,\omega), & (t,s)\in D = [0,2]\times[0,2], \\
w(t,0,\omega) \overset{[0,2]\mathbb{P}.1}{=} \tilde{a}e^t, \\
w(0,s,\omega) \overset{[0,2]\mathbb{P}.1}{=} \tilde{a}e^s,
\end{cases} \qquad (16)$$

where $\mathcal{D}^\sigma_{ts}w(t,s,\omega)$ is the gH-derivative of $w(t,s,\omega)$ in type $\sigma \in \{1,2\}$ and $\tilde{a} = (1,2,3)$. For simplicity, we denote $h_\omega(t,s,w(t,s,\omega)) = w(t,s,\omega)$, $\varphi(t,\omega) = \tilde{a}e^t$ and $\psi(s,\omega) = \tilde{a}e^s$. Additionally, for each $\theta \in [0,1]$, the $\theta-$level sets of $\varphi(t,\omega)$ and $\psi(s,\omega)$ are given by $[\varphi(t,\omega)]^\theta = [(1+\theta)e^t, (3-\theta)e^t]$ and $[\psi(s,\omega)]^\theta = [(1+\theta)e^s, (3-\theta)e^s]$. Therefore,

$$\left[\frac{\partial\varphi(t,\omega)}{\partial t}\right]^\theta = \left[(1+\theta)e^t, (3-\theta)e^t\right] \quad \textit{and} \quad \left[\frac{\partial\psi(s,\omega)}{\partial s}\right]^\theta = \left[(1+\theta)e^s, (3-\theta)e^s\right],$$

which means that $\varphi(t,\omega)$ and $\psi(s,\omega)$ are partial (i)-gH-differentiable w.r.t. variables t and s. Next, we can see that the function $h_\omega : D\times[C^1(D\times\Omega)]^3 \to E$ is a fuzzy stochastic process and $h_\omega(\cdot,\cdot,w(\cdot,\cdot,\omega)) : [0,2]\times[0,2] \to E$ is a continuous on D with $\mathbb{P}.1$. Denote $\mathcal{C}_r(D,E) = \{w(\cdot,\cdot,\omega) \in \mathcal{C}(D,E) : \rho(w(t,s,\omega),\hat{0}) \leq r\}$. Then, we directly obtain

$$\rho(h_\omega(t,s,w(t,s,\omega)),\hat{0}) \overset{D\mathbb{P}.1}{=} \rho(w(t,s,\omega),\hat{0}) \overset{D\mathbb{P}.1}{\leq} r(ts)^0 \overset{D\mathbb{P}.1}{=} A(\omega)(ts)^0.$$

In addition, for each w, $u \in \mathcal{C}_r$ and $(t,s) \in D$, we also have

$$\begin{aligned}
\rho(h_\omega(t,s,w(t,s,\omega)), h_\omega(t,s,u(t,s,\omega))) &\overset{D\mathbb{P}.1}{=} \rho(w(t,s,\omega),u(t,s,\omega)) \overset{D\mathbb{P}.1}{\leq} \frac{4}{ts}\rho(w(t,s,\omega),u(t,s,\omega)) \\
&\overset{D\mathbb{P}.1}{\leq} \frac{4}{ts}\left(\rho(w(t,s,\omega),u(t,s,\omega)) + t\rho(w_t(t,s,\omega),u_t(t,s,\omega)) + s\rho(w_s(t,s,\omega),u_s(t,s,\omega))\right).
\end{aligned}$$

where $C(\omega) = 4$, $p = 0$ *and* $q = r = 1$. *Therefore, the Goursat problem (16) has a unique solution of type 1 in* D. *Moreover, the mild fuzzy random of the problem (16) can be shown in Figure 1.*

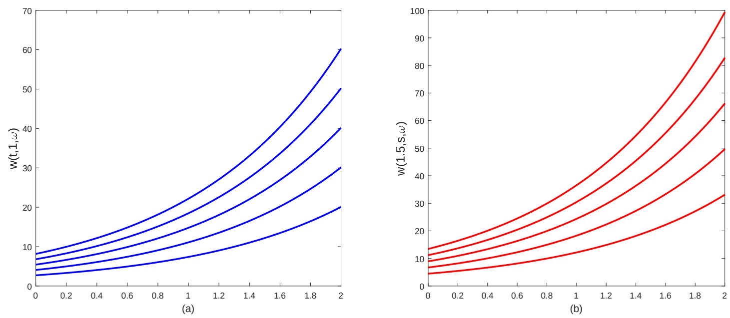

Figure 1: Some trajectories of mild fuzzy random solution in type 1 with $s = 1$ (Fig. 1(a)) and $t = 1.5$ (Fig. 1(b))

5 The existence and uniqueness of mild fuzzy random solution of type 2

Denote by $\widehat{S}(D \times \Omega) = \{w \in S(D \times \Omega, E):$ the gH-differences in (9) exist$\}$ and $\widehat{T}$ is an operator defined by

$$\widehat{T}w(t,s,\omega) \overset{D\mathbb{P}.1}{=} g(t,s,\omega) \ominus (-1)\int_0^t \int_0^s h_\omega(\nu, \tau w(\nu,\tau,\omega), w_\nu(\nu,\tau,\omega), w_\tau(\nu,\tau,\omega))d\nu d\tau. \tag{17}$$

Denote by $\widehat{S^*}(D \times \Omega)$ the space obtained from the completion of $(\widehat{T}\widehat{S}(D \times \Omega), \hat{\rho})$ in sense of $\hat{\rho}$ defined in Remark 4.2.

Lemma 5.1. *The metric space* $(\widehat{S^*}(D \times \Omega), \hat{\rho})$ *is a complete generalized metric space.*

Proof. Let $\{w_n\} \subset \widehat{S^*}(D \times \Omega)$ be Cauchy sequence. Then, it is also Cauchy sequence in $S(D \times \Omega)$, which means that w_n uniformly converges to $u \in S(D \times \Omega)$. Now, our aim is to prove that $u \in \widehat{S^*}(D \times \Omega)$. Indeed, denote

$$G(w)(t,s,\omega) = (-1)\int_0^t \int_0^s h_\omega(\nu, \tau, w(\nu,\tau,\omega), w_\nu(\nu,\tau,\omega), w_\tau(\nu,\tau,\omega))d\nu d\tau.$$

Consequently, we have $G(w_n)(t,s,\omega) = (-1)\int_0^t \int_0^s h_\omega(\nu,\tau, w_n(\nu,\tau,\omega), (w_n)_\nu(\nu,\tau,\omega), (w_n)_\tau(\nu,\tau,\omega))d\nu d\tau$. Next, for all $(t,s,\omega) \in D \times \Omega$, we have

$$\begin{cases} \text{len}[g(t,s,\omega)]^\theta \geq \text{len}[G(w_n)(t,s,\omega)]^\theta & \text{for all } \theta \in [0,1], \\ (g(t,s,\omega))^\theta_- - (G(w_n)(t,s,\omega))^\theta_- & \text{is nondecreasing w.r.t } \theta, \\ (g(t,s,\omega))^\theta_+ - (G(w_n)(t,s,\omega))^\theta_+ & \text{is nonincreasing w.r.t. } \theta. \end{cases}$$

Since h_ω is continuous and $w_n \rightrightarrows u$, it follows $\text{len}\left[\int_0^t \int_0^s h_\omega(\nu,\tau, w_n(\nu,\tau,\omega), (w_n)_\nu(\nu,\tau,\omega), (w_n)_\tau(\nu,\tau,\omega))d\nu d\tau\right]^\theta$ converges to $\text{len}\left[\int_0^t \int_0^s h_\omega(\nu,\tau, w(\nu,\tau,\omega), w_\nu(\nu,\tau,\omega), w_\tau(\nu,\tau,\omega))d\nu d\tau\right]^\theta$ as $n \to \infty$ for all $\theta \in [0,1]$, which means that $\text{len}\left[G(w_n)(t,s,\omega)\right]^\theta \rightrightarrows \text{len}\left[G(w)(t,s,\omega)\right]^\theta$, and hence, we have $\text{len}\left[g(t,s,\omega)\right]^\theta \geq \text{len}\left[G(w)(t,s,\omega)\right]^\theta$. Moreover, for all $0 \leq \theta \leq \alpha' \leq 1$, we have $\left[(g(t,s,\omega))^\theta_- - (G(w_n)(t,s,\omega))^\theta_-\right] \leq \left[(g(t,s,\omega))^{\alpha'}_- - (G(w_n)(t,s,\omega))^{\alpha'}_-\right]$. Then, let $n \to \infty$, one gets $\left[(g(t,s,\omega))^\theta_- - (G(w)(t,s,\omega))^\theta_-\right] \leq \left[(g(t,s,\omega))^\gamma_- - (G(w)(t,s,\omega))^{\alpha'}_-\right]$. By similar arguments, we also receive $\left[(g(t,s,\omega))^\theta_+ - (G(w)(t,s,\omega))^\theta_+\right] \geq \left[(g(t,s,\omega))^{\alpha'}_+ - (G(w)(t,s,\omega))^{\alpha'}_+\right]$. Therefore, the difference $g(t,s,\omega) \ominus (-1)\displaystyle\int_0^t \int_0^s h_\omega(\nu,\tau, w(\nu,\tau,\omega), w_\nu(\nu,\tau,\omega), w_\tau(\nu,\tau,\omega))d\tau d\nu$ exists and hence, it implies $u \in \widehat{S^*}(D \times \Omega)$, or equivalently, $\widehat{S^*}(D \times \Omega)$ is closed in $S(D \times \Omega)$. Since the space $S(D \times \Omega)$ is complete then $\widehat{S^*}(D \times \Omega)$ is also complete. □

Theorem 5.2. *Assume that* $\widehat{S^*}(D \times \Omega) \neq \emptyset$, $\widehat{T}(\widehat{S^*}(D \times \Omega)) \subset \widehat{S^*}(D \times \Omega)$ *and the hypotheses* (**A1**), (**A2**), (**A3**) *are satisfied. Then, the Goursat problem (1) − (2) has a unique mild fuzzy random solution of type 2. The conclusion still holds if we replace condition* (**A3**) *by* (**A4**).

Proof. By similar arguments as in Theorem 4.3, we also have $\widehat{T}(\widehat{S}(D\times\Omega))\subset\widehat{S}(D\times\Omega)$ and the mapping $\widehat{T}$ satisfies all conditions 1^0), 2^0) and 3^0) of Theorem 2.2. Since $\rho(w\ominus v, w\ominus e)\leq\rho(w,\omega)+\rho(v,e)$, the formula (17) follows that

$$\rho\left(\widehat{T}w_1,\widehat{T}w_2\right)\overset{\hat{D}\mathbb{P}.1}{\leq}\int_0^t\int_0^s\rho(h_\omega(\nu,\tau,w_1,(w_1)_\nu,(w_1)_\tau),h_\omega(\nu,\tau,w_2,(w_2)_\nu,(w_2)_\tau))d\nu d\tau.$$

Next, by similar arguments, we also receive $d(\widehat{T}w_1,\widehat{T}w_2)\leq\frac{3}{r}d(w_1,w_2)$ with $0<\dfrac{3}{r}<1$. The condition 2^0) is also achieved based on the same approach as in Theorem 4.3. Next, assume that u, $v\in\widehat{S}(D\times\Omega)$ are two fixed points of the operator $\widehat{T}$. Then, we have

$$\rho(w,v)+\frac{t}{\sqrt{k}}\rho(w_t,v_t)+\frac{s}{\sqrt{k}}\rho(w_s,v_s)\overset{D\mathbb{P}.1}{\leq}(3\lambda)^{\frac{1-\sigma^{n+1}}{1-\sigma}}(2M(\omega)ts)^{\sigma^{n+1}}(ts)^{\frac{1-\beta}{1-\sigma}(1-\sigma^{n+1})}\left(1+\frac{2}{\sqrt{k}}\right)$$

which implies that $d(w,v)<\infty$ and the condition (3^0) is proved. Therefore, the operator $\widehat{T}$ has a unique fixed point by Luxemburg theorem. The proof is completed.

If the hypothesis **(A3)** is replaced by **(A4)** then since $\widehat{S^*}(D\times\Omega)\subset\widehat{S}(D\times\Omega)$, we deduce that $\widehat{T}$ maps $\widehat{S^*}(D\times\Omega)$ into itself. Next, let w_1, $z_2\in\widehat{S^*}(D\times\Omega)$ be arbitrary such that $\hat{d}(w_1,z_2)<\infty$. Then, according to the estimations $\rho(w,v)\overset{\hat{D}\mathbb{P}.1}{\leq}\dfrac{2A(\omega)}{p+1}(ab)^{p+1}$, $\rho(w_t,v_t)\overset{\hat{D}\mathbb{P}.1}{\leq}\dfrac{2A(\omega)}{p+1}a^pb^{p+1}$, $\rho(w_s,v_s)\overset{\hat{D}\mathbb{P}.1}{\leq}\dfrac{2A(\omega)}{p+1}a^{p+1}b^p$, and by using the same approach as in Theorem 4.6, we get

$$\rho(\widehat{T}w_1,\widehat{T}w_2)\overset{\hat{D}\mathbb{P}.1}{\leq}3C(\omega)\left(\frac{2A(\omega)}{p+1}\right)^{q-1}\hat{d}(w_1,w_2)\frac{1}{N_1(\lambda)N_2(\lambda)}(ts)^{p+1}e^{\lambda(t+s)}$$

$$t\rho((\widehat{T}w_1)_t,(\widehat{T}w_2)_t)\overset{\hat{D}\mathbb{P}.1}{\leq}3C(\omega)\left(\frac{2A(\omega)}{p+1}\right)^{q-1}\hat{d}(w_1,w_2)\frac{1}{N_2(\lambda)}(ts)^{p+1}e^{\lambda(t+s)}$$

$$s\rho((\widehat{T}w_1)_s,(\widehat{T}w_2)_s)\overset{\hat{D}\mathbb{P}.1}{\leq}3C(\omega)\left(\frac{2A(\omega)}{p+1}\right)^{q-1}\hat{d}(w_1,w_2)\frac{1}{N_1(\lambda)}(ts)^{p+1}e^{\lambda(t+s)}.$$

Hence, it implies that $\hat{d}(\widehat{T}w_1,\widehat{T}w_2)\overset{\hat{D}\mathbb{P}.1}{\leq}\dfrac{9C(\omega)[2A(\omega)]^{q-1}}{(p+1)^{q-1}}\dfrac{1}{N(\lambda)}\hat{d}(w_1,w_2)$. Finally, by choosing λ large enough such that $\dfrac{9C(\omega)[2A(\omega)]^{q-1}}{(p+1)^{q-1}}\dfrac{1}{N(\lambda)}<1$, the condition ($1^0$) is proved. The rest of proof is to show that the conditions (2^0), (3^0) are fulfilled. Indeed, from (15), we have $\hat{d}(w_n,w_{n+1})\overset{\hat{D}\mathbb{P}.1}{\leq}\dfrac{6A(\omega)}{(p+1)e^{\lambda(t+s)}}<+\infty$ for all $n=1,2,\ldots$ Therefore, the operator $\widehat{T}$ has a unique fixed point. The proof is completed. □

Example 5.3. *In the following, we consider the Goursat problem for following fuzzy random partial differential equation*

$$\begin{cases}\mathcal{D}^\sigma_{ts}w(t,s,\omega)\overset{D\mathbb{P}.1}{=}w(t,s,\omega),\\ w(t,0,\omega)\overset{D\mathbb{P}.1}{=}\tilde{a}e^{-t},\\ w(0,s,\omega)\overset{D\mathbb{P}.1}{=}\tilde{a}e^{-s},\end{cases}\tag{18}$$

where $\mathcal{D}^\sigma_{ts}w(t,s,\omega)$ *is the gH-derivative of* $w(t,s,\omega)$ *of type* $\sigma\in\{1,2\}$ *and* $\tilde{b}=(2,3,5)$. *For simplicity, we denote* $h_\omega(t,s,w(t,s,\omega))=w(t,s,\omega)$, $\varphi(t,\omega)\overset{D\mathbb{P}.1}{=}\tilde{b}e^{-t}$ $\psi(s,\omega)=\tilde{b}e^{-s}$. *Here, for each* $\theta\in[0,1]$, *the* $\theta-$*level sets of* $\varphi(t,\omega)$ *and* $\psi(s,\omega)$ *can be given by* $[\varphi(t,\omega)]^\theta=[(2+\theta)e^{-t},(5-2\theta)e^{-t}]$ *and* $[\psi(s,\omega)]^\theta=[(2+\theta)e^{-s},(5-2\theta)e^{-s}]$. *Then, we have*

$$\left[\frac{\partial\varphi(t,\omega)}{\partial t}\right]^\theta=\left[(-5+2\theta)e^{-t},(-2-\theta)e^{-t}\right]\ \text{and}\ \left[\frac{\partial\psi(s,\omega)}{\partial s}\right]^\theta=\left[(-5+2\theta)e^{-s},(-2-\theta)e^{-s}\right],$$

that means $\varphi(t,\omega)$ *and* $\psi(s,\omega)$ *are gH-differentiable in type 2 w.r.t. variables* t *and* s. *Additionally, we can see that the function* $h_0(t,s,w(t,s,\cdot)):\Omega\to E$ *is a fuzzy stochastic process for each* $(t,s)\in D$ *and* $h_\omega(\cdot,\cdot,w(\cdot,\cdot,\omega)):[0,2]\times[0,2]\to E$ *is continuous with* $\mathbb{P}.1$. *Next, we denote*

$$\mathcal{C}_r(D,E)=\left\{w(\cdot,\cdot,\omega)\in\mathcal{C}(D,E):\rho(w(t,s,\omega),\hat{0})\leq r\right\}.$$

Therefore, we have $\rho(h_\omega(t,s,w(t,s,\omega)),\hat{0}) \overset{D\mathbb{P}.1}{=} A(\omega)(ts)^0$, *where* $A(\omega)=r$ *and* $p=0$. *Moreover, for each* $w,\ u\in\mathcal{C}_r$ *and* $(t,s)\in D$, *we also have*

$$\rho(h_\omega(t,s,w(t,s,\omega)),h_\omega(t,s,u(t,s,\omega))) \overset{D\mathbb{P}.1}{=} \rho(w(t,s,\omega),u(t,s,\omega)) \overset{D\mathbb{P}.1}{\leq} \frac{4}{ts}\rho(w(t,s,\omega),u(t,s,\omega))$$
$$\overset{D\mathbb{P}.1}{\leq} \frac{4}{ts}\left(\rho(w(t,s,\omega),u(t,s,\omega))+t\rho(w_t(t,s,\omega),u_t(t,s,\omega))+s\rho(w_s(t,s,\omega),u_s(t,s,\omega))\right),$$

where $C(\omega)=4$ *and* $q=r=1$. *Hence, since all assumptions of Theorem 5.2 are satisfied, it follows that the Goursat problem (18) has a unique solution of type 2 in* D. *Moreover, the graphical representation of mild fuzzy random solution is shown in Figure 2.*

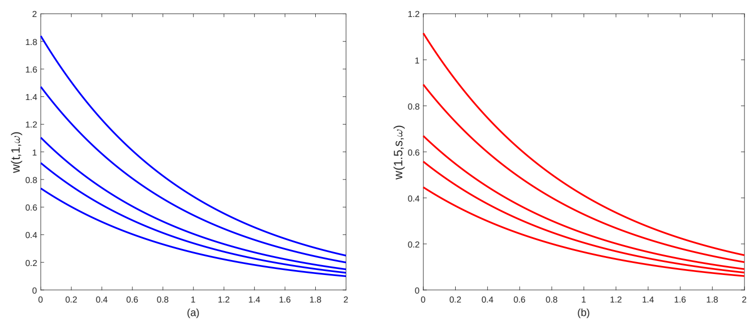

Figure 2: Some trajectories of mild fuzzy random solution in type 2 with $s=1$ (Fig. 2(a)) and $t=1.5$ (Fig. 2(b))

6 The boundedness of mild fuzzy random solutions

Under the generalized Lipschitz conditions, we investigate the boundedness of mild fuzzy random solutions.

Theorem 6.1. *Assume that the hypotheses* (**A1**), (**A2**) *and* (**A3**) *are satisfied. Then the unique mild fuzzy random solutions in both type 1 and type 2 of problem* (1) - (2) *satisfy the following estimation*

$$\rho(w(t,s,\omega),\hat{0}) \overset{D\mathbb{P}.1}{\leq} \rho(g(t,s,\omega),\hat{0})+M(\omega)(ts).$$

If we replace the hypothesis (**A3**) *by* (**A4**) *then* $\rho(w(t,s,\omega),\hat{0}) \overset{D\mathbb{P}.1}{\leq} \rho(g(t,s,\omega),\hat{0})+\dfrac{A(\omega)}{(p+1)^2}(ts)^{p+1}$.

Proof. No loss of generality, if w is a solutions of type 1 or type 2 of the problem (1) - (2) then we have

$$\rho(w(t,s,\omega),\hat{0}) \overset{D\mathbb{P}.1}{\leq} \rho(g(t,s,\omega),\hat{0})+\int_0^t\int_0^s \rho(h_\omega(\tau,\nu,w,w_\tau,w_\nu),\hat{0})d\nu d\tau.$$

By the hypothesis (**A3**), it implies that for both mild fuzzy random solutions of type 1 and type 2, we always receive $\rho(h_\omega(\nu,\tau,w,w_t,w_s),\hat{0}) \overset{D\mathbb{P}.1}{\leq} M(\omega)$. Thus, $\displaystyle\int_0^t\int_0^s \rho(h_\omega(\tau,\nu,w,w_\tau,w_\nu),\hat{0})d\nu d\tau \overset{D\mathbb{P}.1}{\leq} M(\omega)ts$, which proves the inequality of Theorem 6.1. In addition, it is implied directly from (**A4**) that $\rho(h_\omega(\tau,\nu,w,w_\tau,w_\nu),\hat{0}) \overset{D\mathbb{P}.1}{\leq} A(\omega)(ts)^p$. Therefore, we obtain $\rho(w(t,s,\omega),\hat{0}) \overset{D\mathbb{P}.1}{\leq} \rho(g(t,s,\omega),\hat{0})+\frac{A(\omega)}{(p+1)^2}(ts)^{p+1}$. □

Example 6.2. *Assume that* $\Omega=\mathbb{R}$, $\mathcal{F}$ *is a Borel* $\sigma-$ *field on* Ω *and* $\mathbb{P}$ *is Lebesgue measure on* $(\Omega,\mathcal{F})$. *Consider the Goursat problem for following fuzzy random partial differential equation (PDEs)*

$$\mathcal{D}^\sigma_{ts}w(t,s,\omega) \overset{D\mathbb{P}.1}{=} w(t,s,\omega), \qquad (t,s)\in D=[0,2]\times[0,2], \tag{19}$$

where $\mathcal{D}^\sigma_{ts}w(t,s,\omega)$ *is the gH-derivative of* $w(t,s,\omega)$ *of type* $\sigma\in\{1,2\}$, $\tilde{a}=(1,2,3)$ *and* $h_\omega(t,s,w(t,s,\omega))=w(t,s,\omega)$.

Case 1. *Consider the fuzzy random PDEs (19) with the initial conditions* $w(t,0,\omega) \overset{[0,2]\mathbb{P}.1}{=} \tilde{a}e^t$, $w(0,s,\omega) \overset{[0,2]\mathbb{P}.1}{=} \tilde{a}e^s$ *on* $D = [0,2] \times [0,2]$. *Then, the mild fuzzy random solution of type 1 of this problem can be given by*

$$w(t,s,\omega) \overset{D\mathbb{P}.1}{=} \tilde{a}e^t + \tilde{a}e^s \ominus \tilde{a} + \int_0^t \int_0^s w(\nu,\tau,\omega)d\nu d\tau \overset{D\mathbb{P}.1}{=} \left(e^t + e^s + 1\right)\tilde{a} + \int_0^t \int_0^s w(\nu,\tau,\omega)d\nu d\tau.$$

where $\varphi(t,\omega) := \tilde{a}e^t$ *and* $\psi(s,\omega) := \tilde{a}e^s$ *are gH-differentiable of type 1 fuzzy-valued functions. From Example 4.7, it follows that all assumptions* ***(A1)*** $-$ ***(A4)*** *are fulfilled. Hence, the Goursat problem for the fuzzy random PDEs (19) has a unique mild fuzzy random solution in type* 1. *Moreover, we immediately get*

$$\rho\left(w(t,s,\omega),\hat{0}\right) \overset{D\mathbb{P}.1}{\leq} \left(e^t + e^s + 1\right)\rho\left(\tilde{a},\hat{0}\right) + A(\omega)(ts) \overset{D\mathbb{P}.1}{\leq} \left(2e^2 + 1\right)\rho\left(\tilde{a},\hat{0}\right) + A(\omega)(ts)$$

that follows the boundedness of mild fuzzy random solution in type 1.

Case 2. *Consider the fuzzy random PDE (19) with the initial conditions* $w(t,0,\omega) \overset{[0,2]\mathbb{P}.1}{=} \tilde{a}e^{-t}$, $w(0,s,\omega) \overset{[0,2]\mathbb{P}.1}{=} \tilde{a}e^{-s}$ *on* $D = [0,2] \times [0,2]$. *Then, the mild fuzzy random solution of type 2 of this problem can be given by*

$$w(t,s,\omega) \overset{D\mathbb{P}.1}{=} \tilde{a}e^{-t} + \tilde{a}e^{-s} \ominus \tilde{a} \ominus (-1)\int_0^t \int_0^s w(\nu,\tau,\omega)d\nu d\tau.$$

where $\varphi(t,\omega) := \tilde{a}e^{-t}$ *and* $\psi(s,\omega) := \tilde{a}e^{-s}$ *are gH-differentiable in type 2 fuzzy-valued functions. It follows from Example 5.3 that all assumptions* ***(A1)*** $-$ ***(A4)*** *are fulfilled. Hence, the Goursat problem for the fuzzy random PDE (19) has a unique mild fuzzy random solution of type* 2. *Moreover, we immediately get*

$$\begin{aligned}\rho\left(w(t,s,\omega),\hat{0}\right) &\overset{D\mathbb{P}.1}{\leq} \left(e^{-t} + e^{-s} + 1\right)\rho\left(\tilde{a},\hat{0}\right) + \int_0^t \int_0^s \rho\left(w(\nu,\tau,\omega),\hat{0}\right)d\nu d\tau \\ &\overset{D\mathbb{P}.1}{\leq} \left(e^{-t} + e^{-s} + 1\right)\rho\left(\tilde{a},\hat{0}\right) + A(\omega)(ts) \overset{D\mathbb{P}.1}{\leq} \left(2e^{-2} + 1\right)\rho\left(\tilde{a},\hat{0}\right) + A(\omega)(ts)\end{aligned}$$

that follows the boundedness of mild fuzzy random solution in type 2.

Example 6.3. *Assume that* $\Omega = \mathbb{R}$, $\mathcal{F}$ *is a Borel* $\sigma-$ *field on* Ω *and* $\mathbb{P}$ *is Lebesgue measure on* $(\Omega,\mathcal{F})$. *Now, we consider the Goursat problem*

$$\mathcal{D}^{\sigma}_{ts}w(t,s,\omega) \overset{D\mathbb{P}.1}{=} w(t,s,\omega) - s,\ w(t,0,\omega) \overset{[0,2]\mathbb{P}.1}{=} \tilde{a}e^t,\ w(0,s,\omega) \overset{[0,2]\mathbb{P}.1}{=} \tilde{a}e^s + s,\ (t,s) \in D = [0,2]^2, \tag{20}$$

where $\mathcal{D}^{\sigma}_{ts}w(t,s,\omega)$ *is the gH-derivative of* $w(t,s,\omega)$ *of type* $\sigma \in \{1,2\}$ *and* $\tilde{a} = (1,2,3)$ *is a triangular fuzzy number. For convenience, we denote* $h_\omega(t,s,w(t,s,\omega)) = w(t,s,\omega) - s$, $\varphi(t,\omega) = \tilde{a}e^t$ *and* $\psi(s,\omega) = \tilde{a}e^s + s$.

By doing similar arguments as in Example 4.7, we also obtain $\varphi(t,\omega)$ *and* $\psi(s,\omega)$ *are gH-differentiable in type 1 w.r.t. variables t and s. Next, it is easy to see that the function* $h_0(t,s,w(t,s,\cdot)) : \Omega \to E$ *is a fuzzy stochastic process for all* $(t,s) \in D$ *and* $h_\omega(\cdot,\cdot,w(\cdot,\cdot,\omega)) : [0,2] \times [0,2] \to E$ *is a continuous fuzzy-valued function with* $\mathbb{P}.1$. *Next, on the functional space* $\mathcal{C}_r(D,E) = \{w(\cdot,\cdot,\omega) \in \mathcal{C}(D,E) : \rho(w(t,s,\omega),\hat{0}) \leq r\}$, *we have*

$$\rho(h_\omega(t,s,w(t,s,\omega)),\hat{0}) \overset{D\mathbb{P}.1}{=} \rho(w(t,s,\omega) - s,\hat{0}) \overset{D\mathbb{P}.1}{\leq} \rho(w(t,s,\omega),\hat{0}) + \rho(s,\hat{0}) \overset{D\mathbb{P}.1}{\leq} r(ts)^0 + 2(ts)^0 \overset{D\mathbb{P}.1}{=} A(\omega)(ts)^0,$$

where $A(\omega) = r + 2$ *and* $\rho(s,\hat{0}) = \sup_{\theta\in[0,1]} \rho_H\left([s]^\theta,[\hat{0}]^\theta\right) = 2$. *In addition, for each* $w,\ u \in \mathcal{C}_r$ *and* $(t,s) \in D$, *we have*

$$\begin{aligned}\rho(h_\omega(t,s,w(t,s,\omega)),h_\omega(t,s,u(t,s,\omega))) &\overset{D\mathbb{P}.1}{=} \rho(w(t,s,\omega) - s,u(t,s,\omega) - s) \overset{D\mathbb{P}.1}{\leq} \frac{4}{ts}\rho(w(t,s,\omega),u(t,s,\omega)) \\ &\overset{D\mathbb{P}.1}{\leq} \frac{4}{ts}\left(\rho(w(t,s,\omega),u(t,s,\omega)) + t\rho(w_t(t,s,\omega),u_t(t,s,\omega)) + s\rho(w_s(t,s,\omega),u_s(t,s,\omega))\right),\end{aligned}$$

where $C(\omega) = 4$ *and* $q = r = 1$. *Hence, since all hypotheses* ***(A1)***, ***(A2)*** *and* ***(A4)*** *are fulfilled, we deduce that the Goursat problem (20) has a unique mild fuzzy random solution of type 1 on* $D = [0,2] \times [0,2]$. *Additionally, according to the integral equality (3), the formula of mild fuzzy random solution in type 1 of the problem (20) is as follows*

$$w(t,s,\omega) \overset{D\mathbb{P}.1}{=} s + \tilde{a}e^t + \tilde{a}e^s \ominus \tilde{a} + \int_0^t \int_0^s \left[w(\nu,\tau,\omega) - \tau\right]d\nu d\tau.$$

Therefore, we have $\rho\left(w(t,s,\omega),\hat{0}\right) \overset{D\mathbb{P}.1}{\leq} 2+(e^s+e^t+1)\,\rho(\tilde{a},\hat{0})+A(\omega)(ts) \overset{D\mathbb{P}.1}{\leq} 2+\left(2e^2+1\right)\rho(\tilde{a},\hat{0})+A(\omega)(ts)$. *Moreover, the unique solution of this problem is* $w(t,s,\omega)=s+\tilde{a}e^{t+s}$ *whose graphical representation can be shown in Figure 3.*

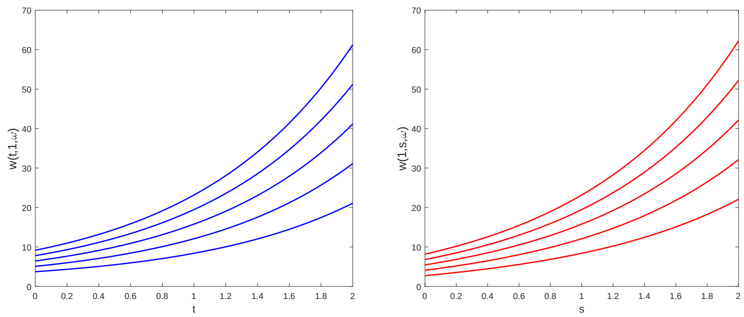

Figure 3: Some trajectories of mild fuzzy random solution in type 1 with $s=1$ (Fig. 3(a)) and $t=1$ (Fig. 3(b))

7 Conclusions

In this paper, we have investigated the existence and uniqueness of solutions for a class of fuzzy random Goursat problems under gH-differentiability and with some kinds of generalized Lipschitz conditions. The approach bases on approximate iterative method and Luxemburg fixed point theorem in generalized metric spaces. It is interesting when we are thinking that, the proofs of main results are constructive. We built some successive approximation sequence converging to mild fuzzy random solutions. Thus our method can be the foundation for many numerical methods dealing with fuzzy random PDEs. This idea will be useful in our future work, when we consider Goursat problems under granular computing introduced and developed in [24]. Furthermore, we can study fuzzy random evolution equations by developing techniques of semigroups and condensing map presented in [23].

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