Iranian Journal of Fuzzy Systems
Volume 18, Number 2, (2021), pp. 51-64
Original paper

Parameter estimation in fuzzy partial univariate linear regression model with non-fuzzy inputs and triangular fuzzy outputs

G. Hesamian[1], M. G. Akbari[2] and M. Shams[3]

[1]Department of Statistics, Payame Noor University, Tehran 19395-3697, Iran
[2]Department of Statistics, University of Birjand, Birjand, Iran.
[3]Department of Statistics, Faculty of Mathematical Sciences, University of Kashan, Kashan, Iran.

gh.hesamian@pnu.ac.ir, g_z_akbari@birjand.ac.ir, mehdishams@kashanu.ac.ir

Abstract

This paper proposed an extension for the classical partial univariate regression model with non-fuzzy inputs and triangular fuzzy output. For this purpose, the popular non-parametric estimator and the conventional arithmetic operations of triangular fuzzy numbers were combined to construct a fuzzy univariate regression model. Then, a hybrid algorithm was developed to estimate the bandwidth and fuzzy regression coefficient. Some common goodness-of-fit criteria were also used to examine the performance of the proposed method. The effectiveness of the proposed method was then illustrated through two numerical examples including a simulation study. The proposed method was also compared with several common fuzzy linear regression models with exact inputs and fuzzy outputs. Compared to the available fuzzy linear regressions models, the numerical results clearly indicated that the proposed fuzzy regression model is capable of exhibiting more accurate performances.

Keywords: Fuzzy partial linear model, kernel method, least absolute deviation, optimal bandwidth, goodness-of-fit measure.

1 Introduction

Regression analysis is the most basic and commonly used statistical technique to estimate the relationships between independent (predictor or explanatory) variables and a dependent (response or outcome) variable. It can also be used to explain the variable relationships and predict actual outcomes. Since the introduction of the fuzzy regression analysis by Tanaka et al. [57], the fuzzy univariate/multivariate parametric regression methods have been widely applied in numerous real applications under fuzzy data (for a comprehensive systematic review see [13]). The observations of the predictors can be either fuzzy [1, 47, 5, 7, 18, 15, 3, 17, 29, 14, 37, 33, 34, 35, 41, 63, 46, 45, 60, 21, 24, 43, 48] or real numbers [43, 12, 10, 9, 36, 42, 23, 30, 32, 38, 40, 44, 49, 55, 57, 61, 63, 2, 52, 31, 8, 23, 16, 27, 28, 50, 64]. In addition, Cheng and Lee [10] investigated the two most basic non-parametric regression techniques, namely, K-NN nearest neighbor smoothing and kernel smoothing for crisp input and fuzzy output. They also suggested an algorithm for selection of the best smoothing parameters based on minimization of cross-validation criteria. Wang et al. [58] proposed a fuzzy non-parametric model with crisp input and LR-fuzzy output based on the local linear smoothing technique with the cross-validation procedure to apt for the optimal value of the smoothing parameter for fitting the model. Moreover, Hesamian and Akbari [26] proposed a univariate kernel-based non-parametric regression model with fuzzy input and output data.

It is worthy noting that the fuzzy linear regression analysis (see [57, 11, 5]) is the most basic and commonly used statistical technique in fuzzy domain which estimates the relationships between predictor and a dependent response variable. However, partial linear models, as a combination of parametric and nonparametric methods, often lead to

Corresponding Author: M. Shams
Received: December 2019; Revised: June 2020; Accepted: September 2020.

higher prediction/estimation accuracies compared to the linear models. A partially linear model is defined as:

$$y_i = \beta x_i + f(t_i) + \epsilon_i,\ i = 1, 2, \ldots, n,$$

where x_is denote the inputs, t_is are additional covariates, f represents an unknown smooth function and ϵ_i represents the error of the model [22, 51, 62]. Note that the relation between the response variable and inputs are assumed to be linear while the relation between the response variable and covariates is taken as non-linear in cases where the effect of inputs are remove from the model. To check this the residuals of the model $y_i = \beta x_i + \epsilon_i$ (e_i) are plotted versus the residuals of $t_i = bx_i + \epsilon_i^{'}$ ($e_i^{'}$) and it is expected to have a functional relationship between e_i and e'_i (see [53], Chap. 6). On the contrary, fuzzy regression analysis extends the classical regression analysis in cases where some elements of the model are represented by fuzzy data. In this regards, the investigation of a fuzzy partial univariate model with exact predictors and fuzzy responses can be mentioned as the main contribution of this paper. For this purpose, two common parametric and nonparametric techniques were employed to estimate unknown fuzzy (nonparametric) function and fuzzy (slope) coefficient. The performance of the proposed method was also compared with several common fuzzy linear regression models in terms of some common extended goodness-of-fit criteria and scatter plots. For the practical reasons, the proposed method was further examined by two numerical example including a simulation study.

The rest of the present paper is organized as follows: an overview of some concepts and results of fuzzy numbers is presented in Section 2. Section 3 offers a fuzzy partial univariate model with non-fuzzy predictors and fuzzy responses. The effectiveness and performance of the proposed method are evaluated and compared with several common fuzzy regression methods based on two numerical examples including a simulation study and a practical example in terms of some common performance measures in Section 4. Finally, the main contributions of this paper will be summarized in Section 5.

2 Preliminary

This section briefly reviews several concepts and terminologies related to fuzzy numbers and a distance between fuzzy numbers used throughout the paper. A fuzzy set $\widetilde{A}$ of $\mathbb{R}$ (the real line) is defined by its membership function $\mu_{\widetilde{A}} : \mathbb{R} \to [0, 1]$. In addition, a fuzzy set $\widetilde{A}$ of $\mathbb{R}$ is called a fuzzy number if it is normal, i.e. there is a unique $x^*_{\widetilde{A}} \in \mathbb{R}$ so that $\mu_{\widetilde{A}}(x^*_{\widetilde{A}}) = 1$, and for every $\alpha \in [0, 1]$, the set $\widetilde{A}[\alpha] = \{x \in \mathbb{R} : \mu_{\widetilde{A}}(x) \geq \alpha\}$ is a nonempty compact interval in $\mathbb{R}$. This interval is denoted by $\widetilde{A}[\alpha] = [\widetilde{A}^L_\alpha, \widetilde{A}^U_\alpha]$, where $\widetilde{A}^L_\alpha = \inf\{x : x \in \widetilde{A}[\alpha]\}$ and $\widetilde{A}^U_\alpha = \sup\{x : x \in \widetilde{A}[\alpha]\}$. In addition, for all $\widetilde{A}, \widetilde{B} \in \mathbb{F}(\mathbb{R})$ and $\alpha \in [0, 1]$, some common arithmetic operations between two fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ can be evaluated as follows [39]:

$$\begin{aligned}
&(\widetilde{A} \oplus \widetilde{B})[\alpha] = [\widetilde{A}^L_\alpha + \widetilde{B}^L_\alpha, \widetilde{A}^U_\alpha + \widetilde{B}^U_\alpha],\\
&(\widetilde{A} \ominus \widetilde{B})[\alpha] = [\widetilde{A}^L_\alpha - \widetilde{B}^U_\alpha, \widetilde{A}^U_\alpha - \widetilde{B}^L_\alpha],\\
&(k \otimes \widetilde{A})[\alpha] = [k\widetilde{A}^L_\alpha I(k \geq 0) + k\widetilde{A}^U_\alpha I(k < 0), k\widetilde{A}^U_\alpha I(k \geq 0) + k\widetilde{A}^L_\alpha I(k < 0)],
\end{aligned}$$

where $I()$ denotes the indicator function. It is worth noting that fuzzy numbers are approximate assessments, given by experts and accepted by decision-makers when obtaining more accurate values is impossible or unnecessary. To simplify the task of representing and handling fuzzy numbers, several authors have captured the information contained in a (unimodal) fuzzy number using a functional parametric form called an LR-fuzzy number $\widetilde{A} = (a; l_a, r_a)_{LR}$ where $l_a, r_a > 0$. The membership function of an LR-fuzzy number $\widetilde{A}$ is defined by:

$$\widetilde{A}(x) = \begin{cases} L(\dfrac{a-x}{l_a}), & x \leq a,\\ R(\dfrac{x-a}{r_a}), & x > a, \end{cases} \tag{1}$$

where $a \in \mathbb{R}$, $l_a > 0$ and $r_a > 0$ are referred to as the center value and the left and right spreads of $\widetilde{A}$, respectively. The shape function L (or R) is a decreasing function from $\mathbb{R}^+ \to [0, 1]$ such that

1. $L(0) = 1$,
2. $L(x) < 1$ for every $x > 0$,
3. $L(x) > 0$ for every $x < 1$,

4. $L(1) = 0$ (or $L(x) > 0$ for any $x \in \mathbb{R}$ and $L(+\infty) = 0$).

A special type of LR-fuzzy number is the Triangular fuzzy numbers (**TFN**s) with the shape functions $L(x) = R(x) = \max\{0, 1 - |x|\}$, $x \in \mathbb{R}$. In addition, a L_p distance between two **TFN**s $\widetilde{A} = (a; l_a, r_a)_T$ and $\widetilde{B} = (b; l_b, r_b)_T$ was utilized in this paper defined as

$$D_p(\widetilde{A}, \widetilde{B}) = \sqrt[p]{\frac{|a-b|^p + |l_a - l_b|^p + |r_a - r_b|^p}{3}}, \ p \geq 1.$$

For any three **TFN**s $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$, it satisfies the following conditions:

1) $D_p(\widetilde{A}, \widetilde{B}) = 0$ if and only if $\widetilde{A} = \widetilde{B}$,

2) $D_p(\widetilde{A}, \widetilde{B}) = D_p(\widetilde{B}, \widetilde{A})$,

3) $D_p(\widetilde{A}, \widetilde{C}) \leq \big(D_p(\widetilde{A}, \widetilde{B}) + D_p(\widetilde{B}, \widetilde{C})\big)$.

We employ D_1 and D_2 to evaluate the unknown components of the proposed fuzzy regression model in the next section.

3 Fuzzy partial linear model

A fuzzy partial linear model with univariate non-fuzzy input and triangular fuzzy output was considered using an absolute error distance in this section. It should be noted that the addition and scalar multiplication of **TFN**s will lead to new **TFN**s. That's why the focus of this study is on such fuzzy data.

Definition 3.1. *Assume that the observed data of n statistical units are denoted by $(\widetilde{y}_i, x_i, t_i)$. Based on the aforementioned data set, following fuzzy partial linear model (**FPLM**) can be considered:*

$$\widetilde{y}_i = (\widetilde{\beta} \otimes x_i) \oplus \widetilde{f}(t_i) \oplus \widetilde{\epsilon}_i, \quad i = 1, 2, \ldots, n, \tag{2}$$

where

1. *$\widetilde{y}_i = (y_i; l_{y_i}, r_{y_i})_T$ represent the fuzzy responses,*
2. *x_i's are covariates for fuzzy linear regression, $i = 1, 2, \ldots, n$,*
3. *$\widetilde{\beta} = (\beta; l_\beta, r_\beta)_T$ is an unknown fuzzy coefficient to be estimated,*
4. *t_i's are additional covariates,*
5. *$\widetilde{f}(t_i) = (f(t_i); l_{t_i}, r_{t_i})_T$, $t_i \in [0, 1]$ is an unknown fuzzy smooth function, and*
6. *$\widetilde{\epsilon}_i$ are **TFN** error terms.*

Here, a two-step procedure was applied to estimate the fuzzy coefficient of $\widetilde{\beta}$ and the unknown bandwidth h.

Step 1. By extending the conventional nonparametric Nadaraya-Watson estimator [54], the unknown fuzzy smooth function $\widetilde{f}$ can be evaluated as

$$\hat{\widetilde{f}}(t_i) = \bigoplus_{j=1}^{n} (w_j(t_i) \otimes (\tilde{y}_j \ominus (x_j \otimes \tilde{\beta}))), \tag{3}$$

where, $w_j(t_i) = \dfrac{K\left(\frac{t_j - t_i}{h}\right)}{\sum_{j=1}^{n} K\left(\dfrac{t_j - t_i}{h}\right)}$, and $h > 0$ controls the amount of smoothing and is known as bandwidth of the kernel $K(.)$ [59]. By replacing Eq. (3) in Eq. (2), we get

$$\begin{aligned}\tilde{y}_i &= (x_i \otimes \tilde{\beta}) \oplus \bigoplus_{j=1}^{n} (w_j(t_i) \otimes (\tilde{y}_j \ominus (x_j \otimes \tilde{\beta}))) \oplus \widetilde{\epsilon}_i = (x_i \otimes \tilde{\beta}) \oplus \bigoplus_{j=1}^{n} (w_j(t_i) \otimes \tilde{y}_j) \ominus \bigoplus_{j=1}^{n} (w_j(t_i) \otimes (x_i \otimes \tilde{\beta})) \oplus \widetilde{\epsilon}_i \\ &= \bigoplus_{j=1}^{n} (w_j(t_i) \otimes \tilde{y}_j) \oplus (x_i \otimes \tilde{\beta}) \ominus ((\sum_{j=1}^{n} w_j(t_i) x_j) \otimes \tilde{\beta}) \oplus \widetilde{\epsilon}_i = \widetilde{u}_i \oplus ((x_i \otimes \tilde{\beta}) \ominus (z_i \otimes \tilde{\beta})) \oplus \widetilde{\epsilon}_i,\end{aligned} \tag{4}$$

where $\widetilde{u}_i = \bigoplus_{j=1}^{n}(w_j(t_i) \otimes \tilde{y}_j)$ and $z_i = \sum_{j=1}^{n} w_j(t_i)x_j$.

Step 2. By adding $\ominus\widetilde{u}_i$ on the both sides of Eq. (4), we get

$$\tilde{y}_i \ominus \widetilde{u}_i = (0; l_{u_i} - r_{u_i}, r_{u_i} - l_{u_i})_T \oplus ((x_i \otimes \tilde{\beta}) \ominus (\widetilde{z}_i \otimes \tilde{\beta})) \oplus \widetilde{\epsilon}_i = (x_i \otimes \tilde{\beta}) \ominus (z_i \otimes \tilde{\beta}) \oplus \widetilde{\xi}_i, \tag{5}$$

where $\widetilde{\xi}_i = (0; l_{u_i} - r_{u_i}, r_{u_i} - l_{u_i})_T \oplus \widetilde{\epsilon}_i$.

According to Steps 1 and 2, we finally obtain a fuzzy linear model as:

$$\widetilde{w}_i = \widetilde{w}_i^* \oplus \widetilde{\xi}_i, \tag{6}$$

where $\widetilde{w}_i = \tilde{y}_i \ominus \widetilde{u}_i = \bigoplus_{j=1}^{n}(w_j(t_i) \otimes (\tilde{y}_i \ominus \tilde{y}_j))$, and

$$\widetilde{w}_i^* = ((x_i \otimes \tilde{\beta}) \ominus (z_i \otimes \tilde{\beta})) = (w_i^*; l_{w_i^*}, r_{w_i^*})_T, \tag{7}$$

in which

1. $w_i^* = (x_i - z_i)\beta$,
2. $l_{w_i^*} = (x_i l_\beta I(x_i \geq 0) - x_i r_\beta I(x_i < 0)) - (z_i r_\beta I(z_i \geq 0) - z_i l_\beta I(z_i < 0))$,
3. $r_{w_i^*} = (x_i r_\beta I(x_i \geq 0) - x_i l_\beta I(x_i < 0)) - (z_i l_\beta I(z_i \geq 0) - z_i r_\beta I(z_i < 0))$,
4. $z_i = \sum_{j=1}^{n} w_j(t_i)x_j$,

and $I(.)$ shows the indicator function.

Table 1: Some common kernel functions

Triangular	$K(y) = \begin{cases} 1 - \lvert y\rvert, & \lvert y\rvert \leq 1, \\ 0, & \lvert y\rvert > 1. \end{cases}$
Triweight	$K(y) = \begin{cases} \frac{35}{32}(1 - y^2)^3, & \lvert y\rvert \leq 1, \\ 0, & \lvert y\rvert > 1. \end{cases}$

3.1 Algorithm for estimating the unknown components of the model

To evaluate the unknown components of fuzzy linear regression model (6), it can be seen that the unknown bandwidth of h and fuzzy coefficient of $\widetilde{\beta}_h$ are confected to each other and thus should be simultaneously estimated. Two popular kernels namely the triweight and triangular kernels (Table 1) were employed to examine their effects on the model performance. For this purpose, the classical (generalized) cross-validation procedure and least absolute error was extended based on the available data. The computational procedure for estimating through optimal value of bandwidth and fuzzy regression coefficient was implemented using Mathematica and Minitab softwares by the following algorithm:

(S1) The unknown bandwidth of h can be evaluated as follows:

(a) Choose a kernel K,

(b) Let $\widehat{h}^{(1)} = 0.01$,

(c) Compute

$$GCV(\widehat{h}^{(1)}) = \frac{1}{n}\sum_{j=1}^{n} \frac{D_2^2(\tilde{w}_j^{\widehat{h}^{(1)}}, \hat{\tilde{w}}_j^{\widehat{h}^{(1)}})}{(1 - tr(W_{\widehat{h}^{(1)}}))^2},$$

where $\hat{\tilde{w}}_j^{\widehat{h}^{(1)}} = ((x_j \otimes \widehat{\tilde{\beta}}_{\widehat{h}^{(1)}}) \ominus (z_j \otimes \widehat{\tilde{\beta}}_{\widehat{h}^{(1)}}))$ in which $\widehat{\tilde{\beta}}^{(i)}_{\widehat{h}^{(1)}} = \arg\min_{\beta\in\mathbb{R}, l_\beta, r_\beta > 0} \sum_{j=1}^{n} D_1(\tilde{w}_j^{\widehat{h}^{(1)}}, \hat{\tilde{w}}_j^{\widehat{h}^{(1)}})$, and $tr(W_{\widehat{h}^{(1)}})$ shows the trace of the matrix of $W_{\widehat{h}^{(1)}} = [w_{ij}]$ where $w_{ij} = K(\frac{t_i - t_j}{\widehat{h}^{(1)}})$, $i, j = 1, 2, \ldots, n$.

(d) Let $h^{(2)} = h^{(1)} + 0.01$ and return to (c). If $|CV(\widehat{h}^{(2)}) - CV(\widehat{h}^{(1)})| < \epsilon$ for a small number of ϵ then $\widehat{h}^{(2)}$ is the optimal bandwidth otherwise; repeat the algorithm until $|CV(\widehat{h}^{(k)}) - CV(\widehat{h}^{(k-1)})| < \epsilon$. Therefore, $\widehat{h}^{(k)}$ is the optimal bandwidth.

(S2) According to the optimal bandwidth $\widehat{h}^{(k)}$, compute the optimal value of fuzzy regression coefficient as:

$$\widehat{\widetilde{\beta}}_{\widehat{h}^{(k)}} = \arg \min_{\beta\in\mathbb{R}, l_\beta, r_\beta>0} \sum_{i=1}^{n} D_1(\tilde{w}_i^{\widehat{h}^{(k)}}, \hat{\tilde{w}}_i^{\widehat{h}^{(k)}}),$$

The sum of the estimated absolute deviation between fuzzy outputs and their corresponding estimated values can be denoted by $SAE = \sum_{i=1}^{n} D_1(\tilde{w}_i^{\widehat{h}^{(k)}}, \hat{\tilde{w}}_i^{\widehat{h}^{(k)}})$.

Remark 3.2. *To examine the goodness-of-fit for the proposed fuzzy univariate regression model, some measures were employed to assess the performance of the fuzzy regression model. In this regard, we used some common goodness-of-fit measures [26] including* $G_1 = \frac{1}{n}\sum_{i=1}^{n} G_i^1$, $G_2 = \frac{1}{n}\sum_{i=1}^{n} G_i^2$ *where*

$$G_i^1 = \frac{1}{1 + E_1(\widetilde{y}_i, \widehat{\widetilde{y}}_i)}, \quad G_i^2 = \frac{1}{1 + E_2(\widetilde{y}_i, \widehat{\widetilde{y}}_i)},$$

in which

$$E_1(\widetilde{y}_i, \widehat{\widetilde{y}}_i) = \frac{\int_0^1 |\widetilde{y}_i(x) - \widehat{\widetilde{y}}_i(x)|dx}{\int_0^1 \widetilde{y}_i(x)dx}, \ \text{and}\ E_2(\widetilde{y}_i, \widehat{\widetilde{y}}_i) = \int_0^1 |\widetilde{y}_i(x) - \widehat{\widetilde{y}}_i(x)|dx.$$

Another performance criterion extensively used in the context of fuzzy regression analysis [5, 29] to investigate the closeness between the fuzzy responses and their corresponding estimated values is also employed in this study: $S = \frac{1}{n}\sum_{j=1}^{n} S_{UI}(\widehat{\widetilde{y}}_j, \tilde{y}_j)$, *where* $S_{UI}(\widehat{\widetilde{y}}_j, \tilde{y}_j) = \dfrac{Card(\widehat{\widetilde{y}}_j \cap \tilde{y}_j)}{Card(\widehat{\widetilde{y}}_j \cup \tilde{y}_j)}$, *in which* $\cap$ *and* $\cup$ *denote the intersection and union operators on the space of fuzzy numbers and* $Card(\widetilde{A}) = \int \widetilde{A}(x)dx$ *denotes the cardinal number of* $\widetilde{A}$. *It is worthy noting that* S *is a similarity measure that is it satisfies the following conditions:*

1. $S(\widetilde{A}, \widetilde{B}) \in [0, 1]$.
2. $S(\widetilde{A}, \widetilde{B}) = S(\widetilde{B}, \widetilde{A})$.
3. $S(\widetilde{A}, \widetilde{B}) = 1$ *if and only if* $\widetilde{A} = \widetilde{B}$.
4. *If* $\widetilde{A} \subseteq \widetilde{B} \subseteq \widetilde{C}$, *then* $S(\widetilde{A}, \widetilde{C}) \leq \min\{S(\widetilde{A}, \widetilde{B}), S(\widetilde{B}, \widetilde{C})\}$.
5. $S(\widetilde{A} \cap \widetilde{B}, \widetilde{A} \cup \widetilde{B}) = S(\widetilde{A}, \widetilde{B})$.
6. $S(\widetilde{A} \cup \widetilde{C}, \widetilde{B} \cup \widetilde{C}) \geq S(\widetilde{A}, \widetilde{B})$,

It can be seen that $G_1, G_2, S \in [0, 1]$. *Therefore, the values of* G_1, G_2 *and* S *above 0.5 indicate the closeness of the responses to their corresponding estimations. In addition, to examine the relation between* $\widetilde{y}$ *and* $\widehat{\widetilde{y}}$ *values based on their scatter plots, the fuzzy responses* ($\widetilde{y}$) *and the estimated value of fuzzy responses* ($\widehat{\widetilde{y}}$) *are converted to defuzzified values of* $M_{\widetilde{y}}$ *and* $M_{\widehat{\widetilde{y}}}$ *according to the sugeno criteria [56]:*

$$M_{\widetilde{y}} = \frac{\int x\widetilde{y}(x)dx}{\int \widetilde{y}(x)dx}, \quad M_{\widehat{\widetilde{y}}} = \frac{\int x\widehat{\widetilde{y}}(x)dx}{\int \widehat{\widetilde{y}}(x)dx}.$$

Note that the center of gravity of a ***TFN*** $\widetilde{A} = (a; l_a, r_a)_T$ *can be determined by* $M_{\widetilde{A}} = a + (r_a - l_a)/3$.

Remark 3.3. *To examine the relationship between* $\widetilde{y}$ *and* t *(in cases where the role of* x *is removed from the model), the following steps extend the classical method:*

T1) Compute $M_{\widetilde{e}_i}$ *where* $\widetilde{e}_i = \widetilde{y}_i \ominus (\widehat{\widetilde{\beta}} \otimes x_i)$ *in which* $\widehat{\widetilde{\beta}} = \arg\min_{\beta\in\mathbb{R}, l_\beta, r_\beta>0} \sum_{i=1}^{n} D_1(\widetilde{y}_i, \widetilde{\beta} \otimes x_i)$,

T2) Compute $e_i^{'} = t_i - \widehat{b}x_i$ *where* $\widehat{b} = \arg\min_{b\in\mathbb{R}} \sum_{i=1}^{n} |t_i - bx_i|$,

T3) Plot $M_{\widetilde{e}}$ *versus* $e^{'}$ *to investigate the functional relationship between* $\widetilde{y}$ *and* t.

Remark 3.4. *It should be noted that Hesamian and Akbari [26] extended the conventional partial linear model for fuzzy input-output data while the proposed method relies on exact inputs. It should be also mentioned that if the fuzzy observations of* $\widetilde{y}_i$ *and the fuzzy coefficient of* $\widetilde{\beta}$ *reduce to the exact values, then it is easy to check that the proposed* ***FPLM*** *will reduce to the classical partial linear model.*

4 Numerical example

The feasibility and effectiveness of the proposed **FPLM** were assessed with some techniques of fuzzy linear regression models with non-fuzzy predictors and fuzzy responses based on two numerical examples. A competitive study can be achieved by application of the criteria given in Remark 3.2 to calculate the model performances.

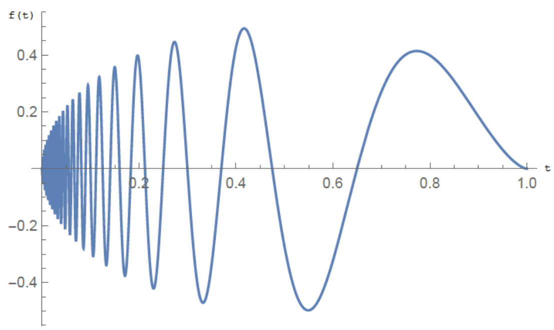

Figure 1: Time series plot of $f(t)$ in Example 4.1.

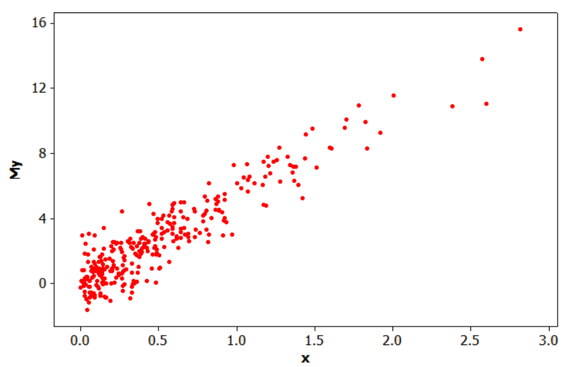

Figure 2: Plot of $M_{\widetilde{y}_i}$ versus x_i (with triweight kernel) based on the 5^{th} simulated data set in Example 4.1.

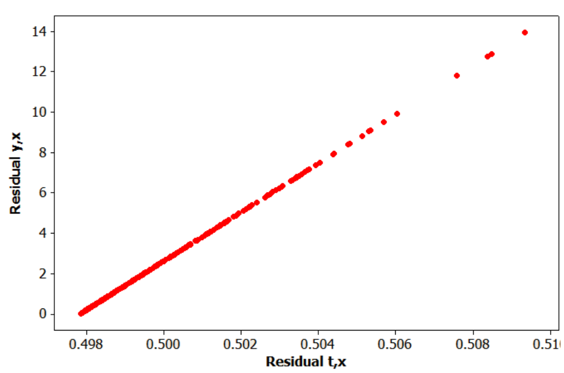

Figure 3: Plot of $M_{\widehat{\widetilde{e}}_i}$ versus $e_i^{'}$ based on the triweight kernel for 5^{th} simulated data set in Example 4.1.

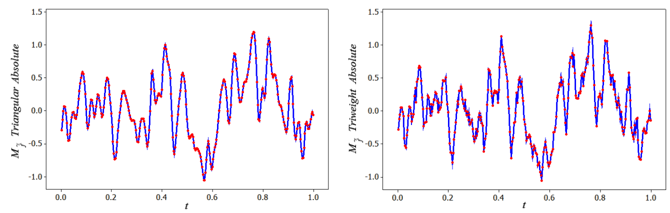

Figure 4: Plots of fuzzy smooth function of $\widehat{\widetilde{f}}$ based on the triweight and triangular kernels in Example 4.1.

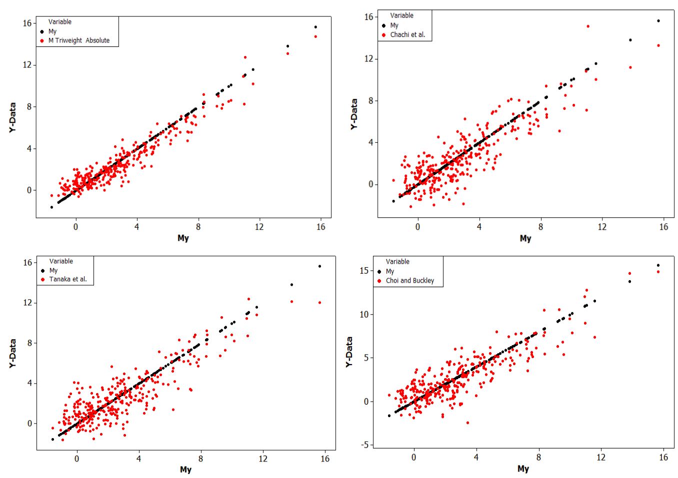

Figure 5: Plot of $M_{\widehat{\widetilde{y}}_i}$ versus $M_{\widetilde{y}_i}$ based on the triweight kernel (relevant to the 5^{th} simulated data set) and some common fuzzy linear regression models in Example 4.1.

Example 4.1. *(**A simulation study**) Consider the following **FPLM** based on* $m = 10$ *simulated data set of size* $n = 100$*:*

$$\widetilde{y}_i = (\widetilde{\beta} \otimes x_i) \oplus \widetilde{f}(t_i) \oplus \epsilon_i, \ \ i = 1, 2, \ldots, n,$$

where

1. $\widetilde{\beta} = (5; 1, 2)_T$,
2. x_i*'s is a simulated random sample from an exponential distribution with mean value of* 0.5,
3. $t_i = (i - 0.5)/100$ *for* $i = 1, 2, \ldots, 100$,
4. $\widetilde{\epsilon}_i = (\epsilon_i; l_{\epsilon_i}, r_{\epsilon_i})_T$*'s are simulated fuzzy error terms, where* $\epsilon_i \sim N(0, 1)$ *and* $l_{\epsilon_i} = r_{\epsilon_i} = 0.1|\epsilon_i|$*, and*
5. $f(t_i) = \sqrt{t_i(1 - t_i)} \sin(\frac{2.1\pi}{t_i + 0.05})$.

The time series plot of $f(t)$ *is shown in Fig. 1. Popular triangular and triweight kernels were applied to estimate* $\widetilde{f}(t)$ *based on the introduced distance recalled in Section 3.1. In this regard, the goodness-of-fit measures SAE,* G_1*,* G_2 *and S were used to evaluate the performance of the proposed method. The mean value of each performance measure relevant to each kernel were calculated as the ultimate performance of the respective model. Accordingly, Table 2 presents the mean values of the performance measures for the triweight and triangular kernels. With* $\overline{S} = \mathbf{0.63}$*,* $\overline{G}_1 = \mathbf{0.66}$*,* $\overline{G}_2 = \mathbf{0.61}$ *and* $\overline{SAE} = \mathbf{103.47}$*, it can be concluded that the triweight kernel led to lower SAE and higher S,* G_1 *and* G_2 *values than those of the triangular kernel. In order to do a comparative study against such fuzzy linear regression methods, here we focused on some common fuzzy linear models introduced by Choi and Buckley [11], Chachi and Taheri [4] and Tanaka et al. [57]. The mean value of performance measures for these fuzzy linear regression models are also listed in Table 2. As per the obtained values of performance measures, the proposed method led to better performance results than the other methods for all kernels. Specifically, consider the* 5^{th} *simulated data set. Fig. 2 shows that the relationship between* $\widetilde{y}$ *and* x *is linear. Using Remark 3.3, the plot of* $M_{\widetilde{e}}$ *versus* $e^{'}$ *is shown in Fig. 3 according to*

Steps (T1)-(T3). The numerical evaluations reveal that $\widetilde{e}_i = \widetilde{y}_i \ominus ((0.027; 0.59, 0.37)_T \oplus ((4.83; 0.39, 0.27)_T \otimes x_i))$ *and* $e_i^{'} = t_i - (-602.9 + 1211x_i)$*. It can be seen that there is a linear relationship between* $M_{\widetilde{e}}$ *and* $e^{'}$*. This clearly indicates that the role of partial part of the model on prediction performance for this data set. The unknown components of the* ***FPLM*** *can be evaluate via Steps (S1) and (S2). The performance measures of other fuzzy linear regression models are also evaluate as shown in Table 2. Comparing the various methods presented in Table 2 in terms of the goodness-of-fit criteria, it can be seen that the proposed* ***FPLM*** *gave rise to better results compared to the other methods for both triangular and triweight kernels. The plots of* $M_{\widehat{\widetilde{f}}}$ *versus* t *for both triangular and triweight kernels are drawn in Fig. 4. The performance of the proposed* ***FPLM*** *was further compared to other methods via scatter plot evaluations for the best model (triweight kernel), as depicted in Fig. 5 . The fact that the corresponding* $M_{\widehat{\widetilde{y}}}$ *values to the proposed method were closer to the* $M_{\widetilde{y}}$ *values, as compared to other methods, verified the superiority of the proposed* ***FPLM*** *over other fuzzy linear regression models based on the studied fuzzy data set.*

Table 2: The mean performance measures of the proposed **FPLM** corresponding to triangular and triweight kernels, the model without $\widetilde{f}$ and some common fuzzy linear regression models in Example 4.1.

Proposed	$\overline{SSE}$	$\overline{S}$	$\overline{G}_1$	$\overline{G}_2$
Triangular	110.65	0.61	0.63	0.57
Triweight	103.47	0.63	0.66	0.61
Method	$\overline{SSE}$	$\overline{S}$	$\overline{G}_1$	$\overline{G}_2$
Choi and Buckley	145.68	0.37	0.43	0.44
Chachi et al.	143.68	0.39	0.46	0.47
Tanaka et al.	148.97	0.35	0.41	0.42

Table 3: Performance measures of the proposed **FPLM** corresponding to 5^{th} simulated sample for triangular and triweight kernels, the model without $\widetilde{f}$ and some common fuzzy linear regression models in Example 4.1.

Proposed	SSE	S	G_1	G_2	Fitted fuzzy regression model
Triangular	108.65	0.63	0.64	0.59	$\widehat{\widetilde{y}}_i = ((4.92; 0.63, 0.83)_T \otimes x_i) \oplus \widehat{\widetilde{f}}(t_i)$
Triweight	101.63	0.65	0.67	0.62	$\widehat{\widetilde{y}}_i = ((4.99; 0.49, 0.99)_T \otimes x_i) \oplus \widehat{\widetilde{f}}(t_i)$
Method	SSE	S	G_1	G_2	Fitted fuzzy regression model
Choi and Buckley	146.83	0.38	0.44	0.45	$\widehat{\widetilde{y}}_i = (0.07; 0.29, 0.41)_T \oplus ((4.58; 1.19, 2.11)_T \otimes x_i)$
Chachi et al.	145.04	0.41	0.47	0.48	$\widehat{\widetilde{y}}_i = (4.86; 1.19, 2.01)_T \oplus ((-0.07; 0.38, 0.35)_T \otimes x_i)$
Tanaka et al.	147.11	0.36	0.42	0.43	$\widehat{\widetilde{y}}_i = (-0.1; 0.44, 0.35)_T \oplus ((4.33; 1.31, 2.12)_T \otimes x_i)$

Example 4.2. *In this example, the proposed* ***FPLM*** *is examined in hydrology studies to measure the suspended load based on discharge in watersheds. According to a study conducted in Darband region (situated between Bojnord and Esferayen Townships, Khorasan province of Iran) some water characteristics were measured using the standard procedures.*

Table 4: Performance measures of the proposed **FPLM** corresponding to the triweight kernel and some common fuzzy linear regression models in Example 4.2.

Proposed	SAE	S	G_1	G_2	Fitted fuzzy regression model
Triweight	129.01	0.73	0.82	0.85	$\widehat{h} = 0.04$, $\widehat{\widetilde{y}}_i = ((1.68; 0.044)_T \otimes x_i) \oplus \widehat{\widetilde{f}}(t_i)$
Method	SAE	S	G_1	G_2	Fitted fuzzy linear regression
Choi and Buckley	177.49	0.68	0.80	0.78	$\widehat{\widetilde{y}}_i = (3.851; 1.322)_T \oplus (1.518; 1.23)_T \otimes x_i$
Chachi et al.	171.57	0.67	0.78	0.72	Equation (30), pp. 7 of [6]
Tanaka et al.	188.74	0.54	0.54	0.55	$\widehat{\widetilde{y}}_i = (3.991; 1.871)_T \oplus (2.084; 1.43)_T \otimes x_i$

*The daily discharge and suspended load of the watershed were measured for 184 days [6] (see Table (1), pp. 18). Here, we shall deal with estimation of the amount of suspended load (as the fuzzy output) based on discharge (as non-fuzzy covariate) in accordance with the proposed **FPLM**. In this example, the **FPLM** is assumed to be* $\widetilde{y}_i = (\widetilde{\beta} \otimes x_i) \oplus \widetilde{f}(t_i) \oplus \epsilon_i$, $i = 1, 2, \ldots, 184$ *where* $\widetilde{\beta}$ *and* $\widetilde{f}$ *are symmetric **TFN**s and* $t_i = i/n$*. Fig. 6 shows that the relationship between* $\widetilde{y}$ *and* x *is linear. According to Steps (T1)-(T3), the results show that* $\widetilde{e}_i = \widetilde{y}_i \ominus ((3.27; 0.27, 0.3)_T \oplus ((1.69; 0.20, 0.39)_T \otimes x_i))$ *and* $e_i^{'} = t_i - (-0.0000001 + 19.81 x_i)$*. The plot of* $M_{\widetilde{e}}$ *versus* $e^{'}$ *is shown in Fig. 7. Accordingly, it can be observed that there is a functional relationship between* $M_{\widetilde{e}}$ *and* $e^{'}$*. This clearly indicates that the partial part of the linear may have a positive effect on model prediction performance. The unknown components of the **FPLM** can be evaluate via Steps (S1) and (S2). Applying the proposed method to the hydrology data set, the unknown bandwidth* h *and fuzzy coefficient regression* $\widetilde{\beta}$ *can be estimated according to Steps (S1) and (S2). For this purpose, we focused on triweight kernel due to its higher performance than the triangular kernel as shown in Example 4.1. The computational procedure show that* $\widehat{h} = 0.04$ *and* $\widetilde{\beta} = (1.68; 0.044)_T$*. The fitted **FPLM** and their corresponding goodness-of-fit measures are listed in Table 3. In another study, Chach et al. been shown that their proposed fuzzy linear regression model was advantageous and superior to some other fuzzy linear regression models (for more, see [6]). Therefore, in order to provide a competitive study, we employed three fuzzy linear regression models proposed by Chachi et al. [6] and two commonly used approaches introduced by Choi and Buckley [11] and Tanaka et al. [57]. The measures* SAE*,* S*,* G_1 *and* G_2 *were adopted to calculate the goodness-of-fit measures of the mentioned methods. The results are listed in Table 4. In terms of the the goodness-of-fit measures* $SAE = 129.01$*,* $G_1 = 0.82$*,* $G_2 = 0.85$ *and* $S = 0.73$*, it can be concluded that the proposed **FPLM** exhibits a better performance to other methods. Investigating the scatter plots for this data set (Fig 9) for the values of* $M_{\widehat{\widetilde{y}}}$ *and* $M_{\widetilde{y}}$*, the superiority of the proposed **FPLM** over other methods was also evident in this example.*

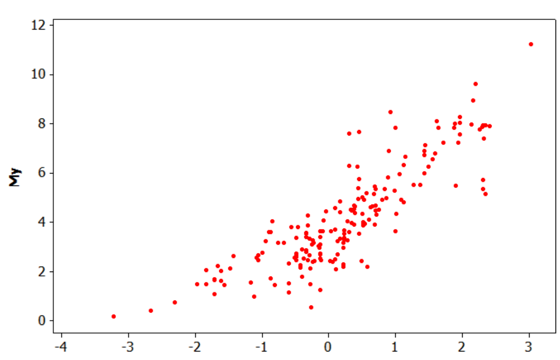

Figure 6: Plot of $M_{\widetilde{y}_i}$ versus x_i in Example 4.2.

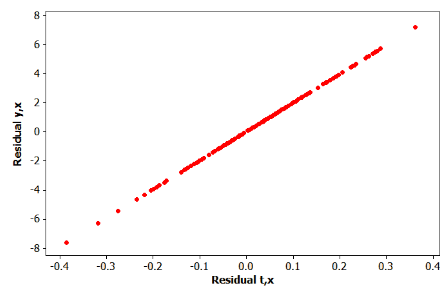

Figure 7: Plot of $M_{\widehat{e}_i}$ versus $e_i^{'}$ in Example 4.2.

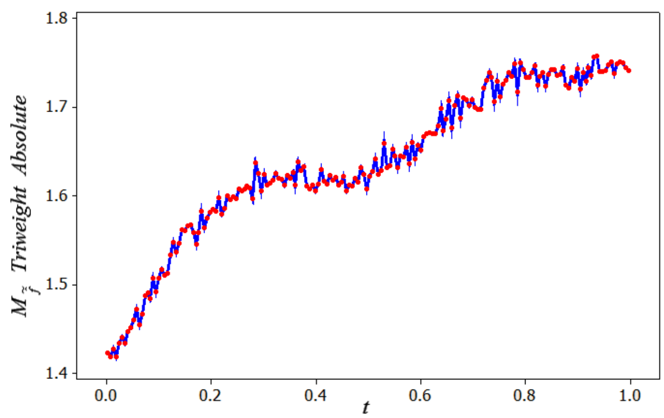

Figure 8: Plot of fuzzy smooth function of $\widehat{\widetilde{f}}$ based on the triweight in Example 4.2.

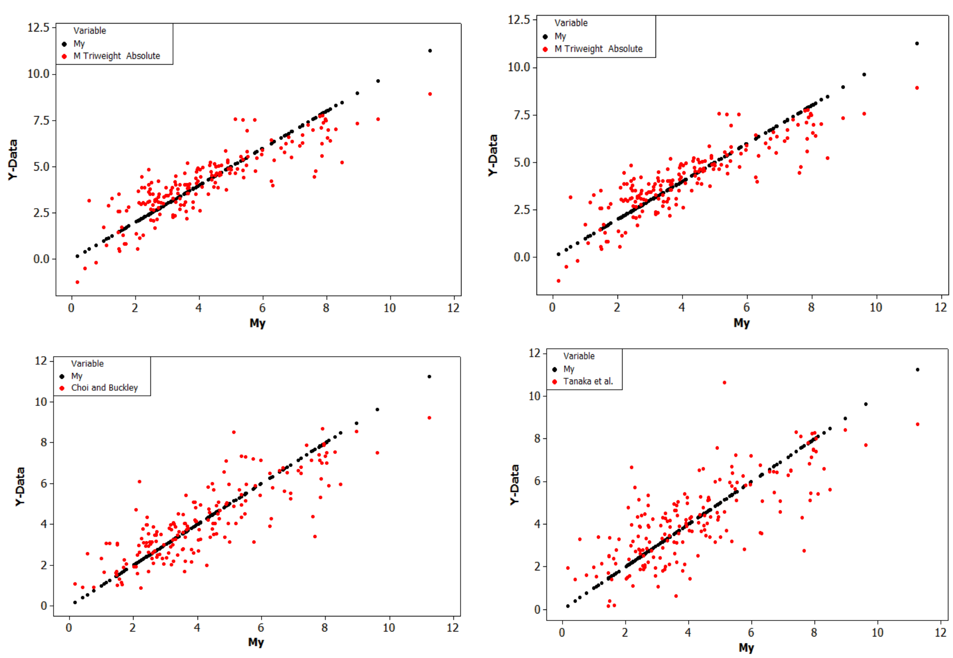

Figure 9: Plot of $\widehat{\widetilde{y}_i}$ versus $\widetilde{y}_i$ for the proposed **FPLM** and some common fuzzy linear regression models in Example 4.2.

5 Conclusion

Partial linear models have recently gained an extensive deal of attention in linear regression models since they combine both parametric and nonparametric regression techniques. In this paper, a classical partial univariate model was extended with non-fuzzy inputs and triangular fuzzy output. For this purpose, the conventional Nadaraya-Watson estimator was first employed to construct a fuzzy linear regression model. Then, a fuzzy least absolute deviation adopted with a generalized cross-validation criterion was utilized to evaluate unknown fuzzy coefficient and bandwidth. The performance of the proposed method was illustrated by two numerical examples via some common goodness-of-fit criteria and a common extended scatter plot based on a common center of gravity. Calculation results revealed that among the studied kernels, triweight kernel can lead to the best results. The effectiveness and advantages of the proposed regression model were also compared with several available fuzzy linear regression methods. The results clearly indicated the superior performance of our method compared to the others. Future researches should be focused on extending the proposed method for other type of LR-fuzzy numbers. The problem of multivariate cases as well as investigating the sensitivity analysis with respect to the outliers could be considered as the other potential topics for further researches.

6 Acknowledgements

The authors would like to thank the referees and editor for their remarks and criticisms that have led to improve the presentation of this work.

References

[1] G. Alfonso, A. F. R. L. De Hierro, C. Roldan, *A fuzzy regression model based on finite fuzzy numbers and its application to real-world financial data*, Journal of Computational and Applied Mathematics, **318** (2017), 47-58.

[2] M. J. Amiri, A. R. Zarei, J. Abedi-Koupai, S. Eslamian, *The performance of fuzzy regression method for estimating of reference evapotranspiration under controlled environment*, International Journal of Hydrology Science and Technology, **9** (2019), 28-38.

[3] K. D. Atalay, E. Eraslan, M. O. Cinar, *A hybrid algorithm based on fuzzy linear regression analysis by quadratic programming for time estimation: An experimental study in manufacturing industry*, Journal of Manufacturing Systems, **36** (2015), 182-188.

[4] J. Chachi, S. M. Taheri, *A least-absolutes regression model for imprecise response based on the generalized Hausdorff-metric*, Journal of Uncertain Systems, **7** (2013), 265-276.

[5] J. Chachi, S. M. Taheri, *Multiple fuzzy regression model for fuzzy input-output data*, Iranian Journal of Fuzzy Systems, **13** (2016), 63-78.

[6] J. Chachi, S. M. Taheri, N. R. Arghami, *A hybrid fuzzy regression model and its application in hydrology engineering*, Applied Soft Computing, **25** (2014), 149-158.

[7] J. Chachi, S. M. Taheri, S. Fattahi, A. H. Ravandi, *Two robust fuzzy regression models and their applications in predicting imperfections of cotton yarn*, The Iranian Textile Association of Science and Technology, **4** (2016), 60-68.

[8] K. Y. Chan, H. K. Lam, C. K. F. Yiu, T. S. Dillon, *A flexible fuzzy regression method for addressing non-linear uncertainty on aesthetic quality assessments*, IEEE Transactions on Systems, Man, and Cybernetics: Systems, **47** (2017), 2363-2377.

[9] L. H. Chen, S. H. Nien, *A new approach to formulate fuzzy regression models*, Applied Soft Computing, **86** (2020), https://doi.org/10.1016/j.asoc.2019.105915.

[10] C. B. Cheng, E. S. Lee, *Nonparametric fuzzy regression k-NN and kernel smoothing techniques*, Computers and Mathematics with Applications, **38** (1999), 239-251.

[11] S. H. Choi, J. J. Buckley, *Fuzzy regression using least absolute deviation estimators*, Soft Computing, **12** (2008), 257-263.

[12] S. H. Choi, H. Jung, H. Kim, *Ridge fuzzy regression model*, International Journal of Fuzzy Systems, **21** (2019), 2077-2090.

[13] N. Chukhrova, A. Johannssen, *Fuzzy regression analysis: Systemtic review and bibliography*, Applied Soft Computing, **84** (2019), https://doi.org/10.1016/j.asoc.2019.105708.

[14] J. De Andres-Sanchez, *Fuzzy regression analysis: An actuarial perspective*, In: Fuzzy statistical decision-making: Theory and applications, Studies in Fuzziness and Soft Computing, 343. Edited by C. Kahraman & O. Kabak. Springer, Berlin, (2016), 175-201.

[15] A. F. R. L. De Hierro, J. Martinez-Moreno, C. Aguilar-Pena, C. R. L. De Hierro, *Estimation of a fuzzy regression model using fuzzy distances*, IEEE Transactions on Fuzzy Systems, **24** (2016), 344-359.

[16] A. F. R. L. De Hierro, J. Martinez-Moreno, C. A. Pena, C. R. L. De Hierro, *A fuzzy regression approach using bernstein polynomials for the spreads*, Mathematics and Computers in Simulation, **128** (2016), 13-25.

[17] P. D'Urso, *Linear regression analysis for fuzzy/crisp input and fuzzy/crisp output data*, Computational Statistics and Data Analysis, **42** (2003), 47-72.

[18] P. D'Urso, R. Massari, A. Santoro, *Robust fuzzy regression analysis*, Information Sciences, **181** (2011), 4154-4174.

[19] R. L. Eubank, J. D. Hart, P. Speckman, *Trigonometric series regression estimators with an application to partially linear models*, Journal of Multivariate Analysis, **32** (1990), 70-83.

[20] M. B. Ferraro, R. Coppi, G. Gonzalez-Rodriguez, A. Colubi, *A linear regression model for imprecise response*, International Journal of Approximate Reasoning, **51** (2010), 759-770.

[21] H. Gholizadeh, N. Javadian, H. Fazlollahtabar, *Fuzzy regression integrated with Genetic-Tabu algorithm for prediction and optimization of a turning process*, The International Journal of Advanced Manufacturing Technology, **96** (2018), 2781-2790.

[22] S. A. Hamilton, Y. K. Truong, *Local linear estimation in partly linear models*, Journal of Multivariate Analysis, **60** (1997), 1-19.

[23] P. Y. Hao, J. H. Chiang, *Fuzzy regression analysis by support vector learning approach*, IEEE Transactions on Fuzzy Systems, **16** (2008), 428-441.

[24] Y. L. He, C. H. Wei, H. Long, R. A. R. Ashfaq, J. Z. Huang, *Random weight network-based fuzzy non-linear regression for trapezoidal fuzzy number data*, Applied Soft Computing, **70** (2017), 959-979.

[25] S. Heilpern, *Representation and application of fuzzy numbers*, Fuzzy Sets and Systems, **91** (1997), 259-268.

[26] G. Hesamian, M. G. Akbari, *Non-parametric kernel estimation based on fuzzy random variables*, IEEE Transactions on Fuzzy Systems, **25** (2017), 84-99.

[27] G. Hesamian, M. G. Akbari, *Fuzzy spline univariate regression with exact predictors and fuzzy responses*, Journal of Computational and Applied Mathematics, **375** (2020), 1-13.

[28] G. Hesamian, M. G. Akbari, *A robust varying coefficient approach to fuzzy multiple regression model*, Journal of Computational and Applied Mathematics, **371** (2020), 1-13.

[29] G. Hesamian, M. G. Akbari, M. Asadollahi, *Fuzzy semi-parametric partially linear model with fuzzy inputs and fuzzy outputs*, Expert Systems with Applications, **71** (2017), 230-239.

[30] M. Hojati, C. Bector, K. Smimou, *A simple method for computation of fuzzy linear regression*, European Journal of Operational Research, **166** (2005), 172-184.

[31] H. Jiang, C. K. Kwong, W. Y. Park, *Probabilistic fuzzy regression approach for preference modeling*, Engineering Applications of Artifcial Intelligence, **64** (2017), 286-294.

[32] H. Y. Jung, J. H. Yoon, S. H. Choi, *Fuzzy linear regression using rank transform method*, Fuzzy Sets and Systems, **274** (2015), 97-108.

[33] M. Kelkinnama, S. M. Taheri, *Fuzzy least-absolutes regression using shape preserving operations*, Information Sciences, **214** (2012), 105-120.

[34] U. T. Khan, C. Valeo, *A new fuzzy linear regression approach for dissolved oxygen prediction*, Hydrological Sciences Journal, **60** (2015), 1096-1119

[35] K. J. Kim, D. H. Kim, S. H. Choi, *Least absolute deviation estimator in fuzzy regression*, Journal of Applied Mathematics and Computing, **18** (2005), 649-656.

[36] I. K. Kim, W. J. Lee, J. H. Yoon, S. H. Choi, *Fuzzy regression model using trapezoidal fuzzy numbers for re-auction data*, International Journal of Fuzzy Logic and Intelligent Systems, **16** (2016), 72-80.

[37] K. S. Kula, A. Apaydin, *Fuzzy robust regression analysis based on the ranking of fuzzy sets*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, **16** (2008), 663-681.

[38] K. S. Kula, F. S. H. Tank, T. E. Dalkyly, *A study on fuzzy robust regression and its application to insurance*, Mathematical and Computational Applications, **17** (2012), 223-234.

[39] K. H. Lee, *First course on fuzzy theory and applications*, Springer-Verlag: Berlin, 2005.

[40] W. J. Lee, H. Y. Jung, J. H. Yoon, S. H. Choi, *The statistical inferences of fuzzy regression based on bootstrap techniques*, Soft Computing, **19** (2015), 883-890.

[41] J. Li, W. Zeng, J. S. H. Xie, Q. Yin, *A new fuzzy regression model based on least absolute deviation*, Engineering Applications of Artificial Intelligence, **52** (2016), 54-64.

[42] Y. Liu, Y. Chen, J. Zhou, S. Zhong, *Fuzzy linear regression models for QFD using optimized h values*, Engineering Applications of Artifcial Intelligence, **39** (2015), 45-54.

[43] H. T. Liu, J. Wang, Y. L. He, R. A. R. Ashfaq, *Extreme learning machine with fuzzy input and fuzzy output for fuzzy regression*, Neural Computing and Applications, **28** (2017), 3465-3476.

[44] J. S. H. Lu, R. Wang, *An enhanced fuzzy linear regression model with more flexible spreads*, Fuzzy Sets and Systems, **160** (2009), 2505-2523.

[45] F. Megri, A. C. Megri, R. Djabri, *An integrated fuzzy support vector regression and the particle swarm optimization algorithm to predict indoor thermal comfort*, Indoor and Built Environment, **25** (2016), 1248-1258.

[46] M. M. Nasrabadi, E. Nasrabadi, A. R. Nasrabadi, *Fuzzy linear regression analysis: A multi-objective programming approach*, Applied Mathematics and Computation, **163** (2005), 245-251.

[47] M. Otadi, *Fully fuzzy polynomial regression with fuzzy neural networks*, Neurocomputing, **142** (2014), 486-493.

[48] N. Y. Pehlivan, A. Apaydin, *Fuzzy radial basis function network for fuzzy regression with fuzzy input and fuzzy output*, Complex and Intelligent Systems, **2** (2016), 61-73.

[49] S. S. H. Petit-Renaud, T. Denoeuxa, *Nonparametric regression analysis of uncertain and imprecise data using belief functions*, International Journal of Approximate Reasoning, **35** (2004), 1-28.

[50] A. F. Roldan Lopez de Hierro, J. Martinez-Moreno, C. Aguilar-Pena, C. Roldan Lopez de Hierro, *Estimation of a fuzzy regression model using fuzzy distances*, IEEE Transaction on Fuzzy Systems, **24** (2016), 344-359.

[51] M. Roozbeh, M. Arashi, *Feasible ridge estimator in partially linear models*, Journal of Multivariate Analysis, **116** (2013), 35-44.

[52] H. Shakouri Ganjavi, R. Nadimi, S. F. Ghaderi, *Investigation on objective function and assessment rule in fuzzy regressions based on equality possibility, fuzzy union and intersection concepts*, Computers and Industrial Engineering, **110** (2017), 207-215.

[53] S. J. Sheather, *A modern approach to regression with R*, Springer Science+Business Media, (2009), doi. 10.1007/978-0-387-09608-7.

[54] J. Simonoff, *Smoothing methods in statistics*, Springer, New York, 1996.

[55] B. Y. Sohn, *Robust fuzzy linear regression based on M-estimators*, Journal of Applied Mathematics and Computing, **18** (2005), 596-597.

[56] M. Sugeno, *An introductory survey of fuzzy control*, Information Sciences, **36** (1985), 59-83.

[57] H. Tanaka, I. Hayashi, J. Watada, *Possibilistic linear regression analysis for fuzzy data*, European Journal of Operational Research, **40** (1989), 389-396.

[58] N. Wang, W. X. Zhang, C. L. Mei, *Fuzzy nonparametric regression based on local linear smoothing technique*, Information Sciences, **177** (2007), 3882-3900.

[59] L. Wasserman, *All of nonparametric statistics*, New York, Springer, 2007.

[60] H. C. Wu, *The construction of fuzzy least squares estimators in fuzzy linear regression models*, Expert Systems with Applications, **38** (2011), 632-640.

[61] K. Yen, S. Ghoshray, G. Roig, *A linear regression model using triangular fuzzy number coefficients*, Fuzzy Sets and Systems, **106** (1999), 167-177.

[62] J. You, G. Chen, *Semiparametric generalized least-squares estimation in partially linear regression models with correlated errors*, Journal of Statistical Planning and Inference, **137** (2007), 117-132.

[63] W. Zeng, Q. Feng, J. Li, *Fuzzy least absolute linear regression*, Applied Soft Computing, **52** (2017), 1009-1019.

[64] J. Zhou, H. Zhang, Y. Gu, A. A. Pantelous, *Affordable levels of house prices using fuzzy linear regression analysis: The case of shanghai*, Soft Computing, **22** (2018), 5407-5418.