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Original paper

Global dissipativity of fuzzy bidirectional associative memory neural networks with proportional delays

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Abstract

This article aimed to investigate the problem of global dissipativity of Fuzzy Bidirectional Associative Memory Neural Networks (FBAMNNs- for short) with proportional delays. Via Lyapunov Functionals (LFs- for short) and Linear Matrix Inequality (LMI- for short) approach, we obtained new sufficient conditions to guarantee the global dissipativity and global exponential dissipativity of the proposed model. In addition, two different types of activation functions are considered, including general bounded and Lipschitz-type activation functions. Moreover, the globally attractive and globally exponentially attractive sets are presented. Lastly, two numerical examples are given to illustrate the effectiveness of the developed results.

Keywords: FBAMNNs, global dissipativity, global exponential dissipativity, Lyapunov function, LMIs, proportional delays.

1 Introduction

Artificial Neural Network (ANN) may be thought of as simplified models of the networks of neurons that occur naturally in the biological brain [5, 3, 4, 9, 10, 16]. From mathematical point of view, it is considered as one of the modern mathematical-computational methods which are used to solve dynamic problems in developed behavioral systems during a time period. One of the most powerful type of ANN called Bidirectional Associative Memory Neural Networks (BAMNNs) [1]. The first model of BAM was introduced in 1988 by Kosko ([11, 12]), it is composed of neurons arranged in two layers, the A-layer and B-layer. The neurons in one layer are fully interconnected to the one in the other layer, while there are no interconnections among neurons in the same layer. The BAMNN attracted the attention of many researchers: In paper [2], the authors derived the stability condition for impulsive BAMNNs subject to the distributed and leakage time-varying delays; in article [8], Aouiti, C., *et al.,* studied the existence, uniqueness and global stability of pseudo periodic solution for neutral BAMNNs subject to time-varying and leakage delay, in [17], Rajivganthi, C., *et al.,* investigated the dissipativity analysis of complex-valued BAMNNs (for more details see papers [4, 7, 26]). BAMNNs are gaining prominence in various applications: pattern recognition, weather prediction, handwriting recognition, face recognition, autopilot, robotics also this kind of ANN is being extensively researched in load forecasting, processing substation alarms and predicting weather for solar radiation and wind farms. However, in every phenomena and practical problem there exist many uncertainties par example in image processing the loss of information when 3D shapes or scenes are projected into 2D images and ambiguity in representations and interpretations of complex scenes. Fuzzy set theory provides the mathematical strength to capture these uncertainties. This theory provides an inference mechanism under cognitive uncertainty. Then, it is reasonable to integrate fuzzy set theory into the BAM paradigm to give birth to Fuzzy BAM Neural Network (FBAMNN), which will become a powerful tool to solve complex problems and will have a much wide scope of applicability. By the way, it's important to remember that in 1996 Yang *et al.,* are

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the first who introduced the Fuzzy Cellular Neural Networks (FCNNs), which integrates fuzzy logic into the structure of cellular neural networks (CNNs) [29, 30]. It has fuzzy logic between its template input and/or output besides the sum of product operations. Sine then, there have been extensive results on the problems of their dynamical behaviors: In paper [22], Song, Q., *et al.,* derived stability conditions to guarantee the existence of periodic solutions and the exponential convergence rate index for the discrete-time FCNNs. Article [32], dealt with the dynamical oscillations of periodic solutions for delayed FCNNs. In [31], Yu, F.,*et al.,* studied the global exponential synchronization of FCNNs with delays and reaction–diffusion terms.

The time delay exists in ANNs because neurone can not respond instantly. It is one of main sources of instability or bad performances. Many researchers have investigated the dynamics of neural networks with constant delays, time-varying delays, distributed delays and leakage delays (see [6, 8, 7, 13, 21, 33]). In this work we dealt with a new type of delays called proportional delays. The combination of the two previous ANNs models and the last new dalays gives FBAMNNs with proportional delays which is our central model in this manuscript.

In 1972, Willems introduced the theory of dissipativity [27, 28]. The dissipativity problem is an important subject for physical systems, such as the design of group coordination, hybrid systems, nonlinear time-delay systems and Internet based control. Since then, the subject of dissipativity has received much attention because it is an important property in synchronization theory, stability theory, chaos, and robust control. Further, many researchers have devoted their attention towards dissipativity problem for different types of NNs with delays. For instance: In [14], Liao, X.,*et al.,* investigated the global dissipativity of a class of continuous-time recurrent neural networks. In paper [23], Song, Q. and Zhao, Z. discussed the global dissipativity of NNs subject to both variable and unbounded delays. In paper [26], Wang, L., *et al.,* studied the global dissipativity of BAMNNs subject to mixed delays. The authors in [13] reported the stability conditions by utilizing a new generalized Halanay inequalities to ensure the global dissipativity for a more general class of NNs. The dissipative-based repetitive controller is proposed in [19] for the class of network-based singular systems by utilizing Lyapunov stability theory and so on ([18, 17, 20, 19, 21, 24] and the references cited therein).

Motivated by the above, in this paper, we establish the global dissipativity of the following model:

$$\begin{cases} \dot{x}_i(t) &= -b_i x_i(t) + \sum\limits_{j=1}^{m} c_{ij} f_j(y_j(t)) + \sum\limits_{j=1}^{m} d_{ij} f_j(y_j(q_1 t)) + \sum\limits_{j=1}^{m} e_{ij}\eta_j(t) + \bigwedge\limits_{j=1}^{m} T_{ij}\eta_j(t) \\ & \quad + \bigwedge\limits_{j=1}^{m} \alpha_{ij} f_j(y_j(q_1 t)) + \bigvee\limits_{j=1}^{m} \beta_{ij} f_j(y_j(q_1 t)) + \bigvee\limits_{j=1}^{m} S_{ij}\eta_j(t) + u_i(t), \\ \dot{y}_j(t) &= -b_j^1 y_j(t) + \sum\limits_{i=1}^{n} c_{ji}^1 g_i(x_i(t)) + \sum\limits_{i=1}^{n} d_{ji}^1 g_i(x_i(q_2 t)) + \sum\limits_{i=1}^{n} e_{ji}^1 \eta_i^1(t) + \bigwedge\limits_{i=1}^{n} T_{ji}^1 \eta_i^1(t) \\ & \quad + \bigwedge\limits_{i=1}^{n} \alpha_{ji}^1 g_i(x_i(q_2 t)) + \bigvee\limits_{i=1}^{n} \beta_{ji}^1 g_i(x_i(q_2 t)) + \bigvee\limits_{i=1}^{n} S_{ji}^1 \eta_i^1(t) + v_j(t). \end{cases} \tag{1}$$

Model description :

- $n \geq 1,\ m \geq 1$ define the number of units in the system.
- $x(\cdot) = (x_1(\cdot),\ x_2(\cdot), \cdots, x_n(\cdot))^T$ and $y(\cdot) = (y_1(\cdot),\ y_2(\cdot), \cdots, y_m(\cdot))^T$ denote to the membrane potential.
- $B = diag(b_1,\ b_2, \cdots, b_n)$ and $B^1 = diag(b_1^1,\ b_2^1, \cdots, b_m^1)$ are positives constants corresponds to the passive decay rate.
- $C = (c_{ij})_{n\times m}$, $C^1 = (c_{ji}^1)_{m\times n}$, $D = (d_{ij})_{n\times m}$ and $D^1 = (d_{ji}^1)_{m\times n}$ are elements of feedback templates.
- $E = (e_{ij})_{n\times m},\ E^1 = (e_{ji}^1)_{m\times n}$ are feed-forward template.
- $\alpha = (\alpha_{ij})_{n\times m}$, $\alpha^1 = (\alpha_{ji}^1)_{m\times n}$, $\beta = (\beta_{ij})_{n\times m}$, $\beta^1 = (\beta_{ji}^1)_{m\times n}$, $T = (T_{ij})_{n\times m}$, $T^1 = (T_{ji}^1)_{m\times n}$, $S = (S_{ij})_{n\times m}$ and $S^1 = (S_{ji}^1)_{m\times n}$ are the elements of the fuzzy: feedback MIN template, feedback MAX template, feed-forward MIN template and feed-forward MAX template, respectively.
- $\bigvee$ symbolizes the fuzzy OR operation and $\bigwedge$ is the fuzzy AND operation.
- $\eta(\cdot) = (\eta_1(\cdot),\ \eta_2(\cdot), \cdots, \eta_m(\cdot))^T$ and $\eta^1(\cdot) = (\eta_1^1(\cdot),\ \eta_2^1(\cdot), \cdots, \eta_n^1(\cdot))^T$ denote the input with $|\eta_j(t)| \leq \tilde{\eta}_j,\ |\eta_i^1(t)| \leq \tilde{\eta}_i^1$, and $\Gamma = [\tilde{\eta}_1,\ \tilde{\eta}_2, \cdots, \tilde{\eta}_m]^T,\ \tilde{\Gamma} = [\tilde{\eta}_1^1,\ \tilde{\eta}_2^1, \cdots, \tilde{\eta}_n^1]^T$.
- $U(\cdot) = (u_1(\cdot),\ u_2(\cdot), \cdots, u_n(\cdot))^T$ and $V(\cdot) = (v_1(\cdot),\ v_2(\cdot), \cdots, v_m(\cdot))^T$ are continuous bounded external inputs functions with $|u_i(t)| \leq \tilde{u}_i,\ |v_j(t)| \leq \tilde{v}_j$ and $\tilde{U} = [\tilde{u}_1,\ \tilde{u}_2, \cdots, \tilde{u}_n]^T,\ \tilde{V} = [\tilde{v}_1,\ \tilde{v}_2, \cdots, \tilde{v}_m]^T$.
- $f(\cdot) = (f_1(\cdot),\ f_2(\cdot), \cdots, f_m(\cdot))^T$ and $g(\cdot) = (g_1(\cdot),\ g_2(\cdot), \cdots, g_n(\cdot))^T$ denote the activation function.

- q_1 and q_2 are proportional delay factors.

The initial values of system (1) are

$$\begin{cases} x_i(s) = \varphi_i(s), \;\; s \in [-\rho t_0, t_0], \; i = 1, 2, \cdots, n, \\ y_j(s) = \psi_j(s), \; s \in [-\rho t_0, t_0], \; j = 1, 2, \cdots, m, \end{cases} \tag{2}$$

where $\rho = \max\{q_1, \; q_2\}$, $\varphi_i(s) \in C([-\rho t_0, t_0], \mathbb{R}^n)$, $\psi_j(s) \in C([-\rho t_0, t_0], \mathbb{R}^m)$.

Motivated by the above-stated discussions, in this paper we discuss the global dissipativity for FBAMNNs with proportional delays. More preciously, our deliberation is focused on developing a positive invariant and globally attractive set to guarantee the global dissipativity. Further, by utilizing LMI approach and some advanced inequality technique, new sufficient conditions are established to achieve the global exponential dissipativity of the FBAMNNs subject to proportional delays.

The main contributions of this article lies in the following directions.

✓ The choice of FBAMNNs model is not arbitrary since our system (1) includes Hopfield neural networks, BAM neural networks, cellular neural networks and fuzzy cellular neural networks as special cases. We generalize the most results exist in litterateur [9, 23, 24, 25, 26].

✓ This paper is one of the first papers that attempts to study the global dissipativity problem of FBAMNNs with proportional delays. Two classes of activation functions are considered. First, we utilized the class of bounded activation functions to show the global dissipativity in Theorem 3.1 and the global exponential dissipativity in Theorem 3.2. Second, we utilized the class of Lipschitz-continuous activation functions to show the global dissipativity in Theorem 3.3.

✓ The FBAMNNs are subject to proportional delays. The proportional delay $\tau(t) = (1-q)t$ ($\to +\infty$ as $t \to +\infty$, $0 < q < 1$) is among the delay exist in the dynamic of ANNs models is fewer conservative and more widely used in the recent years. It is unlimited contrary to distributed delay, constant delay and the limited delay.

✓ New sufficient conditions are given to assuring the global dissipativity and global exponential dissipativity.

✓ The global dissipativity criteria are a more general case of passivity and stability analyses. Thus, it is essential to consider dissipativity problems for FBAMNNs.

✓ Two numerical examples with their simulations are given to illustrate the effectiveness of the theoretical results.

Remark 1.1. *The proportional delay factors satisfy* $0 < q_1 < 1, \; 0 < q_2 < 1$ *and* $q_1 t = t - (1-q_1)t, \; q_2 t = t - (1-q_2)t$. $(1-q_1)t$ *and* $(1-q_2)t$ *are the continuous time-varying functions satisfying* $(1-q_1)t \to +\infty, \; (1-q_2)t \to +\infty$ *as* $t \to +\infty$.

The contents of this article is organized as follows.
In Section 2, we will introduce basic hypotheses, helpful definitions and essential lemmas. In Section 3, we will present our main results. In section 4, two numerical examples are given to demonstrate the validity of our result. We will address the conclusions in section 5.

2 Assumptions, definitions and lemmas

For the sake of simplicity, we adapt the following notations:

Notation 2.1. • $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$ *stand the set of real numbers, the set of all n-dimensional real vectors and the set of* $n \times m$ *real matrices respectively.*

- *For a symmetric matrix* $M \in \mathbb{R}^{n\times m}$, $M > 0, \; M \geq 0, \; M^{-1}, \; M^T$ *means that* M *is a positive defined matrix, positive semi-defined matrix, inverse matrix, transpose matrix respectively.*

- $\star$ *means a symmetric matrix block for the symmetric block.*

- $x \in \mathbb{R}^n \setminus \{0\}$ *implies* $x \notin \{0\}$.

Further, the following assumptions, lemmas and definitions are significant to ensure the main results.

2.1 Basic assumptions

Throughout this paper, we give the following assumptions:

Assumption 2.2. *If there exists a nonnegative constants M_j and M_i^1, then the activation functions satisfying*

$$|f_j(\wp)| \leq M_j \text{ and } |g_i(\wp)| \leq M_i^1,\ \forall \wp \in \mathbb{R}, i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m.$$

Assumption 2.3. *If there exists a positive constants l_j^f and $l_i^g > 0$, then the activation functions satisfying*

$$|f_j(\wp) - f_j(\wp_1)| \leq l_j^f |\wp - \wp_1| \text{ and } |g_i(\wp) - g_i(\wp_1)| \leq l_i^g |\wp - \wp_1|, \forall \wp, \wp_1 \in \mathbb{R}, i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, m.$$

2.2 Helpful definitions

Definition 2.4. *System* (1) *is globally dissipative if there exists a compact set $\Upsilon \subset \mathbb{R}^{n+m}$ such that $\forall \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^{n+m}$, $\exists\ \mathcal{T}$, when $t \geq t_0 + \mathcal{T}$, $\begin{pmatrix} x(t,\ t_0,\ x_0) \\ y(t,\ t_0,\ y_0) \end{pmatrix} \in \Upsilon$, where $\begin{pmatrix} x(t,\ t_0,\ x_0) \\ y(t,\ t_0,\ y_0) \end{pmatrix} \in \mathbb{R}^{n+m}$ indicates the solution of* (1) *for the initial state $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^{n+m}$ and initial time t_0. Therefore, the set Υ is named globally attractive. For $t \geq t_0$, if $\forall \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \Upsilon$ involves $\begin{pmatrix} x(t,\ t_0,\ x_0) \\ y(t,\ t_0,\ y_0) \end{pmatrix} \subseteq \Upsilon \subseteq \mathbb{R}^{n+m}$, then the set Υ is named as positive invariant.*

Definition 2.5. *Let Υ be a globally attractive set of model (1). Then, we say that the system (1) is globally exponentially dissipative, if there exists a compact set $\Upsilon^* \supset \Upsilon$ in $\mathbb{R}^{n+m}$ such as $\forall \omega \in \mathbb{R}^{n+m} \setminus \Upsilon^*$, there exist constants $\mathcal{M}(\omega) > 0$ and $\alpha > 0$ such that*

$$\inf_{(x^T(t), y^T(t))^T \in \mathbb{R}^{n+m} \setminus \Upsilon^*} \left\{ \left\| \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} - \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \right\| \Big\backslash \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \in \Upsilon^* \right\} \leq \mathcal{M}(\omega) e^{-\alpha(t - t_0)},$$

then the set Υ^* is said globally exponentially attractive set, where $\omega = (x_0^T,\ y_0^T)^T$, $x_0 \in \mathbb{R}^n$, $y_0 \in \mathbb{R}^m$ and $(x^T,\ y^T)^T \in \mathbb{R}^{n+m} \setminus \Upsilon^*$ means $(x^T,\ y^T)^T \in \mathbb{R}^{n+m}$ while $(x^T,\ y^T)^T \notin \Upsilon^*$.

Lemma 2.6. [32] *For $i, j = 1, 2, \cdots, n$, let $u_j,\ u_j^*,\ \sqsupset_{ij},\ \neg_{ij} \in \mathbb{R}$, $k_j : \mathbb{R} \to \mathbb{R}$ be continuous functions, then the following inequalities satisfy*

$$\begin{aligned} \left| \bigwedge_{j=1}^{n} \sqsupset_{ij} k_j(u_j) - \bigwedge_{j=1}^{n} \sqsupset_{ij} k_j(u_j^*) \right| &\leq \sum_{j=1}^{n} |\sqsupset_{ij}| |k_j(u_j) - k_j(u_j^*)|, \\ \left| \bigvee_{j=1}^{n} \neg_{ij} k_j(u_j) - \bigvee_{j=1}^{n} \neg_{ij} k_j(u_j^*) \right| &\leq \sum_{j=1}^{n} |\neg_{ij}| |k_j(u_j) - k_j(u_j^*)|. \end{aligned}$$

Lemma 2.7. [9] *For a positive definite matrix B and any $\digamma,\ \digamma_1 \in \mathbb{R}^n$, the following inequalities satisfy*

$$2\digamma^T \digamma_1 \leq \digamma^T B^{-1} \digamma + \digamma_1^T B \digamma_1.$$

Lemma 2.8. [9] *For a given constant matrices $\Xi_1,\ \Xi_3$ and Ξ_2, where $\Xi_1 = \Xi_1^T,\ \Xi_2 = \Xi_2^T > 0$. Then*

$$\Xi_1 + \Xi_3^T \Xi_2^{-1} \Xi_3 < 0,$$

if and only if

$$\begin{pmatrix} \Xi_1 & \Xi_3^T \\ \Xi_3 & -\Xi_2 \end{pmatrix} < 0 \ \text{ or } \begin{pmatrix} -\Xi_2 & \Xi_3 \\ \Xi_3^T & \Xi_1 \end{pmatrix} < 0.$$

3 Main results

In this section, we discuss the problems of the global dissipativity and global exponential dissipativity for system (1). To ease the analysis, we consider the following factors:

$$\begin{aligned}
\pi &= \sum_{i=1}^{n}\sum_{j=1}^{m}\big(|c_{ij}|+|d_{ij}|+|\alpha_{ij}|+|\beta_{ij}|\big)|M_j| + \sum_{i=1}^{n}\sum_{j=1}^{m}\big(|e_{ij}|+|T_{ij}|+|S_{ij}|\big)\tilde{\eta}_j + \sum_{i=1}^{n}\tilde{u}_i,\\
\tilde{\pi} &= \sum_{j=1}^{m}\sum_{i=1}^{n}\big(|c^1_{ji}|+|d^1_{ji}|+|\alpha^1_{ji}|+|\beta^1_{ji}|\big)|M^1_i| + \sum_{j=1}^{m}\sum_{i=1}^{n}\big(|e^1_{ji}|+|T^1_{ji}|+|S^1_{ji}|\big)\tilde{\eta}^1_i + \sum_{j=1}^{m}\tilde{v}_j,\\
\theta &= (\theta_{ij})_{n\times m} = d_{ij}+\alpha_{ij}+\beta_{ij}, \quad \vartheta = (\vartheta_{ji})_{m\times n} = d^1_{ji}+\alpha^1_{ji}+\beta^1_{ji},\\
\kappa &= (\kappa_{ij})_{n\times m} = e_{ij}+T_{ij}+S_{ij}, \quad \gamma = (\gamma_{ji})_{m\times n} = e^1_{ji}+T^1_{ji}+S^1_{ji}.
\end{aligned}$$

Theorem 3.1. *Suppose that Assumption 2.2 holds. Then, FBAMNNs in equation* (1) *is dissipative and the set*

$$\Upsilon_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{n+m} \Big\backslash \sum_{i=1}^{n}|x_i(t)|^2 + \sum_{j=1}^{m}|y_j(t)|^2 \le \frac{1}{\delta}\Big[\frac{1}{2}(\pi^2+\tilde{\pi}^2)\Big] \right\},$$

is a positive invariant and globally attractive set.

Proof. In order to achieve the desired result, we consider the following Lyapunov-Krasovskii functional for the addressed system (1) as given below

$$V(t) = \frac{1}{2}\sum_{i=1}^{n}|x_i(t)|^2 + \frac{1}{2}\sum_{j=1}^{m}|y_j(t)|^2. \tag{3}$$

The upper right-hand derivative $D^+V(\cdot)$ along the trajectories of system (1) is given by

$$\begin{aligned}
D^+V(t) \quad \le \quad & \sum_{i=1}^{n}|x_i(t)|\Big[-b_i|x_i(t)| + \sum_{j=1}^{m}|c_{ij}||f_j(y_j(t))| + \sum_{j=1}^{m}|d_{ij}||f_j(y_j(q_1t))| + \sum_{j=1}^{m}|e_{ij}||\eta_j(t)|\\
+ & \bigwedge_{j=1}^{m}|T_{ij}||\eta_j(t)| + \bigwedge_{j=1}^{m}|\alpha_{ij}||f_j(y_j(q_1t))| + \bigvee_{j=1}^{m}|\beta_{ij}||f_j(y_j(q_1t))| + \bigvee_{j=1}^{m}|S_{ij}||\eta_j(t)| + |u_i(t)|\Big]\\
+ & \sum_{j=1}^{m}|y_j(t)|\Big[-b^1_j|y_j(t)| + \sum_{i=1}^{n}|c^1_{ji}||g_i(x_i(t))| + \sum_{i=1}^{n}|d^1_{ji}||g_i(x_i(q_2t))| + \sum_{i=1}^{n}|e^1_{ji}||\eta^1_i(t)|\\
+ & \bigwedge_{i=1}^{n}|T^1_{ji}||\eta^1_i(t)| + \bigwedge_{i=1}^{n}|\alpha^1_{ji}||g_i(x_i(q_2t))| + \bigvee_{i=1}^{n}|\beta^1_{ji}||g_i(x_i(q_2t))| + \bigvee_{i=1}^{n}|S^1_{ji}||\eta^1_i(t)| + |v_j(t)|\Big].
\end{aligned} \tag{4}$$

By using Lemma 2.6, one obtains

$$\begin{aligned}
&\bigwedge_{j=1}^{m}|T_{ij}||\eta_j(t)| \le \sum_{j=1}^{m}|T_{ij}||\eta_j(t)|, \ \bigwedge_{j=1}^{m}|\alpha_{ij}||f_j(y_j(q_1t))| \le \sum_{j=1}^{m}|\alpha_{ij}||f_j(y_j(q_1t))|,\\
&\bigvee_{j=1}^{m}|\beta_{ij}||f_j(y_j(q_1t))| \le \sum_{j=1}^{m}|\beta_{ij}||f_j(y_j(q_1t))|, \ \bigvee_{j=1}^{m}|S_{ij}||\eta_j(t)| \le \sum_{j=1}^{m}|S_{ij}||\eta_j(t)|,\\
&\bigwedge_{i=1}^{n}|T^1_{ji}||\eta^1_i(t)| \le \sum_{i=1}^{n}|T^1_{ji}||\eta^1_i(t)|, \ \bigwedge_{i=1}^{n}|\alpha^1_{ji}||g_i(x_i(q_2t))| \le \sum_{i=1}^{n}|\alpha^1_{ji}||g_i(x_i(q_2t))|,\\
&\bigvee_{i=1}^{n}|\beta^1_{ji}||g_i(x_i(q_2t))| \le \sum_{i=1}^{n}|\beta^1_{ji}||g_i(x_i(q_2t))|, \ \bigvee_{i=1}^{n}|S^1_{ji}||\eta^1_i(t)| \le \sum_{i=1}^{n}|S^1_{ji}||\eta^1_i(t)|.
\end{aligned}$$

Then, we can write

$$\begin{aligned}
D^+V(t) \quad \leq \quad & \sum_{i=1}^{n}|x_i(t)|\Big[-b_i|x_i(t)|+\sum_{j=1}^{m}|c_{ij}||f_j(y_j(t))|+\sum_{j=1}^{m}|d_{ij}||f_j(y_j(q_1t))| \\
+ \quad & \sum_{j=1}^{m}|e_{ij}||\eta_j(t)|+\sum_{j=1}^{m}|T_{ij}||\eta_j(t)|+\sum_{j=1}^{m}|\alpha_{ij}||f_j(y_j(q_1t))|+\sum_{j=1}^{m}|\beta_{ij}||f_j(y_j(q_1t))|+\sum_{j=1}^{m}|S_{ij}||\eta_j(t)| \\
+ \quad & |u_i(t)|\Big]+\sum_{j=1}^{m}|y_j(t)|\Big[-b_j^1|y_j(t)|+\sum_{i=1}^{n}|c_{ji}^1||g_i(x_i(t))|+\sum_{i=1}^{n}|d_{ji}^1||g_i(x_i(q_2t))|+\sum_{i=1}^{n}|e_{ji}^1||\eta_i^1(t)| \\
+ \quad & \sum_{i=1}^{n}|T_{ji}^1||\eta_i^1(t)|+\sum_{i=1}^{n}|\alpha_{ji}^1||g_i(x_i(q_2t))|+\sum_{i=1}^{n}|\beta_{ji}^1||g_i(x_i(q_2t))|+\sum_{i=1}^{n}|S_{ji}^1||\eta_i^1(t)|+|v_j(t)|\Big] \\
\leq \quad & -\sum_{i=1}^{n}b_i|x_i(t)|^2+\sum_{i=1}^{n}|x_i(t)|\Big[\sum_{j=1}^{m}\big(|c_{ij}|+|d_{ij}|+|\alpha_{ij}|+|\beta_{ij}|\big)|M_j|+\sum_{j=1}^{m}\big(|e_{ij}|+|T_{ij}| \\
+ \quad & |S_{ij}|\big)\tilde{\eta}_j+\tilde{u}_i\Big]-\sum_{j=1}^{m}b_j^1|y_j(t)|^2+\sum_{j=1}^{m}|y_j(t)|\Big[\sum_{i=1}^{n}\big(|c_{ji}^1|+|d_{ji}^1|+|\alpha_{ji}^1|+|\beta_{ji}^1|\big)|M_i^1| \\
+ \quad & \sum_{i=1}^{n}\big(|e_{ji}^1|+|T_{ji}^1|+|S_{ji}^1|\big)\tilde{\eta}_i^1+\tilde{v}_j\Big] \\
= \quad & -\sum_{i=1}^{n}b_i|x_i(t)|^2+\sum_{i=1}^{n}|x_i(t)|\pi-\sum_{j=1}^{m}b_j^1|y_j(t)|^2+\sum_{j=1}^{m}|y_j(t)|\tilde{\pi} \\
\leq \quad & -\sum_{i=1}^{n}b_i|x_i(t)|^2+\frac{1}{2}\sum_{i=1}^{n}(|x_i(t)|^2+\pi^2)-\sum_{j=1}^{m}b_j^1|y_j(t)|^2+\frac{1}{2}\sum_{j=1}^{m}(|y_j(t)|^2+\tilde{\pi}^2) \\
= \quad & -\Big\{\sum_{i=1}^{n}(b_i-\frac{1}{2})|x_i(t)|^2+\sum_{j=1}^{m}(b_j^1-\frac{1}{2})|y_j(t)|^2\Big\}+\frac{1}{2}\pi^2+\frac{1}{2}\tilde{\pi}^2 \\
\leq \quad & -\delta\Big\{\sum_{i=1}^{n}|x_i(t)|^2+\sum_{j=1}^{m}|y_j(t)|^2\Big\}+\frac{1}{2}\pi^2+\frac{1}{2}\tilde{\pi}^2, \qquad (5)
\end{aligned}$$

when $(x^T(t),\ y^T(t))^T\in\mathbb{R}^{n+m}\backslash\Upsilon_1$ for $(\varphi^T,\ \psi^T)^T\in\Upsilon_1$ and $\big(x^T(t,\ t_0,\ \varphi),\ y^T(t,\ t_0,\ \psi)\big)^T\in\Upsilon_1,\quad t\geq t_0$. For $(\varphi^T,\ \psi^T)^T\notin\Upsilon_1$, there exists $\mathcal{T}>0$ such that $\big(x^T(t,\ t_0,\ \varphi),\ y^T(t,\ t_0,\ \psi)\big)^T\in\Upsilon_1$ holds for all $t>t_0+\mathcal{T}$. Based on Definition 2.4, we deduce that the system (1) is dissipative and Υ_1 is a positive invariant and globally attractive set of model (1). □

Theorem 3.2. (I) *Suppose that Assumption 2.2 holds. Then, FBAMNNs in equation* (1) *is globally dissipative and*

$$\Upsilon_2=\left\{\begin{pmatrix} x \\ y \end{pmatrix}\in\mathbb{R}^{n+m}\Big\backslash\sum_{i=1}^{n}|x_i(t)|\leq\frac{1}{\min\limits_{1\leq i\leq n}(b_i)}\pi,\ \sum_{j=1}^{m}|y_j(t)|\leq\frac{1}{\min\limits_{1\leq j\leq m}(b_j^1)}\tilde{\pi}\right\},$$

is a positive invariant and globally attractive set.

(II) *Suppose that Assumption 2.3 holds. Then, FBAMNNs in equation (1) is globally exponential dissipative and*

$$\tilde{\Upsilon}_2=\left\{\begin{pmatrix} x \\ y \end{pmatrix}\in\mathbb{R}^{n+m}\Big\backslash\sum_{i=1}^{n}|x_i(t)|\leq\frac{1}{\min\limits_{1\leq i\leq n}\{b_i-\xi\}}\pi,\ \sum_{j=1}^{m}|y_j(t)|\leq\frac{1}{\min\limits_{1\leq j\leq m}\{b_j^1-\xi\}}\tilde{\pi}\right\},$$

is a globally exponential attractive set.

Proof. Firstly, we choose the Lyapunov-Krasovskii functional in the following form:

$$V(t)=\sum_{i=1}^{n}|x_i(t)|+\sum_{j=1}^{m}|y_j(t)|. \qquad (6)$$

Calculating the upper right-hand derivative of $V(\cdot)$ along the positive half trajectory of system and by using Lemma 2.6 we have

$$\begin{aligned} D^+V(t) \quad \leq \quad & \sum_{i=1}^{n}\Big[-b_i|x_i(t)| + \sum_{j=1}^{m}|c_{ij}||f_j(y_j(t))| + \sum_{j=1}^{m}|d_{ij}||f_j(y_j(q_1t))| + \sum_{j=1}^{m}|e_{ij}||\eta_j(t)| + \sum_{j=1}^{m}|T_{ij}||\eta_j(t)| \\ + \quad & \sum_{j=1}^{m}|\alpha_{ij}||f_j(y_j(q_1t))| + \sum_{j=1}^{m}|\beta_{ij}||f_j(y_j(q_1t))| + \sum_{j=1}^{m}|S_{ij}||\eta_j(t)| + |u_i(t)|\Big] \\ + \quad & \sum_{j=1}^{m}\Big[-b_j^1|y_j(t)| + \sum_{i=1}^{n}|c_{ji}^1||g_i(x_i(t))| + \sum_{i=1}^{n}|d_{ji}^1||g_i(x_i(q_2t))| + \sum_{i=1}^{n}|e_{ji}^1||\eta_i^1(t)| + \sum_{i=1}^{n}|T_{ji}^1||\eta_i^1(t)| \\ + \quad & \sum_{i=1}^{n}|\alpha_{ji}^1||g_i(x_i(q_2t))| + \sum_{i=1}^{n}|\beta_{ji}^1||g_i(x_i(q_2t))| + \sum_{i=1}^{n}|S_{ji}^1||\eta_i^1(t)| + |v_j(t)|\Big] \\ \leq \quad & \Big[\sum_{i=1}^{n}-b_i|x_i(t)| + \sum_{j=1}^{m}\big(|c_{ij}| + |d_{ij}| + |\alpha_{ij}| + |\beta_{ij}|\big)|M_j| + \sum_{j=1}^{m}\big(|e_{ij}| + |T_{ij}| + |S_{ij}|\big)\tilde{\eta}_j + \tilde{u}_i\Big] \\ + \quad & \Big[\sum_{j=1}^{m}b_j^1|y_j(t)| + \sum_{i=1}^{n}\big(|c_{ji}^1| + |d_{ji}^1| + |\alpha_{ji}^1| + |\beta_{ji}^1|\big)|M_i^1| + \sum_{i=1}^{n}\big(|e_{ji}^1| + |T_{ji}^1| + |S_{ji}^1|\big)\tilde{\eta}_i^1 + \tilde{v}_j\Big] \\ \leq \quad & \big[\sum_{i=1}^{n}-b_i|x_i(t)| + \pi\big] + \big[\sum_{j=1}^{m}-b_j^1|y_j(t)| + \tilde{\pi}\big] < 0, \end{aligned} \tag{7}$$

when $(x^T(t),\ y^T(t)) \in \mathbb{R}^{n+m} \setminus \Upsilon_2,\ D^+V(t) < 0$, which involves that for $(\varphi^T,\ \psi^T)^T \in \Upsilon_2,\ t \geq t_0$, $\big((x^T(t,\ t_0,\ \varphi),\ y(t,\ t_0,\ \psi)\big)^T \in \Upsilon_2$ satisfy. For $(\varphi^T,\ \psi^T)^T \notin \Upsilon_2$, there exists $\mathcal{T} > 0$ such that $\big((x^T(t,\ t_0,\ \varphi),\ y(t,\ t_0,\ \psi)\big)^T \in \Upsilon_2$ satisfy for all $t > t_0 + \mathcal{T}$ holds. So Υ_2 is a positive invariant and globally attractive set.

Secondly, we choose ξ such that

$$0 < \xi < \min_{1\leq\leq n, 1\leq j\leq m}\{b_i,\ b_j^1\},$$

and we choose another Lyapunov-Krasovskii functional in the following form:

$$V(t) = e^{\xi t}\sum_{i=1}^{n}|x_i(t)| + e^{\xi t}\sum_{j=1}^{m}|y_j(t)|. \tag{8}$$

Calculating the upper right-hand derivative of $V(\cdot)$ along the positive half trajectory of system (1) and by using Lemma 2.6 one obtains.

$$\begin{aligned} D^+V(t) \quad \leq \quad & \sum_{i=1}^{n}e^{\xi t}\Big[(\xi - b_i)|x_i(t)| + \sum_{j=1}^{m}|c_{ij}||f_j(y_j(t))| + \sum_{j=1}^{m}|d_{ij}||f_j(y_j(q_1t))| + \sum_{j=1}^{m}|e_{ij}||\eta_j(t)| \\ + \quad & \sum_{j=1}^{m}|T_{ij}||\eta_j(t)| + \sum_{j=1}^{m}|\alpha_{ij}||f_j(y_j(q_1t))| + \sum_{j=1}^{m}|\beta_{ij}||f_j(y_j(q_1t))| + \sum_{j=1}^{m}|S_{ij}||\eta_j(t)| + |u_i(t)|\Big] \\ + \quad & \sum_{j=1}^{m}e^{\xi t}\Big[(\xi - b_j^1)|y_j(t)| + \sum_{i=1}^{n}|c_{ji}^1||g_i(x_i(t))| + \sum_{i=1}^{n}|d_{ji}^1||g_i(x_i(q_2t))| + \sum_{i=1}^{n}|e_{ji}^1||\eta_i^1(t)| \\ + \quad & \sum_{i=1}^{n}|T_{ji}^1||\eta_i^1(t)| + \sum_{i=1}^{n}|\alpha_{ji}^1||g_i(x_i(q_2t))| + \sum_{i=1}^{n}|\beta_{ji}^1||g_i(x_i(q_2t))| + \sum_{i=1}^{n}|S_{ji}^1||\eta_i^1(t)| + |v_j(t)|\Big] \\ \leq \quad & e^{\xi t}\Big[\sum_{i=1}^{n}(\xi - b_i)|x_i(t)| + \sum_{j=1}^{m}\big(|c_{ij}| + |d_{ij}| + |\alpha_{ij}| + |\beta_{ij}|\big)|M_j| + \sum_{j=1}^{m}\big(|e_{ij}| + |T_{ij}| + |S_{ij}|\big)\tilde{\eta}_j + \tilde{u}_i\Big] \\ + \quad & e^{\xi t}\Big[\sum_{j=1}^{m}(\xi - b_j^1)|y_j(t)| + \sum_{i=1}^{n}\big(|c_{ji}^1| + |d_{ji}^1| + |\alpha_{ji}^1| + |\beta_{ji}^1|\big)|M_i^1| + \sum_{i=1}^{n}\big(|e_{ji}^1| + |T_{ji}^1| + |S_{ji}^1|\big)\tilde{\eta}_i^1 + \tilde{v}_j\Big] \\ \leq \quad & e^{\xi t}\big[\sum_{i=1}^{n}(\xi - b_i)|x_i(t)| + \pi\big] + e^{\xi t}\big[\sum_{j=1}^{m}(\xi - b_j^1)|y_j(t)| + \tilde{\pi}\big] < 0. \end{aligned} \tag{9}$$

When $(x^T(t),\ y^T(t))^T \in \mathbb{R}^{n+m} \setminus \tilde{\Upsilon}_2$, integrating the both sides of Equation (9) from 0 to t ($t > 0$), we come to $V(t) \leq V(0)$, which can result that

$$\sum_{i=1}^{n} |x_i(t)| + \sum_{j=1}^{m} |y_j(t)| \leq e^{-\xi t} \sum_{i=1}^{n} |x_i(0)| + \sum_{j=1}^{m} |y_j(0)|. \tag{10}$$

Then

$$\sum_{i=1}^{n} |x_i(t)| + \sum_{j=1}^{m} |y_j(t)| \leq \sup_{\rho t_0 \leq s \leq t_0} \{V(s)\} e^{-\xi t}. \tag{11}$$

Let $\mathcal{M} = \sup\limits_{\rho t_0 \leq s \leq t_0} \{V(s)\}$, one can derive

$$\inf_{(x^T(t), y^T(t)) \in \mathbb{R}^{n+m} \setminus \Upsilon^*} \left\{ \left\| \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} - \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \right\| \Big\backslash \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \in \tilde{\Upsilon}_2^* \right\} \leq \mathcal{M} e^{-\xi t}. \tag{12}$$

According to Definition 2.4, we get FBAMNNs in equation (1) is a globally exponential dissipative system and $\tilde{\Upsilon}_2$ is a globally exponential attractive set. □

Theorem 3.3. *Assume that the Assumption 2.3 holds. Then there exists a symmetric positive definite matrix P with an appropriate dimensions such that the following LMIs hold:*

$$\Xi = \begin{pmatrix} N + I & C^1 + C^T & \vartheta \\ \star & -P & 0 \\ \star & \star & -q_2 I \end{pmatrix} < 0, \tag{13}$$

and

$$\tilde{N} + P + \frac{1}{q_1} \theta^T \theta + I < 0, \tag{14}$$

where

$$N = diag\left\{ \frac{-2b_1\lambda_2}{l_1^f}, \frac{-2b_2\lambda_2}{l_2^f}, \cdots, \frac{-b_n\lambda_2}{l_n^f} \right\},\ \tilde{N} = diag\left\{ \frac{-2b_1^1\chi_2}{l_1^g}, \frac{-2b_2^1\chi_2}{l_2^g}, \cdots, \frac{-2b_n^1\chi_2}{l_n^g} \right\},$$

$\chi_1,\ \chi_2,\ \lambda_1$ and λ_2 are positive constants such that $\chi_1 + \chi_2 = 1,\ \lambda_1 + \lambda_2 = 1,\ \chi_1 \neq 0,\ \lambda_1 \neq 0$. Then, FBAMNNs in equation (1) a globally dissipative and

$$\Upsilon_3 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{n+m} \Big\backslash |g^T(x(t))| \leq \frac{l^g(|\kappa|\Gamma + \tilde{U})}{B\chi_1},\ |f^T(y(t))| \leq \frac{l^f(|\gamma|\tilde{\Gamma} + \tilde{V})}{B^1\lambda_1} \right\},$$

is a positive invariant and globally attractive set.

Proof. Choose the following Lyapunov-Krasovskii functional as given below

$$V(t) = V_1(t) + V_2(t), \tag{15}$$

where

$$\begin{aligned} V_1(t) &= 2\sum_{i=1}^{n} \int_0^{x_i(t)} g_i(s)ds + \sum_{i=1}^{n} \int_{q_2 t}^{t} g_i^2(x_i(s))ds, \\ V_2(t) &= 2\sum_{j=1}^{m} \int_0^{y_j(t)} f_j(s)ds + \sum_{j=1}^{m} \int_{q_1 t}^{t} f_j^2(y_j(s))ds. \end{aligned}$$

The upper right-hand derivative of $V(\cdot)$ along the trajectory of system (1) is given by

$$\begin{aligned}\dot{V}_1(t) &= 2\sum_{i=1}^{n} g_i(x_i(t))\Big[-b_ix_i(t)+\sum_{j=1}^{m}c_{ij}f_j(y_j(t))+\sum_{j=1}^{m}d_{ij}f_j(y_j(q_1t))+\sum_{j=1}^{m}e_{ij}\eta_j(t)\\&+\bigwedge_{j=1}^{m}T_{ij}\eta_j(t)+\bigwedge_{j=1}^{m}\alpha_{ij}f_j(y_j(q_1t))+\bigvee_{j=1}^{m}\beta_{ij}f_j(y_j(q_1t))+\bigvee_{j=1}^{m}S_{ij}\eta_j(t)+u_i(t)\Big]\\&+\sum_{i=1}^{n}g_i^2(x_i(t))-q_2\sum_{i=1}^{n}g_i^2(x_i(q_2t)).\end{aligned}$$

By using Lemma 2.6, we have

$$\begin{aligned}\dot{V}_1(t) &\le 2\sum_{i=1}^{n} g_i(x_i(t))\Big[-b_ix_i(t)+\sum_{j=1}^{m}c_{ij}f_j(y_j(t))+\sum_{j=1}^{m}d_{ij}f_j(y_j(q_1t))+\sum_{j=1}^{m}e_{ij}\eta_j(t)\\&+\sum_{j=1}^{m}T_{ij}\eta_j(t)+\sum_{j=1}^{m}\alpha_{ij}f_j(y_j(q_1t))+\sum_{j=1}^{m}\beta_{ij}f_j(y_j(q_1t))+\sum_{j=1}^{m}S_{ij}\eta_j(t)+u_i(t)\Big]\\&+\sum_{i=1}^{n}g_i^2(x_i(t))-q_2\sum_{i=1}^{n}g_i^2(x_i(q_2t))\\&= 2g^T(x(t))\Big[-Bx(t)+Cf(y(t))+Df(y(q_1t))+E\eta(t)+T\eta(t)+\alpha f(y(q_1t))\\&+\beta f(y(q_1t))+S\eta(t)+U(t)\Big]+g^2(x(t))-q_2g^2(x(q_2t)),\end{aligned}\tag{16}$$

$$\begin{aligned}\dot{V}_2(t) &= 2\sum_{j=1}^{m} f_j(y_j(t))\Big[-b_j^1y_j(t)+\sum_{i=1}^{n}c_{ji}^1g_i(x_i(t))+\sum_{i=1}^{n}d_{ji}^1g_i(x_i(q_2t))+\sum_{i=1}^{n}e_{ji}^1\eta_i^1(t)+\bigwedge_{i=1}^{n}T_{ji}^1\eta_i^1(t)\\&+\bigwedge_{i=1}^{n}\alpha_{ji}^1g_i(x_i(q_2t))+\bigvee_{i=1}^{n}\beta_{ji}^1g_i(x_i(q_2t))+\bigvee_{i=1}^{n}S_{ji}^1\eta_i^1(t)+v_j(t)\Big]+\sum_{j=1}^{m}f_j^2(y_j(t))-q_1\sum_{j=1}^{m}f_j^2(y_j(q_1t)).\end{aligned}\tag{17}$$

Similarly, by using Lemma 2.6, one obtains

$$\begin{aligned}\dot{V}_2(t) &\le 2\sum_{j=1}^{m} f_j(y_j(t))\Big[-b_j^1y_j(t)+\sum_{i=1}^{n}c_{ji}^1g_i(x_i(t))+\sum_{i=1}^{n}d_{ji}^1g_i(x_i(q_2t))+\sum_{i=1}^{n}e_{ji}^1\eta_i^1(t)\\&+\sum_{i=1}^{n}T_{ji}^1\eta_i^1(t)+\sum_{i=1}^{n}\alpha_{ji}^1g_i(x_i(q_2t))+\sum_{i=1}^{n}\beta_{ji}^1g_i(x_i(q_2t))+\sum_{i=1}^{n}S_{ji}^1\eta_i^1(t)+v_j(t)\Big]\\&+\sum_{j=1}^{m}f_j^2(y_j(t))-q_1\sum_{j=1}^{m}f_j^2(y_j(q_1t))\\&= 2f^T(y(t))\Big[-B^1y(t)+C^1g(x(t))+D^1g(x(q_2t))+E^1\eta^1(t)+T^1\eta^1(t)+\alpha^1g(x(q_2t))\\&+\beta^1g(x(q_2t))+S^1\eta^1(t)+V(t)\Big]+f^2(y(t))-q_1f^2(y(q_1t)).\end{aligned}\tag{18}$$

Applying Lemma 2.7 to the above inequality, we have

$$\begin{aligned}f^T(t)(C^1+C)g(x(t))+g^T(x(t))(C^1+C)f(y(t)) &\le f^T(y(t))(C^1+C)^TP^{-1}(C^1+C)f(y(t))+g^T(x(t))Pg(x(t)),\\ g^T(x(t))\theta f(y(q_1t))+f^T(y(q_1t))\theta^Tg(x(t)) &\le \frac{1}{q_1}g^T(x(t))\theta\theta^Tg(x(t))+q_1f^T(y(q_1t))f(y(q_1t)),\\ f^T(y(t))\vartheta g(x(q_2t))+g^T(x(q_2t))\vartheta^Tf(y(t)) &\le \frac{1}{q_2}f^T(y(t))\vartheta^T\vartheta f(y(t))+q_2g^T(x(q_2t))g(x(q_2t)).\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\dot{V}(t) \quad &\leq \quad -2g^T(x(t))Bx(t) + f^T(y(t))(C^1+C)^T P^{-1}(C^1+C)f(y(t)) + g^T(x(t))Pg(x(t))\\
&+ \quad \frac{1}{q_2} f^T(y(t))\vartheta^T\vartheta f(y(t)) + q_2 g^T(x(q_2t))g(x(q_2t)) + g^T(x(t))\kappa\eta(t) + g^T(x(t))U(t)\\
&+ \quad g^2(x(t)) - q_2 g^2(x(q_2t)) - 2f^T(y(t))B^1y(t) + \frac{1}{q_1} g^T(x(t))\theta^T\theta g(x(t)) + q_1 f^T(y(q_1t))f(y(q_1t))\\
&+ \quad 2f^T(y(t))\gamma\eta^1(t) + 2f^T(y(t))V(t) + f^2(y(t)) - q_1 f^2(y(q_1t))\\
&\leq \quad -2g^T(x(t))\frac{B(\chi_1+\chi_2)}{l^g} g(x(t)) + f^T(y(t))\big[(C^1+C)^T P^{-1}(C^1+C) + \frac{1}{q_2}\vartheta^T\vartheta + I\big]f(y(t))\\
&+ \quad g^T(x(t))\big[P + \frac{1}{q_1}\theta^T\theta + I\big]g(x(t)) + g^T(x(t))\kappa\eta(t) + g^T(x(t))U(t) - 2f^T(y(t))\frac{B^1(\lambda_1+\lambda_2)}{l^f} f(y(t))\\
&+ \quad 2f^T(y(t))\gamma\eta^1(t) + 2f^T(y(t))V(t)\\
&= \quad -2\frac{B\chi_1}{l^g}|g^T(x(t))|\big[|g(x(t))| - \frac{l^g(|\kappa|\Gamma + \tilde{U})}{B\chi_1}\big] + f^T(y(t))\big[N + (C^1+C)^T P^{-1}(C^1+C) + \frac{1}{q_2}\vartheta^T\vartheta\\
&+ \quad I\big]f(y(t))g^T(x(t)) + \big[\tilde{N} + P + \frac{1}{q_1}\theta^T\theta + I\big]g(x(t)) - 2\frac{B^1\lambda_1}{l^f}|f^T(y(t))|\big[|f(y(t))| - \frac{l^f(|\gamma|\tilde{\Gamma} + \tilde{V})}{B^1\lambda_1}\big]. \quad (19)
\end{aligned}$$

In accordance with equation (13) and (14), we obtain

$$\dot{V}(t) \quad \leq \quad -2|g^T(x(t))|\big[\frac{B\chi_1}{l^g} g(x(t)) - \frac{l^g(|\kappa|\Gamma + \tilde{U})}{B\chi_1}\big] - 2|f^T(y(t))|\big[\frac{B^1\lambda_1}{l^f} f(y(t)) - \frac{l^f(|\gamma|\tilde{\Gamma} + \tilde{V})}{B^1\lambda_1}\big], \qquad (20)$$

when $(x^T(t),\ y^T(t)) \in \mathbb{R}^{n+m} \setminus \Upsilon_3$. Therefore, we conclude from equation (20) that system (1) is dissipative and Υ_3 is a positive invariant and globally attractive set of model (1). □

Remark 3.4. *Our manuscript offers a theoretical basis for the design of the fuzzy cellular neural networks with proportional delays more effective in the resolution of many problem thanks to the template input and/or output besides the sum of product operation. Hence, the obtained results can enrich the study on dynamical characteristics of a class of FBAMNNs.*

Remark 3.5. *Our main result on the global dissipativity of FBAMNNs in equation (1) is new. So, it is the first article where the global dissipativity and global exponential dissipativity of the system (1) have been studied. Compared with the results investigated in [23, 26], our results are totally different from them.*

Remark 3.6. *In paper[15] the problem of the global dissipativity of a class of quaternion-valued BAMNNs with time delay is investigated; in [9], the authors discussed the global dissipativity of high-order Hopfield BAMNNs with time-varying coefficients and distributed delays; in article [25] the authors studied the global dissipativity of a class of BAMNNs with both time-varying and unbound delays and in [26] the global dissipativity of BAMNNs with both time-varying and continuously distributed delays have been investigated.*

All above results cannot be applied to model (1) to obtain the global dissipativity of our fuzzy bidirectional associative memory neural networks. In this paper, we take into consideration the influences of the fuzzy logic on the dynamics BAMNNs. The obtained results complement some earlier publications.

Remark 3.7. *In paper [14] the authors mainly investigated the global dissipativity of a general class of continuous-time recurrent neural networks, in [17] the authors concerned the dissipativity analysis of complex-valued BAMNNs with time delay, article [24] dealt with the problem of global dissipativity for memristor-based inertial networks with time varying delay.*

These papers did not analyze the effect of proportional delays on the neural networks. All the results cannot be applied to (1) to obtain the global dissipativity of fuzzy BAM neural networks with proportional delays. From this viewpoint, our results of this paper also complement some earlier publications. We can conclude that the proportional delay effects on the dynamics system (1).

4 Examples and simulations results

In this section, we present two examples to illustrate the feasibility of the previous main results. We consider the following FBAMNNs system:

$$
\begin{cases}
\dot{x}_1(t) &= -b_1x_1(t) + \sum_{j=1}^{2} c_{1j}f_j(y_j(t)) + \sum_{j=2}^{n} d_{1j}f_j(y_j(q_1t)) + \sum_{j=1}^{2} e_{1j}\eta_j(t) + \bigwedge_{j=1}^{2} T_{1j}\eta_j(t) \\
&+ \bigwedge_{j=1}^{2} \alpha_{1j}f_j(y_j(q_1t)) + \bigvee_{j=1}^{2} \beta_{1j}f_j(y_j(q_1t)) + \bigvee_{j=1}^{2} S_{1j}\eta_j(t) + u_1(t) \\
\dot{x}_2(t) &= -b_2x_2(t) + \sum_{j=1}^{2} c_{2j}f_j(y_j(t)) + \sum_{j=1}^{2} d_{2j}f_j(y_j(q_1t)) + \sum_{j=1}^{2} e_{2j}\eta_j(t) + \bigwedge_{j=1}^{2} T_{2j}\eta_j(t) \\
&+ \bigwedge_{j=1}^{2} \alpha_{2j}f_j(y_j(q_1t)) + \bigvee_{j=1}^{2} \beta_{2j}f_j(y_j(q_1t)) + \bigvee_{j=1}^{2} S_{2j}\eta_j(t) + u_2(t) \\
\dot{y}_1(t) &= -b_1^1y_1(t) + \sum_{i=1}^{2} c_{1i}^1g_i(x_i(t)) + \sum_{i=1}^{2} d_{1i}^1g_i(x_i(q_2t)) + \sum_{i=1}^{2} e_{1i}^1\eta_i^1(t) + \bigwedge_{i=1}^{2} T_{1i}^1\eta_i^1(t) \\
&+ \bigwedge_{i=1}^{2} \alpha_{1i}^1g_i(x_i(q_2t)) + \bigvee_{i=1}^{2} \beta_{1i}^1g_i(x_i(q_2t)) + \bigvee_{i=1}^{2} S_{1i}^1\eta_i^1(t) + v_1(t) \\
\dot{y}_2(t) &= -b_2^1y_2(t) + \sum_{i=1}^{2} c_{2i}^1g_i(x_i(t)) + \sum_{i=1}^{2} d_{2i}^1g_i(x_i(q_2t)) + \sum_{i=1}^{2} e_{2i}^1\eta_i^1(t) + \bigwedge_{i=1}^{2} T_{2i}^1\eta_i^1(t) \\
&+ \bigwedge_{i=1}^{2} \alpha_{2i}^1g_i(x_i(q_2t)) + \bigvee_{i=1}^{2} \beta_{2i}^1g_i(x_i(q_2t)) + \bigvee_{i=1}^{2} S_{2i}^1\eta_i^1(t) + v_2(t).
\end{cases}
\tag{21}
$$

Example 4.1. *In system (21), we take $n = m = 2$.*
Let

$$
\begin{aligned}
&f_j(y_j) = 0.5(|y_j+1| - |y_j-1|),\ g_i(x_i) = 0.6(|x_i+1| - |x_i-1|),\\
&\eta_j(t) = \eta_i^1(t) = 1,\ q_1 = q_2 = 0.5,\ u_i(t) = 0.5e^{-0.6t}\tanh(t),\\
&v_j(t) = 0.6e^{-0.7t}\tanh(t),\ i,\ j = 1,\ 2.
\end{aligned}
$$

The rest of the elements are defined as follows

$$
\begin{aligned}
&B = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix},\ C = \begin{pmatrix} 0.6 & 0.4 \\ 1 & 0.9 \end{pmatrix},\ D = \begin{pmatrix} 0.2 & 0.4 \\ 0 & 1.2 \end{pmatrix},\ E = \begin{pmatrix} 0.004 & 0.001 \\ 0.006 & 0.002 \end{pmatrix},\\
&T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \alpha = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix},\ \beta = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix},\ S = \begin{pmatrix} 0.012 & 0.010 \\ 0.010 & 0.012 \end{pmatrix},\\
&B^1 = \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix},\ C^1 = \begin{pmatrix} 0.1 & 1.5 \\ 0 & 0.2 \end{pmatrix},\ D^1 = \begin{pmatrix} -1 & 0 \\ 0.6 & -1 \end{pmatrix},\ E^1 = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.02 \end{pmatrix},\\
&T^1 = \begin{pmatrix} 0.01 & 0.03 \\ 0.03 & 0.01 \end{pmatrix},\ \alpha^1 = \begin{pmatrix} 0.6 & 0.3 \\ 0.2 & 0.1 \end{pmatrix},\ \beta^1 = \begin{pmatrix} 0.5 & 0.7 \\ 0.6 & 0.8 \end{pmatrix},\ S^1 = \begin{pmatrix} 0.002 & 0.001 \\ 0.005 & 0.004 \end{pmatrix}.
\end{aligned}
$$

Assumption 2.2 is satisfied. From the Theorem 3.2, we see that system (21) is globally exponential dissipative and

$$
\tilde{\Upsilon}_2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^4 \Big\backslash |x_1| + |x_2| \le \frac{13.757}{3-\xi}, |y_1| + |y_2| \le \frac{11.162}{4-\xi} \right\}
$$

is a globally exponential attractive set of system (21), where $0 < \xi < 3$. Moreover the simulation result for the considered system (21) is presented in Fig. 1. Fig. 1 depicts the state trajectories performance with respect to the set $\tilde{\Upsilon}_2$ and also reveals the effective performance of the proposed result to stabilize the system.

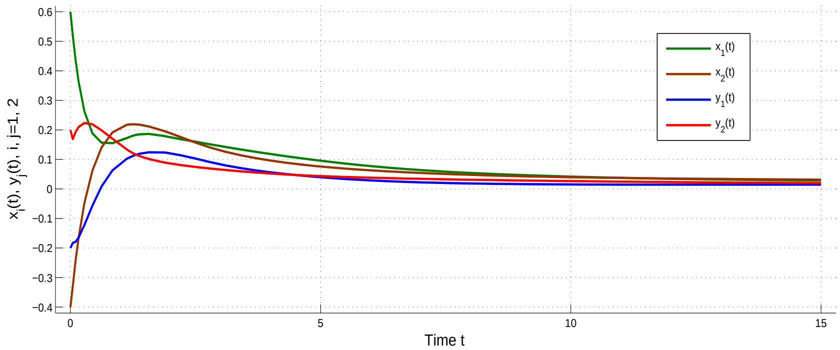

Figure 1: State trajectories of the FBAMNNs (21) in Example 4.1

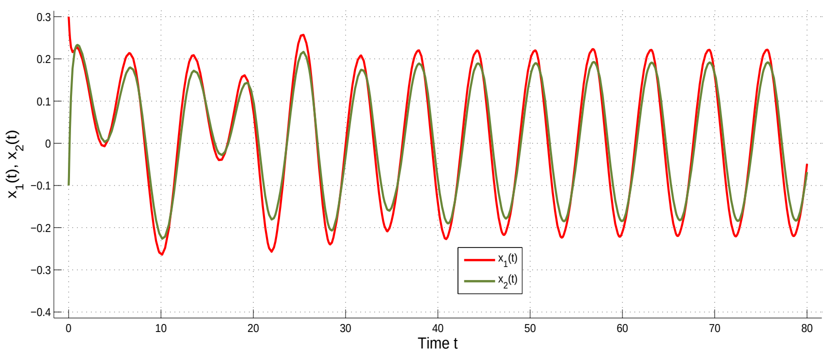

Figure 2: State trajectories of the FBAMNNs (21) with $q_1 = q_2 = 0.5$.

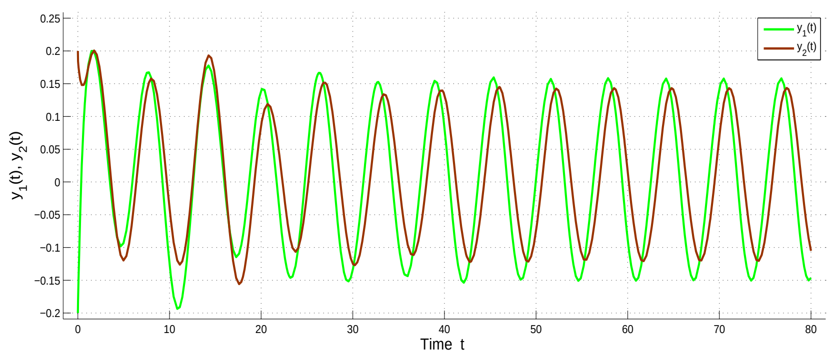

Figure 3: State trajectories of the FBAMNNs (21) with $q_1 = q_2 = 0.5$.

Example 4.2. *Consider system (21) with $n = m = 2$ and*

$$
\begin{aligned}
&g_i(x_i) = 0.3(x_i + \tanh(x_i)),\ f_j(y_j) = 0.6(y_j + \tanh(y_j)),\\
&\eta_j(t) = \eta_i^1(t) = 1,\ i,\ j = 1,\ 2,\\
&q_1 = q_2 = 0.5,\ u_1(t) = e^{-0.2t}\sin(t),\ u_2(t)e^{-0.3t}\sin(t),\\
&v_1(t) = e^{-0.4t}\cos(t),\ v_2(t) = e^{-0.5t}\cos(t).
\end{aligned}
$$

The rest of the elements are defined as follows

$$B = \begin{pmatrix} 11 & 0 \\ 0 & 9 \end{pmatrix},\ C = \begin{pmatrix} 1.6 & -0.1 \\ -2 & 0.3 \end{pmatrix},\ D = \begin{pmatrix} -1.4 & -0.1 \\ -0.2 & -2.3 \end{pmatrix},\ E = \begin{pmatrix} 0.04 & 0.03 \\ 0.04 & 0.03 \end{pmatrix},$$
$$T = \begin{pmatrix} 0.02 & 0.03 \\ 0.04 & 0.01 \end{pmatrix},\ \alpha = \begin{pmatrix} 1.5 & -1 \\ 1.5 & 1.5 \end{pmatrix},\ \beta = \begin{pmatrix} 0 & -0.5 \\ 0.5 & -0.1 \end{pmatrix},\ S = \begin{pmatrix} 0.01 & 0.02 \\ 0.01 & 0.02 \end{pmatrix},$$
$$B^1 = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix},\ C^1 = \begin{pmatrix} 1.1 & 0.5 \\ -0.3 & 2.2 \end{pmatrix},\ D^1 = \begin{pmatrix} -3 & -2.5 \\ 0.3 & -1.2 \end{pmatrix},\ E^1 = \begin{pmatrix} 0.01 & 0.05 \\ 0.05 & 0.02 \end{pmatrix},$$
$$T^1 = \begin{pmatrix} 0.01 & 0.01 \\ 0.02 & 0.02 \end{pmatrix},\ \alpha^1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},\ \beta^1 = \begin{pmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{pmatrix},\ S^1 = \begin{pmatrix} 0.004 & 0.003 \\ 0.001 & 0.002 \end{pmatrix}.$$

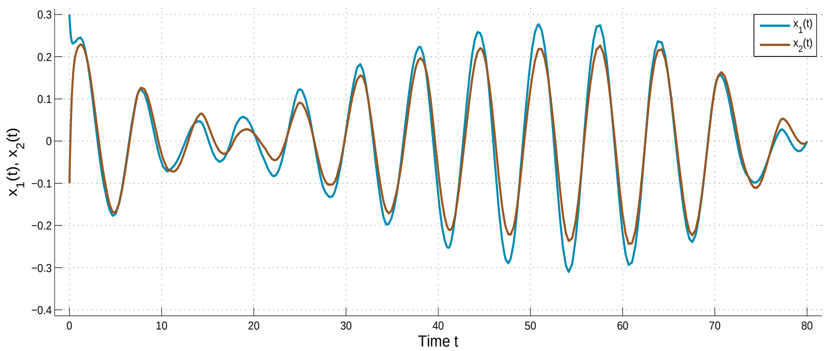

Figure 4: State trajectories of the FBAMNNs (21) with $q_1 = q_2 = 0.1$.

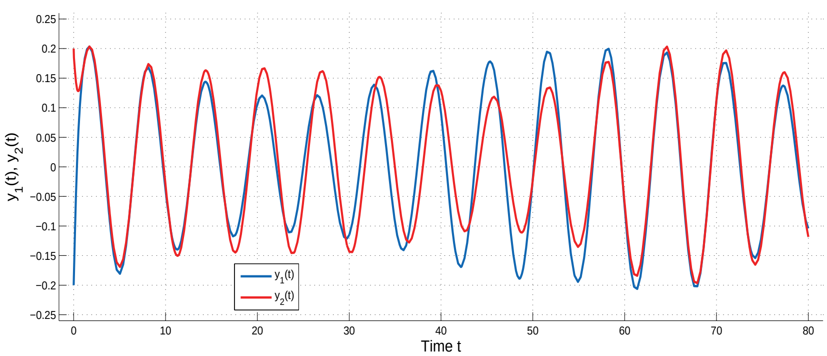

Figure 5: State trajectories of the FBAMNNs (21) with $q_1 = q_2 = 0.1$.

Now, by considering $0 < \frac{d\tanh(t)}{dt} < 1$, *it can be seen that the Assumption 2.3 is satisfied with* $l_i^g = 0.6$, $l_j^f = 1.2$. *Also, we choose* $\lambda_1 = 0.1,\ \lambda_2 = 0.9,\ \chi_1 = 0.2 \quad \chi_2 = 0.8,$

$$N = \begin{pmatrix} -16.5 & 0 \\ 0 & -18 \end{pmatrix},\ \tilde{N} = \begin{pmatrix} -24 & 0 \\ 0 & -24 \end{pmatrix}.$$

For simulation purpose, we consider the remaining parameters are same as above, with the above considered parameter values and by solving the LMIs (13)-(14) *in Theorem 3.3 via the Matlab LMI toolbox then we get a feasible solutions and the obtained positive definite matrix is given by*

$$P = \begin{pmatrix} 10.783 & 0.610 \\ 0.610 & 10.991 \end{pmatrix}.$$

Also the considered parameters and the obtained result satisfies the conditions of Theorem 3.3 and we conclude that system (21) is globally dissipative and

$$\Upsilon_3 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^4 \Big\backslash \begin{pmatrix} |x_1 + \tanh(x_1)| \\ |x_2 + \tanh(x_2)| \end{pmatrix} \leq \begin{pmatrix} 1.045 \\ 1.300 \end{pmatrix}, \begin{pmatrix} |y_1 + \tanh(y_1)| \\ |y_2 + \tanh(y_2)| \end{pmatrix} \leq \begin{pmatrix} 2.415 \\ 2.473 \end{pmatrix} \right\}$$

is a positive invariant and globally attractive set of (21). In order to show the effectiveness of the proposed result, the simulation result obtained for system (21) with the above parameter values is provided. To be precise, Fig. 6 depicts the state response of system (21) with respect to the set Υ_3 which ensures the globally dissipativity analysis of the addressed system.

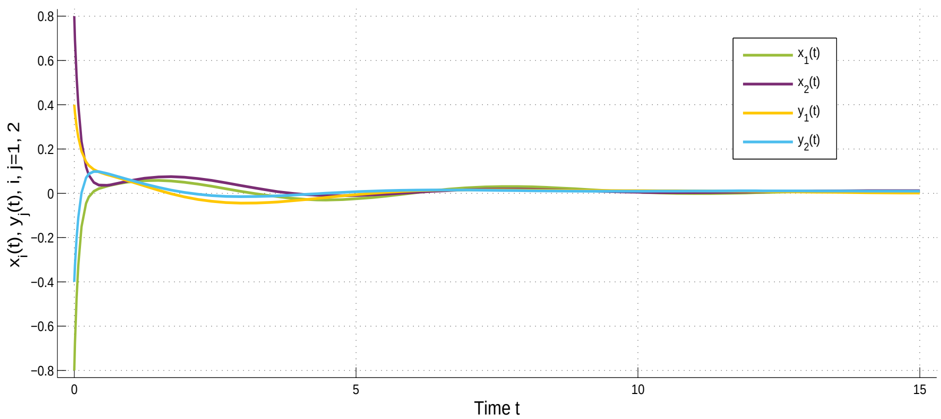

Figure 6: State trajectories of the FBAMNNs (21) in Example 4.2

Remark 4.3. *Consider system (21) with $n = m = 2$*
Let

$$\begin{aligned} &f_j(y_j) = 0.5(|y_j + 1| - |y_j - 1|),\ g_i(x_i) = 0.6(|x_i + 1| - |x_i - 1|), \\ &\eta_j(t) = \eta_i^1(t) = 1, \\ &q_1 = q_2 = 0.5, u_i(t) = 0.5\cos(t), \\ &v_j(t) = 0.6\sin(t),\ i,\ j = 1,\ 2. \end{aligned}$$

The rest of the elements are similar to Example 4.1.
Conclusion: all the conditions and Theorem 3.2 are satisfied. Therefore, system (21) is globally exponential dissipative and

$$\tilde{\Upsilon}_2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^4 \Big\backslash |x_1| + |x_2| \leq \frac{13.757}{3-\xi}, |y_1| + |y_2| \leq \frac{11.162}{4-\xi} \right\}$$

is a globally exponential attractive set of system (21), where $0 < \xi < 3$.
Figures 2, 3 shows state trajectories of $x(t)$ and $y(t)$ respectively of system (21) with $q_1 = q_2 = 0.5$ and figures 4, 5 shows state trajectories of $x(t)$ and $y(t)$ respectively of system (21) with $q_1 = q_2 = 0.1$ Underlining a very remarkable difference between the figures 2,3. In the figures 2, 3, the state trajectories of $x(t)$ and $y(t)$ are stable. Opposite of that, in the figures 4, 5, the state trajectories of $x(t)$ and $y(t)$ are unstable. Therefore, the effects of the delay value q_1 and q_2 can make the system unstable.

5 Conclusions

In this paper, we investigated the global dissipativity problem for a class of FBAMNNs with proportional delays. Our approach is based on Lyapunov functionals which simplifies the resolution of our aforementioned model and LMIs approach which is only need tuning of parameters and matrices and can be easily checked by resorting to the Matlab LMI toolbox. Then, we obtained new sufficient conditions guarantee the global dissipativity and global exponential dissipativity of the proposed model. Finally, numerical examples with their simulations are presented to prove the applicability of the new mains results. To our best knowledge, there are no work about the global dissipativity of FBAMNNs with proportional delays. The method of this manuscript can be used to study other biological and economic models.

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