Iranian Journal of Fuzzy Systems

Volume 18, Number 2, (2021), pp. 109-125

Original paper

University of Sistan and Baluchestan

Semi-t-operators and idempotent semi-t-operators on bounded lattices

Y. M. Wang[1], Z. B. Li[2] and H. W. Liu[3]

[1,3]*School of Mathematics, Shandong University, 250100 Jinan, PR China*
[2]*School of Statistics, Shandong University of Finance and Economics, 250014 Jinan, PR China*

623073044@qq.com, lizhenbo@sdufe.edu.cn, hw.liu@sdu.edu.cn

Abstract

Recently, Fang and Hu introduced the definition of semi-t-operators on bounded lattices. In this study, we propose several construction methods to obtain such a semi-t-operator from a given semi-t-conorm and semi-t-norm on bounded lattices. Furthermore, we discuss the presence of idempotent semi-t-operators on bounded lattices, and show several different methods for construction of idempotent semi-t-operators on bounded lattices with additional constraints.

Keywords: Semi-t-operators, bounded lattices, idempotent semi-t-operators.

1 Introduction

1.1 A brief review on semi-t-operators

T-operators and nullnorms as aggregation operators were introduced in [2, 3], respectively, and it was shown that they generalize the notions of t-norms and t-conorms. The idea of t-operators is to reduce the border conditions requiring only continuity of the partial functions on 0 and 1. Whereas the idea of nullnorms is to allow an absorbing element to lie anywhere in the unit interval. Curiously, both generalizations coincide as it is deduced from their corresponding characterizations given in [20] and [2], respectively. In fact, it is proved in [20] that t-operators must have an absorbing element in the unit interval and then their characterization becomes equal to the characterization of nullnorms in [2].

Later, motivated by the generalizations of t-norms and t-conorms, Drygaś generalized t-operators (nullnorms) and introduced semi-t-operators (semi-nullnorms) by removing the commutativity from the set of axioms of t-operators (nullnorms). However, these two generalizations [8] are different since they have different block structures on $[0, 1]^2$. These operators are interesting not only from a theoretical point of view (since their structures are given by a combination of a t-norm and a t-conorm), but also for their applications, since they have proved to be useful in several fields as expert system, neural networks, decision making or fuzzy quantifiers [10, 14, 19]. Furthermore, it is also interesting that semi-t-operators that are used as aggregation operators or in fuzzy logic maintain as much logical properties as possible. Based on this idea, many authors have been interested in studying these logical properties of semi-t-operators [8, 9, 22, 23, 26, 27, 25].

1.2 The motivation of our work

In recent years, due to wider application scope compared with their counterparts on the real unit interval, binary aggregation functions on bounded lattices have received more and more attention. The classes of aggregation operators on bounded lattices have more complicated structures. It is worth noting that unlike the certainty of the structures of binary aggregation operators on the unit interval, their structures on any bounded lattice are uncertain and diverse. There are many methods for constructing t-norms, t-conorms, uninorms and nullnorms on bounded lattices presented by different authors [4, 5, 6, 7, 11, 12, 15, 16, 17, 18, 21, 24].

Corresponding Author: H. W. Liu

Received: April 2020; Revised: September 2020; Accepted: November 2020.

However, it is difficult to generalize semi-t-operators to bounded lattices because continuity cannot be defined on bounded lattices. Recently, Fang and Hu [13] introduced the definition of a semi-t-operator on a bounded lattice in a very clever way using surjective mappings instead of continuity, and illustrated the presence of such a semi-t-operator by showing some construction methods. Then naturally there is a question: Are there any other construction methods for a semi-t-operator on the bounded lattice different from the known ones? Fortunately, we do find that there are other methods for constructing semi-t-operators on the bounded lattice. We replace the value of semi-t-operators on $I_b^a \times L$ ($I_a^b \times L$) with other elements in L. In addition, Çaylı and Karaçal [6] have proved that an idempotent nullnorm need not always exist on an arbitrary bounded lattice. Then there is another question: Does there exist an idempotent semi-t-operator on any bounded lattice? We prove that an idempotent semi-t-operator need not always exist on an arbitrary bounded lattice, and introduce the construction methods for idempotent semi-t-operators on bounded lattices with some additional constraints.

The remainder of this paper is organized as follows. In Section 2, we mainly recall the definition and properties of semi-t-operators on bounded lattices. In Section 3, we present some construction methods to obtain a semi-t-operator on a bounded lattice, and give the relations of all semi-t-operators on the bounded lattice showed in this section. We illustrate the presence of idempotent semi-t-operators on bounded lattices with additional constraints in Section 4, and some methods for constructing such semi-t-operators are then proposed. Section 5 contains our conclusions.

2 Preliminaries

We assume that the basic theory of bounded lattices, semi-t-norms, semi-t-conorms, semi-nullnorms and semi-t-operators are familiar to the readers. One can find the definitions, notions and results on them in [1, 8, 13]. In this section we just recall a few definitions concerning bounded lattices and semi-t-operators on them.

A lattice $(L, \leq, \wedge, \vee)$ is called *bounded* if it has the top element (written as 1_L) and the bottom element (written as 0_L), that is, there exist two elements $1_L, 0_L \in L$ such that $0_L \leq x \leq 1_L$ for all $x \in L$.

From now on, we will assume that $(L, \leq, 0_L, 1_L)$ is a bounded lattice. Let $s, t \in L$ and $s \leq t$, the subinterval $[s,t]$ of L is defined by $[s,t] = \{x \in L \mid s \leq x \leq t\}$. Other subintervals of L such as $(s,t]$, $[s,t)$ and (s,t) can be similarly defined.

Definition 2.1. [1] *Let $s, t \in L$, if s and t are incomparable, we use the notation $s \parallel t$. The other case we denote by $s \nparallel t$. The set of elements in L incomparable with s is denoted by I_s, that is, $I_s = \{x \in L \mid x \parallel s\}$.*

Definition 2.2. [13] *A binary operation $T : L^2 \to L$ ($S : L^2 \to L$) is called a semi-t-norm (semi-t-conorm) if it is associative, non-decreasing in each variable and has a neutral 1_L (0_L), i.e., $T(1_L, x) = T(x, 1_L) = x$ ($S(0_L, x) = S(x, 0_L) = x$) for all $x \in L$.*

Definition 2.3. [13] *A binary operation $V : L^2 \to L$ is called a semi-nullnorm if it is associative, non-decreasing in each variable and has an absorbing element $k \in L$ such that*
(i) $V(0_L, x) = V(x, 0_L) = x$ for all $x \leqslant k$,
(ii) $V(1_L, x) = V(x, 1_L) = x$ for all $x \geqslant k$.

Definition 2.4. [13] *A binary operation $F : L^2 \to L$ is called a semi-t-operator if it is associative, non-decreasing in each variable, fulfills $F(0_L, 0_L) = 0_L, F(1_L, 1_L) = 1_L$ and such that the functions $F_{0_L} \in \mathcal{M}(L, [0_L, a])$, $F^{0_L} \in \mathcal{M}(L, [0_L, b])$, $F_{1_L} \in \mathcal{M}(L, [b, 1_L])$, $F^{1_L} \in \mathcal{M}(L, [a, 1_L])$, where $\mathcal{M}(X, Y)$ denotes the family of all surjective mappings from X to Y and $F_{0_L}(x) = F(0_L, x), F_{1_L}(x) = F(1_L, x), F^{0_L}(x) = F(x, 0_L)$, $F^{1_L}(x) = F(x, 1_L)$, $F(0_L, 1_L) = a$ and $F(1_L, 0_L) = b$.*

By $I_a^b = \{x \in L \mid x \parallel a \text{ and } x \nparallel b\}$ we denote the set of elements which are incomparable with a but comparable with b. Similarly, by $I_b^a = \{x \in L \mid x \parallel b \text{ and } x \nparallel a\}$ we denote the set of elements which are incomparable with b but comparable with a. By $I_{a,b} = \{x \in L \mid x \parallel a \text{ and } x \parallel b\}$ we denote the set of elements which are incomparable with a and b

Proposition 2.5. [13] *Let $a, b \in L \backslash \{0_L, 1_L\}$, F be a semi-t-operator on L with $F(0_L, 1_L) = a$ and $F(1_L, 0_L) = b$.*
(i) $F(x, y) = a$ for $(x, y) \in [0_L, a] \times [a, 1_L]$.
(ii) $F(x, y) = b$ for $(x, y) \in [b, 1_L] \times [0_L, b]$.

Proposition 2.6. [13] *Let $a, b \in L \backslash \{0_L, 1_L\}$, F be a semi-t-operator on L with $F(0_L, 1_L) = a$ and $F(1_L, 0_L) = b$.*
1. *If $a \leq b$, then*
(a) $F(x, y) = x$ for $(x, y) \in [a, b] \times L$.
(b) $F |_{[0_L, a]^2} : [0_L, a]^2 \to [0_L, a]$ is a semi-t-conorm on $[0_L, a]$.

(c) $F\mid_{[b,1_L]^2}\colon [b,1_L]^2 \to [b,1_L]$ *is a semi-t-norm on* $[b,1_L]$.

2. *If* $b \le a$, *then*
   (a) $F(x,y) = y$ *for* $(x,y) \in L \times [b,a]$.
   (b) $F\mid_{[0_L,b]^2}\colon [0_L,b]^2 \to [0_L,b]$ *is a semi-t-conorm on* $[0_L,b]$.
   (c) $F\mid_{[a,1_L]^2}\colon [a,1_L]^2 \to [a,1_L]$ *is a semi-t-norm on* $[a,1_L]$.

We can easily obtain the following proposition from the monotonicity of semi-t-operators on bounded lattices.

Proposition 2.7. *Let* $a,b \in L\backslash\{0_L,1_L\}$, F *be a semi-t-operator on* L *with* $F(0_L,1_L)=a$ *and* $F(1_L,0_L)=b$.

1. *If* $a \le b$, *then*
   (a) $F(x,y)=a$ *for* $(x,y) \in [0_L,a] \times I_b^a$.
   (b) $F(x,y)=b$ *for* $(x,y) \in [b,1_L] \times I_a^b$.
   (c) $a \le F(x,y) \le b$ *for* $(x,y) \in I_{a,b} \times [a,b]$.
   (d) $(x \wedge a) \vee (y \wedge a) \le F(x,y) \le a$ *for* $(x,y) \in [0_L,a] \times I_a^b \cup [0_L,a] \times I_{a,b}$.
   (e) $b \le F(x,y) \le (x \vee b) \wedge (y \vee b)$ *for* $(x,y) \in [b,1_L] \times I_b^a \cup [b,1_L] \times I_{a,b}$.
   (f) $(x \wedge a) \vee (y \wedge a) \le F(x,y) \le x \vee a$ *for* $(x,y) \in I_a^b \times [0_L,a] \cup I_a^b \times I_a^b \cup I_a^b \times I_{a,b}$.
   (g) $a \le F(x,y) \le x \vee a$ *for* $(x,y) \in I_a^b \times [a,b] \cup I_a^b \times [b,1_L] \cup I_a^b \times I_b^a$.
   (h) $x \wedge b \le F(x,y) \le b$ *for* $(x,y) \in I_b^a \times [0_L,a] \cup I_b^a \times [a,b] \cup I_b^a \times I_a^b$.
   (i) $x \wedge b \le F(x,y) \le (x \vee b) \wedge (y \vee b)$ *for* $(x,y) \in I_b^a \times [b,1_L] \cup I_b^a \times I_b^a \cup I_b^a \times I_{a,b}$.
   (j) $(x \wedge a) \vee (y \wedge a) \le F(x,y) \le b$ *for* $(x,y) \in I_{a,b} \times [0_L,a] \cup I_{a,b} \times I_a^b$.
   (k) $a \le F(x,y) \le (x \vee b) \wedge (y \vee b)$ *for* $(x,y) \in I_{a,b} \times [b,1_L] \cup I_{a,b} \times I_b^a$.
2. *If* $b \le a$, *then*
   (a) $F(x,y)=b$ *for* $(x,y) \in I_a^b \times [0_L,b]$.
   (b) $F(x,y)=a$ *for* $(x,y) \in I_b^a \times [a,1_L]$.
   (c) $b \le F(x,y) \le a$ *for* $(x,y) \in [b,a] \times I_{a,b}$.
   (d) $(x \wedge b) \vee (y \wedge b) \le F(x,y) \le b$ *for* $(x,y) \in I_b^a \times [0_L,b] \cup I_{a,b} \times [0_L,b]$.
   (e) $a \le F(x,y) \le (x \vee a) \wedge (y \vee a)$ *for* $(x,y) \in I_a^b \times [a,1_L] \cup I_{a,b} \times [a,1_L]$.
   (f) $(x \wedge b) \vee (y \wedge b) \le F(x,y) \le y \vee b$ *for* $(x,y) \in [0_L,b] \times I_b^a \cup I_b^a \times I_b^a \cup I_{a,b} \times I_b^a$.
   (g) $b \le F(x,y) \le y \vee b$ *for* $(x,y) \in [b,a] \times I_b^a \cup [a,1_L] \times I_b^a \cup I_a^b \times I_b^a$.
   (h) $y \wedge a \le F(x,y) \le a$ *for* $(x,y) \in [0_L,b] \times I_a^b \cup [b,a] \times I_a^b \cup I_b^a \times I_a^b$.
   (i) $y \wedge a \le F(x,y) \le (x \vee a) \wedge (y \vee a)$ *for* $(x,y) \in [a,1_L] \times I_a^b \cup I_a^b \times I_a^b \cup I_{a,b} \times I_a^b$.
   (j) $(x \wedge b) \vee (y \wedge b) \le F(x,y) \le a$ *for* $(x,y) \in [0_L,b] \times I_{a,b} \cup I_b^a \times I_{a,b}$.
   (k) $b \le F(x,y) \le (x \vee a) \wedge (y \vee a)$ *for* $(x,y) \in [a,1_L] \times I_{a,b} \cup I_a^b \times I_{a,b}$.

3 Construction of semi-t-operators on L

In this section, we propose some construction methods to obtain a semi-t-operator from a given semi-t-conorm and semi-t-norm on bounded lattices. At first we will assume that all points in L are comparable with a and b.

Theorem 3.1. *Let* $a,b \in L\backslash\{0_L,1_L\}$, $I_a^b=\emptyset$, $I_b^a=\emptyset$ *and* $I_{a,b}=\emptyset$. *A binary operation* $F: L^2 \to L$ *is a semi-t-operator on* L *with* $F(0_L,1_L)=a$ *and* $F(1_L,0_L)=b$ *if and only if there exist a semi-t-norm* T *and a semi-t-conorm* S *such that*

$$F(x,y)=\begin{cases} S(x,y) & \text{if } (x,y)\in[0_L,a]^2,\\ T(x,y) & \text{if } (x,y)\in[b,1_L]^2,\\ x & \text{if } (x,y)\in[a,b]\times L,\\ a & \text{if } (x,y)\in[0_L,a]\times[a,1_L],\\ b & \text{if } (x,y)\in[b,1_L]\times[0_L,b],\end{cases} \tag{1}$$

when $a \le b$ *and*

$$F(x,y)=\begin{cases} S(x,y) & \text{if } (x,y)\in[0_L,a]^2,\\ T(x,y) & \text{if } (x,y)\in[b,1_L]^2,\\ y & \text{if } (x,y)\in L\times[a,b],\\ a & \text{if } (x,y)\in[0_L,a]\times[a,1_L],\\ b & \text{if } (x,y)\in[b,1_L]\times[0_L,b],\end{cases} \tag{2}$$

when $b \le a$.

Proof. The proof is obvious from Theorem 4.1 in [13]. □

3.1 When $a \le b$

We will introduce several new constructions of semi-t-operators with $a \le b$ on an arbitrary bounded lattice L in this subsection. Compared with the semi-t-operators given in Theorem 4.1 in [13], it should be pointed out that we try to replace the corresponding value a on $I_b^a \times L$ (resp. b on $I_a^b \times L$) with b (resp. a) in the following theorem.

Theorem 3.2. *Let $a, b \in L \backslash \{0_L, 1_L\}$ and $a \le b$. The following functions $F_S^1, F_T^1 : L^2 \to L$ are semi-t-operators on L.*

$$F_S^1(x,y) = \begin{cases} S(x,y) & \text{if } (x,y) \in [0_L, a]^2, \\ T(x,y) & \text{if } (x,y) \in [b, 1_L]^2, \\ S(x \wedge a, y \wedge a) & \text{if } (x,y) \in [0_L, a] \times I_a^b \cup [0_L, a] \times I_{a,b} \cup I_a^b \times [0_L, a] \cup I_a^b \times I_a^b \\ & \quad \cup I_a^b \times I_{a,b} \cup I_{a,b} \times [0_L, a] \cup I_{a,b} \times I_a^b \cup I_{a,b} \times I_{a,b}, \\ b & \text{if } (x,y) \in [b, 1_L] \times \{L \backslash [b, 1_L]\} \cup I_b^a \times L, \\ x & \text{if } (x,y) \in [a,b] \times L, \\ a & \text{otherwise}, \end{cases} \tag{3}$$

$$F_T^1(x,y) = \begin{cases} S(x,y) & \text{if } (x,y) \in [0_L, a]^2, \\ T(x,y) & \text{if } (x,y) \in [b, 1_L]^2, \\ T(x \vee b, y \vee b) & \text{if } (x,y) \in [b, 1_L] \times I_b^a \cup [b, 1_L] \times I_{a,b} \cup I_b^a \times [b, 1_L] \cup I_b^a \times I_b^a \\ & \quad \cup I_b^a \times I_{a,b} \cup I_{a,b} \times [b, 1_L] \cup I_{a,b} \times I_b^a \cup I_{a,b} \times I_{a,b}, \\ a & \text{if } (x,y) \in [0_L, a] \times \{L \backslash [0, a]\} \cup I_a^b \times L, \\ x & \text{if } (x,y) \in [a,b] \times L, \\ b & \text{otherwise}, \end{cases} \tag{4}$$

where S is a semi-t-conorm on $[0_L, a]^2$ and T is a semi-t-norm on $[b, 1_L]^2$.

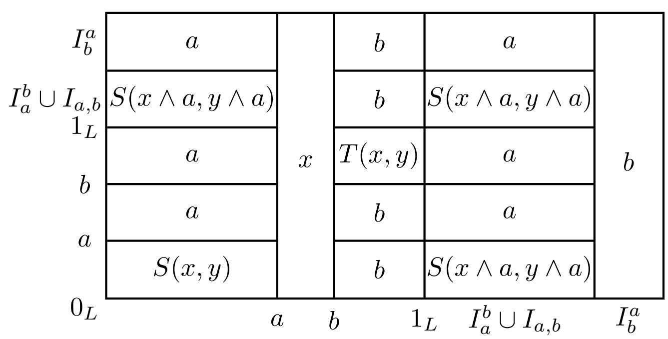

Fig.1 F_S^1 on L

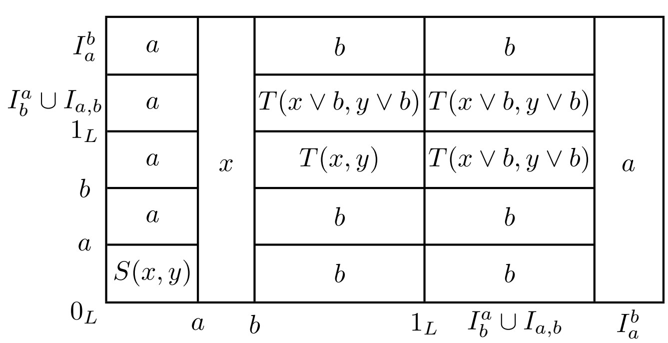

Fig.2 F_T^1 on L

Proof. We prove only that the first function is a semi-t-operator on L, as the proof for the second one is analogous. It is obvious that $F_S^1(0_L, 0_L) = 0_L$ and $F_S^1(1_L, 1_L) = 1_L$. We can easily obtain that $(F_S^1)_{0_L} \in \mathcal{M}(L, [0_L, a])$, $(F_S^1)^{0_L} \in \mathcal{M}(L, [0_L, b])$, $(F_S^1)_{1_L} \in \mathcal{M}(L, [b, 1_L])$, $(F_S^1)^{1_L} \in \mathcal{M}(L, [a, 1_L])$. Now let us verify that F_S^1 is non-decreasing in each variable. Due to Theorem 4.1 in [13], we just need to consider the following cases. Firstly, let $x, y, z \in L$ and $x \leq y$.

1. $x \in [0_L, a]$ and $y \in I_b^a$,
$F_S^1(x,z) \leq a \leq b = F_S^1(y,z)$ for all $z \in L$.
2. $x \in [a,b]$ and $y \in I_b^a$,
$F_S^1(x,z) = x \leq b = F_S^1(y,z)$ for all $z \in L$.
3. $x \in I_a^b$ and $y \in I_b^a$,
$F_S^1(x,z) \leq a \leq b = F_S^1(y,z)$ for all $z \in L$.
4. $x \in I_b^a$ and $y \in (b, 1_L]$,
$F_S^1(x,z) = b \leq F_S^1(y,z)$ for all $z \in L$.
5. $x \in I_{a,b}$ and $y \in I_b^a$,
$F_S^1(x,z) \leq a \leq b = F_S^1(y,z)$ for all $z \in L$.

Secondly, let $y, z \in L$ and $y \leq z$, then $F_S^1(x,y) = b = F_S^1(x,z)$ for $x \in I_b^a$.

Next, we will prove that F_S^1 satisfies the associativity, that is, $F_S^1(x, F_S^1(y,z)) = F_S^1(F_S^1(x,y),z)$ for all $x, y, z \in L$. Due to Theorem 4.1 in [13], we just need to consider the following cases.

1. $x \in [0_L, a]$
   1.1. $y \in [0_L, a]$ and $z \in I_b^a$,
   $F_S^1(x, F_S^1(y,z)) = F_S^1(x,a) = a = F_S^1(S(x,y),z) = F_S^1(F_S^1(x,y),z)$.
   1.2. $y \in [a,b]$ and $z \in I_b^a$,
   $F_S^1(x, F_S^1(y,z)) = F_S^1(x,y) = a = F_S^1(a,z) = F_S^1(F_S^1(x,y),z)$.
   1.3. $y \in [b, 1_L]$ and $z \in I_b^a$,
   $F_S^1(x, F_S^1(y,z)) = F_S^1(x,b) = a = F_S^1(a,z) = F_S^1(F_S^1(x,y),z)$.
   1.4. $y \in I_a^b \cup I_{a,b}$ and $z \in I_b^a$,
   $F_S^1(x, F_S^1(y,z)) = F_S^1(x,a) = a = F_S^1(S(x \wedge a, y \wedge a),z) = F_S^1(F_S^1(x,y),z)$.
   1.5. $y \in I_b^a$,
   $F_S^1(x, F_S^1(y,z)) = F_S^1(x,b) = a = F_S^1(a,z) = F_S^1(F_S^1(x,y),z)$ for all $z \in L$.
2. $x \in [a,b]$,
$F_S^1(x, F_S^1(y,z)) = x = F_S^1(x,z) = F_S^1(F_S^1(x,y),z)$ for all $y, z \in L$.
3. $x \in [b, 1_L]$
   3.1. $y \in [0_L, a]$ and $z \in I_b^a$,
   $F_S^1(x, F_S^1(y,z)) = F_S^1(x,a) = b = F_S^1(b,z) = F_S^1(F_S^1(x,y),z)$.
   3.2. $y \in [a,b]$ and $z \in I_b^a$,
   $F_S^1(x, F_S^1(y,z)) = F_S^1(x,y) = b = F_S^1(b,z) = F_S^1(F_S^1(x,y),z)$.
   3.3. $y \in [b, 1_L]$ and $z \in I_b^a$,
   $F_S^1(x, F_S^1(y,z)) = F_S^1(x,b) = b = F_S^1(T(x,y),z) = F_S^1(F_S^1(x,y),z)$.
   3.4. $y \in I_a^b \cup I_{a,b}$ and $z \in I_b^a$,
   $F_S^1(x, F_S^1(y,z)) = F_S^1(x,a) = b = F_S^1(b,z) = F_S^1(F_S^1(x,y),z)$.
   3.5. $y \in I_b^a$,
   $F_S^1(x, F_S^1(y,z)) = F_S^1(x,b) = b = F_S^1(b,z) = F_S^1(F_S^1(x,y),z)$ for all $z \in L$.
4. $x \in I_a^b \cup I_{a,b}$
   4.1. $y \in [0_L, a]$ and $z \in I_b^a$
   $F_S^1(x, F_S^1(y,z)) = F_S^1(x,a) = a = F_S^1(S(x \wedge a, y \wedge a),z) = F_S^1(F_S^1(x,y),z)$.
   4.2. $y \in [a,b]$ and $z \in I_b^a$,
   $F_S^1(x, F_S^1(y,z)) = F_S^1(x,y) = a = F_S^1(a,z) = F_S^1(F_S^1(x,y),z)$.
   4.3. $y \in [b, 1_L]$ and $z \in I_b^a$,
   $F_S^1(x, F_S^1(y,z)) = F_S^1(x,b) = a = F_S^1(a,z) = F_S^1(F_S^1(x,y),z)$.
   4.4. $y \in I_a^b \cup I_{a,b}$ and $z \in I_b^a$,
   $F_S^1(x, F_S^1(y,z)) = F_S^1(x,a) = a = F_S^1(S(x \wedge a, y \wedge a),z) = F_S^1(F_S^1(x,y),z)$.
   4.5. $y \in I_b^a$,
   $F_S^1(x, F_S^1(y,z)) = F_S^1(x,b) = a = F_S^1(a,z) = F_S^1(F_S^1(x,y),z)$ for all $z \in L$.
5. $x \in I_b^a$,
$F_S^1(x, F_S^1(y,z)) = b = F_S^1(b,z) = F_S^1(F_S^1(x,y),z)$ for all $y, z \in L$.

Summarizing, we know that F_S^1 is a semi-t-operator on L with $F_S^1(0_L, 1_L) = a \leq b = F_S^1(1_L, 0_L)$. □

In the previous theorem, the expression of semi-t-operators on $I_{a,b} \times L$ is either the same as the one on $I_a^b \times L$ or the one on $I_b^a \times L$. Now we introduce two constructions of semi-t-operators whose expression on $I_{a,b} \times L$ is different from the one on $I_a^b \times L$ and the one on $I_b^a \times L$.

Theorem 3.3. *Let $a, b \in L\backslash\{0_L, 1_L\}$ and $a \leq b$. The following functions $F_S^2, F_T^2 : L^2 \to L$ are semi-t-operators on L.*

$$F_S^2(x,y) = \begin{cases} S(x,y) & \text{if } (x,y) \in [0_L, a]^2, \\ T(x,y) & \text{if } (x,y) \in [b, 1_L]^2, \\ S(x \wedge a, y \wedge a) & \text{if } (x,y) \in [0_L, a] \times I_a^b \cup I_a^b \times [0_L, a] \cup I_a^b \times I_a^b, \\ b & \text{if } (x,y) \in [b, 1_L] \times \{L\backslash[b, 1_L]\} \cup I_b^a \times L, \\ x & \text{if } (x,y) \in [a, b] \times L, \\ a & \text{otherwise,} \end{cases} \tag{5}$$

$$F_T^2(x,y) = \begin{cases} S(x,y) & \text{if } (x,y) \in [0_L, a]^2, \\ T(x,y) & \text{if } (x,y) \in [b, 1_L]^2, \\ T(x \vee b, y \vee b) & \text{if } (x,y) \in [b, 1_L] \times I_b^a \cup I_b^a \times [b, 1_L] \cup I_b^a \times I_b^a, \\ a & \text{if } (x,y) \in [0_L, a] \times \{L\backslash[0_L, a]\} \cup I_a^b \times L, \\ x & \text{if } (x,y) \in [a, b] \times L, \\ b & \text{otherwise,} \end{cases} \tag{6}$$

where S is a semi-t-conorm on $[0_L, a]^2$ and T is a semi-t-norm on $[b, 1_L]^2$.

Proof. The proof is similar to that of Theorem 3.2. □

Replacing the corresponding value $S(x \wedge a, y \wedge a)$ (resp. $T(x \vee b, y \vee b)$) in Theorem 3.3 with element a (resp. b), we obtain the following two construction methods of semi-t-operators on L.

Theorem 3.4. *Let $a, b \in L\backslash\{0_L, 1_L\}$ and $a \leq b$. The following functions $F_S, F_T : L^2 \to L$ are semi-t-operators on L.*

$$F_S(x,y) = \begin{cases} S(x,y) & \text{if } (x,y) \in [0_L, a]^2, \\ T(x,y) & \text{if } (x,y) \in [b, 1_L]^2, \\ b & \text{if } (x,y) \in [b, 1_L] \times \{L\backslash[b, 1_L]\} \cup I_b^a \times L, \\ x & \text{if } (x,y) \in [a, b] \times L, \\ a & \text{otherwise,} \end{cases} \tag{7}$$

$$F_T(x,y) = \begin{cases} S(x,y) & \text{if } (x,y) \in [0_L, a]^2, \\ T(x,y) & \text{if } (x,y) \in [b, 1_L]^2, \\ a & \text{if } (x,y) \in [0_L, a] \times \{L\backslash[0_L, a]\} \cup I_a^b \times L, \\ x & \text{if } (x,y) \in [a, b] \times L, \\ b & \text{otherwise,} \end{cases} \tag{8}$$

where S is a semi-t-conorm on $[0_L, a]^2$ and T is a semi-t-norm on $[b, 1_L]^2$.

Proof. The proof is similar to that of Theorem 3.2. □

Proposition 3.5. *Let $a, b \in L\backslash\{0_L, 1_L\}$ and $a \leq b$. Then for each semi-t-conorm S on $[0_L, a]^2$ and semi-t-norm T on $[b, 1_L]^2$ it holds*

$$F_S^1 \leq F_S^2 \leq F_S \leq F_T \leq F_T^2 \leq F_T^1.$$

3.2 When $b \leq a$

We will introduce some new constructions of semi-t-operators with $b \leq a$ on an arbitrary bounded lattice L in this subsection. The following theorem provides us with a way to obtain the construction methods for semi-t-operators on L with $b \leq a$ from the known ones in subsection 3.1.

Theorem 3.6. *The function $F : L^2 \to L$ is a semi-t-operator on L with $a = F(0_L, 1_L)$ and $b = F(1_L, 0_L)$ if and only if the function $G : L^2 \to L$ given by $G(x,y) = F(y,x)$ for all $x, y \in L$ is a semi-t-operator on L with $b = G(0_L, 1_L)$ and $a = G(1_L, 0_L)$.*

Proof. Without loss of generality, we suppose that $a \leq b$. Let $a_1 = G(0_L, 1_L)$ and $b_1 = G(1_L, 0_L)$, then we have $a_1 = G(0_L, 1_L) = F(1_L, 0_L) = b \geq a = F(0_L, 1_L) = G(1_L, 0_L) = b_1$.

It is clear that $G(0_L, 0_L) = 0_L$ and $G(1_L, 1_L) = 1_L$. Since $F_{0_L} \in \mathcal{M}(L, [0_L, a])$ for all $x \in [0_L, a]$ there exists $z \in L$ such that $F(0_L, z) = x$. Thus for all $x \in [0, b_1]$ there exists $z \in L$ such that $G(z, 0_L) = F(0_L, z) = x$, that is $G^{0_L} \in \mathcal{M}(L, [0_L, b_1])$. Similarly, we can obtain that $G_{0_L} \in \mathcal{M}(L, [0_L, a_1])$, $G_{1_L} \in \mathcal{M}(L, [b_1, 1_L])$ and $G^{1_L} \in \mathcal{M}(L, [a_1, 1_L])$. The monotonicity of G is obvious from the monotonicity of F in both coordinates. It follows from the associativity of F that $G(x, G(y, z)) = F(F(z, y), x) = F(z, F(y, x)) = G(G(x, y), z)$ for all $x, y, z \in L$, that is, G satisfies the associativity.

Summarizing, we know that G is a semi-t-operator on L with $G(0_L, 1_L) = a_1 = b \geq a = b_1 = G(1_L, 0_L)$.

Conversely, the sufficiency can be similarly shown. □

According to the conclusion of the previous theorem, we can get the following corresponding construction methods for semi-t-operators on L with $b \leq a$ from the ones in subsection 3.1.

Theorem 3.7. *Let $a, b \in L\backslash\{0_L, 1_L\}$ and $b \leq a$. Then the following functions $F_1^S, F_1^T : L^2 \to L$ are semi-t-operators on L.*

$$F_1^S(x,y) = \begin{cases} S(x,y) & \text{if } (x,y) \in [0_L,b]^2, \\ T(x,y) & \text{if } (x,y) \in [a,1_L]^2, \\ S(x\wedge b, y\wedge b) & \text{if } (x,y) \in [0_L,b]\times I_b^a \cup [0_L,b]\times I_{a,b} \cup I_b^a \times [0_L,b] \cup I_b^a \times I_b^a \\ & \qquad \cup I_b^a \times I_{a,b} \cup I_{a,b}\times[0_L,b] \cup I_{a,b}\times I_b^a \cup I_{a,b}\times I_{a,b}, \\ a & \text{if } (x,y) \in \{L\backslash[a,1_L]\}\times[a,1_L] \cup L\times I_a^b, \\ y & \text{if } (x,y) \in L\times[b,a], \\ b & \text{otherwise,} \end{cases} \tag{9}$$

$$F_1^T(x,y) = \begin{cases} S(x,y) & \text{if } (x,y) \in [0_L,b]^2, \\ T(x,y) & \text{if } (x,y) \in [a,1_L]^2, \\ T(x\vee a, y\vee a) & \text{if } (x,y) \in [a,1_L]\times I_a^b \cup [a,1_L]\times I_{a,b} \cup I_a^b \times [a,1_L] \cup I_a^b \times I_a^b \\ & \qquad \cup I_a^b \times I_{a,b} \cup I_{a,b}\times[a,1_L] \cup I_{a,b}\times I_a^b \cup I_{a,b}\times I_{a,b}, \\ b & \text{if } (x,y) \in \{L\backslash[0_L,b]\}\times[0_L,b] \cup L\times I_b^a, \\ y & \text{if } (x,y) \in L\times[b,a], \\ a & \text{otherwise,} \end{cases} \tag{10}$$

where S is a semi-t-conorm on $[0_L, b]^2$ and T is a semi-t-norm on $[a, 1_L]^2$.

Theorem 3.8. *Let $a, b \in L\backslash\{0_L, 1_L\}$ and $b \leq a$. Then the following functions $F_2^S, F_2^T : L^2 \to L$ are semi-t-operators on L.*

$$F_2^S(x,y) = \begin{cases} S(x,y) & \text{if } (x,y) \in [0_L,b]^2, \\ T(x,y) & \text{if } (x,y) \in [a,1_L]^2, \\ S(x\wedge b, y\wedge b) & \text{if } (x,y) \in [0_L,b]\times I_b^a \cup I_b^a\times[0_L,b] \cup I_b^a\times I_b^a, \\ a & \text{if } (x,y) \in \{L\backslash[a,1_L]\}\times[a,1_L] \cup L\times I_a^b, \\ y & \text{if } (x,y) \in L\times[b,a], \\ b & \text{otherwise,} \end{cases} \tag{11}$$

$$F_2^T(x,y) = \begin{cases} S(x,y) & \text{if } (x,y) \in [0_L,b]^2, \\ T(x,y) & \text{if } (x,y) \in [a,1_L]^2, \\ T(x\vee a, y\vee a) & \text{if } (x,y) \in [a,1_L]\times I_a^b \cup I_a^b\times[a,1_L] \cup I_a^b\times I_a^b, \\ b & \text{if } (x,y) \in \{L\backslash[0_L,b]\}\times[0_L,b] \cup L\times I_b^a, \\ y & \text{if } (x,y) \in L\times[b,a], \\ a & \text{otherwise,} \end{cases} \tag{12}$$

where S is a semi-t-conorm on $[0_L, b]^2$ and T is a semi-t-norm on $[a, 1_L]^2$.

Theorem 3.9. *Let $a, b \in L\backslash\{0_L, 1_L\}$ and $b \leq a$. Then the following functions $F^S, F^T : L^2 \to L$ are semi-t-operators on L.*

$$F^S(x,y) = \begin{cases} S(x,y) & \text{if } (x,y) \in [0_L, b]^2, \\ T(x,y) & \text{if } (x,y) \in [a, 1_L]^2, \\ a & \text{if } (x,y) \in \{L\backslash[a,1_L]\} \times [a,1_L] \cup L \times I_a^b, \\ y & \text{if } (x,y) \in L \times [b,a], \\ b & \text{otherwise}, \end{cases} \tag{13}$$

$$F^T(x,y) = \begin{cases} S(x,y) & \text{if } (x,y) \in [0_L, b]^2, \\ T(x,y) & \text{if } (x,y) \in [a, 1_L]^2, \\ b & \text{if } (x,y) \in \{L\backslash[0_L,b]\} \times [0_L,b] \cup L \times I_b^a, \\ y & \text{if } (x,y) \in L \times [b,a], \\ a & \text{otherwise}, \end{cases} \tag{14}$$

where S is a semi-t-conorm on $[0_L, b]^2$ and T is a semi-t-norm on $[a, 1_L]^2$.

Example 3.10. *Let $(L = \{0_L, b, x_1, x_2, a, x_3, n, m, p, 1_L\}, \leq, 0_L, 1_L)$ be a bounded lattices given by Fig.3.*

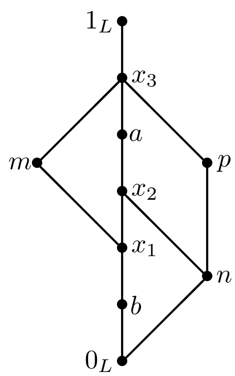

Fig.3 The lattice L

Table 1 S *on* $[0_L, b]$

S	0_L	b
0_L	0_L	b
b	b	b

Table 2 T *on* $[a, 1_L]$

T	a	x_3	1_L
a	a	a	a
x_3	a	a	x_3
1_L	a	x_3	1_L

First, we define the semi-t-conorm S on $[0_L, b]$ as shown in Table 1 and the semi-t-norm T on $[a, 1_L]$ as shown in Table 2. It follows from Theorem 3.7 that F_1^S on L in Table 3 and F_1^T on L in Table 4 are semi-t-operators.

Table 3 F_1^S *on* L

F_1^S	0_L	b	x_1	x_2	a	x_3	1_L	n	m	p
0_L	0_L	b	b	b	b	b	b	0_L	b	0_L
b	b	b	b	b	b	b	b	b	b	b
x_1	x_1	x_1	x_1	x_1	x_1	x_1	x_1	x_1	x_1	x_1
x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2
a	a	a	a	a	a	a	a	a	a	a
x_3	a	a	a	a	a	a	x_3	a	a	a
1_L	a	a	a	a	a	x_3	1_L	a	a	a
n	0_L	b	b	b	b	b	b	0_L	b	0_L
m	a	a	a	a	a	a	a	a	a	a
p	0_L	b	b	b	b	b	b	0_L	b	0_L

Table 4 F_1^T *on* L

F_1^T	0_L	b	x_1	x_2	a	x_3	1_L	n	m	p
0_L	0_L	b	b	b	b	b	b	b	b	b
b	b	b	b	b	b	b	b	b	b	b
x_1	x_1	x_1	x_1	x_1	x_1	x_1	x_1	x_1	x_1	x_1
x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2
a	a	a	a	a	a	a	a	a	a	a
x_3	a	a	a	a	a	a	x_3	a	a	a
1_L	a	a	a	a	a	x_3	1_L	a	x_3	x_3
n	b	b	b	b	b	b	b	b	b	b
m	a	a	a	a	a	a	x_3	a	a	a
p	a	a	a	a	a	a	x_3	a	a	a

Proposition 3.11. *Let $a, b \in L\backslash\{0_L, 1_L\}$ and $b \leq a$. Then for each semi-t-conorm S on $[0_L, b]^2$ and semi-t-norm T on $[a, 1_L]^2$ it holds*

$$F_1^S \leq F_2^S \leq F^S \leq F^T \leq F_2^T \leq F_1^T.$$

4 Construction of idempotent semi-t-operators on L

In this section, we will investigate the presence of idempotent semi-t-operators on bounded lattices.

Definition 4.1. *Let $a, b \in L \backslash \{0_L, 1_L\}$ and F be a semi-t-operator on L with $F(0_L, 1_L) = a$ and $F(1_L, 0_L) = b$. F is called an idempotent semi-t-operator if $F(x, x) = x$ for all $x \in L$.*

4.1 When $a \le b$

Theorem 4.2. *Let $a, b \in L \backslash \{0_L, 1_L\}$ and $a \le b$. If L contains a sublattice which is isomorphic to one of the sublattices*

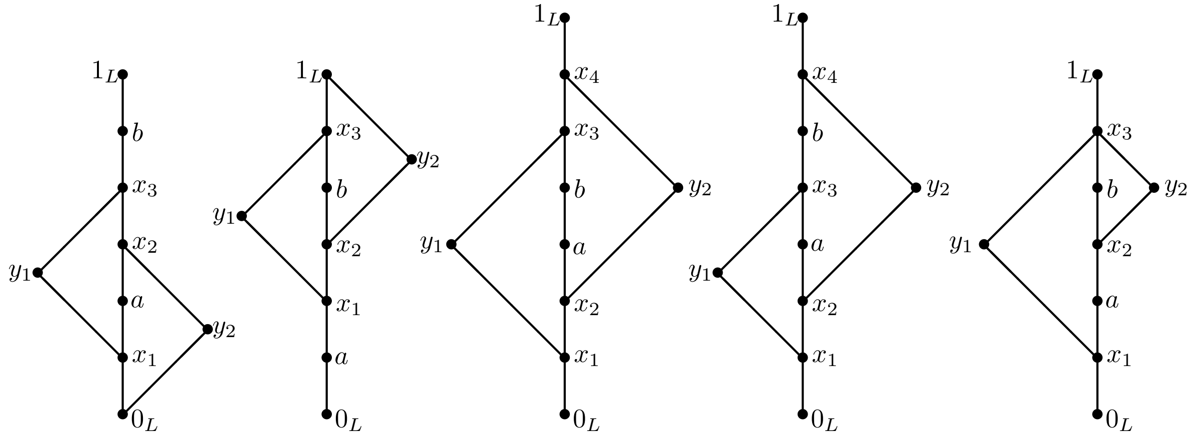

Fig.4 *Fig.5* *Fig.6* *Fig.7* *Fig.8*

characterized by Hasse diagram in $Fig.4-Fig.8$, then there is no idempotent semi-t-operator on L.

Proof. Suppose that there exists an idempotent semi-t-operator F on L which is characterized by Hasse diagram in Fig.5. From Proposition 2.7(i) it follows that $x_2 = y_2 \wedge b \le F(y_2, y_1) \le (y_2 \vee b) \wedge (y_1 \vee b) = x_3$, which means $F(y_2, y_1) \in \{x_2, b, x_3\}$.

(1) Suppose that $F(y_2, y_1) = x_2$. From Proposition 2.7(e) and the idempotency of F, it follows that $b \le F(1, y_2) \le 1$ and $F(1, y_2) \ge F(y_2, y_2) = y_2$, that is, $F(1, y_2) = 1$. Similarly, we have $F(1, y_1) = x_3$. Thus,

$b = F(1, x_2) = F(1, F(y_2, y_1)) = F(F(1, y_2), y_1) = F(1, y_1) = x_3,$

which is a contradiction.

(2) Suppose that $F(y_2, y_1) = b$. From Proposition 2.7(h) and the idempotency of F, it follows that $x_1 \le F(y_1, 0) \le b$ and $F(y_1, 0) \le F(y_1, y_1) = y_1$, that is, $F(y_1, 0) = x_1$. Similarly, we have $F(y_2, x_1) = x_2$. Thus,

$b = F(b, 0) = F(F(y_2, y_1), 0) = F(y_2, F(y_1, 0)) = F(y_2, x_1) = x_2,$

which is a contradiction.

(3) Suppose that $F(y_2, y_1) = x_3$. Similarly as above we can get $F(y_2, x_1) = x_2$. Thus,

$b = F(x_3, 0) = F(F(y_2, y_1), 0) = F(y_2, F(y_1, 0)) = F(y_2, x_1) = x_2,$

which is a contradiction.

That is, there is no idempotent semi-t-operator on L which contains a sublattice depicted in Fig.5. The proofs for L characterized by the other four Hasse diagrams are similar. □

In the previous theorem one can see that there does not exist an idempotent semi-t-operator on every bounded lattice with $F(0_L, 1_L) = a$ and $F(1_L, 0_L) = b$. So it is important for us to investigate under which additional conditions there always exists an idempotent semi-t-operator on L. For this purpose, we present the following theorems to characterize idempotent semi-t-operators on L.

Theorem 4.3. *Let $a, b \in L \backslash \{0_L, 1_L\}$ and $a \le b$. If $I_b^a = \emptyset$ and $I_{a,b} = \emptyset$, and there is only one element m in I_a^b, then*

$F_m : L^2 \to L$ given by (15) is an idempotent semi-t-operator on L.

$$F_m(x,y) = \begin{cases} x \vee y & \text{if } (x,y) \in [0_L, a]^2, \\ x \wedge y & \text{if } (x,y) \in [b, 1_L]^2, \\ (x \wedge a) \vee (m \wedge a) & \text{if } (x,y) \in [0_L, a] \times \{m\}, \\ a & \text{if } (x,y) \in [0_L, a] \times [a,b] \cup [0_L, a] \times [b, 1_L], \\ x & \text{if } (x,y) \in [a,b] \times L, \\ m & \text{if } x = m \text{ and } y = m, \\ m \vee a & \text{if } (x,y) \in \{m\} \times (a,b] \cup \{m\} \times [b, 1_L], \\ (m \wedge a) \vee (y \wedge a) & \text{if } (x,y) \in \{m\} \times [0_L, a], \\ b & \text{otherwise.} \end{cases} \tag{15}$$

Proof. It is easy to see that F_m is idempotent, $F_m(0_L, 0_L) = 0_L$ and $F_m(1_L, 1_L) = 1_L$. We can easily obtain that $(F_m)_{0_L} \in \mathcal{M}(L, [0_L, a])$, $(F_m)^{0_L} \in \mathcal{M}(L, [0_L, b])$, $(F_m)_{1_L} \in \mathcal{M}(L, [b, 1_L])$, $(F_m)^{1_L} \in \mathcal{M}(L, [a, 1_L])$. Now let us verify that F_m is non-decreasing in each variable. Due to Theorem 3.1 we just need to consider the following cases. Firstly, let $x, y, z \in L$ and $x \leq y$.

1. $x \in [0_L, a]$
   1.1. $y \in [a, b]$ and $z = m$,
      $F_m(x, m) = (x \wedge a) \vee (m \wedge a) \leq a \leq y = F_m(y, m)$.
   1.2. $y \in [b, 1_L]$ and $z = m$,
      $F_m(x, m) = (x \wedge a) \vee (m \wedge a) \leq a \leq b \leq F_m(y, m)$.
   1.3. $y = m$
      1.3.1. $z \in [0_L, a]$,
         $F_m(x, z) = x \vee z = (x \wedge a) \vee (z \wedge a) \leq (m \wedge a) \vee (z \wedge a) = F_m(m, z)$.
      1.3.2. $z \in (a, b] \cup [b, 1_L]$,
         $F_m(x, z) = a \leq m \vee a = F_m(m, z)$.
      1.3.3. $z = m$,
         $F_m(x, m) = (x \wedge a) \vee (m \wedge a) = m \wedge a \leq m = F_m(m, m)$.
2. $x \in [a, b]$, $y \in [b, 1_L]$ and $z = m$,
   $F_m(x, m) = x \leq b \leq F_m(y, m)$.
3. $x = m$
   3.1. $y \in (a, b]$
      3.1.1. $z \in [0_L, a]$,
         $F_m(m, z) = (m \wedge a) \vee (z \wedge a) \leq a < y = F_m(y, z)$.
      3.1.2. $z \in (a, b] \cup [b, 1_L]$,
         $F_m(m, z) = m \vee a \leq y \vee a = y = F_m(y, z)$.
      3.1.3. $z = m$,
         $F_m(m, m) = m \leq y = F_m(y, m)$.
   3.2. $y \in [b, 1_L]$
      3.2.1. $z \in [0_L, a]$,
         $F_m(m, z) = (m \wedge a) \vee (z \wedge a) \leq a \leq b = F_m(y, z)$.
      3.2.2. $z \in (a, b] \cup [b, 1_L]$,
         $F_m(m, z) = m \vee a \leq m \vee b = b \leq F_m(y, z)$.
      3.2.3. $z = m$,
         $F_m(m, m) = m \leq b = F_m(y, m)$.

Secondly, let $x, y, z \in L$ and $y \leq z$.

1. $y \in [0_L, a]$
   1.1. $z \in (a, b] \cup [b, 1_L]$ and $x = m$,
      $F_m(m, y) = (m \wedge a) \vee (y \wedge a) \leq a \leq m \vee a = F_m(m, z)$.
   1.2. $z = m$
      1.2.1. $x \in [0_L, a]$,
         $F_m(x, y) = x \vee y = (x \wedge a) \vee (y \wedge a) \leq (x \wedge a) \vee (m \wedge a) = F_m(x, m)$.
      1.2.2. $x \in [a, b]$,
         $F_m(x, y) = x = F_m(x, m)$.
      1.2.3. $x \in [b, 1_L]$,

$F_m(x,y) = b = F_m(x,m)$.

1.2.4. $x = m$,

$F_m(m,y) = (m \wedge a) \vee (y \wedge a) \leq (m \wedge a) \vee (m \wedge a) = m \wedge a \leq m = F_m(m,m)$.

2. $y \in (a,b]$, $z \in [b,1_L]$ and $x = m$,

$F_m(m,y) = m \vee a = F_m(m,z)$.

3. $y = m$

3.1. $z \in (a,b]$

3.1.1. $x \in [0_L,a]$,

$F_m(x,m) = (x \wedge a) \vee (m \wedge a) \leq a = F_m(x,z)$.

3.1.2. $x \in [a,b] \cup [b,1_L]$,

$F_m(x,m) = F_m(x,z)$.

3.1.3. $x = m$,

$F_m(m,m) = m \leq m \vee a = F_m(m,z)$.

3.2. $z \in [b,1_L]$

3.2.1. $x \in [0_L,a]$,

$F_m(x,m) = (x \wedge a) \vee (m \wedge a) \leq a = F_m(x,z)$.

3.2.2. $x \in [a,b]$,

$F_m(x,m) = x = F_m(x,z)$.

3.2.3. $x \in [b,1_L]$,

$F_m(x,m) = b \leq x \wedge z = F_m(x,z)$.

3.2.4. $x = m$,

$F_m(m,m) = m \leq m \vee a = F_m(m,z)$.

Next, we will prove that F_m satisfies the associativity, that is, $F_m(x,F_m(y,z)) = F_m(F_m(x,y),z)$ for all $x,y,z \in L$. Due to Theorem 3.1 we just need to consider the following cases.

1. $x \in [0_L,a]$

1.1. $y \in [0_L,a]$ and $z = m$,

$F_m(x,F_m(y,m)) = F_m(x,(y \wedge a) \vee (m \wedge a)) = x \vee (y \wedge a) \vee (m \wedge a) = x \vee y \vee (m \wedge a) = ((x \vee y) \wedge a) \vee (m \wedge a) = F_m(x \vee y, m) = F_m(F_m(x,y),m)$.

1.2. $y \in [a,b]$ and $z = m$,

$F_m(x,F_m(y,z)) = F_m(x,y) = a = F_m(a,z) = F_m(F_m(x,y),z)$.

1.3. $y \in [b,1_L]$ and $z = m$,

$F_m(x,F_m(y,z)) = F_m(x,b) = a = F_m(a,z) = F_m(F_m(x,y),z)$.

1.4. $y = m$

1.4.1. $z \in [0_L,a]$,

$F_m(x,F_m(m,z)) = F_m(x,(m \wedge a) \vee (z \wedge a)) = x \vee (m \wedge a) \vee (z \wedge a) = x \vee (m \wedge a) \vee z = (x \wedge a) \vee (m \wedge a) \vee z = F_m((x \wedge a) \vee (m \wedge a), z) = F_m(F_m(x,m),z)$.

1.4.2. $z \in (a,b] \cup [b,1_L]$,

$F_m(x,F_m(m,z)) = F_m(x, m \vee a) = a = F_m((x \wedge a) \vee (m \wedge a), z) = F_m(F_m(x,m),z)$ since $a < m \vee a \leq b$.

1.4.3. $z = m$,

$F_m(x,F_m(m,m)) = F_m(x,m) = (x \wedge a) \vee (m \wedge a) = (((x \wedge a) \vee (m \wedge a)) \wedge a) \vee (m \wedge a) = F_m((x \wedge a) \vee (m \wedge a), m) = F_m(F_m(x,m),m)$.

2. $x \in [a,b]$,

$F_m(x,F_m(y,z)) = x = F_m(x,z) = F_m(F_m(x,y),z)$ for all $y,z \in L$.

3. $x \in [b,1_L]$

3.1. $y \in [0_L,a]$ and $z = m$,

$F_m(x,F_m(y,m)) = F_m(x,(y \wedge a) \vee (m \wedge a)) = b = F_m(b,m) = F_m(F_m(x,y),m)$.

3.2. $y \in [a,b]$ and $z = m$,

$F_m(x,F_m(y,m)) = F_m(x,y) = b = F_m(b,m) = F_m(F_m(x,y),m)$.

3.3. $y \in [b,1_L]$ and $z = m$,

$F_m(x,F_m(y,m)) = F_m(x,b) = b = F_m(x \wedge y, m) = F_m(F_m(x,y),m)$.

3.4. $y = m$

3.4.1. $z \in [0_L,a]$,

$F_m(x,F_m(m,z)) = F_m(x,(m \wedge a) \vee (z \wedge a)) = b = F_m(b,z) = F_m(F_m(x,m),z)$.

3.4.2. $z \in (a,b] \cup [b,1_L]$,

$F_m(x,F_m(m,z)) = F_m(x, m \vee a) = b = F_m(b,z) = F_m(F_m(x,m),z)$ since $a < m \vee a \leq b$.

3.4.3. $z = m$,

$F_m(x, F_m(m,m)) = F_m(x,m) = b = F_m(b,m) = F_m(F_m(x,m),m)$.

4. $x = m$

4.1. $y \in [0_L, a]$

4.1.1. $z \in [0_L, a]$,

$F_m(m, F_m(y,z)) = F_m(m, y \vee z) = (m \wedge a) \vee ((y \vee z) \wedge a) = (m \wedge a) \vee y \vee z = (m \wedge a) \vee (y \wedge a) \vee z = F_m((m \wedge a) \vee (y \wedge a), z) = F_m(F_m(m,y),z)$.

4.1.2. $z \in (a,b] \cup [b, 1_L]$,

$F_m(m, F_m(y,z)) = F_m(m,a) = a = F_m((m \wedge a) \vee (y \wedge a), z) = F_m(F_m(m,y),z)$.

4.1.3. $z = m$,

$F_m(m, F_m(y,m)) = F_m(m, (y \wedge a) \vee (m \wedge a)) = (m \wedge a) \vee (((y \wedge a) \vee (m \wedge a)) \wedge a) = (m \wedge a) \vee (y \wedge a) = (((m \wedge a) \vee (y \wedge a)) \wedge a) \vee (m \wedge a) = F_m((m \wedge a) \vee (y \wedge a), m) = F_m(F_m(m,y),m)$.

4.2. $y \in (a,b]$,

$F_m(m, F_m(y,z)) = F_m(m,y) = m \vee a = F_m(m \vee a, z) = F_m(F_m(m,y),z)$ for all $z \in L$ since $a < m \vee a \leq b$.

4.3. $y \in [b, 1_L]$,

$F_m(m, F_m(y,z)) = m \vee a = F_m(m \vee a, z) = F_m(F_m(m,y),z)$ for all $z \in L$.

4.4. $y = m$

4.4.1. $z \in [0_L, a]$,

$F_m(m, F_m(m,z)) = F_m(m, (m \wedge a) \vee (z \wedge a)) = (m \wedge a) \vee (((m \wedge a) \vee (z \wedge a)) \wedge a) = (m \wedge a) \vee (z \wedge a) = F_m(m,z) = F_m(F_m(m,m),z)$.

4.4.2. $z \in (a,b] \cup [b, 1_L]$,

$F_m(m, F_m(m,z)) = F_m(m, m \vee a) = m \vee a = F_m(m,z) = F_m(F_m(m,m),z)$.

4.4.3. $z = m$,

$F_m(m, F_m(m,m)) = F_m(m,m) = m = F_m(m,m) = F_m(F_m(m,m),m)$. □

Similarly, we can prove that construction methods introduced in the following three theorems yield idempotent semi-t-operators on bounded lattice L.

Theorem 4.4. *Let $a, b \in L \backslash \{0_L, 1_L\}$ and $a \leq b$. If $I_a^b = \emptyset$ and $I_{a,b} = \emptyset$, and there is only one element n in I_b^a, then $F_n : L^2 \to L$ given by (16) is an idempotent semi-t-operator on L.*

$$F_n(x,y) = \begin{cases} x \vee y & \text{if } (x,y) \in [0_L, a]^2, \\ x \wedge y & \text{if } (x,y) \in [b, 1_L]^2, \\ a & \text{if } (x,y) \in [0_L, a] \times \{L \backslash [0_L, a]\}, \\ x & \text{if } (x,y) \in [a,b] \times L, \\ (x \vee b) \wedge (n \vee b) & \text{if } (x,y) \in [b, 1_L] \times \{n\}, \\ n \wedge b & \text{if } (x,y) \in \{n\} \times [0_L, a] \cup \{n\} \times [a,b), \\ n & \text{if } x = n \text{ and } y = n, \\ (n \vee b) \wedge (y \vee b) & \text{if } (x,y) \in \{n\} \times [b, 1_L], \\ b & \text{otherwise.} \end{cases} \tag{16}$$

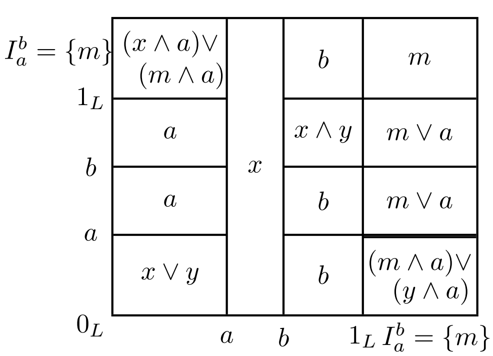

Fig.9 F_m on L

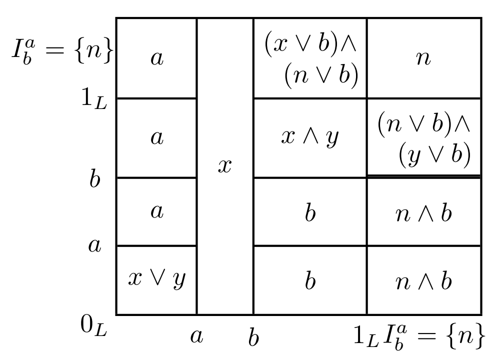

Fig.10 F_n on L

Theorem 4.5. *Let $a, b \in L\backslash\{0_L, 1_L\}$ and $a \leq b$. If $I_a^b = \emptyset$ and $I_b^a = \emptyset$, and there is only one element p in $I_{a,b}$, then $F_p^a : L^2 \to L$ given by (17) and $F_p^b : L^2 \to L$ given by (18) are idempotent semi-t-operators on L.*

$$F_p^a(x,y) = \begin{cases} x \vee y & \text{if } (x,y) \in [0_L, a]^2, \\ x \wedge y & \text{if } (x,y) \in [b, 1_L]^2, \\ a & \text{if } (x,y) \in [0_L, a] \times [a,b] \cup [0_L, a] \times [b, 1_L] \cup \{p\} \times (a,b), \\ (x \wedge a) \vee (p \wedge a) & \text{if } (x,y) \in [0_L, a] \times \{p\}, \\ x & \text{if } (x,y) \in [a,b] \times L, \\ (x \vee b) \wedge (p \vee b) & \text{if } (x,y) \in [b, 1_L] \times \{p\}, \\ (p \wedge a) \vee (y \wedge a) & \text{if } (x,y) \in \{p\} \times [0_L, a], \\ p & \text{if } x = p \text{ and } y = p, \\ (p \vee b) \wedge (y \vee b) & \text{if } (x,y) \in \{p\} \times [b, 1_L], \\ b & \text{otherwise,} \end{cases} \tag{17}$$

$$F_p^b(x,y) = \begin{cases} x \vee y & \text{if } (x,y) \in [0_L, a]^2, \\ x \wedge y & \text{if } (x,y) \in [b, 1_L]^2, \\ a & \text{if } (x,y) \in [0_L, a] \times [a,b] \cup [0_L, a] \times [b, 1_L], \\ (x \wedge a) \vee (p \wedge a) & \text{if } (x,y) \in [0_L, a] \times \{p\}, \\ x & \text{if } (x,y) \in [a,b] \times L, \\ (x \vee b) \wedge (p \vee b) & \text{if } (x,y) \in [b, 1_L] \times \{p\}, \\ (p \wedge a) \vee (y \wedge a) & \text{if } (x,y) \in \{p\} \times [0_L, a], \\ p & \text{if } x = p \text{ and } y = p, \\ (p \vee b) \wedge (y \vee b) & \text{if } (x,y) \in \{p\} \times [b, 1_L], \\ b & \text{otherwise.} \end{cases} \tag{18}$$

Theorem 4.6. *Let $a, b \in L\backslash\{0_L, 1_L\}$, $a \leq b$, $I_{a,b} = \emptyset$, and assume that there is only one element m in I_a^b and one element n in I_b^a. Suppose that one of the following two cases holds*
(i) $m \parallel n$, $m \vee a < b$ and $n \wedge b > a$.
(ii) $m \nparallel n$.
Then $F_{m,n} : L^2 \to L$ given by (19) is an idempotent semi-t-operator on L.

$$F_{m,n}(x,y) = \begin{cases} x \vee y & \text{if } (x,y) \in [0_L, a]^2, \\ x \wedge y & \text{if } (x,y) \in [b, 1_L]^2, \\ a & \text{if } (x,y) \in [0_L, a] \times [a,b] \cup [0_L, a] \times [b, 1_L] \cup [0_L, a] \times \{n\}, \\ (x \wedge a) \vee (m \wedge a) & \text{if } (x,y) \in [0_L, a] \times \{m\}, \\ x & \text{if } (x,y) \in [a,b] \times L, \\ (x \vee b) \wedge (n \vee b) & \text{if } (x,y) \in [b, 1_L] \times \{n\}, \\ (m \wedge a) \vee (y \wedge a) & \text{if } (x,y) \in \{m\} \times [0_L, a], \\ m \vee a & \text{if } (x,y) \in \{m\} \times (a,b] \cup \{m\} \times [b, 1_L] \cup \{m\} \times \{n\}, \\ m & \text{if } x = m \text{ and } y = m, \\ n \wedge b & \text{if } (x,y) \in \{n\} \times [0_L, a] \cup \{n\} \times [a,b) \cup \{n\} \times \{m\}, \\ n & \text{if } x = n \text{ and } y = n, \\ (n \vee b) \wedge (y \vee b) & \text{if } (x,y) \in \{n\} \times [b, 1_L], \\ b & \text{otherwise.} \end{cases} \tag{19}$$

4.2 When $b \leq a$

According to the conclusion of Theorem 3.6, we can get the following corresponding construction methods for idempotent semi-t-operators on L with $b \leq a$ from the ones in subsection 4.1.

Theorem 4.7. *Let $a, b \in L\backslash\{0_L, 1_L\}$ and $b \leq a$. If $I_a^b = \emptyset$ and $I_{a,b} = \emptyset$, and there is only one element n in I_b^a, then $F^n : L^2 \to L$ given by (20) is an idempotent semi-t-operator on L.*

$$F^n(x,y) = \begin{cases} x \vee y & \text{if } (x,y) \in [0_L, b]^2, \\ x \wedge y & \text{if } (x,y) \in [a, 1_L]^2, \\ a & \text{if } (x,y) \in \{L\backslash[a,1_L]\} \times [a, 1_L], \\ y & \text{if } (x,y) \in L \times [b,a], \\ (x \wedge b) \vee (n \wedge b) & \text{if } (x,y) \in [0_L, b] \times \{n\}, \\ n \vee b & \text{if } (x,y) \in (b,a] \times \{n\} \cup [a, 1_L] \times \{n\}, \\ n & \text{if } x = n \text{ and } y = n, \\ (n \wedge b) \vee (y \wedge b) & \text{if } (x,y) \in \{n\} \times [0_L, b], \\ b & \text{otherwise.} \end{cases} \tag{20}$$

Theorem 4.8. *Let $a, b \in L\backslash\{0_L, 1_L\}$ and $b \leq a$. If $I_b^a = \emptyset$ and $I_{a,b} = \emptyset$, and there is only one element m in I_a^b, then $F^m : L^2 \to L$ given by (21) is an idempotent semi-t-operator on L.*

$$F^m(x,y) = \begin{cases} x \vee y & \text{if } (x,y) \in [0_L, b]^2, \\ x \wedge y & \text{if } (x,y) \in [a, 1_L]^2, \\ b & \text{if } (x,y) \in \{L\backslash[0_L,b]\} \times [0_L, b], \\ y & \text{if } (x,y) \in L \times [b,a], \\ (m \vee a) \wedge (y \vee a) & \text{if } (x,y) \in \{m\} \times [a, 1_L], \\ m \wedge a & \text{if } (x,y) \in [0_L, b] \times \{m\} \cup [b,a) \times \{m\}, \\ m & \text{if } x = m \text{ and } y = m, \\ (x \vee a) \wedge (m \vee a) & \text{if } (x,y) \in [a, 1_L] \times \{m\}, \\ a & \text{otherwise.} \end{cases} \tag{21}$$

Theorem 4.9. *Let $a, b \in L\backslash\{0_L, 1_L\}$ and $b \leq a$. If $I_b^a = \emptyset$ and $I_a^b = \emptyset$, and there is only one element p in $I_{a,b}$, then $F_a^p : L^2 \to L$ given by (22) and $F_b^p : L^2 \to L$ given by (23) are idempotent semi-t-operators on L.*

$$F_a^p(x,y) = \begin{cases} x \vee y & \text{if } (x,y) \in [0_L, b]^2, \\ x \wedge y & \text{if } (x,y) \in [a, 1_L]^2, \\ a & \text{if } (x,y) \in [0_L, b] \times [a, 1_L] \cup [b,a] \times [a, 1_L] \cup (b,a) \times \{p\}, \\ (x \wedge b) \vee (p \wedge b) & \text{if } (x,y) \in [0_L, b] \times \{p\}, \\ y & \text{if } (x,y) \in L \times [b,a], \\ (x \vee a) \wedge (p \vee a) & \text{if } (x,y) \in [a, 1_L] \times \{p\}, \\ (p \wedge b) \vee (y \wedge b) & \text{if } (x,y) \in \{p\} \times [0_L, b], \\ p & \text{if } x = p \text{ and } y = p, \\ (p \vee a) \wedge (y \vee a) & \text{if } (x,y) \in \{p\} \times [a, 1_L], \\ b & \text{otherwise,} \end{cases} \tag{22}$$

$$F_b^p(x,y) = \begin{cases} x \vee y & \text{if } (x,y) \in [0_L, b]^2, \\ x \wedge y & \text{if } (x,y) \in [a, 1_L]^2, \\ b & \text{if } (x,y) \in [b,a] \times [0_L, b] \cup [a, 1_L] \times [0_L, b] \cup (b,a) \times \{p\}, \\ (x \wedge b) \vee (p \wedge b) & \text{if } (x,y) \in [0_L, b] \times \{p\}, \\ y & \text{if } (x,y) \in L \times [b,a], \\ (x \vee a) \wedge (p \vee a) & \text{if } (x,y) \in [a, 1_L] \times \{p\}, \\ (p \wedge b) \vee (y \wedge b) & \text{if } (x,y) \in \{p\} \times [0_L, b], \\ p & \text{if } x = p \text{ and } y = p, \\ (p \vee a) \wedge (y \vee a) & \text{if } (x,y) \in \{p\} \times [a, 1_L], \\ a & \text{otherwise.} \end{cases} \tag{23}$$

Theorem 4.10. *Let $a, b \in L\backslash\{0_L, 1_L\}$, $b \leq a$, $I_{a,b} = \emptyset$, and assume that there is only one element m in I_a^b and one element n in I_b^a. Suppose that one of the following two cases holds*

(i) $m \parallel n$, $n \vee b < a$ and $m \wedge a > b$.

(ii) $m \nparallel n$.

Then $F^{n,m} : L^2 \to L$ given by (24) is an idempotent semi-t-operator on L.

$$F^{n,m}(x,y) = \begin{cases} x \vee y & \text{if } (x,y) \in [0_L, b]^2, \\ x \wedge y & \text{if } (x,y) \in [a, 1_L]^2, \\ a & \text{if } (x,y) \in [0_L, b] \times [a, 1_L] \cup [b,a] \times [a, 1_L] \cup \{n\} \times [a, 1_L], \\ (x \wedge b) \vee (n \wedge b) & \text{if } (x,y) \in [0_L, b] \times \{n\}, \\ y & \text{if } (x,y) \in L \times [b,a], \\ (x \vee a) \wedge (m \vee a) & \text{if } (x,y) \in [a, 1_L] \times \{m\}, \\ (n \wedge b) \vee (y \wedge b) & \text{if } (x,y) \in \{n\} \times [0_L, b], \\ n \vee b & \text{if } (x,y) \in (b,a] \times \{n\} \cup [a, 1_L] \times \{n\} \cup \{m\} \times \{n\}, \\ m & \text{if } x = m \text{ and } y = m, \\ m \wedge a & \text{if } (x,y) \in [0_L, b] \times \{m\} \cup [b,a) \times \{m\} \cup \{n\} \times \{m\}, \\ n & \text{if } x = n \text{ and } y = n, \\ (m \vee a) \wedge (y \vee a) & \text{if } (x,y) \in \{m\} \times [a, 1_L], \\ b & \text{otherwise.} \end{cases} \quad (24)$$

Example 4.11. *Let $(L^1 = \{0_{L^1}, x_1, b, x_2, a, x_3, n, m, 1_{L^1}\}, \leq, 0_{L^1}, 1_{L^1})$ and $(L^2 = \{0_{L^2}, b, y_1, y_2, a, y_3, n, m, 1_{L^2}\}, \leq, 0_{L^2}, 1_{L^2})$ be two bounded lattices whose lattices diagrams can be showed in Fig.11 and Fig.12, respectively.*

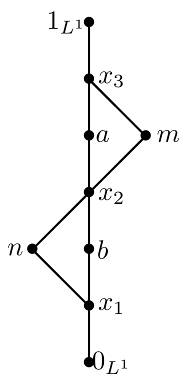

Fig.11 The lattice L^1

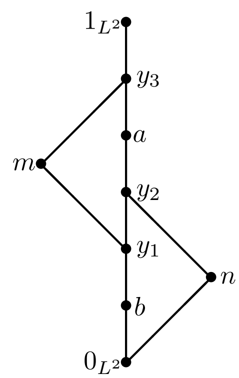

Fig.12 The lattice L^2

It follows from Theorem 4.10 that $F^{n,m}$ on L^1 in Table 5 and $F^{n,m}$ on L^2 in Table 6 are idempotent semi-t-operators.

Table 5 $F^{n,m}$ *on* L^1

$F^{n,m}$	0_{L^1}	x_1	b	x_2	a	x_3	1_{L^1}	n	m
0_{L^1}	0_{L^1}	x_1	b	b	b	b	b	x_1	b
x_1	x_1	x_1	b	b	b	b	b	x_1	b
b	b	b	b	b	b	b	b	b	b
x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2
a	a	a	a	a	a	a	a	a	a
x_3	a	a	a	a	a	x_3	x_3	a	x_3
1_{L^1}	a	a	a	a	a	x_3	1_{L^1}	a	x_3
n	x_1	x_1	b	x_2	x_2	x_2	x_2	n	x_2
m	x_2	x_2	x_2	x_2	a	x_3	x_3	x_2	m

Table 6 $F^{n,m}$ *on* L^2

$F^{n,m}$	0_{L^2}	b	y_1	y_2	a	y_3	1_{L^2}	n	m
0_{L^2}	0_{L^2}	b	b	b	b	b	b	0_{L^2}	b
b	b	b	b	b	b	b	b	b	b
y_1	y_1	y_1	y_1	y_1	y_1	y_1	y_1	y_1	y_1
y_2	y_2	y_2	y_2	y_2	y_2	y_2	y_2	y_2	y_2
a	a	a	a	a	a	a	a	a	a
y_3	a	a	a	a	a	y_3	y_3	a	y_3
1_{L^2}	a	a	a	a	a	y_3	1_{L^2}	a	y_3
n	0_{L^2}	b	y_2	y_2	y_2	y_2	y_2	n	y_2
m	y_1	y_1	y_1	y_1	a	y_3	y_3	y_1	m

5 Conclusion

In this paper, we presented several methods for construction of semi-t-operators on bounded lattices, and showed the relations between these semi-t-operators in Section 3. Furthermore, we illustrated the presence of idempotent semi-t-operators on bounded lattices with additional constraints: If $I_a^b = \emptyset$, $I_{a,b} = \emptyset$ and there is only one element in I_b^a; If $I_b^a = \emptyset$, $I_{a,b} = \emptyset$ and there is only one element in I_a^b; If $I_a^b = \emptyset$, $I_b^a = \emptyset$ and there is only one element in $I_{a,b}$; If $I_{a,b} = \emptyset$ and there is only one element in I_b^a and one element in I_a^b. The methods for the construction of corresponding idempotent semi-t-operators on bounded lattices were introduced in Section 4.

Acknowledgment

This work was supported by the National Natural Science Foundation of China (No. 61573211). The authors wish to express their appreciation for several excellent suggestions for improvements in this paper made by the referees.

References

[1] G. Birkhoff, *Lattice theory*, American Mathematical Society Colloquium Publications, Rhode Island, 1967.

[2] T. Calvo, B. De Baets, J. C. Fodor, *The functional equations of Frank and Alsina for uninorms and nullnorms*, Fuzzy Sets and Systems, **120** (2001), 385-394.

[3] T. Calvo, A. Fraile, G. Mayor, *Algunes consideracions sobre connectius generalitzats*, Actes del VI Congres Catala de Logica, Barcelona, (1986), 45-46.

[4] G. D. Çaylı, *On a new class of t-norms and t-conorms on bounded lattices*, Fuzzy Sets and Systems, **332** (2018), 129-143.

[5] G. D. Çaylı, P. Drygaś, *Some properties of idempotent uninorms on a special class of bounded lattices*, Information Sciences, **422** (2018), 352-363.

[6] G. D. Çaylı, F. Karaçal, *Idempotent nullnorms on bounded lattices*, Information Sciences, **425** (2018), 154-163.

[7] G. D. Çaylı, F. Karaçal, R. Mesiar, *On a new class of uninorms on bounded lattices*, Information Sciences, **367-368** (2016), 221-231.

[8] P. Drygaś, *Distributivity between semi-t-operators and semi-nullnorms*, Fuzzy Sets and Systems, **264** (2015), 100-109.

[9] P. Drygaś, E. Rak, *Distributivity equations in the class of semi-t-operators*, Fuzzy Sets and Systems, **291** (2016), 66-81.

[10] D. Dubois, H. Prade, *Fundamentals of fuzzy sets*, Springer Science and Business Media, LLC, 2000.

[11] Ü. Ertuğrul, *Construction of nullnorms on bounded lattices and an equivalence relation on nullnorms*, Fuzzy Sets and Systems, **334** (2018), 94-109.

[12] Ü. Ertuğrul, F. Karaçal, R. Mesiar, *Modified ordinal sums of triangular norms and triangular conorms on bounded lattices*, International Journal of Intelligent Systems, **30** (2015), 807-817.

[13] B. W. Fang, B. Q. Hu, *Semi-t-operators on bounded lattices*, Information Sciences, **490** (2019), 191-209.

[14] J. Fodor, M. Roubens, *Fuzzy preference modeling and multicriteria decision support*, Springer Netherlands, Springer Science and Business Media Dordrecht, 1994.

[15] M. A. Ince, F. Karaçal, R. Mesiar, *Medians and nullnorms on bounded lattices*, Fuzzy Sets and Systems, **289** (2016), 74-81.

[16] F. Karaçal, Ü. Ertuğrul, R. Mesiar, *Characterization of uninorms on bounded lattices*, Fuzzy Sets and Systems, **308** (2017), 54-71.

[17] F. Karaçal, M. A. Ince, R. Mesiar, *Nullnorms on bounded lattices*, Information Sciences, **325** (2015), 227-236.

[18] F. Karaçal, R. Mesiar, *Uninorms on bounded lattices*, Fuzzy Sets and Systems, **261** (2015), 33-43.

[19] G. J. Klir, B. Yuan, *Fuzzy sets and fuzzy logic, theory and application*, Prentice Hall PTR, Upper Saddle River, New Jersey, 1995.

[20] M. Mas, G. Mayor, J. Torrens, *T-operators*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, **7** (1999), 31-50.

[21] J. Medina, *Characterizing when an ordinal sum of t-norms is a t-norm on bounded lattices*, Fuzzy Sets and Systems, **202** (2012), 75-88.

[22] F. Qin, *Distributivity between semi-uninorms and semi-t-operators*, Fuzzy Sets and Systems, **299** (2016), 66-88.

[23] F. Qin, Y. M. Wang, *Distributivity between semi-t-operators and Mayor's aggregation operators*, Information Sciences, **346-347** (2016), 6-16.

[24] S. Saminger, *On ordinal sums of triangular norms on bounded lattices*, Fuzzy Sets and Systems, **157** (2006), 1403-1416.

[25] Y. Su, W. Zong, H. W. Liu, *On distributivity equations for uninorms over semi-t-operators*, Fuzzy Sets and Systems, **299** (2016), 41-65.

[26] Y. Su, W. Zong, H. W. Liu, P. Xue, *Migrativity property for uninorms and semi-t-operators*, Information Sciences, **325** (2015), 455-465.

[27] Y. Su, W. Zong, H. W. Liu, P. Xue, *The distributivity equations for semi-t-operators over uninorms*, Fuzzy Sets and Systems, **287** (2016), 172-183.