Iranian Journal of Fuzzy Systems

Volume 18, Number 2, (2021), pp. 175-186

Original paper

Note on "Fuzzy universal algebras on L-sets"

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Abstract

This note is the continuation of the paper entitled "Fuzzy universal algebras on L-sets"(IJFS, Volume 16, Number 4, (2019), pp. 175–187) and it focuses on $\mathcal{Q}$-valued (universal) algebras on $\mathcal{Q}$-typed sets. When $\mathcal{Q}$ is an involutive quantaloid, some basic related notions in $\mathcal{Q}$-valued algebra such as subalgebra, quotient algebra, homomorphism, congruence, direct product and variety etc are given and the properties of them are studied. When $\mathcal{Q}$ is still a symmetric quantaloid, the $\mathcal{Q}$-valued algebra is lifted to $\mathcal{Q}$-valued power algebra, and the power isomorphism theorem is given. As a special case of $\mathcal{Q}$-valued algebra, the fuzzy universal algebra on L-set is presented naturally when L is a commutative and divisible quantale.

Keywords: $\mathcal{Q}$-typed set, universal algebra, $\mathcal{Q}$-valued algebra, congruence, power algebra.

1 Introduction

The research of fuzzy universal algebras draws much attention in fuzzy community. Since Rosenfeld's fuzzy groups [26], many authors have studied the theories of fuzzy algebras (See [2, 3, 4, 5, 9, 10, 11, 12, 13, 14, 15, 21, 22, 23, 24, 28, 30]). The existed fuzzy universal algebraic theories are based on classical sets. Recently, Wei and Yue in [32] established a basic theory of fuzzy universal algebras based on the L-sets (or fuzzy sets). This is to say the universe of this kind of fuzzy algebra is fuzzy set.

In [31], Stubbe gave an introduction of the basic notions in quantaloid-enriched category. From [31], we know that some origin ideas in fuzzy set theory can be obtained by the results in quantaloid-enriched category. For example, the contravariant presheaf on a discrete $[0, 1]$-category can be considered as a fuzzy set. In the Ph.D thesis [29], Shen gave a thorough investigation of quantaloid-enriched categories and studied adjunctions in quantaloid-enriched categories. For the quantaloid with special properties, Heymans in [16] studied $\mathcal{Q}$-*-categories in an involutive quantaloid and Ackerman in [1] studied quantaloid-enriched categories associated to a frame in a symmetric quantaloid. Based on quantaloid-enriched category, Höhle and Kubiak in [20] studied non-commutative and non-idempotent theory of quantale sets, Pu and Zhang in [25] studied L-valued preordered sets and demonstrated that L-valued set (introduced by Höhle in [17, 19]) can be viewed as an L-set with an L-valued equivalence relation. In other words, L-valued sets can be considered as L-valued relations on L-sets. This is just one motivation of this paper to generalize the results in [32] to $\mathcal{Q}$-typed set by using quantaloid-enriched categories. A $\mathcal{Q}$-typed set is the underlying set of a $\mathcal{Q}$-category and it can be regarded as a discrete $\mathcal{Q}$-category. All contravariant presheaves on a $\mathcal{Q}$-typed set A form a $\mathcal{Q}$-typed set $\mathcal{P}A$. $\mathcal{Q}$-matrices and $\mathcal{Q}$-typed sets are analogous to relations and sets, and $\mathcal{P}A$ is analogous to the power set of an set.

This paper is the continuation of [32](entitled "Fuzzy universal algebras on L-sets"). We give a systematic study on $\mathcal{Q}$-valued (universal) algebras on $\mathcal{Q}$-typed sets. We show that the basic theory of fuzzy universal algebras on L-sets can be generalized to $\mathcal{Q}$-typed sets setting when $\mathcal{Q}$ is an involutive quantaloid. In fact, we will see that the universal algebra on $\mathcal{D}(L)$-typed set is just the fuzzy universal algebra on L-set in [32] when L is a commutative and divisible quantale.

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Received: September 2019; Revised: August 2020; Accepted: November 2020.

2 Preliminaries

We first introduce some basic concepts about quantaloid.

Definition 2.1. [27] *A quantaloid is a category $\mathcal{Q}$ such that the set $\mathcal{Q}(X,Y)$ of the arrows from X to Y is a sup-lattice for all objects $X,Y\in\mathcal{Q}$, and the composition $\circ$ preserves suprema in both variables, that is,*

$$f\circ\bigvee_{i\in I}g_i=\bigvee_{i\in I}(f\circ g_i),\qquad \bigvee_{j\in J}f_j\circ g=\bigvee_{j\in J}(f_j\circ g),$$

for all $f,f_j\in\mathcal{Q}(Y,Z)$ and $g,g_i\in\mathcal{Q}(X,Y)$.

The bottom and top element of $\mathcal{Q}(X,Y)$ are denoted by $\bot_{X,Y}$ and $\top_{X,Y}$, respectively. The identity arrow on an object X is denoted by 1_X. We use $\mathcal{Q}_0$ denote the objects of $\mathcal{Q}$ and use $\mathcal{Q}_1$ denote the morphisms of $\mathcal{Q}$.

A quantaloid $\mathcal{Q}$ is called involutive if it has an isomorphism $(-)^*:\mathcal{Q}^{op}\to\mathcal{Q}$, which is the identity on objects and for each morphism $f\in\mathcal{Q}(X,Y)$, we have a morphism $f^*\in\mathcal{Q}(Y,X)$ such that

$$(f\circ g)^*=g^*\circ f^*,\quad 1_X^*=1_X,\quad f^{**}=f\quad\text{and}\quad (\bigvee_{i\in I}f_i)^*=\bigvee_{i\in I}f_i^*.$$

In this paper, we always assume that $\mathcal{Q}$ is an involutive quantaloid.

Given a quantaloid $\mathcal{Q}$ and $\mathcal{Q}$-arrows $g\in\mathcal{Q}(Y,Z)$ and $f\in\mathcal{Q}(X,Y)$, there are two adjunctions

$$-\circ f\dashv -\swarrow f:\mathcal{Q}(X,Z)\to\mathcal{Q}(Y,Z),$$

$$g\circ -\dashv g\searrow -:\mathcal{Q}(X,Y)\to\mathcal{Q}(X,Z),$$

determined by the adjoint property

$$g\circ f\le h\Leftrightarrow g\le h\swarrow f\Leftrightarrow f\le g\searrow h.$$

Definition 2.2. [16] *Let $\mathcal{Q}$ be a quantaloid. A $\mathcal{Q}$-typed set A is a pair (A_0,t), where A_0 is a set and $t:A_0\to\mathcal{Q}_0$ assigns to each element of A_0 an object of $\mathcal{Q}$, and t is called the type function of A.*

For convenience, in this paper, the type functions symbols of all $\mathcal{Q}$-typed sets are used by t. A morphism of $\mathcal{Q}$-typed sets $F:A\dashrightarrow B$, or a type-preserving function, is a function $F:A_0\dashrightarrow B_0$ such that $t(x)=t(F(x))$ for all $x\in A_0$. A $\mathcal{Q}$-typed set B is called a subset of A if $B_0\subseteq A_0$ and the type function of B is the type function of A restricted to B_0. In this paper we always assume A_0 is not empty.

Let A and B be two $\mathcal{Q}$-typed sets. A $\mathcal{Q}$-matrix from A to B is a function $P:A_0\times B_0\to\mathcal{Q}_1$ satisfying $P(x,y)\in\mathcal{Q}(t(x),t(y))$ for all $x\in A_0$ and $y\in B_0$, denoted by $P:A\nrightarrow B$. For two $\mathcal{Q}$-matrices ϕ and ψ from A to B, the natural order $\phi\le\psi$ is defined by $\phi(x,y)\le\psi(x,y)$ for all $x\in A_0$ and $y\in B_0$.

Let $\phi:A\nrightarrow B$ and $\psi:B\nrightarrow C$ be two $\mathcal{Q}$-matrices, the composition of $\mathcal{Q}$-matrices $\psi\circ\phi:A\nrightarrow C$ is given by

$$(\psi\circ\phi)(x,z)=\bigvee_{y\in B_0}\psi(y,z)\circ\phi(x,y).$$

It is easy to check that this composition is associative and it distributes over arbitrary joins. The composition has a unit—the identity $\mathcal{Q}$-matrix $id_A:A\nrightarrow A$ given by: $\forall x,y\in A_0,\quad id_A(x,y)=\begin{cases}1_{t(x)}, & x=y,\\ \bot_{t(x),t(y)}, & \text{others.}\end{cases}$

In this way, we get a quantaloid $\mathcal{Q}$-**Matr**. Furthermore, since $\mathcal{Q}$ is involutive, then it can induce an involution on $\mathcal{Q}$-**Matr** in the following way: $P^*:A\nrightarrow B$ is define by $P^*(x,y)=(P(y,x))^*$ for $P:B\nrightarrow A$.

For $\mathcal{Q}$-matrices $\phi:A\nrightarrow B$, $\psi:B\nrightarrow C$ and $\eta:A\nrightarrow C$, the left implication $\eta\swarrow\phi:B\nrightarrow C$ and right implication $\psi\searrow\eta:A\nrightarrow B$ are calculated as follows:

$$\eta\swarrow\phi(y,z)=\bigwedge_{x\in A_0}\eta(x,z)\swarrow\phi(x,y),$$

$$\psi\searrow\eta(x,y)=\bigwedge_{z\in C_0}\psi(y,z)\searrow\eta(x,z).$$

Each type-preserving function $F:A\dashrightarrow B$ can induce a pair of adjoint $\mathcal{Q}$-matrices $F_\sharp\dashv F^\sharp$ in the sense of $id_A\le F^\sharp\circ F_\sharp$ and $id_B\ge F_\sharp\circ F^\sharp$, where $F_\sharp:A\nrightarrow B$ and $F^\sharp:B\nrightarrow A$ are defined by $F_\sharp(x,y)=id_B(F(x),y)$ and $F^\sharp(y,x)=id_B(y,F(x))$ for all $x\in A_0$, $y\in B_0$, respectively.

Definition 2.3. [16] *Let A be a $\mathcal{Q}$-typed set and $P : A \nrightarrow A$ be a $\mathcal{Q}$-matrix (we usually say P is a $\mathcal{Q}$-matrix on A).*
(1) *P is called reflexive if $id_A \leq P$;*
(2) *P is called transitive if $P \circ P \leq P$;*
(3) *P is called symmetric if $P^* = P$.*

If P is reflexive and transitive, then P is called a $\mathcal{Q}$-valued preorder on A. If P is reflexive, transitive and symmetric, then P is called a $\mathcal{Q}$-valued equivalence.

Let A be a $\mathcal{Q}$-typed set. Define two $\mathcal{Q}$-matrices $\triangle_A : A \nrightarrow A$ and $\triangledown_A : A \nrightarrow A$ by

$$\forall x, y \in A_0,\ \triangle_A(x, y) = \begin{cases} 1_{t(x)}, & x = y, \\ \perp_{t(x),t(y)}, & \text{others.} \end{cases}$$

and

$$\forall x, y \in A_0,\ \triangledown_A(x, y) = \begin{cases} 1_{t(x)}, & t(x) = t(y), \\ \perp_{t(x),t(y)}, & \text{others.} \end{cases}$$

respectively. Then it is easy to check that $\triangle_A$ and $\triangledown_A$ are two $\mathcal{Q}$-valued equivalences on A.

Let $P_1 : A \nrightarrow A$ and $P_2 : A \nrightarrow A$ be two $\mathcal{Q}$-valued equivalences. The supremum of P_1 and P_2 can be given by

$$(P_1 \vee P_2)(x, y) = \bigvee_{i_1, i_2, \ldots, i_k \in \{1,2\}, k < +\infty} (P_{i_1} \circ P_{i_2} \circ \cdots \circ P_{i_k})(x, y).$$

Furthermore, we have the following Lemma 2.4.

Lemma 2.4. *Let $P_1 : A \nrightarrow A$ and $P_2 : A \nrightarrow A$ be two $\mathcal{Q}$-valued equivalence. Then the following are equivalent:*
(1) *$P_1 \circ P_2 = P_2 \circ P_1$;*
(2) *$P_1 \circ P_2 = P_1 \vee P_2$.*

Definition 2.5. [8] *An algebra $\mathbf{X}$ of type T is an ordered pair $< X, \mathcal{F} >$, where X is a nonempty set and $\mathcal{F}$ is a family of finitary operations on X indexed by T such that corresponding to each n-ary function symbol f in T there is an n-ary operation $f^{\mathbf{X}}$ on X.*

3 Universal algebras on $\mathcal{Q}$-typed sets

In this section and the next section, we will give some concepts and results in universal algebra on $\mathcal{Q}$-typed sets in parallel with those in [32]. Since the proofs of some results are similar to those in [32], we omit the proofs of them and just list the results.

For a family of $\mathcal{Q}$-typed sets $\{A_i\}_{i\in I}$, the product of $\{A_i\}_{i\in I}$ is $\prod_{i\in I} A_i = ((\prod_{i\in I} A_i)_0, t)$, where

$$(\prod_{i\in I} A_i)_0 = \{(x_i)_{i\in I} \mid x_i \in (A_i)_0, \text{ and } t(x_i) = t(x_j) \text{ for all } i, j \in I\},$$

and $t : (\prod_{i\in I} A_i)_0 \to \mathcal{Q}_0$ is defined by $t((x_i)_{i\in I}) = t(x_i)$. Specially, for a nonnegative integer n, the n-product of A is denoted by A^n.

For a $\mathcal{Q}$-typed set A and a nonnegative integer n, a n-ray operation $\phi : A^n \dashrightarrow A$ is any type-preserving function from A^n to A. When $n = 0$, the operation ϕ is called a nullary operation and it is completely determined by the image of $\phi(\infty)$(in this way, $A^0 = (\{\infty\}, t)$ and $t(\infty) = t(\phi(\infty))$).

Definition 3.1. *Let A be a $\mathcal{Q}$-typed set and $\mathbb{F}$ be a set of operations on A indexed by the type T. The pair $\mathbf{A} =< A, \mathbb{F} >$ is called a $\mathcal{Q}$-valued (universal) algebra on A.*

Definition 3.2. *Let $\mathbf{A}$ and $\mathbf{B}$ be two $\mathcal{Q}$-valued algebras of the same type T. A type-preserving function $\alpha : A \dashrightarrow B$ is called a homomorphism from $\mathbf{A}$ to $\mathbf{B}$ if*

$$\alpha(\phi^{\mathbf{A}}(x_1, x_2, \ldots, x_n)) = \phi^{\mathbf{B}}(\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_n)),$$

for each n-ray function symbol $\phi \in \mathrm{T}$ and $(x_1, x_2, \ldots, x_n) \in A_0^n$.

If the homomorphism $\alpha : A \dashrightarrow B$ is one-to-one, then α is called an embedding. And if the homomorphism $\alpha : A \dashrightarrow B$ is one-to-one and onto, then α is called an isomorphism from $\mathbf{A}$ to $\mathbf{B}$.

Definition 3.3. *Let* $\mathbf{A}$ *be a* $\mathcal{Q}$*-valued algebra on* A *and* $\mathbf{B}$ *be a* $\mathcal{Q}$*-valued algebras on* B*.* $\mathbf{B}$ *is called a subalgebra of* $\mathbf{A}$ *if* B *is a subset of* A *and* $\phi^{\mathbf{B}}$ *is* $\phi^{\mathbf{A}}$ *restricted to* B_0 *for each function symbol* ϕ*. A subset* B *of* A *is called a subuniverse of* $\mathbf{A}$ *if* $\phi^{\mathbf{A}}(x_1, x_2, \cdots, x_n) \in B_0$ *for all* $x_1, x_2, \cdots, x_n \in B_0$ *and* $\phi \in \mathrm{T}$*.*

Definition 3.4. *Let* P *be a* $\mathcal{Q}$*-matrix on* A *and* $\mathbf{A} =< A, \mathbb{F} >$ *be a* $\mathcal{Q}$*-valued algebra.* P *is said to be compatible on* $\mathbf{A}$ *if*

$$\wedge_{i=1}^{i=n} P(x_i, y_i) \leq P(\phi^{\mathbf{A}}(x_1, x_2, ..., x_n), \phi^{\mathbf{A}}(y_1, y_2, ..., y_n)),$$

for all $\phi \in \mathrm{T}$ *and* $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in (A^n)_0$*.*
A compatible $\mathcal{Q}$*-valued equivalence on* $\mathbf{A}$ *is called a congruence on* $\mathbf{A}$*.*

Let $\mathbf{A} =< A, \mathbb{F} >$ be a $\mathcal{Q}$-valued algebra on A and P be a congruence on $\mathbf{A}$. Define the $\mathcal{Q}$-typed set $A/P = ((A/P)_0, t)$ by $t(x/P) = t(x)$, where

$$(A/P)_0 = \{x/P \mid x \in A_0\},$$

and

$$x/P = \{y \in A_0 \mid t(x) = t(y),\ P(x, y) \geq 1_{t(x)}\}.$$

For a n-ray operation $\phi^{\mathbf{A}} \in \mathbb{F}$, we can extend it to $\phi^{\mathbf{A}/P} : (A/p)^n \dashrightarrow A/p$ as follows:

$$\phi^{\mathbf{A}/P}(x_1/P, x_2/P, \ldots, x_n/P) = \phi^{\mathbf{A}}(x_1, x_2, \ldots, x_n)/P.$$

We can assert this definition is well-defined. In fact, if $x'_i \in x_i/P$ for $1 \leq i \leq n$, then

$$1_{t(x_i)} \leq \bigwedge_{i=1}^{i=n} P(x_i, x'_i) \leq P(\phi^{\mathbf{A}}(x_1, x_2, \ldots, x_n), \phi^{\mathbf{A}}(x'_1, x'_2, \ldots, x'_n)).$$

It follows that $\phi^{\mathbf{A}}(x'_1, x'_2, \ldots, x'_n) \in \phi^{\mathbf{A}}(x_1, x_2, \ldots, x_n)/P$. The pair $< A/P, \mathbb{F}/P >$ is called the $\mathcal{Q}$-valued quotient algebra of $< A, \mathbb{F} >$ by P, denoted by $\mathbf{A}/P$, where $\mathbb{F}/P = \{\phi^{\mathbf{A}/P} \mid \phi^{\mathbf{A}} \in \mathbb{F}\}$.

If $\mathbf{A} =< A, \mathbb{F} >$ is a $\mathcal{Q}$-valued algebra on A and P be a congruence on $\mathbf{A}$, then we can also define a congruence $P/A : A/P \nrightarrow A/P$ on $\mathbf{A}/P$ by $P/A(x/P, y/P) = P(x, y)$.

It is easy to check that P/A is a congruence on $\mathbf{A}/P$. Furthermore, $\alpha : A \dashrightarrow A/p$ defined by $\alpha(x) = x/P$ is a homomorphism from $\mathbf{A}$ to $\mathbf{A}/P$.

Lemma 3.5. *Let* $\alpha : A \dashrightarrow B$ *be a homomorphism from* $\mathbf{A}$ *to* $\mathbf{B}$ *and* Q *is a congruence on* $\mathbf{B}$*. Then* $\alpha^{\Leftarrow}(Q)$ *be a congruence on* $\mathbf{A}$*, where* $\alpha^{\Leftarrow}(Q) : A \nrightarrow A$ *is defined by* $\alpha^{\Leftarrow}(Q)(x, x') = Q(\alpha(x), \alpha(x'))$*.*

In Lemma 3.5, when $Q = id_B$, $\alpha^{\Leftarrow}(Q)$ is called the kernel of α, denoted by $ker(\alpha)$. $ker(\alpha)$ is as follows:

$$\forall x, y \in A_0,\ ker(\alpha)(x, y) = \begin{cases} 1_{t(x)}, & \alpha(x) = \alpha(y), \\ \perp_{t(x), t(y)}, & \text{others.} \end{cases}$$

Theorem 3.6. *(First Isomorphism Theorem) Let* $\alpha : A \dashrightarrow B$ *be an onto homomorphism from* $\mathbf{A}$ *to* $\mathbf{B}$*. Then* $\mathbf{A}/ker(\alpha) \cong \mathbf{B}$*.*

Let $\mathbf{A}$ be an algebra on A and Q, P be two congruences on $\mathbf{A}$ such that $Q \leq P$. Define $P/Q : A/Q \nrightarrow A/Q$ by $P/Q(x/Q, y/Q) = P(x, y)$. Then P/Q is a $\mathcal{Q}$-valued equivalence on $((A/Q)_0, t)$ and P/Q is a congruence on $\mathbf{A}/Q$.

Theorem 3.7. *(Second Isomorphism Theorem) Let* P, Q *be congruences on* $\mathbf{A}$ *and* $Q \leq P$*. Then* $(\mathbf{A}/Q)/(P/Q) \cong \mathbf{A}/P$*.*

Suppose $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ and P is a congruence on $\mathbf{A}$, and define $P|B : B \nrightarrow B$ by $P|B(x, y) = P(x, y)$ for $x, y \in B_0$. Then $P|B$ is a congruence on $\mathbf{B}$. Let $B^P = (B_0^P, t)$ and

$$B_0^P = \{x \in A_0 |\ B_0 \cap x/P \neq \emptyset\}.$$

It is easy to check that B^P is a subuniverse of $\mathbf{A}$.

Let $\mathbf{B}^P$ be the generated $\mathcal{Q}$-valued algebra by $\mathbf{A}$ on B^P. Then we have the following theorem.

Theorem 3.8. *(Third Isomorphism Theorem) Let* $\mathbf{B}$ *be a subalgebra of* $\mathbf{A}$ *and* P *be a congruence on* $\mathbf{A}$*. Then* $\mathbf{B}/(P|B) \cong \mathbf{B}^P/(P|B^P)$*.*

Let $\{\mathbf{A_i}\}_{i\in I}$ be a family of $\mathcal{Q}$-valued algebras of the same type T. The direct product $\mathbf{A} = \bigoplus_{i\in I} \mathbf{A_i}$ is a $\mathcal{Q}$-valued algebra on the universe $\prod_{i\in I} A_i$ and such that for $\phi \in \mathrm{T}$ and $a_1, a_2, \ldots, a_n \in (\prod_{i\in I} A_i)_0$,

$$\phi^{\mathbf{A}}(a_1, a_2, \ldots, a_n)(i) = \phi^{\mathbf{A_i}}(a_1(i), a_2(i), \ldots, a_n(i)),$$

for $i \in I$ (it is easy to check this definition is well defined). Define $\pi_j : \prod_{i\in I} A_i \dashrightarrow A_j$ by $\pi_j(a) = a(j)$ for $j \in I$, and we call π_j the projection map of $\prod_{i\in I} A_i$.

Theorem 3.9. *Let $\mathbf{A_1} \times \mathbf{A_2}$ be the direct product of $\mathbf{A_1}$ and $\mathbf{A_2}$. Then the projection map π_i is a homomorphism from $\mathbf{A_1} \times \mathbf{A_2}$ to $\mathbf{A_i}$. Furthermore, $ker(\pi_1) \wedge ker(\pi_2) = \triangle_{A_1\times A_2}$, $ker(\pi_1) \vee ker(\pi_2) = \triangledown_{A_1\times A_2}$ and $ker(\pi_1) \circ ker(\pi_2) = ker(\pi_2) \circ ker(\pi_1)$ hold.*

Proof. Let $\phi \in \mathrm{T}$ and $\big((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\big) \in ((A_1 \times A_2)^n)_0$. Then

$$\begin{aligned} & \pi_1(\phi^{\mathbf{A_1}\times\mathbf{A_2}}((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n))) \\ = \; & \pi_1((\phi^{\mathbf{A_1}}(x_1, x_2, \ldots, x_n), \phi^{\mathbf{A_2}}(y_1, y_2, \ldots, y_n)) \\ = \; & \phi^{\mathbf{A_1}}(x_1, x_2, \ldots, x_n) \\ = \; & \phi^{\mathbf{A_1}}(\pi_1(x_1, y_1), \pi_1(x_2, y_2), \ldots, \pi_1(x_n, y_n)). \end{aligned}$$

So π_1 is a homomorphism. Similarly, π_2 is a homomorphism.

Since

$$ker(\pi_1)((x_1, y_1), (x_2, y_2)) = \begin{cases} 1_{t(x_1,y_1)}, & x_1 = x_2, \\ \bot_{t(x_1,y_1),t(x_2,y_2)}, & \text{others.} \end{cases}$$

and

$$ker(\pi_2)((x_1, y_1), (x_2, y_2)) = \begin{cases} 1_{t(x_1,y_1)}, & y_1 = y_2, \\ \bot_{t(x_1,y_1),t(x_2,y_2)}, & \text{others.} \end{cases}$$

then we have

$$\begin{aligned} ker(\pi_1) \wedge ker(\pi_2)((x_1, y_1), (x_2, y_2)) &= \begin{cases} 1_{t(x_1,y_1)} \wedge 1_{t(x_1,y_1)}, & x_1 = x_2, y_1 = y_2 \\ \bot_{t(x_1,y_1),t(x_2,y_2)}, & \text{others.} \end{cases} \\ &= \begin{cases} 1_{t(x_1,y_1)}, & (x_1, y_1) = (x_2, y_2), \\ \bot_{t(x_1,y_1),t(x_2,y_2)}, & \text{others.} \end{cases} \end{aligned}$$

This is to say that $ker(\pi_1) \wedge ker(\pi_2) = \triangle_{A_1\times A_2}$.

$$\begin{aligned} & ker(\pi_1) \circ ker(\pi_2)((x_1, y_1), (x_2, y_2)) \\ = \; & \bigvee_{(x_3,y_3)\in(A_1\times A_2)_0} ker(\pi_1)((x_3, y_3), (x_2, y_2)) \circ ker(\pi_2)((x_1, y_1)(x_3, y_3)) \\ = \; & \begin{cases} ker(\pi_1)((x_2, y_1), (x_2, y_2)) \circ ker(\pi_2)((x_1, y_1)(x_2, y_1)), & t(x_1, y_1) = t(x_2, y_2), \\ \bot_{t(x_1,y_1),t(x_2,y_2)}, & \text{others.} \end{cases} \\ = \; & \begin{cases} 1_{t(x_1,y_1)}, & t(x_1, y_1) = t(x_2, y_2), \\ \bot_{t(x_1,y_1),t(x_2,y_2)}, & \text{others.} \end{cases} \end{aligned}$$

$$\begin{aligned} & ker(\pi_2) \circ ker(\pi_1)((x_1, y_1), (x_2, y_2)) \\ = \; & \bigvee_{(x_3,y_3)\in(A_1\times A_2)_0} ker(\pi_2)((x_3, y_3), (x_2, y_2)) \circ ker(\pi_1)((x_1, y_1), (x_3, y_3))) \\ = \; & \begin{cases} ker(\pi_2)((x_1, y_2), (x_2, y_2)) \circ ker(\pi_1)((x_1, y_1), (x_1, y_2))), & t(x_1, y_1) = t(x_2, y_2), \\ \bot_{t(x_1,y_1),t(x_2,y_2)}, & \text{others.} \end{cases} \\ = \; & \begin{cases} 1_{t(x_1,y_1)}, & t(x_1, y_1) = t(x_2, y_2), \\ \bot_{t(x_1,y_1),t(x_2,y_2)}, & \text{others.} \end{cases} \end{aligned}$$

Hence, $ker(\pi_1) \circ ker(\pi_2) = ker(\pi_2) \circ ker(\pi_1) = \triangledown_{A_1\times A_2}$. From Lemma 2.4, we have $ker(\pi_1) \vee ker(\pi_2) = \triangledown_{A_1\times A_2}$. □

Theorem 3.10. *Let $\mathbf{A}$ be the direct product of $\mathbf{A_1}$ and $\mathbf{A_2}$. Then $\mathbf{A} \cong \mathbf{A}/ker(\pi_1) \times \mathbf{A}/ker(\pi_2)$.*

Proof. Define $\theta : A_1 \times A_2 \dashrightarrow (A_1 \times A_2)/ker(\pi_1) \times (A_1 \times A_2)/ker(\pi_2)$ by

$$\forall (x,y) \in (A_1 \times A_2)_0,\ \theta((x,y)) = ((x,y)/ker(\pi_1), (x,y)/ker(\pi_2)).$$

Now we check that θ is the isomorphism between $\mathbf{A}$ and $\mathbf{A}/ker(\pi_1) \times \mathbf{A}/ker(\pi_2)$.

θ is injective: Let $((x,y)/ker(\pi_1), (x,y)/ker(\pi_2)) = ((x_1,y_1)/ker(\pi_1), (x_1,y_1)/ker(\pi_2))$. Then $(x,y)/ker(\pi_1) = (x_1,y_1)/ker(\pi_1)$ and $(x,y)/ker(\pi_2) = (x_1,y_1)/ker(\pi_2)$. Hence $x = x_1$ and $y = y_1$, i.e., $(x,y) = (x_1,y_1)$.

θ is surjective: If $((x,y)/ker(\pi_1), (x_1,y_1)/ker(\pi_2)) \in \big((A_1 \times A_2)/ker(\pi_1) \times (A_1 \times A_2)/ker(\pi_2)\big)_0$, then $t((x,y)/ker(\pi_1)) = t((x_1,y_1)/ker(\pi_2)) = t(x,y) = t(x_1,y_1) = t(x) = t(x_1) = t(y) = t(y_1)$. Since

$$ker(\pi_1)((x,y_1),(x,y)) = 1_{t(x)},$$

it holds that $(x,y_1) \in (x,y)/ker(\pi_1)$. Similarly, we have $(x,y_1) \in (x_1,y_1)/ker(\pi_2)$. Hence $(x,y_1)/ker(\pi_1) = (x,y)/ker(\pi_1)$ and $(x,y_1)/ker(\pi_2) = (x_1,y_1)/ker(\pi_2)$. So

$$\theta((x,y_1)) = ((x,y_1)/ker(\pi_1), (x,y_1)/ker(\pi_2)) = ((x,y)/ker(\pi_1), (x_1,y_1)/ker(\pi_2)).$$

θ is a homomorphism:

$$\begin{aligned}
& \phi^{\mathbf{A}/ker(\pi_1) \times \mathbf{A}/ker(\pi_2)}(\theta(x_1,y_1), \ldots, \theta(x_n,y_n)) \\
= \; & \phi^{\mathbf{A}/ker(\pi_1) \times \mathbf{A}/ker(\pi_2)}(((x_1,y_1)/ker(\pi_1), (x_1,y_1)/ker(\pi_2)), \ldots, \\
& ((x_n,y_n)/ker(\pi_1), (x_n,y_n)/ker(\pi_2))) \\
= \; & \big(\phi^{\mathbf{A}/ker(\pi_1)}((x_1,y_1)/ker(\pi_1), \ldots, (x_n,y_n)/ker(\pi_1)), \\
& \phi^{\mathbf{A}/ker(\pi_2)}((x_1,y_1)/ker(\pi_2), \ldots, (x_n,y_n)/ker(\pi_2))\big) \\
= \; & \big(\phi^{\mathbf{A}}((x_1,y_1), \ldots, (x_n,y_n))/ker(\pi_1), \phi^{\mathbf{A}}((x_1,y_1), \ldots, (x_n,y_n))/ker(\pi_2)\big) \\
= \; & \big((\phi^{\mathbf{A_1}}(x_1, \ldots, x_n), \phi^{\mathbf{A_2}}(y_1, \ldots, y_n))/ker(\pi_1), \\
& (\phi^{\mathbf{A_1}}(x_1, \ldots, x_n), \phi^{\mathbf{A_2}}(y_1, \ldots, y_n))/ker(\pi_2)\big) \\
= \; & \theta(\phi^{\mathbf{A}}((x_1,y_1), \ldots, (x_n,y_n))),
\end{aligned}$$

as desired. □

Definition 3.11. *A congruence P on $\mathbf{A}$ is a factor congruence if there exists a congruence $P^{\Box}$ on $\mathbf{A}$ such that $P \wedge P^{\Box} = \triangle_A$ and $x/P \cap y/P^{\Box} \neq \emptyset$ for all $(x,y) \in (A \times A)_0$.*

We will have the following theorem.

Theorem 3.12. *If P and $P^{\Box}$ is a pair of factor congruence on $\mathbf{A}$, then $\mathbf{A} \cong \mathbf{A}/P \times \mathbf{A}/P^{\Box}$.*

Let $\mathcal{K}$ be a class of $\mathcal{Q}$-valued algebras with the same type and some class operators are given as follwows:
$\mathbf{A} \in I(\mathcal{K})$ iff $\mathbf{A}$ is isomorphic to some member of $\mathcal{K}$;
$\mathbf{A} \in S(\mathcal{K})$ iff $\mathbf{A}$ is a subalgebra of some member of $\mathcal{K}$;
$\mathbf{A} \in H(\mathcal{K})$ iff $\mathbf{A}$ is a homomorphism image to some member of $\mathcal{K}$;
$\mathbf{A} \in P(\mathcal{K})$ iff $\mathbf{A}$ is the direct product of a nonempty family of $\mathcal{K}$.

Definition 3.13. *A nonempty class $\mathcal{K}$ of $\mathcal{Q}$-valued algebras with the same type is called a variety if it is closed under subalgebras, homomorphic images, and direct products.*

Let $V(\mathcal{K})$ denote the smallest variety containing $\mathcal{K}$. Similar to the classical setting of varieties, we have the following properties.

Theorem 3.14. *$SH \leq HS$, $PS \leq SP$ and $PH \leq HP$. H, S and IP are idempotent. Furthermore, $V = HSP$.*

4 $\mathcal{Q}$-valued power algebra

For each object X of $\mathcal{Q}$, write $*_X$ for the $\mathcal{Q}$-typed set with exactly one object $*$ such that $t* = X$. Since a $\mathcal{Q}$-typed set A with the identity $\mathcal{Q}$-matrix id_A is a discrete $\mathcal{Q}$-category, we can define presheaf and co-presheaf on $\mathcal{Q}$-typed set A similar to those on $\mathcal{Q}$-category. A presheaf with type $t\lambda$ on a $\mathcal{Q}$-typed set A is a $\mathcal{Q}$-matrix $\lambda : A \nrightarrow *_{t\lambda}$. We often write $\lambda(x)$ instead of $\lambda(x,*)$ for short. All presheaves on A is denoted by $\mathcal{P}A_0$. $\mathcal{P}A_0$ and t form a $\mathcal{Q}$-typed set $\mathcal{P}A$. Dually, a co-presheaf with type $t\rho$ on A is a $\mathcal{Q}$-matrix $\rho : *_X \nrightarrow A$. All co-presheaves on A form a $\mathcal{Q}$-typed set $\mathcal{P}^\dagger A$.

Let $P : A \nrightarrow A$ be a $\mathcal{Q}$-matrix on A and define $P_l : \mathcal{P}A_0 \to \mathcal{P}A_0$ by $P_l(\lambda) = \lambda \circ P$, where $\lambda \circ P(x) = \bigvee_{y\in A_0} \lambda(y) \circ P(x,y)$. Similarly, $P_u : \mathcal{P}^\dagger A_0 \to \mathcal{P}^\dagger A_0$ is defined by $P_u(\eta) = P \circ \eta$, where $P \circ \eta(x) = \bigvee_{y\in A_0} P(y,x) \circ \eta(y)$. Then we have the following Theorems.

Theorem 4.1. *If $P : A \nrightarrow A$ is a $\mathcal{Q}$-matrix on A, then P is $\mathcal{Q}$-valued preorder on A if and only if $P_l : \mathcal{P}A_0 \to \mathcal{P}A_0$ is a $\mathcal{Q}$-valued closure operator on A in the sense of $id_{\mathcal{P}A_0} \le P_l$ and $P_l \circ P_l \le P_l$.*

Theorem 4.2. *P is a $\mathcal{Q}$-valued preorder on A if and only if P_u is a $\mathcal{Q}$-valued closure operator on A.*

Let $P : A \nrightarrow A$ be a $\mathcal{Q}$-matrix on A and define $P^{\rightarrow} : \mathcal{P}A \nrightarrow \mathcal{P}A$ and $P^{\leftarrow} : \mathcal{P}^\dagger A \nrightarrow \mathcal{P}^\dagger A$ by

$$P^{\rightarrow}(\lambda,\mu) = P_l(\mu) \swarrow \lambda \quad\text{and}\quad P^{\leftarrow}(\eta,\rho) = \rho \searrow P_u(\eta),$$

respectively.

Theorem 4.3. *The following statements are equivalent:*
(1) P is a $\mathcal{Q}$-valued preorder on A;
(2) $P^{\rightarrow}$ is a $\mathcal{Q}$-valued preorder on $\mathcal{P}A$.

Theorem 4.4. *The following statements are equivalent:*
(1) P is a $\mathcal{Q}$-valued preorder on A;
(2) $P^{\leftarrow}$ is a $\mathcal{Q}$-valued preorder on $\mathcal{P}^\dagger A$.

Let $\mathbf{A}$ be a $\mathcal{Q}$-valued algebra on A. Given a n-ray operation $\phi^{\mathbf{A}} : A^n \dashrightarrow A$, it can be lifted to the $\mathcal{Q}$-valued power set $\mathcal{P}A$ by $\phi^{\mathcal{P}(\mathbf{A})} : \mathcal{P}A^n \dashrightarrow \mathcal{P}A$ as follows:

$$\phi^{\mathcal{P}(\mathbf{A})}(\mu_1,\mu_2,\cdots,\mu_n) = (\sqcap_{i=1}^{i=n}\mu_i) \circ (\phi^{\mathbf{A}})^\sharp,$$

where $\sqcap_{i=1}^{i=n}\mu_i \in \mathcal{P}(A^n)_0$ is given by $\sqcap_{i=1}^{i=n}\mu_i(x_1,\dots,x_n) = \wedge_{i=1}^{i=n}\mu_i(x_i)$.

Similarly, we can lift $\phi^{\mathbf{A}} : A^n \dashrightarrow A$ to $\phi^{\mathcal{P}^\dagger(\mathbf{A})} : \mathcal{P}^\dagger A^n \dashrightarrow \mathcal{P}^\dagger A$ by

$$\phi^{\mathcal{P}^\dagger(\mathbf{A})}(\rho_1,\rho_2,\cdots,\rho_n) = (\phi^{\mathbf{A}})_\sharp \circ (\sqcap_{i=1}^{i=n}\rho_i),$$

where $\sqcap_{i=1}^{i=n}\rho_i \in \mathcal{P}^\dagger(A^n)_0$ is given by $\sqcap_{i=1}^{i=n}\rho_i(x_1,\dots,x_n) = \wedge_{i=1}^{i=n}\rho_i(x_i)$.

$\mathcal{P}(\mathbf{A}) =< \mathcal{P}\mathrm{A}, \{\phi^{\mathcal{P}(\mathbf{A})} | \phi^{\mathbf{A}} \in \mathbf{A}\} >$ and $\mathcal{P}^\dagger(\mathbf{A}) =< \mathcal{P}^\dagger\mathrm{A}, \{\phi^{\mathcal{P}^\dagger(\mathbf{A})} | \phi^{\mathbf{A}} \in \mathbf{A}\} >$ are called $\mathcal{Q}$-valued algebras on $\mathcal{P}A$ and $\mathcal{P}^\dagger A$, respectively.

From [31], we know that $\mathcal{Q}$ is integral if each identity arrow in $\mathcal{Q}$ is the biggest endomorphism on its domain, and $\mathcal{Q}$ is divisible if it satisfies one of the following conditions:
(1) $\mathcal{Q}$ is integral and for all $d,e \in \mathcal{Q}(X,Y)$, $e \circ (e \searrow d) = e \wedge d = (d \swarrow e) \circ e$;
(2) $\mathcal{Q}$ is integral and for all $d \le e \in \mathcal{Q}(X,Y)$, $e \circ (e \searrow d) = d = (d \swarrow e) \circ e$;
(3) for all $d,e \in \mathcal{Q}(X,Y)$, $e \circ (e \searrow d) = d = (d \swarrow e) \circ e \Longleftrightarrow d \le e$;
(4) for all $e \in \mathcal{Q}(X,Y)$, $\mathcal{D}(\mathcal{Q})(e,e) =\downarrow e$, where $\mathcal{D}(\mathcal{Q})(e,e)$ is the set of diagonals from e to e.

Lemma 4.5. *Suppose $\mathcal{Q}$ is a divisible quantaloid. Then the following statements hold:*
(1) $\forall e,d,f \in \mathcal{Q}(X,Y)$ with $e \le f$ and $d \le f$, $e \circ (f \searrow d) = (e \swarrow f) \circ d$;
(2) $\forall e,d_i \in \mathcal{Q}(X,Y)$, $\forall \{d_i\}_{i\in I} \in \mathcal{Q}(X,Y)$, $e \wedge \bigvee_{i\in I} d_i = \bigvee_{i\in I}(e \wedge d_i)$;
(3) $\forall f \in \mathcal{Q}(Y,Z)$, $\forall e,d \in \mathcal{Q}(X,Y)$, $f \circ (e \wedge d) = (f \circ e) \wedge (f \circ d)$.

Proof. (1) Since $\mathcal{Q}$ is divisible, it follows that $e = (e \swarrow f) \circ f$ and $f \circ (f \searrow d) = d$ by $e \le f$ and $d \le f$. Then

$$e \circ (f \searrow d) = (e \swarrow f) \circ f \circ (f \searrow d) = (e \swarrow f) \circ d.$$

The proofs of (2) and (3) are similar to those in divisible quantale in [18]. Here, we give the proof of (3) as follows:
(3) Since $f \circ (e \wedge d) \le (f \circ e) \wedge (f \circ d)$ is obvious, it suffices to show $f \circ (e \wedge d) \ge (f \circ e) \wedge (f \circ d)$. In fact, we have

$$\begin{aligned}
(f \circ e) \wedge (f \circ d) &= (f \circ e) \circ \big((f \circ e) \searrow (f \circ d)\big) \\
&= f \circ \big(e \circ (e \searrow (f \searrow (f \circ d)))\big) \\
&= f \circ \big(e \wedge (f \searrow (f \circ d))\big) \\
&= f \circ \big((f \searrow (f \circ d)) \circ \big((f \searrow (f \circ d)) \searrow e\big)\big) \\
&\le f \circ d \circ (d \searrow e) \\
&= f \circ (e \wedge d),
\end{aligned}$$

as desired. □

Theorem 4.6. *Let $\alpha : A \dashrightarrow B$ be a homomorphism from $\mathbf{A}$ to $\mathbf{B}$. Then*
(1) $\alpha^\star : \mathcal{P}A \dashrightarrow \mathcal{P}B$ defined by $\alpha^\star(\mu) = \mu \circ \alpha^\sharp$ is a homomorphism from $\mathcal{P}(\mathbf{A})$ to $\mathcal{P}(\mathbf{B})$.
(2) $\alpha^\diamond : \mathcal{P}^\dagger A \dashrightarrow \mathcal{P}^\dagger B$ defined by $\alpha^\diamond(\lambda) = \alpha_\sharp \circ \lambda$ is a homomorphism from $\mathcal{P}^\dagger(\mathbf{A})$ to $\mathcal{P}^\dagger(\mathbf{B})$.

Proof. Let $\phi \in \mathrm{T}$ be a n-ray operation and $\mu_i \in \mathcal{P}\mathrm{A}_0 (1 \leq i \leq n)$. From Lemma 4.5 (2), it is easy to check that $\sqcap_{i=1}^{i=n}(\mu_i \circ \alpha^\sharp) = (\sqcap_{i=1}^{i=n}\mu_i) \circ (\alpha^n)^\sharp$. Then we have

$$\begin{aligned}
\alpha^\star(\phi^{\mathcal{P}(\mathbf{A})}(\mu_1, \mu_2, \cdots, \mu_n)) &= (\sqcap_{i=1}^{i=n}\mu_i) \circ (\phi^{\mathbf{A}})^\sharp \circ \alpha^\sharp \\
&= (\sqcap_{i=1}^{i=n}\mu_i) \circ (\alpha \circ \phi^{\mathbf{A}})^\sharp \\
&= (\sqcap_{i=1}^{i=n}\mu_i) \circ (\phi^{\mathbf{B}} \circ \alpha^n)^\sharp \\
&= (\sqcap_{i=1}^{i=n}\mu_i) \circ (\alpha^n)^\sharp \circ (\phi^{\mathbf{B}})^\sharp \\
&= (\sqcap_{i=1}^{i=n}(\mu_i \circ \alpha^\sharp)) \circ (\phi^{\mathbf{B}})^\sharp \\
&= \phi^{\mathcal{P}(\mathbf{B})}(\mu_1 \circ \alpha^\sharp, \mu_2 \circ \alpha^\sharp, \cdots, \mu_n \circ \alpha^\sharp) \\
&= \phi^{\mathcal{P}(\mathbf{B})}(\alpha^\star(\mu_1), \alpha^\star(\mu_2), \cdots, \alpha^\star(\mu_n)).
\end{aligned}$$

This is to say that $\alpha^\star$ is a homomorphism from $\mathcal{P}(\mathbf{A})$ to $\mathcal{P}(\mathbf{B})$.
(2) The proof is similar to (1). □

Remark 4.7. *From the above discussion, we ask whether P is compatible on $\mathbf{A}$ if and only if $P^\rightarrow$ is compatible on $\mathcal{P}(\mathbf{A})$? Since the answer of the question is not affirmative for the fuzzy power algebra based on Cartesian product when the valued lattice is a complete residuated lattice, this is also not valid in $\mathcal{Q}$-valued power algebra if there is no other requirement for the quantaloid. In [5], there is another kind of fuzzy power algebra based on tensor product except for the fuzzy power algebra based on Cartesian product. At the end of this section, we will give another kind of $\mathcal{Q}$-valued power algebras on $\mathcal{Q}$-typed sets and give an affirmative answer to the question.*

Considering the underling set $(A^n)_0$ of A^n is not the Cartesian product $(A_0)^n$ of A_0 generally and each coordinate of the element in $(A^n)_0$ has the same type, we need to modify the definition of transitivity of $\mathcal{Q}$-matrix and require $\mathcal{Q}$ satisfying specific properties. A quantaloid $\mathcal{Q}$ is called symmetric if $\mathcal{Q}$ satisfies the following two conditions:
(1) For all $X, Y \in \mathcal{Q}_0$, $\mathcal{Q}(X,Y) = \mathcal{Q}(Y,X)$;
(2) For all $f \in \mathcal{Q}(X,Y), g \in \mathcal{Q}(Y,Z)$, $f \circ g = g \circ f$.
A symmetric quantaloid can be considered as an involutive quantaloid in the sense of $f^* = f$. In the following, we assume that $\mathcal{Q}$ is symmetric.

Definition 4.8. *Let A be a $\mathcal{Q}$-typed set and $P : A \nrightarrow A$ be an $\mathcal{Q}$-matrix on A. P is called weak transitive if $P(y,z) \circ P(x,y) \leq P(x,z)$ for all $x, y, z \in A_0$ with $t(x) = t(y) = t(z)$. P is called strong symmetric if $P(x,y) = P(y,x)$ for all $x, y \in A_0$.*

This is to say the that weak transitivity just requires the points with the same type satisfying the transitivity. P is called a weak $\mathcal{Q}$-valued preorder on A if P is reflexive and weak transitive. P is called a weak $\mathcal{Q}$-valued equivalence on A if it is reflexive, weak transitive and strong symmetric.

Definition 4.9. *Let P be a $\mathcal{Q}$-matrix on A and $\mathbf{A} =< A, \mathbb{F} >$ be an algebra. P is said to be $\circ$-compatible on $\mathbf{A}$ if*

$$P(x_1,y_1) \circ P(x_2,y_2) \circ \cdots \circ P(x_n,y_n) \leq P(\phi^{\mathbf{A}}(x_1,x_2,...,x_n), \phi^{\mathbf{A}}(y_1,y_2,...,y_n)),$$

for all $\phi \in \mathbb{F}$ and $(x_1,\dots,x_n),(y_1,\dots,y_n) \in (A^n)_0$.

$P(x_1,y_1) \circ P(x_2,y_2) \circ \cdots \circ P(x_n,y_n)$ is denoted by $\bigcirc_{i=1}^{i=n} P(x_i,y_i)$. A $\circ$-compatible weak $\mathcal{Q}$-valued equivalence on $\mathbf{A}$ is called a weak congruence on $\mathbf{A}$.

It is routine to check that the results in Section 3 are also valid for weak congruences.
Similar to the lifting of $\phi^{\mathbf{A}} : A^n \dashrightarrow A$ to $\phi^{\mathcal{P}(\mathbf{A})} : \mathcal{P}A^n \dashrightarrow \mathcal{P}A$, we can extend $\phi^{\mathbf{A}} : A^n \dashrightarrow A$ to $\phi^{\mathcal{P}^\circ(\mathbf{A})} : \mathcal{P}A^n \dashrightarrow \mathcal{P}A$ as follows:

$$\phi^{\mathcal{P}^\circ(\mathbf{A})}(\mu_1, \mu_2, \cdots, \mu_n) = (\odot_{i=1}^{i=n}\mu_i) \circ (\phi^{\mathbf{A}})^\sharp,$$

where $\odot_{i=1}^{i=n}\mu_i(x_1,\dots,x_n) \in \mathcal{P}(A^n)_0$ is given by

$$\odot_{i=1}^{i=n}\mu_i(x_1,\dots,x_n) = \bigcirc_{i=1}^{i=n}\mu_i(x_i) = \mu_1(x_1) \circ \mu_2(x_2) \circ \cdots \circ \mu_n(x_n).$$

Let $\mathbf{A}$ be a $\mathcal{Q}$-valued algebra on A. Then $< \mathcal{P}\mathrm{A}, \{\phi^{\mathcal{P}^\circ(\mathbf{A})} | \phi^{\mathbf{A}} \in \mathbf{A}\} >$ is still a $\mathcal{Q}$-valued algebra and is called the $\circ$-$\mathcal{Q}$-valued power algebra of $\mathbf{A}$, denoted by $\mathcal{P}^\circ(\mathbf{A})$.

For a $\mathcal{Q}$-typed set A and $x \in A_0$, set $A_x = \{y \in A_0 \mid t(y) = t(x)\}$. Define $P^l : \mathcal{P}A_0 \to \mathcal{P}A_0$ by

$$P^l(\lambda)(x) = \lambda \bullet P(x) = \bigvee_{y \in A_x} \lambda(y) \circ P(x, y),$$

and $P^u : \mathcal{P}^\dagger A_0 \to \mathcal{P}^\dagger A_0$ by

$$P_u(\rho)(x) = P \bullet \rho(x) = \bigvee_{y \in A_x} P(y, x) \circ \rho(y).$$

Since $\mathcal{Q}$ is symmetric, the operation $\swarrow$ is accordance with $\searrow$. In the following, $\to$ is used to replace $\swarrow$ and $\searrow$. At this point, a presheaf on A is also a co-presheaf on A. Therefore, $\mathcal{P}A = \mathcal{P}^\dagger A$. From [15], we know there are three extensions of fuzzy relation to fuzzy powerset. This idea can be also used for $\mathcal{Q}$-matrices. Let $P : A \nrightarrow A$ be an $\mathcal{Q}$-matrix on A and define $P^\to : \mathcal{P}A \nrightarrow \mathcal{P}A$, $P^\leftarrow : \mathcal{P}A \nrightarrow \mathcal{P}A$ and $P^+ : \mathcal{P}A \nrightarrow \mathcal{P}A$ by

$$P^\to(\lambda, \mu) = \lambda \to P^l(\mu), \quad P^\leftarrow(\lambda, \mu) = \mu \to P^u(\lambda),$$

and

$$P^+(\lambda, \mu) = P^\to(\lambda, \mu) \wedge P^\leftarrow(\lambda, \mu),$$

respectively.

It is also routine to check that Theorem 4.3 and Theorem 4.4 are also valid for weak $\mathcal{Q}$-valued preorder. Furthermore, we have the following results.

Theorem 4.10. *Let $\mathbf{A} =< A, \mathbb{F} >$ be a $\mathcal{Q}$-valued algebra on A and P be an $\mathcal{Q}$-matrix on A. The following statements are equivalent:*
(1) P is compatible on $\mathbf{A}$;
(2) $P^\to$ is compatible on $\mathcal{P}^\circ(\mathbf{A})$;
(3) $P^\leftarrow$ is compatible on $\mathcal{P}^\circ(\mathbf{A})$.

Theorem 4.11. *Let $\mathbf{A} =< A, \mathbb{F} >$ be a $\mathcal{Q}$-valued algebra on A and P be a weak $\mathcal{Q}$-valued equivalence on A. If P is a weak congruence on $\mathbf{A}$, then P^+ is a weak congruence on $\mathcal{P}^\circ(\mathbf{A})$.*

Theorem 4.12. *Let $\mathbf{A} =< A, \mathbb{F} >$ be a $\mathcal{Q}$-valued algebra on A and P be a weak congruence on $\mathbf{A}$. Then there is an onto homomorphism from $\mathcal{P}^\circ(\mathbf{A}/P)$ to $\mathcal{P}^\circ(\mathbf{A})/P^+$.*

Theorem 4.13. $\mathcal{P}^\circ(\mathbf{A}/P)/ker(\theta) \cong \mathcal{P}^\circ(\mathbf{A})/P^+$.

Remark 4.14. *In classical universal algebra, we know the power algebra of the quotient algebra is isomorphic to the quotient algebra of the power algebra from [6, 7]. But, in $\mathcal{Q}$-typed sets setting, this is not valid. In Theorem 4.12, we show that there is an onto homomorphism from $\mathcal{P}^\circ(\mathbf{A}/P)$ to $\mathcal{P}^\circ(\mathbf{A})/P^+$. In general, the onto homomorphism is not necessary an isomorphism due to the definition of $\mathcal{P}A$. In the next section, we will give an example to show this.*

5 Algebras on fuzzy sets

In this section, as a special case of $\mathcal{Q}$-valued algebra, we focus on algebra on fuzzy set valued in a quantale. A quantale is exactly a quantaloid with only one object. From [31], given a quantale $L = (L, \&, 1)$, one can construct a quantaloid $\mathcal{D}(L)$, called the quantaloid of diagonals in L, as follows:

- objects: elements $a, b, c, \ldots$ in L;
- morphisms: $\mathcal{D}(L)(a, b) = \{d \in L : (d \swarrow a)\&a = d = b\&(b \searrow d)\}$ for all objects a, b.
- composition: $\beta \circ \alpha = \beta\&(b \searrow \alpha) = (\alpha \swarrow b)\&\beta$ for all $\alpha \in \mathcal{D}(L)(a, b)$ and $\beta \in \mathcal{D}(L)(b, c)$;
- the unit 1_a of $\mathcal{D}(L)(a, a)$ is a;
- the partial order on $\mathcal{D}(L)(a, b)$ is inherited from L.

In the following, we suppose L is a commutative and divisible quantale. Then it is easy to see that

$$\mathcal{D}(L)(a, b) = \mathcal{D}(L)(b, a) = \{d \in Q : d \leq a \wedge b\},$$

and

$$\beta \circ \alpha = \beta\&(b \to \alpha) = \alpha\&(b \to \beta) = \alpha \circ \beta,$$

for $\alpha \in \mathcal{D}(L)(a,b)$ and $\beta \in \mathcal{D}(L)(b,c)$. Hence $\mathcal{D}(L)$ is also a symmetric quantaloid.

As in [25]: there is an important fact about $\mathcal{D}(L)$ should not be overlooked: the elements of L paly, at the same time, the role of objects and morphisms in $\mathcal{D}(L)$. This might leads to refinement or enhancement of the results in the theory of quantaloid-enriched categories that make sense only for $\mathcal{D}(L)$-categories.

An L-set is a function $A : A_0 \to L$ and the valued $A(x)$ is interpreted as the degree to which the element x belongs to A. From the construction of $\mathcal{D}(L)$, we know that an L-set can be considered as a $\mathcal{D}(L)$-typed set. Hence the results in [32] can be viewed as a special case of $\mathcal{Q}$-typed set.

Remark 5.1. *(1) When $A = 1_{A_0}$, the fuzzy power algebra on A will reduce to the algebra of fuzzy set introduced in [5].*

(2) Let $E_A : \mathcal{P}A \nrightarrow \mathcal{P}A$ be defined as

$$\forall (f,\delta),(g,\eta) \in \mathcal{P}A_0,\ E_A((f,\delta),(g,\eta)) = S_A((f,\delta),(g,\eta)) \wedge S_A((g,\eta),(f,\delta)),$$

where

$$S_A((f,\delta),(g,\eta)) = \delta \wedge \eta \wedge \bigwedge_{x \in A_0} \big((\delta \to f(x)) \to g(x)\big).$$

Then it is easy to check that E_A is an L-valued equivalence on $\mathcal{P}A$. Given an algebra $\mathbf{A}$ on A, E_A will be a congruence on $\mathcal{P}^\circ(\mathbf{A})$. Hence $\mathcal{P}^\circ(\mathbf{A})$ with E_A can be viewed as a special generalization of an algebra with fuzzy equality in the sense of Bělohlávek and V. Vychodil [3].

Finally, we see the following example to explain Remark 4.14.

Example 5.2. *Let $L = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ and $\& = \wedge$. Suppose $A : A_0 \to L$ is the constant function from A_0 to L with the value $\frac{2}{3}$, where $A_0 = \{x,y,z\}$. Define $P : A \nrightarrow A$ by*

$$P(x,x) = P(y,y) = P(z,z) = P(x,y) = P(y,x) = \frac{2}{3},$$

and

$$P(x,z) = P(z,x) = P(y,z) = P(z,y) = \frac{1}{3}.$$

Then it is easy to check that P is an L-valued equivalence on A. Hence $(A/P)_0 = \{x/P, z/P\}$ and $A/P : (A/P)_0 \to L$ is

$$A/P(x/P) = A/P(z/P) = \frac{2}{3}.$$

There are 23 elements in $\mathcal{P}(A/\mathcal{P})_0$ as follows: $(\frac{0}{x/P} + \frac{0}{z/P}, 0)$, $(\frac{0}{x/P} + \frac{0}{z/P}, \frac{1}{3})$, $(\frac{0}{x/P} + \frac{0}{z/P}, \frac{2}{3})$, $(\frac{0}{x/P} + \frac{0}{z/P}, 1)$, $(\frac{0}{x/P} + \frac{\frac{1}{3}}{z/P}, \frac{1}{3})$, $(\frac{0}{x/P} + \frac{\frac{1}{3}}{z/P}, \frac{2}{3})$, $(\frac{0}{x/P} + \frac{\frac{1}{3}}{z/P}, 1)$, $(\frac{0}{x/P} + \frac{\frac{2}{3}}{z/P}, \frac{2}{3})$, $(\frac{0}{x/P} + \frac{\frac{2}{3}}{z/P}, 1)$, $(\frac{\frac{1}{3}}{x/P} + \frac{0}{z/P}, \frac{1}{3})$, $(\frac{\frac{1}{3}}{x/P} + \frac{0}{z/P}, \frac{2}{3})$, $(\frac{\frac{1}{3}}{x/P} + \frac{0}{z/P}, 1)$, $(\frac{\frac{1}{3}}{x/P} + \frac{\frac{1}{3}}{z/P}, \frac{1}{3})$, $(\frac{\frac{1}{3}}{x/P} + \frac{\frac{1}{3}}{z/P}, \frac{2}{3})$, $(\frac{\frac{1}{3}}{x/P} + \frac{\frac{1}{3}}{z/P}, 1)$, $(\frac{\frac{1}{3}}{x/P} + \frac{\frac{2}{3}}{z/P}, \frac{2}{3})$, $(\frac{\frac{1}{3}}{x/P} + \frac{\frac{2}{3}}{z/P}, 1)$, $(\frac{\frac{2}{3}}{x/P} + \frac{0}{z/P}, \frac{2}{3})$, $(\frac{\frac{2}{3}}{x/P} + \frac{0}{z/P}, 1)$, $(\frac{\frac{2}{3}}{x/P} + \frac{\frac{1}{3}}{z/P}, \frac{2}{3})$, $(\frac{\frac{2}{3}}{x/P} + \frac{\frac{1}{3}}{z/P}, 1)$, $(\frac{\frac{2}{3}}{x/P} + \frac{\frac{2}{3}}{z/P}, \frac{2}{3})$, $(\frac{\frac{2}{3}}{x/P} + \frac{\frac{2}{3}}{z/P}, 1)$.

The element in $(f,\delta)/P^+$ is (g,δ) satisfying $P^l(g) = P^l(f)$. For convenience, we only compute the following as examples $P^l(\frac{0}{x}+\frac{0}{y}+\frac{0}{z}) = \frac{0}{x}+\frac{0}{y}+\frac{0}{z}$, $P^l(\frac{0}{x}+\frac{\frac{1}{3}}{y}+\frac{0}{z}) = \frac{\frac{1}{3}}{x}+\frac{\frac{1}{3}}{y}+\frac{\frac{1}{3}}{z}$, $P^l(\frac{0}{x}+\frac{\frac{2}{3}}{y}+\frac{0}{z}) = \frac{\frac{2}{3}}{x}+\frac{\frac{2}{3}}{y}+\frac{\frac{1}{3}}{z}$ $P^l(\frac{0}{x}+\frac{0}{y}+\frac{\frac{2}{3}}{z}) = \frac{\frac{1}{3}}{x}+\frac{\frac{1}{3}}{y}+\frac{\frac{2}{3}}{z}$ and $P^l(\frac{0}{x}+\frac{\frac{2}{3}}{y}+\frac{\frac{2}{3}}{z}) = \frac{\frac{2}{3}}{x}+\frac{\frac{2}{3}}{y}+\frac{\frac{2}{3}}{z}$. It is routine to list the 13 elements in $(\mathcal{P}A/P^+)_0$ as follows: $(\frac{0}{x}+\frac{0}{y}+\frac{0}{z}, 0)$, $(\frac{0}{x}+\frac{0}{y}+\frac{0}{z}, \frac{1}{3})$, $(\frac{0}{x}+\frac{0}{y}+\frac{0}{z}, \frac{2}{3})$, $(\frac{0}{x}+\frac{0}{y}+\frac{0}{z}, 1)$, $(\frac{\frac{1}{3}}{x}+\frac{\frac{1}{3}}{y}+\frac{\frac{1}{3}}{z}, \frac{1}{3})$, $(\frac{\frac{1}{3}}{x}+\frac{\frac{1}{3}}{y}+\frac{\frac{1}{3}}{z}, \frac{2}{3})$, $(\frac{\frac{1}{3}}{x}+\frac{\frac{1}{3}}{y}+\frac{\frac{1}{3}}{z}, 1)$, $(\frac{\frac{1}{3}}{x}+\frac{\frac{1}{3}}{y}+\frac{\frac{2}{3}}{z}, \frac{2}{3})$, $(\frac{\frac{1}{3}}{x}+\frac{\frac{1}{3}}{y}+\frac{\frac{2}{3}}{z}, 1)$, $(\frac{\frac{2}{3}}{x}+\frac{\frac{2}{3}}{y}+\frac{\frac{1}{3}}{z}, \frac{2}{3})$, $(\frac{\frac{2}{3}}{x}+\frac{\frac{2}{3}}{y}+\frac{\frac{1}{3}}{z}, 1)$, $(\frac{\frac{2}{3}}{x}+\frac{\frac{2}{3}}{y}+\frac{\frac{2}{3}}{z}, \frac{2}{3})$ and $(\frac{\frac{2}{3}}{x}+\frac{\frac{2}{3}}{y}+\frac{\frac{2}{3}}{z}, 1)$. It is obvious we can not establish the one-to-one correspondence between $\mathcal{P}(A/P)_0$ and $(\mathcal{P}A/P^+)_0$.

6 Conclusions

In this paper, we establish a basic theory of universal algebras on $\mathcal{Q}$-typed sets. When L is a commutative and divisible quantale, $\mathcal{D}(L)$–the diagonal of L—will be a quantaloid, and $\mathcal{D}(L)$-typed set will be reduced to L-set. Hence fuzzy universal algebra on L-set introduced in [32] is a special case of $\mathcal{Q}$-valued algebras introduced in this paper. We give a kind of $\mathcal{Q}$-valued power algebra under certain conditions and an example is given to show the difference between power algebras on classical sets and power algebras on L-sets. This tells us that it is very meaningful and valuable to study universal algebras on fuzzy sets. We want to know whether there are other more appropriate constructions of $\mathcal{Q}$-valued power algebra? We leave it as a question.

Acknowledgement

The authors wish to express their appreciation for several excellent suggestions for improvements in this paper made by the referees. This work was supported by National Natural Science Foundation of China (11971448) and Natural Science Foundations of Shandong Province (ZR2017MA017).

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