A consistency-driven approach to construction of Z-number-valued pairwise comparison matrices

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Abstract

The notion of consistency is used to estimate the quality of preference knowledge and its stability for reliable evaluation of decision alternatives. It is well-known that a set of strict consistency conditions are used to keep the rationality of preference intensities between compared elements. These requirements are not achievable in the real situations when decision maker has limited rationality and partially reliable preferences. In this study, we propose an approach to deriving consistency-driven preference degrees for such kind of situations. A preference degree is described by a Z-number to reflect imprecision and partial reliability of preference knowledge. An optimization problem with Z-number valued variables is used to formulate design of consistent preferences. A real-world decision making problem is considered to illustrate application of the proposed method and conduct comparison with an existing technique.

Keywords: Z-number, Z-number-valued preference, decision making, pariwise comparison matrix, consistency.

1 Introduction

Decision making is based on preferences over alternatives and criteria. The fundamental way to model preferences is the pairwise comparisons method (PCM), it underlies development of an optimal choice function (decision-making function). The PCM method can be described as follows. Information on degrees of pairwise comparisons of alternatives (criteria) is represented in a form of a matrix. Formally, this is a matrix (a_{ij}), where an entry a_{ij} denotes a degree to which an *i-th* alternative (criterion) is preferred to *j*-th one[17]. Natural conditions used for a_{ij} are $a_{ii} = 1$ and $a_{ji} = 1/a_{ij}$ (reciprocity), $\forall i, j = 1, \cdots, n$. Traditionally, consistency of (a_{ij}) is based on multiplicative transitivity condition (though different constructs are also used [51, 54]):

$$a_{ij}a_{jk} = a_{ik}, \forall i, j, k. \tag{1}$$

This implies that degree of preference a_{ik} is equal to the product of preferences degrees staying at all possible ways from i to k through j. Fullfillment of this condition is often problematic due to low computational abilities of a human brain. An inconsistency is related to violation of this multiplicative transitivity condition. In view of this, various indices were introduced to measure inconsistency directly or indirectly with triad a_{ij}, a_{jk}, a_{ik}. An inconsistency index is a function that maps a set of comparions matrices $\mathcal{A}$ to a set of real numbers $\mathcal{R}$, $I : \mathcal{A} \to \mathcal{R}$. In some works, a term consistency index is used in the same sense. The most famous consistency index was proposed by Saaty [56]. It is based on an affine transformation of maximal eigenvalue of a matrix. Among well known indexes are those proposed by Kou

and Lin [46], Cavallo and D'Apuzzo [19, 20], Koczkodaj and colleagues [43]-[40] and others. A systematic overview and analysis of inconsistency indexes is given in [17]. Particularly, inconsistency indexes are divided informally into three groups: The indices tightened to weight vector w, the indices related to the most inconsistent triad in pairwise comparisons matrix, and those having a closed form or an analytical solution.

An important area of study is devoted to inconsistency reduction, also referred to as construction of consistency-driven pairwise comparisons matrix. In [30] a sensitivity analysis is used to uncover entries of pairwise comparisons matrix which are most influential for Saaty's index. In [23] it is proposed a more general framework based on l^p distance ($p = 2$ is the Euclidean distance case). A goal programming approach is used to derive a consistent pairwise comparisons matrix closest to a given inconsistent one. Ishizaka, Alessio, and Markus Lusti [35] proposed an expert module to aid a DM in forming consistent pairwise comparisons matrix. A simple and mathematically proven method for improving consistency is proposed in [25]. In contrast to many analogs, the method can be applied also to analytic network process approach (ANP, a more general form of AHP methodology) and better preserves initial information provided by a DM. Paper [16] is devoted to determination of minimum number of entries of pairwise comparisons matrix for inconsistency reduction by using optimization problems. An analysis of convergence of inconsistency reduction algorithms that are based on distance metric is proposed in [33]. A comparative study of various techniques is given in [61].

Initially, degrees of preference were described as crisp numbers from a predefined scale. Further, fuzzy sets were used to account for uncertainty of preference degrees [39, 47, 49, 52, 58, 63, 64]. One of the most famous approaches in this stream was proposed in [63]. The authors consider that a DM may assign preference degrees in form of predefined fuzzy numbers. Further, such fuzzy numbers are used to find fuzzy weights of criteria and fuzzy criteria evaluation of alternatives. In [55, 57, 70] a criticism of the use of fuzzy approaches for formalizing information on preference degrees is provided. The authors argue that fuzzifying preference degrees is characterized by a lack of mathematical soundness and inability to provide validity of results (while consistency can be improved). A series of approaches were proposed to analysis and resolving such problems (see, for example those given in [48, 58, 64]). Existing works devoted to use of interval, fuzzy and probabilistic approaches to deal with inconsistency of uncertain preferences proceed with generalization or adaptation of indices proposed for classical pairwise comparisons matrix [18, 21, 47, 53, 69]. The motivation of the study in [53] is modeling limited rationality of DMs resulted from natural vagueness of human preferences. The authors conduct a wide analysis of consistency of interval-valued pairwise comparisons matrix. Axiomatization of consistency is formulated by weakening some properties of classical pairwise comparisons matrices. Two kinds of interval-valued preferences with weak consistency are proposed. A comparison with existing approaches is conducted. In [18], a general approach to analysis of interval-valued pairwise comparisons matrix is proposed. The authors deals with consistency conditions of multiplicative and fuzzy interval-valued pairwise comparisons matrices. A new inconsistency and indeterminancy indices are introduced for various kinds of interval-valued pairwise comparisons matrices in a unified framework. A novel preference structure were proposed for hesitant fuzzy preference relation in [69]. The structure allows to account for natural uncertainty of a DM's preference degree. They propose an inconsistency index and an approach to arrive at the closest consistent hesitant pairwise comparisons matrix given initially inconsistenc one. Book [47] is the first one devoted to systematic analysis of existing fuzzy pairwise comparison methods. New extensions of these methods are proposed to reduce existing drawback. A real-world case study with incomplete large-scale pairwise comparisons matrix is considered to illustrate the proposed approach.

Let us note that one of the most famous decision techniques relying on PCM method is the AHP methodology. However, this methodology is criticized due to several problems as rank reversal, the use of additive aggregation method (and other axiomatic restrictions), problems related to normalization etc [14, 12, 29, 28, 31]. Rank reversal problem is a change of order of alternatives (or criteria) that occurs if one adds a copy of an existing alternative (criterion). This issue caused intensive debates as to whether this is a fundamental problem or a secondary one (e.g. works [15, 62]). Another problem is that the additive aggregation method (based on the expectation axiom) can be used only when criteria are independent, which is a quite rare case in real-world problems. At the same time, aggregation leads to loss of information. Indeed, any decision model that utilizes information from pairwise comparisons adds its own restrictions. It is important to analyze pairwise comparisons 'as is', prior to process them and use in preference quantification approaches.

The motivation of the proposed research is to develop a more adequate decision-making function. As a rule, decision-making function is hierarchic, it is a vector of choice criteria used in a decision problem. A decision maker (DM) is interviewed to obtain information on a degree to which one criterion is more important than another one in a pair. However, due to imperfect information and other restrictions, a DM finds difficulties to provide exact and accurate preference degrees. In this case, fuzzy degrees of preference are used to account for imprecision. Being imprecise, real-world preferences are also partially reliable. The reasons are restricted competence of a DM, complexity of alternatives, imperfect decision-relevant information, psychological biases etc. So, in line with describing imprecision, it becomes relevant to account of partial reliability of preferences. Up to day, no works have been proposed on consistency of

partially reliable preferences. In order to handle imprecision and partial reliability, Zadeh introduced the Z-number concept [67]. A Z-number $Z = (A, B)$ is an ordered pair of fuzzy numbers used to describe a value of a random variable X. A is a fuzzy restriction on values of X. B is a fuzzy *reliability* of A, defined as a value of probability measure of A. A series of works is devoted to theoretical studies and practical applications of Z-numbers [2]-[13], [22, 27, 26, 32, 36, 37, 38, 50, 59, 65, 66, 68]. In this paper we propose a new approach for construction of consistency-driven partially reliable preferences described by pairwise comparisons matrix with Z-number-valued entries. An entry $Z = (A, B)$ stores imprecise information about preference degree A and a related level of belief B. This is a more adequate way to describe preference levels that the use of pure fuzzy degrees. A statement of problem for inconsistency reduction of Z-number-valued pairwise comparisons matrix is formulated. The problem is formulated as a complex optimization problem involving membership functions and probability distributions. In veiw of this, a differential evolution (DE) optimization technique-based solution approach is proposed. An application is given to illustrate validity of the proposed method and to compare the results with those obtained for fuzzy setting. Indeed, generalization of information contained in pairwise comparisons matrix by using Z-numbers, does not rule out such problems as rank reversal and other problems. The issue is that some of these problems are related to the structure of decison function, not to character of information. We try to reduce inconsistency of preferences when actual information is characterized by fuzziness and probabilistic uncertainty. It allows to account for partial reliability of information contained pairwise comparisons matrix.

The paper is structured as follows. In Section 2, necessary definitions including those of a Z-number, a Z-number-valued pairwise comparisons matrix and others are provided. In Section 3, a statement of problem of construction of a consistent Z-number-valued pairwise comparisons matrix that describes partially reliable preference is formulated. The solution method for the considered problem is given in Section 4. A solution algorithm stemming from the solution method is given in Section 5. An application of the proposed method to a real-world decision problem and comparison with a fuzzy analog is considered in Section 6. Section 7 concludes.

2 Preliminaries

The concept of Z-number is used to describe a value of real-valued random variable X under combination of fuzzy and probabilistic uncertainties. Such kind of combination is usually referred to as a bimodal distribution [5, 7].

Definition 2.1. [5, 7] *A continuous Z-number is an ordered pair* $Z = (A, B)$. *A is a continuous fuzzy number with membership function* $\mu_A(x)$ *playing a role of a fuzzy restriction on a value that a random variable X may take:*

$$\text{Value of } X \text{ is } A.$$

In other words, A is used to describe imprecise information about a value of X. A degree of reliability of A is described as a value of probability measure $P(A) = \int_R \mu_A(x)p(x)dx$, where p is probability distribution of X. If p is precisely known, the $P(A)$ is a crisp number. However, in real-world problems an actual p may not be precisely known, and one has to consider a set of distributions. This requires to deal with fuzzy restriction on a value of $P(A)$. Thus, to describe uncertainty related to p, a continuous fuzzy number B with a membership function $\mu_B : [0,1] \to [0,1]$ is used as a fuzzy restriction:

$$\text{The value of } P(A) \text{ is } B. \tag{2}$$

This implies that fuzzy uncertainty related to p induces fuzziness of a value of $P(A)$. Thus, due to fuzzines and probabilistic uncertainty of information, a value of random variable X can be described as a Z-number $Z = (A, B)$: A is a fuzzy estimation of a value and B is a fuzzy reliability of this estimation.

Definition 2.2. *A Z-number-valued pairwise comparisons matrix* (Z_{ij}) *is a square matrix of Z-numbers:*

$$(Z_{ij} = (A_{ij}, B_{ij})) = \begin{pmatrix} Z_{11} = (A_{11}, B_{11}) & ... & Z_{1n} = (A_{1n}, B_{1n}) \\ . & ... & . \\ Z_{n1} = (A_{n1}, B_{n1}) & ... & Z_{nn} = (A_{nn}, B_{nn}) \end{pmatrix}. \tag{3}$$

A Z-number $Z_{ij} = (A_{ij}, B_{ij})$, $i, j = 1, ..., n$ *describes partially reliable information on degree of preference for i-th alternative (criterion) against j-th one.*

Definition 2.3. *As a Z-number* $Z = (A, B)$ *is characterized by fuzzy number A, fuzzy number B and underlying set of probability distributions G [59], we propose to define distance between Z-numbers* $D(Z_1, Z_2)$ *as follows.*

Distance between A_1 and A_2 is computed as

$$D(A_1, A_2) = \sup_{\alpha \in (0,1]} D(A_1^\alpha, A_2^\alpha), \tag{4}$$

$$D(A_1^\alpha, A_2^\alpha) = \left| \frac{A_{1_1}^\alpha + A_{1_2}^\alpha}{2} - \frac{A_{2_1}^\alpha + A_{2_2}^\alpha}{2} \right|. \tag{5}$$

A_1^α and A_2^α denote α-cuts of A_1 and A_2 respectively, $A_{1_1}^\alpha, A_{1_2}^\alpha$ denote lower and upper bounds of A_1^α ($A_{2_1}^\alpha, A_{2_2}^\alpha$ are those of A_2^α). Distance between B_1 and B_2 is computed analogously.

We also have to find distance between the sets G_1 and G_2 of probability distributions p_1 and p_2 underlying Z_1 and Z_2. The distance between p_1 and p_2 can be expressed as *[59].*

$$D(G_1, G_2) = \inf_{p_1 \in G_1, p_2 \in G_2} \left\{ \left(1 - \int_R (p_1 p_2)^{\frac{1}{2}} dx \right)^{\frac{1}{2}} \right\} \tag{6}$$

In (5), the expression in figure brackets is the Heellinger distance between two probability distributions p_1 and p_2. The inf operator is used to define the distance between the sets G_1 and G_2 as that of between the closest $p_1 \in G_1$ and $p_2 \in G_2$ is found among all the possible pairs of distributions to define distance $D(G_1, G_2)$.

Given $D(A_1, A_2)$, $D(B_1, B_2)$ and $D(G_1, G_2)$, the distance for Z-numbers is defined as

$$D(Z_1, Z_2) = \beta D(A_1, A_2) + (1 - \beta) D_{total}(B_1, B_2), \tag{7}$$

where $D_{total}(B_1, B_2)$ is a distance for reliability restriction computed as

$$D_{total}(B_1, B_2) = w D(B_1, B_2) + (1 - w) D(G_1, G_2). \tag{8}$$

$\beta, w \in [0, 1]$ are DM's assigned importance degrees.

Below we propose a definition of inconsistency index for Z-number-valued matrix adopted from the index introduced for real-valued matrix in [23].

Definition 2.4. *An inconsistency index K for Z-number-valued pairwise comparisons matrix (Z_{ij}) is defined as follows:*

$$K((Z_{ij})) = \max_{i<j<k} \min \left\{ D\left(Z(1), \left(\frac{Z_{ik}}{Z_{ij} Z_{jk}} \right) \right) D\left(Z(1), \left(\frac{Z_{ij} Z_{jk}}{Z_{ik}} \right) \right) \right\}, \tag{9}$$

where the components of Z-number $Z(1) = (A, B)$, are fuzzy singletons $A = 1$ and $B = 1$. The operations of multilication and division of Z-numbers in (9) are defined as in [7].

As compared to many existing indices, index (9) describes inconsistency level of a matrix as that of the most inconsistent triad (not as some average of inconsistency levels of all triads). This is the reason we use this index for Z-number-valued case. On the one hand, it allows to reduce loss of information resulted from averaging of Z-number-valued inconsistency levels. On the other hand, it is characterized by a lower computational complexity.

3   Statement of the problem

Let us consider a problem of generation of consistent pairwise comparisons matrix (Z'_{ij}) closest to a given inconsistent pairwise comparisons matrix (Z_{ij}). The elements of inconsistent Z-matrix (Z_{ij}) will be considered as a perturbation of the elements of matrix (Z'_{ij}) for which reciprocity and consistency are verified. We have to change elements of (Z_{ij}) in order to arrive at (Z'_{ij}). The problem is formulated as follows.

The objective function:

$$J = \sum_{i=1}^{n} \sum_{j=1}^{n} D(Z_{ij}, Z'_{ij}) \to \min \tag{10}$$

The entries Z'_{ij} of pairwise comparisons matrix are considered as Z-number-valued decision variables. The use of (10) implies that one needs to minimize distance between elements of matrix (Z') and those of initial inconsistent matrix (Z).

The constraints are described below.

Multiplicative reciprocity constraints:

$$Z'_{ij} Z'_{ji} = Z(1). \tag{11}$$

Constraints (11) are conditions of multiplicative reciprocity, where $Z(1) = (A, B)$ is such that A, B are fuzzy singletons $A = 1, B = 1$. These are necessary conditions implying that if i-th alternative (criterion) is *more important* than j-th one to a *Z-number-valued degree* Z'_{ij}, then the latter is *less important* than i-th one to the *same Z-number-valued degree*. The product at the left hand side is computed on the basis of the method proposed in [7].

Multiplicative transitivity constraints:

$$Z'_{ij} Z'_{jk} = Z'_{ik}. \tag{12}$$

Constraints (12) are multiplicative transitivity conditions. These imply: If i-th alternative (criterion) is Z'_{ij} times more important than j-th one, and the latter is Z'_{jk} times more important than k-th one, then preference of i-th alternative (criterion) against k-th one should be $Z'_{ik} = Z'_{ij} Z'_{jk}$ (in the ideal case).

Non-negativity constraints:

$$Z'_{ij} \geq Z(0), i, j = 1, \ldots, n. \tag{13}$$

Constraints (13) imply that preference degrees are non-negative, where $Z(0) = (A, B)$ is such that A, B are fuzzy singletons $A = 0, B = 1$. Once a Z-number-valued pairwise comparisons matrix as a solution of the problem is obtained, it is necessary to evaluate its level of inconsistency by using Definition 2.4.

4 Solution method

Problem (10)-(13) is a non-linear optimization problem characterized by fuzzy and probabilistic uncertainties. Taking into account these features, it is needed to develop a solution approach relying on DE optimization technique *[60]*.

DE optimization is a population based evolutionary method, in which population of individuals (vectors each representing candidate solution) with each generation progresses towards a predefined desired property (fitness) for all individuals. To improve standard version of DE, the value of desired property for any individual can evaluated by more than one special functions (including e.g. fitness/ objective/ error/ feasibility/ constraint satisfaction). After a number of generations, one or more solutions are selected from the population based on the value of their property.

The vectors are made of parameters the values of which need to be optimized. The DE parameters usually are numerical (real) values. Therefore, for some problems that use non-numerical (e.g. fuzzy-number valued) parameters or parameters constrained in some other way, some procedures may be required for mapping from the space of problem to space of DE vectors (R^N, where N is the dimensionality of vectors).

In the suggested version of the algorithm, the state property of each vector is evaluated by two functions: *Cost Function (CF)* and *Error Function (EF) (degree of constraint dissatisfaction)*. A value meeting all requirements is assumed to produce a non-negative value of the *EF*, the lower the better. The preference of solution V_1 over solution V_2 ($V_1 \succ V_2$) is defined as follows:

$$IF \ CF(V_1) > CF(V_2)$$
$$THEN \ IF \ EF(V_1) = 0 \ OR \ EF(V_1) \leq EF(V_2) \ THEN \ V_1 \ IS \ BETTER \ THAN \ V_2$$
$$ELSE \ IF \ CF(V_1) < CF(V_2)$$
$$THEN \ IF \ EF(V_2) = 0 \ OR \ EF(V_1) \leq EF(V_2) \ THEN \ V_1 \ IS \ NOT \ BETTER \ THAN \ V_2 \tag{14}$$
$$ELSE$$
$$IF \ EF(V_1) \leq EF(V_2) \ THEN \ V_1 \ IS \ BETTER \ THAN \ V_2$$
$$ELSE \ V_1 \ IS \ NOT \ BETTER \ THAN V_2$$

CF evaluates quality of a solution (in this particular implementation the lower the better):

$$CF\left((Z'_{ij}), (Z_{ij})\right) = \sum_{i,j} \text{Distance}\left(Z'_{ij}, Z_{ij}\right).$$

EF expresses the state of vector in terms of meeting all numerical restrictions over its components (i.e. individual parameters):

$$EF\left((Z'_{ij}), (Z_{ij})\right) = (1 - \alpha) IF\left(\left(Z'_{ij}\right)\right) + \alpha CF\left((Z'_{ij}), (Z_{ij})\right), \tag{15}$$

where

$$IF\left(\left(Z'_{ij}\right)\right) = \sum_i \text{Distance}(Z'_{ii}, Z(1)) + \sum_{i,j>i} \text{Distance}\left(Z'_{ij}Z'_{ji}, Z(1)\right) + \sum_{i\neq j\neq k} \text{Distance}\left(Z'_{ij}, Z'_{ik} \times Z'_{kj}\right).(16)$$

Here $\alpha = \frac{1}{\left(\frac{generation}{\varepsilon \times max\_generations+1}+1\right)}$, $generation$ is current generation (step) of DE, $max\_generations$ –maximum number of generations to obtain the solution (e.g. 20000), ε=0.01 (see Fig. 1). In formula (15), the parameter α is chosen for a desired property of a candidate solution to balance between the closeness to initial PCM and consistency satisfaction at a particular step of DE optimization. At initial steps of optimization, we require DE to pay more attention to satisfaction of condition of closeness to initial pairwise comparisons matrix, since $EF(.) \sim CF(.)$. In the considered version of DE algorithm the value of $EF(.)$ is more important than $CF(.)$ when comparing 2 candidate solutions for preference (refer to formula (15)). Therefore, we gradually adjust satisfaction of consistency restrictions as iterations progress. For large generation numbers (close to $max\_generations$) $EF(.) \sim IF(.)$, which provides for full satisfaction of consistency requirement for solutions at final generations (while still keeping maximum closeness to initial pairwise comparisons matrix).
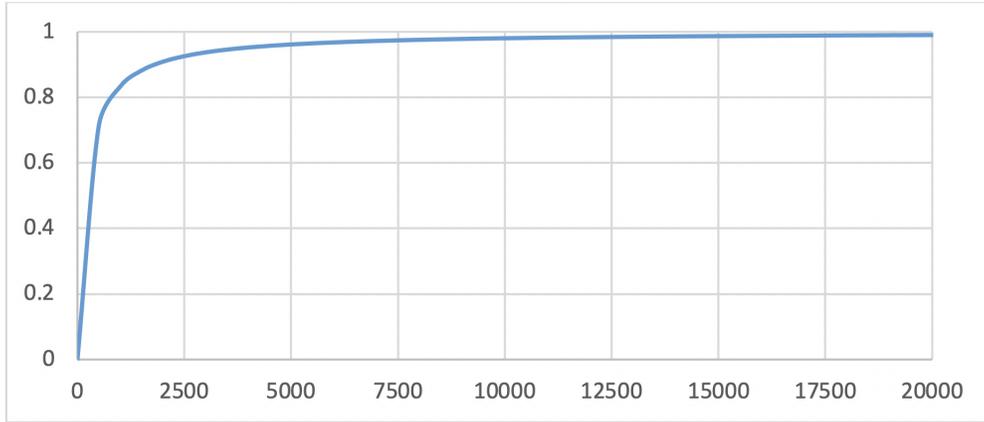

Figure 1: Dependence of parameter $1 - \alpha$ on number of generations.

For the considered problem of finding an optimal pairwise comparisons matrix, any potential solution V represents a set of parameters that can hold a possible symmetric Z-number-valued matrix $(Z_{ij} = (A_{ij}, B_{ij})), i, j = 1, ..., n$. For the case of triangular fuzy numbers (TFNs) A_{ij}, B_{ij}, the required dimension of vector V would be $2\times3\times n\times n$. For example, for pairwise comparisons matrix 3×3, i.e. n=3, vectors V will be of dimension 54. Thus, a candidate pairwise comparisons matrix $(Z'_{ij} = (A'_{ij}, B'_{ij})), i, j = 1, ..., n$ is represented by vector $V = (v_1, ..., v_{6n^2})$. DE optimization algorithm has a few control parameters: ps (Population Size) parameter (usually set to 100 or more), cr (Crossover rate) and f (Mutation rates), the standard values for which are: cr=1 and f=0.9.

5   Solution algorithm

The solution algorithm for the problem of finding an optimal consistent Z-valued pairwise comparisons matrix can be represented algorithmically as follows:

1. Get initial pairwise comparisons matrix (Z_{ij});

2. Selection of distance measure $D(Z'_{ij}, Z_{ij})$ for Z-number-valued matrices;

3. Form Cost and Error functions for DE method on the basis of Distance measure: set $CF((Z'_{ij}), (Z_{ij}))$ as in formula (10) and $EF((Z'_{ij}), (Z_{ij}))$, where $IF((Z'_{ij}), (Z_{ij}))$ is as in (16);

4. Construct mapping procedure (MP) to represent Z-number-valued matrix (Z'_{ij}) as a DE vector $(Z'_{ij}) = MP(V')$;

5. Set DE parameters: f, cr, ps. Initialize population P members (ps number of V' vectors: $V'_1, V'_2, \cdots, V'_{ps}$ with allowable random values;

6. Compute values of $CF(MP(V'),(Z_{ij}))$ and $EF(MP(V'),(Z_{ij}))$ for all vectors $i = 1,\cdots,ps$ in population P. Define maximum number of generations. Set *generation* to 1.

7. Process vectors in population in accordance with DE crossover and mutation logic:

   (a) Choose a next vector $V_i, (i = 1,\cdots,ps)$;

   (b) Choose randomly different 3 vectors from P: V_{r1}, V_{r2}, V_{r3} each of which is different from current V_i

   (c) Generate trial vector $V_t = V_{r1} + f \cdot (V_{r2} - V_{r3})$;

   (d) Generate a new vector from trial vector V_t. Individual vector parameters of V_t are inherited with probability cr into the new vector V_{new}. If the property of V_{new} is better than the property of V_t, (i.e. $V_{new} \succ V_t$ in formula (14)) in population P the vector V_t is replaced by V_{new}.

   The above process updates population P;

8. Recompute properties of all vectors in new population P. Set *generation=generation*+1;

9. Until a predefined number of generations is not performed go to Step 7;

10. Locate from the population P the vector V_{new} with best values for CF and EF. Retrieve the optimal matrix (Z'_{ij}) from this vector: $(Z'_{ij}) = MP(V'_{best})$.

11. At the *final stage*, an index is chosen to measure inconsistency of the obtained matrix (Z'_{ij}) (Definition 2.4). If $K((Z'_{ij}))$ does not exceed a predefined threshold θ_K, then an obtained matrix (Z'_{ij}) is considered consistent. If not, a DM is asked to reconsider his preferences to form new initial matrix (Z_{ij}). Then a problem (10)-(13) is solved again.

The algorithm chart is given below (Fig. 2).

6   An application. Preferences over criteria in a market selection problem

Let us consider extraction of a consistent Z-number-valued matrix to describe preferences over multiple criteria in a foreign market selection problem (country selection for doing business [34]). We will deal with three criteria that describe a series of economical and institutional characteristics: Institutional Proximity, C_1, Economic Proximity, C_2, and Social and Cultural Proximity, C_3. Criterion C_1 represents governance performance and economic freedom. Criterion C_2 describes both domestic development (such as socioeconomic progress, household's standard of living) and global competetitiviness issues. Criterion C_3 concerns cultural characteristics (an extent to which collectivism or individualism-based behavior is intrinsic to a country, issues related to communication with customers etc). Due to complexity of the considered criteria, a DM's preferences may be characterized by fuzziness and partial reliability. In view of this, we use partially reliable preference degrees of the Saaty scale to represent comparative importance of criteria (Table 1):

Table 1. Z-number-valued preference knowledge about criteria importance

	C_1	C_2	C_3
C_1	(equally important, very sure)	(slightly less important, sure)	(between slightly and more important, sure)
C_2	(slightly more important, sure)	(equally important, very sure)	(absolutely more important, almost sure)
C_3	(between slightly and less important, sure)	(absolutely less important, almost sure)	(equally important, very sure)

This information can be formalized by a 3×3 matrix of Z-numbers with TFNs-based components:

$$\begin{pmatrix} Z_{11} = ((0.93, 0.95, 0.97), (0.95, 0.98, 1)) & Z_{12} = ((0.327, 0.333, 0.34), (0.7, 0.8, 0.9)) & Z_{13} = ((3.92, 4, 4.08), (0.7, 0.8, 0.9)) \\ Z_{21} = ((2.94, 3, 3.06), (0.7, 0.8, 0.9)) & Z_{22} = ((0.93, 0.95, 0.97), (0.95, 0.98, 1)) & Z_{23} = ((8.82, 9, 9.02), (0.6, 0.7, 0.8)) \\ Z_{31} = ((0.245, 0.25, 0.255), (0.7, 0.8, 0.9)) & Z_{32} = ((0.1108, 0.111, 0.113), (0.6, 0.7, 0.8)) & Z_{33} = ((0.93, 0.95, 0.97), (0.95, 0.98, 1)) \end{pmatrix}$$

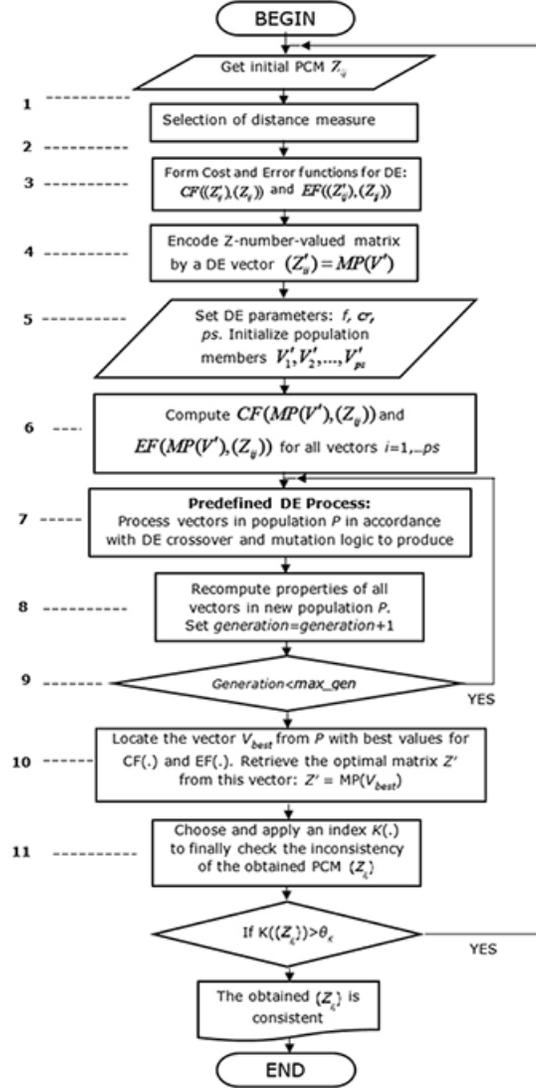Z_{ij} *denotes a* $Z - number - valued$ *degree to which the i-th criterion is preferred to the j-th one.*

Figure 2: The solution algorithm chart.

For example, $Z_{21} = ((2.94, 3, 3.06), (0.7, 0.8, 0.9))$ is a Z-number-valued degree to which C_2 is prefered to C_1. The value of inconsistency index K for the considered matrix (Z_{ij}) is $K((Z_{ij})) = 0.31$ (computed on the basis of Definition 2.4). Let us consider extraction of consistent Z-number-valued matrix (Z'_{ij}) closest to the initial one. For this purpose optimization problem (10)-(13) will be solved by using the method proposed in Section 4. According to the solution algorithm, an initial Z-number-valued matrix (Z_{ij}) is obtained (step 1). Next, distance measure D is chosen for objective (10) as given in Definition 2.3 (step 2). Accordingly, Cost function $CF((Z'_{ij}), (Z_{ij}))$ and error function $EF((Z'_{ij}), (Z_{ij}))$ are defined based on formulas (10) and (15) (step 3). At step 4, MP is chosen to encode Z-number-valued matrix $(Z'_{ij}) = MP(V')$. Next, DE control parameters are set as $ps=5000$, $cr=1$, $f=0.9$, search space dimension is 54 and population members $V'_1, V'_2, ..., V'_{ps}$ are initialized (step 5). At step 6, initial values of $CF(MP(V'), (Z_{ij}))$ and $EF(MP(V'), (Z_{ij}))$ are computed for all $V_i, (i = 1, ..., ps)$ and $generation$ number is set to 1. Then the population vectors $V_i, (i = 1, ..., ps)$ are processed to generate better vectors V_{new} on the basis of procedure in step 7. The properties of new vectors are recomputed (step 8). Steps 7 and 8 are repeated until prefedined generation number is done (step 9). The optimization plot is given in Fig. 3.

At step 10, the vector V_{new} with best values for CF and EF was located and the optimal Z-number-valued matrix $(Z'_{ij}) = MP(V'_{best})$ is retrieved:

$$
\begin{pmatrix}
Z_{11} = ((1.0008, 1.0028, 1.0028), (0.9996, 1, 1)) & Z_{12} = ((0.4029, 0.4029, 0.402935), (0.72, 0.996, 0.996)) & Z_{13} = ((3.65, 3.651, 3.6512), (0.997, 0.9973, 0.9973)) \\
Z_{21} = ((2.4827, 2.483, 2.483), (0.78, 0.78, 0.991447)) & Z_{22} = ((0.995452, 1, 1), (0.95, 0.98, 1)) & Z_{23} = ((9.06, 9.06, 9.064), (0.9947, 0.9947, 0.995)) \\
Z_{31} = ((0.278, 0.278, 0.278), (0.0078, 0.008, 0.50)) & Z_{32} = ((0.110, 0.1104, 0.1104), (0.499, 0.999, 0.999)) & Z_{33} = ((0.999, 1.003, 1.003), (0.999, 1, 1))
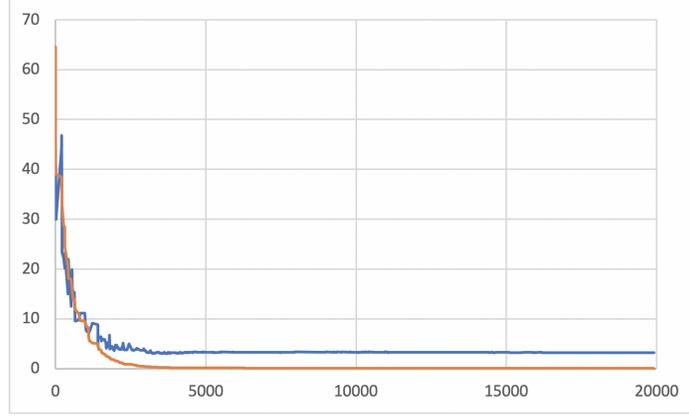\end{pmatrix}
$$

Figure 3: DE Optimization plots for Cost (blue curve) and Error (red curve) functions.

At the final step, we have to verify whether the value of K for the obtained (Z'_{ij}) exceeds a predefined treshold $\theta_K = 0.1$. The computed value of K is $K((Z'_{ij})) = 0.003$ which does not exceed θ_K. Thus, the obtained matrix can be considered as consistent.

Let us compare the obtained results with the case of fuzzy information. In view of this, we assume that preference information is completely reliable and fuzzy preference degrees are equal to A parts of Z-number-valued degrees:

$$\begin{pmatrix} A_{11} = (0.93, 0.95, 0.97) & A_{12} = (0.327, 0.333, 0.34) & A_{13} = (3.92, 4, 4.08) \\ A_{21} = (2.94, 3, 3.06) & A_{22} = (0.93, 0.95, 0.97) & A_{23} = (8.82, 9, 9.02) \\ A_{31} = (0.245, 0.25, 0.255) & A_{32} = (0.1108, 0.111, 0.113) & A_{33} = (0.93, 0.95, 0.97) \end{pmatrix}$$

The obtained optimal fuzzy matrix (A'_{ij}) is

$$\begin{pmatrix} A'_{11} = (0.999, 0.999, 0.999) & A'_{12} = (0.4, 0.4, 0.4) & A'_{13} = (3.65, 3.66, 3.66) \\ A'_{21} = (2.487, 2.489, 2.9) & A'_{22} = (0.999, \ 1, \ 1) & A'_{23} = (9.07, 9.1, 9.1) \\ A'_{31} = (0.27, 0.27, 0.27) & A'_{32} = (0.11, 0.11, 0.11) & A'_{33} = (0.999, 0.999, 0.999) \end{pmatrix}$$

As one can see, the fuzzy preference degrees in this optimal fuzzy matrix are approximately the same as A parts of the optimal Z-number-valued matrix. For example, $A'_{21} = (2.487, 2.489, 2.9)$ is approximately equal to the A part of Z'_{21}. However, information on partial reliability is lost in the fuzzy matrix. $A'_{21} = (2.487, 2.489, 2.9)$ is not the same as $Z'_{21} = ((2.48, 2.48, 2.48), (0.7, 0.7, 0.99))$, it is considered that A'_{21} is completely reliable, whereas the reliability of the second one is about 70%.

We also computed inconsistency index (9) for initial and optimal fuzzy matrices (fuzzy numbers are used instead of Z-numbers in definition 2.4). The value for initial matrix is $K\left((A_{ij})\right) = 0.25$, and for optimal one it is $K\left((A'_{ij})\right) = 0.0002$. As one can see, inconsistency of the Z-number-valued case is higher than that of the fuzzy case (both for initial and optimal matrices). This is related to partial reliability in Z-number-valued matrices. In fuzzy case, fuzzy preference degrees should be found that satisfy all the constraints (11)-(13). In the Z-number-valued case, one needs to find for each fuzzy preference degree a related fuzzy set of probability distributions. Thus, attaining satisfaction of the mentioned constraints becomes a more difficult task. At the same time, computational complexity is higher due to the use of both fuzzy arithmetic and probabilistic arithmetic.

7   Conclusions

Real-world preferences are characterized by imprecise and partially reliable information. A Z-number is a formal construct used to describe such kind of information. In this paper, we propose an approach to construct a consistency-driven Z-number-valued pairwise comparisons matrix characterized by imprecision and partial reliability of a DMs preferences. The approach is based on formalism of Z-numbers. An inconsistency reduction of Z-number-valued pairwise comparisons matrix is formulated as an optimization problem with Z-number valued decision variables. A DE optimization-based solution method is proposed. An application of the proposed approach to a real-world problem of

foreign market selection is considered. In this problem, partially reliable preferences over choice criteria are described by Z-number-valued pairwise comparisons matrix. By using the proposed approach, a consistent Z-number-valued pairwise comparisons matrix is obtained closest to the initial one. A comparison of this approach with inconsistency reduction of pure fuzzy pairwise comparisons matrix is also conducted. The results show that missing Z-number valued information about preferences may lead to improper evaluation of inconsistency level. Indeed, partial reliability of provided pairwise comparisons matrix increases its inconsistency level as compared to completely reliable one. Thus, consideration of partial relaibility of preferences allows for a more adequate analysis of their inconsistency.

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