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Preference implication-based approach to ranking fuzzy numbers

J. Dombi[1] and T. Jónás[2]

[1] *Institute of Informatics, University of Szeged, Szeged, Hungary*
[2] *Institute of Business Economics, Eötvös Loránd University, Budapest, Hungary*

dombi@inf.u-szeged.hu, jonas@gti.elte.hu

Abstract

Dombi and Baczyński presented a new approach to the problem of implication operation by introducing the preference implication, which has very advantageous properties. In this paper, it is presented how the preference implication is connected with soft inequalities and with sigmoid functions. Utilizing this connection the preference implication-based preference measure for two fuzzy numbers is introduced and its key properties, including the reciprocity, are described. Then, the exact expression for computing the new preference measure for trapezoidal fuzzy numbers is presented. Here, using the new preference measure, two crisp relations over trapezoidal fuzzy numbers are introduced. It is shown that one of them is a strict (but not a total) order relation, and the other one is an equivalence relation. The strict order relation can be used to rank comparable fuzzy numbers, while the equivalence relation, which we call the indifference relation, expresses that the order of some fuzzy numbers is indifferent. These two crisp relations can be used to rank a collection of trapezoidal fuzzy numbers. Lastly, the proposed ranking method is compared with some well-known existing fuzzy number ranking methods.

Keywords: Preference implication, preference measure, trapezoidal fuzzy numbers, strict order relation, equivalence relation.

1 Introduction

In many situations, outcomes of multi-criteria decision making are fuzzy numbers. Therefore, the ranking of fuzzy numbers is undoubtedly one of the most important topics in the theory and practice of fuzzy decision-making. Since it is a challenging problem, it has attracted the interest of many researchers. There is a large number and a high diversity of methods and approaches available for tackling the problem of raking fuzzy numbers (see, e.g. [26, 38, 10, 11, 14, 6, 37, 1, 36]). Zumelzu et al. studied more than two hundred partial order relations for fuzzy numbers (see [44]). Huynh et al. [24] introduced a probability-based methodology for comparing and ranking fuzzy numbers. Wang [35] utilized a relative preference relation for ranking triangular and trapezoidal fuzzy numbers. An inclusion index and bitset encoding-based approach was presented by Boulmakoul et al. [8]. By using a synthesis of fuzzy targets and the application of Dempster–Shafer theory, Chai et al. [9] developed an extended ranking method for fuzzy numbers. Chutia and Chutia [16] presented a value and ambiguity-based method for ranking parametric forms of fuzzy numbers. Akbari and Hesamian [4] proposed a class of signed-distance measures for interval-valued fuzzy numbers based on specific bivariate functions, called kernels, and β-values of interval-valued fuzzy numbers. Yu et al. [41] proposed a new epsilon-deviation degree approach which is based on the left and right areas of a fuzzy number and the concept of a centroid point. This method was developed further by Chutia [15]. Yatsalo and Martínez [40] proposed an approach to ranking fuzzy numbers and a fuzzy rank acceptability analysis that provides a degree of confidence for all ranks. Hesamian and Bahrami [23] introduced a credibility theory oriented preference index for ranking fuzzy numbers. Roldán López de Hierro et al. [33] demonstrated how a ranking method for fuzzy numbers can be applied to economic data. Gu and Xuan [22] proposed a possibility theory-based approach for ranking fuzzy numbers. Khorshidi and Nikfalazar [27] used a similarity measure for generalized fuzzy numbers and its application to fuzzy risk analysis. Rahmani et al.

Corresponding Author: T. Jónás
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[31] introduced a method for the defuzzification and ranking of fuzzy numbers based on the statistical beta distribution. Also, there is a lot of interest in ranking intuitionistic fuzzy numbers (see, e.g. [21, 43, 34, 30, 29, 32]).

In our view, the fuzzy number ranking methods can be categorized into two main classes. One of these two classes contains the methods that are founded on defuzzification, while the other class includes procedures that compare fuzzy numbers using preference relations. Both approaches have their advantages and disadvantages. The defuzzification-based approaches are the simpler and easier, but the defuzzification, which is often founded on heuristics, leads to the loss of fuzzy messages. The preference relation-based pair-wise comparisons are more complex procedures, but they preserve the fuzzy messages. Yuan [42] supposed that a fuzzy ranking method had to present preference relation in fuzzy terms. Wang [35] also emphasized that a fuzzy preference relation, through a membership function, represents a preference degree. It should be added that in a recent seminal paper by Zumelzu et al. [44], the authors pointed out that among more than two hundred partial order relations for fuzzy numbers studied, they found just a few that are total orders. They introduced and analyzed the notion of admissible orders for fuzzy numbers with respect to a partial order.

In this study, we will utilize the preference implication that has been recently introduced by Dombi and Baczyński (see [18]). They indicated that the implication is closely related to the preference relation used in multi-criteria decision making. Here, we will show how the preference implication is connected with soft inequalities and explain why we will model soft inequalities by sigmoid functions. Utilizing this connection, we will propose the preference implication-based preference measure for two fuzzy numbers, demonstrate its key properties including the reciprocity, and apply it to trapezoidal fuzzy numbers. Next, we will give the formulas which the new preference measure for trapezoidal fuzzy numbers can be readily computed with. It is worth noting that comparing two fuzzy numbers may not merely mean that one of them is preferred over the other. The comparison should also be able to capture the situation where the order of two fuzzy numbers cannot be judged; and so, the order of these two fuzzy numbers may be considered as being indifferent. In our approach, using the proposed preference measure, we will introduce two crisp relations over a collection of trapezoidal fuzzy numbers and discuss their main algebraic properties. We will show that one of these two crisp relations is a strict, but not a total, order relation; and the other one is an equivalence relation. Here, we will consider two fuzzy numbers as being comparable, when their order can be unambiguously determined. Also, we will show that our strict order relation can be used to rank comparable fuzzy numbers, while our equivalence relation, which we will call the indifference relation, can be utilized to express the view that the order of some fuzzy numbers is indifferent. Hence, these two relations together may be treated as an alternative to admissible orders for fuzzy numbers [44]. Here, we will show how these two crisp relations can be used to rank a collection of trapezoidal fuzzy numbers. Also, we will compare our ranking method with some well-know existing methods.

This paper is organized as follows. In Section 2, the notions of continuous-valued logic, which we will use in this study, are briefly described. The theoretical background of the preference implication-based preference measure is presented in Section 3. In Section 4, the preference implication-based preference measure for two fuzzy numbers is introduced and its key properties are discussed. Then, we apply this new preference measure to trapezoidal fuzzy numbers. Next, in Section 5, we present two crisp relations over a collection of trapezoidal fuzzy numbers, discuss their main algebraic properties and demonstrate how these relations can be used to rank trapezoidal fuzzy numbers. In this section, our method is compared with some well-known existing ranking methods. Lastly, in Section 6, a short summary of our findings is provided.

Here, we will briefly overview the notions of continuous-valued logic that will be used in this study. We will make use of the following representation theorem of Aczél (see Chapter 6.2 in [3]), which is a characterization of associative functions.

Theorem 1.1. *A continuous and strictly increasing function $F\colon [a,b]^2 \to [a,b]$ is associative if and only if*

$$F(x,y) = f^{-1}\left(f(x) + f(y)\right),$$

where $f\colon [a,b] \to [0,\infty]$ is a strictly decreasing or strictly increasing continuous function.

Here, f is called a generator function of F, and F is uniquely determined up to a constant multiplier of f. The following definition of strict t-norms and strict t-conorms is based on Theorem 1.1.

Definition 1.2. *We say that the function $o\colon [0,1]^2 \to [0,1]$ is a strict t-norm (strict t-conorm, respectively) if and only if o is continuous, and there exists a continuous and strictly decreasing (increasing, respectively) function $f\colon [0,1] \to [0,\infty]$, which is uniquely determined up to a positive multiplicative constant, such that*

$$o(x,y) = f^{-1}\left(f(x) + f(y)\right),$$

for any $x, y \in [0,1]$, and

(a) for a strict t-norm c, $f = f_c$ is strictly decreasing with $f_c(1) = 0$ and $\lim_{x\to 0} f_c(x) = \infty$ and

(b) for a strict t-conorm d, $f = f_d$ is strictly increasing with $f_d(0) = 0$ and $\lim_{x\to 1} f_d(x) = \infty$.

In Definition 1.2, function f, which is uniquely determined up to a positive multiplicative constant, is called a generator function of the operator o.

Dombi and Baczyński [18] pointed out that there is an important connection between the implication operator and the preference operator. Namely,

$$(x \to y) = 1 \text{ if and only if } x \leq y,$$

which is the ordering property of the implication. They also argued that in everyday life, the preference and the implication operators are used with the same reasoning method, as suggested in the following two examples:

$$\text{If } x < y \text{ and } y < z, \text{ then } x < z. \qquad \text{If } x \to y \text{ and } y \to z, \text{ then } x \to z.$$

Here, the first example describes the transitivity property of the preference relation $<$. The second example, which is based on the implication operator, demonstrates the reasoning method called the hypothetical reasoning (or hypothetical syllogism, see [28] or [25]). Dombi and Baczyński [18] also refer to the fact that in everyday language, we do not distinguish between these two types of reasoning. According to [18], the preference implication operator is defined as follows.

Definition 1.3. *Let $\nu \in (0, 1)$ and let f be a generator function of a strict t-norm; that is, $f\colon [0, 1] \to [0, \infty]$ is a strictly decreasing function. The preference implication $p_\nu\colon [0, 1]^2 \to [0, 1]$ is given by*

$$p_\nu(x, y) = \begin{cases} 1, & \text{if } (x, y) \in \{(0, 0), (1, 1)\} \\ f^{-1}\left(f(\nu)\frac{f(y)}{f(x)}\right), & \text{otherwise,} \end{cases} \tag{1}$$

where $x, y \in [0, 1]$.

In the preference implication, the parameter ν can be interpreted as a threshold. That is, if the value of the implication is greater than ν, then it is true. Also, based on the following proposition (see Proposition 5 in [18]), $p_\nu(x, y)$ can be interpreted as the continuous logical value of the preference $x < y$.

Proposition 1.4. *For any $x, y \in [0, 1]$, we have $p_\nu(x, y) > \nu$ if and only if $x < y$.*

In continuous-valued logic, the generator function of the Dombi operators (see [17]) $f_\alpha\colon (0, 1) \to (0, \infty)$ is given by

$$f_\alpha(x) = \left(\frac{1 - x}{x}\right)^\alpha, \tag{2}$$

where $\alpha \neq 0$. If $\alpha > 0$, then f_α is strictly decreasing; that is, f_α is a generator function for a strict t-norm. Now, let us utilize this function, with a fixed $\alpha = 1$ parameter value, as a generator function for the preference implication operator given in Definition 1.3. Then, after direct calculation, we get the following form of the preference implication that we call the Dombi form of preference implication: $p_{\nu,D}\colon [0, 1]^2 \to [0, 1]$,

$$p_{\nu,D}(x, y) = \begin{cases} 1, & \text{if } (x, y) \in \{(0, 0), (1, 1)\} \\ \frac{\nu y(1-x)}{\nu y(1-x)+(1-\nu)(1-y)x}, & \text{otherwise,} \end{cases} \tag{3}$$

where $\nu \in (0, 1)$, $x, y \in [0, 1]$. Figure 1 shows some example plots of the preference implication $p_{\nu,D}$.

We will use the sigmoid function, which is defined as follows.

Definition 1.5. *The sigmoid function $\sigma^{(\lambda)}\colon \mathbb{R} \to (0, 1)$ with the parameter λ is given by*

$$\sigma^{(\lambda)}(x) = \frac{1}{1 + e^{-\lambda x}}, \tag{4}$$

where $x, \lambda \in \mathbb{R}$ and λ is nonzero.

Note that the sigmoid function in Eq. (4) is identical with that in Eq. (11) with $c = -\lambda$. The main properties, such as the continuity, monotonicity, limits, role of the parameter and convexity of the sigmoid function $\sigma^{(\lambda)}$ are as follows.

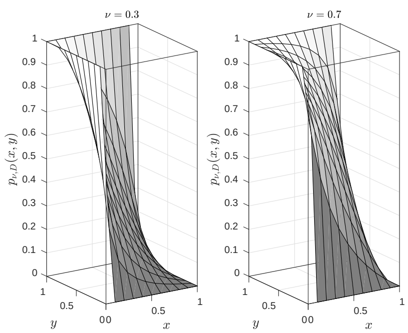

Figure 1: Plots of the preference implication.

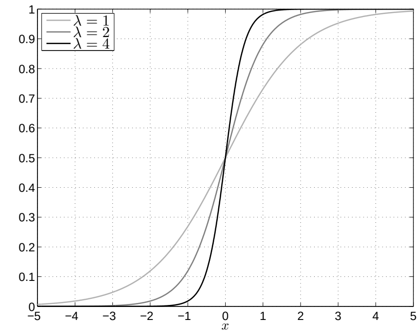

Figure 2: Plots of sigmoid functions.

Continuity. $\sigma^{(\lambda)}$ is a continuous function in $\mathbb{R}$.

Monotonicity. If $\lambda > 0$ ($\lambda < 0$, respectively), then $\sigma^{(\lambda)}$ is strictly increasing (decreasing, respectively)

Limits. The $\sigma^{(\lambda)}$ function takes neither the value zero, nor the value 1, as these are its limits:

$$\lim_{x\to+\infty} \sigma^{(\lambda)}(x) = \begin{cases} 1, & \text{if } \lambda > 0 \\ 0, & \text{if } \lambda < 0, \end{cases} \qquad \lim_{x\to-\infty} \sigma^{(\lambda)}(x) = \begin{cases} 1, & \text{if } \lambda < 0 \\ 0, & \text{if } \lambda > 0. \end{cases}$$

Role of the parameter. The parameter λ of $\sigma^{(\lambda)}$ has a semantic meaning related to the shape of the function curve. The first derivative of $\sigma^{(\lambda)}(x)$ at $x = 0$ is $\frac{\lambda}{4}$; that is, the λ parameter determines the slope of $\sigma^{(\lambda)}(x)$ at $x = 0$. Also,

$$\lim_{\lambda\to\infty} \sigma^{(\lambda)}(x) = \chi(x),$$

where $\chi \colon \mathbb{R} \to [0, 1]$ is the Heaviside function with the half-maximum convention; that is, $\chi(0) = \frac{1}{2}$.

Convexity. $\sigma^{(\lambda)}(x)$ has a single inflection point that is at $x = 0$. If $\lambda > 0$ ($\lambda < 0$, respectively), then $\sigma^{(\lambda)}(x)$ changes from concave to convex (convex to concave, respectively) at $x = 0$.

Figure 2 shows some example plots of sigmoid functions that have positive λ parameter values.

2 The theoretical background of the preference implication-based preference measure

In this section, we give the theoretical explanation for why we will propose a sigmoid function-based approach (see Definition 2.5) for modeling preferences.

2.1 The generator function-dependent sigmoid function and the preference implication

Here, we will introduce the concept of the generator function-dependent sigmoid function and show how it is connected with the preference implication. We will also demonstrate that the well-known sigmoid function is just a special case of the generator function-dependent sigmoid function. Namely, we will show that when the generator function is that of the Dombi operators, then the generator function-dependent sigmoid function coincides with the traditional sigmoid function.

Theorem 2.1. *Let $\nu \in (0, 1)$ and let f be the generator function of the preference implication p_ν given in Definition 1.3. If $\mu : \mathbb{R} \to (0, 1)$ is a function such that*

$$p_\nu(\mu(x), \mu(y)) = \mu(y - x), \tag{5}$$

holds for any $x, y \in \mathbb{R}$, then function μ has the form

$$\mu(x) = f^{-1}\left(f(\nu)\mathrm{e}^{cx}\right),$$

where $c \in \mathbb{R}$ is a constant.

Proof. Let $\nu \in (0,1)$ and let $\mu\colon \mathbb{R} \to (0,1)$ be a function such that Eq. (5) holds for any $x, y \in \mathbb{R}$. Then, based on the definition for the preference implication (see Definition 1.3), we have

$$f^{-1}\left(f(\nu)\frac{f(\mu(y))}{f(\mu(x))}\right) = \mu(y-x),$$

where f is the generator function of the preference implication p_ν. Applying f to both sides of the previous equation, we get

$$f(\nu)\frac{f(\mu(y))}{f(\mu(x))} = f\left(\mu(y-x)\right),$$

which can also be written as

$$\frac{\frac{f(\mu(y))}{f(\nu)}}{\frac{f(\mu(x))}{f(\nu)}} = \frac{f\left(\mu(y-x)\right)}{f(\nu)}. \tag{6}$$

Since both members of Eq. (6) are positive, by taking the logarithm of them, we get

$$\ln\left(\frac{f(\mu(y))}{f(\nu)}\right) - \ln\left(\frac{f(\mu(x))}{f(\nu)}\right) = \ln\left(\frac{f\left(\mu(y-x)\right)}{f(\nu)}\right). \tag{7}$$

Now, let the function $g\colon \mathbb{R} \to \mathbb{R}$ be given by

$$g(x) = \ln\left(\frac{f(\mu(x))}{f(\nu)}\right). \tag{8}$$

Then, by utilizing function g, Eq. (7) can be written as $g(y) - g(x) = g(y-x)$ or alternatively,

$$g((y-x)+x) = g(y-x) + g(x). \tag{9}$$

Notice that Eq. (9) is Cauchy's functional equation; and so, g has the form $g(x) = cx$, where c is an arbitrary fixed constant. Next, by substituting $g(x) = cx$ into Eq. (8), after direct calculation, we get $f(\mu(x)) = f(\nu)\mathrm{e}^{cx}$ and by applying f^{-1} to both sides of this equation, we have

$$\mu(x) = f^{-1}\left(f(\nu)\mathrm{e}^{cx}\right). \tag{10}$$

□

Remark 2.2. *We should emphasize that the function in Eq.* (10) *may be viewed as the general form of the sigmoid function. This function depends on the generator function f. If $f = f_\alpha$ is the generator function of the Dombi operators given in Eq.* (2) *with $\alpha = 1$, then for $\nu = \frac{1}{2}$, Eq.* (10) *gives*

$$\mu(x) = \frac{1}{1+\mathrm{e}^{cx}}, \tag{11}$$

which is the same as the sigmoid function in Definition 1.5 with $\lambda = -c$.

From now on, we will use Definition 1.5 for a sigmoid function.

2.2 Soft inequalities and the preference implication

In this subsection, we will show how the preference implication is connected with soft inequalities. We will interpret the soft inequalities as follows.

Interpretation 2.3. *A continuous function $\sigma\colon \mathbb{R} \to [0,1]$ is said to be a representation of the soft inequality $0 < x$; that is,*

$$\sigma(x) = truth(0 < x),$$

if σ satisfies the following requirements:

$$(a)\ \sigma(x) < \frac{1}{2} \text{ if and only if } x < 0, \quad (b)\ \sigma(x) = \frac{1}{2} \text{ if and only if } x = 0, \quad (c)\ \sigma(x) > \frac{1}{2} \text{ if and only if } 0 < x.$$

The following theorem states an important connection between the preference implication and the soft inequalities, and explains why we will propose the sigmoid function in Eq. (4) for modeling soft inequalities.

Theorem 2.4. *Let $\nu = 1/2$. Then,*

1) a function $\mu\colon \mathbb{R} \to (0,1)$ is a representation of the soft inequality $0 < x$ and

2) for any $x, y \in \mathbb{R}$,

$$p_{\nu,D}(\mu(x), \mu(y)) = \mu(y - x), \tag{12}$$

holds

if and only if there is $\lambda > 0$ such that $\mu(x) = \sigma^{(\lambda)}(x)$.

Proof. Let $\nu = 1/2$. Firstly, we will show that if $\mu = \sigma^{(\lambda)}$ with a fixed $\lambda > 0$ constant, then μ satisfies 1) and 2). Let $\mu = \sigma^{(\lambda)}$ with a fixed $\lambda > 0$ constant. Then, by noting the properties of the sigmoid function $\sigma^{(\lambda)}$, we immediately get that function μ satisfies the requirements for a representation of the soft inequality $0 < x$ (see Interpretation 2.3). That is, μ satisfies 1). Next, since $\mu = \sigma^{(\lambda)}$, we have $\mu(x), \mu(y) \in (0,1)$ for any $x, y \in \mathbb{R}$. Now, by applying the Dombi form of preference implication in Eq. (3) with the parameter value $\nu = 1/2$, after direct calculation, we get

$$p_{\nu,D}(\mu(x), \mu(y)) = \frac{1}{1 + \frac{1-\frac{1}{2}}{\frac{1}{2}} \frac{1-\mu(y)}{\mu(y)} \frac{\mu(x)}{1-\mu(x)}} = \frac{1}{1 + \mathrm{e}^{-\lambda(y-x)}} = \sigma^{(\lambda)}(y - x) = \mu(y - x). \tag{13}$$

This means that μ satisfies 2).

Now, we will show that if μ satisfies 1) and 2), then for any $x \in \mathbb{R}$, $\mu(x) = \sigma^{(\lambda)}(x)$. Let $\nu = 1/2$ and $\mu\colon \mathbb{R} \to (0,1)$ be a function that satisfies 1) and 2). Then, based on Theorem 2.1 and Remark 2.2, the function μ has the form

$$\mu(x) = \frac{1}{1 + \mathrm{e}^{cx}}, \tag{14}$$

where $c \in \mathbb{R}$ is a constant. As function μ satisfies 1) as well, μ is a strictly increasing function (see Interpretation 2.3). Hence, by noting the properties of the sigmoid function in Eq. (4), parameter c in Eq. (14) is necessarily negative. That is, c can be written as $c = -\lambda$, where $\lambda > 0$; and so $\mu(x) = \sigma^{(\lambda)}(x)$ for any $x \in \mathbb{R}$. □

2.3 The preference implication-based preference measure for two numbers

Now, following Theorem 2.4, we will introduce the preference implication-based measure of the preference for two real numbers. Utilizing the interpretation $\sigma(x) = truth(0 < x)$ (see Interpretation 2.3), and the fact that

$$truth(x < y) = truth(0 < y - x) = \sigma(y - x),$$

the result of Theorem 2.4 can be stated as follows. For any $x, y \in \mathbb{R}$,

$$p_{\nu,D}(truth(0 < x), truth(0 < y)) = truth(x < y)$$

holds if and only if

$$truth(x < y) = \sigma^{(\lambda)}(y - x),$$

where $\lambda > 0$ is a fixed constant. Hence, we will define the preference implication-based measure of the preference for two real numbers as follows.

Definition 2.5. *Let $\lambda > 0$ and $x, y \in \mathbb{R}$. The preference implication-based measure $\mu^{(\lambda)}\colon \mathbb{R}^2 \to (0,1)$ of the preference $x < y$ is given by*

$$\mu^{(\lambda)}(x, y) = \frac{1}{1 + \mathrm{e}^{-\lambda(y-x)}}. \tag{15}$$

Let $d_{yx} = y - x$; that is, d_{yx} is the signed distance between y and x. Also, $d_{xy} := x - y$. Using the signed distance d_{yx}, the preference measure $\mu^{(\lambda)}$ in Eq. (15) can be written as

$$\mu^{(\lambda)}(0, d_{yx}) = \frac{1}{1 + \mathrm{e}^{-\lambda(d_{yx}-0)}} = \frac{1}{1 + \mathrm{e}^{-\lambda d_{yx}}} \tag{16}$$

and similarly,

$$\mu^{(\lambda)}(d_{yx},0) = \frac{1}{1+\mathrm{e}^{-\lambda(0-d_{yx})}} = \frac{1}{1+\mathrm{e}^{\lambda d_{yx}}}.$$

The following properties of the preference measure $\mu^{(\lambda)}$ in Eq. (16) immediately follow from its definition and from the properties of the sigmoid function:

$$\mu^{(\lambda)}(0,d_{yx}) < \frac{1}{2} \text{ iff } d_{yx} < 0, \quad \mu^{(\lambda)}(0,d_{yx}) = \frac{1}{2} \text{ iff } d_{yx} = 0, \quad \mu^{(\lambda)}(0,d_{yx}) > \frac{1}{2} \text{ iff } d_{yx} > 0,$$

$$\lim_{d_{yx}\to-\infty} \mu^{(\lambda)}(0,d_{yx}) = 0, \quad \lim_{d_{yx}\to+\infty} \mu^{(\lambda)}(0,d_{yx}) = 1$$

$$\mu^{(\lambda)}(0,d_{yx}) = \mu^{(\lambda)}(d_{xy},0) \tag{17}$$

$$\mu^{(\lambda)}(0,d_{yx}) + \mu^{(\lambda)}(d_{yx},0) = 1. \tag{18}$$

Note that Eq. (18) is called the reciprocity property of $\mu^{(\lambda)}$.

3 The preference implication-based preference measure for fuzzy numbers

Here, we will present a concept for measuring the preference between two fuzzy numbers by using the preference implication-based preference measure that we introduced in the previous section. From now on, we will use the following definition for an LR fuzzy number.

Definition 3.1. *The LR fuzzy number A is given by the membership function $\mu_A\colon \mathbb{R} \to [0,1]$,*

$$\mu_A\left(x;\underline{x}_A^L,\overline{x}_A^L,\overline{x}_A^R,\underline{x}_A^R\right) = \begin{cases} 0, & \text{if } x < \underline{x}_A^L \\ l_A(x), & \text{if } \underline{x}_A^L \le x < \overline{x}_A^L \\ 1, & \text{if } \overline{x}_A^L \le x < \overline{x}_A^R \\ r_A(x), & \text{if } \overline{x}_A^R \le x < \underline{x}_A^R \\ 0, & \text{if } \underline{x}_A^R \ge x, \end{cases} \tag{19}$$

where $\underline{x}_A^L < \overline{x}_A^L \le \overline{x}_A^R < \underline{x}_A^R$, and $l_A\colon [\underline{x}_A^L,\overline{x}_A^L) \to [0,1)$ and $r_A\colon [\overline{x}_A^R,\underline{x}_A^R) \to [0,1)$ are continuous, strictly increasing and decreasing functions with the inverse functions $l_A^{-1}\colon [0,1) \to [\underline{x}_A^L,\overline{x}_A^L)$ and $r_A^{-1}\colon [0,1) \to [\overline{x}_A^R,\underline{x}_A^R)$, respectively.

Note that in Definition 3.1, the functions l_A and r_A determine the left hand side and the right hand side of the membership function of fuzzy number A. We should add that the notion of fuzzy number is more general than that in Definition 3.1 (see, e.g., [7, 19, 5]). In our study, for the sake of simple application to real-world data, under the notion of fuzzy number, we will always mean a LR fuzzy number.

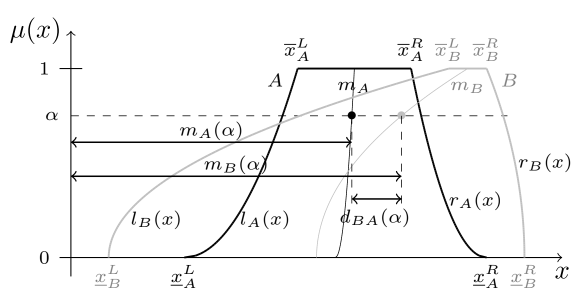

Figure 3: Two fuzzy numbers with their middle curves.

Suppose that A and B are two fuzzy numbers given by Definition 3.1 (see Figure 3). Then, the α-cut intervals $I_A(\alpha)$ and $I_B(\alpha)$ of the membership functions of A and B, respectively, are

$$I_A(\alpha) = [a_A(\alpha), b_A(\alpha)], \quad I_B(\alpha) = [a_B(\alpha), b_B(\alpha)], \tag{20}$$

where

$$a_A(\alpha) = l_A^{-1}(\alpha), \quad b_A(\alpha) = r_A^{-1}(\alpha), \quad a_B(\alpha) = l_B^{-1}(\alpha), \quad b_B(\alpha) = r_B^{-1}(\alpha), \tag{21}$$

and $\alpha \in [0,1]$. Let $m_A(\alpha)$ and $m_B(\alpha)$ be the midpoints of the α-cut intervals $I_A(\alpha)$ and $I_B(\alpha)$, respectively. That is,

$$m_A(\alpha) = \frac{l_A^{-1}(\alpha) + r_A^{-1}(\alpha)}{2}, \quad m_B(\alpha) = \frac{l_B^{-1}(\alpha) + r_B^{-1}(\alpha)}{2},$$

where $\alpha \in [0,1]$. By having $m_A(\alpha)$ and $m_B(\alpha)$ for all $\alpha \in [0,1]$, we get the middle curves m_A and m_B of the fuzzy numbers A and B, respectively (see Figure 3). Here, we will characterize the relative position of the fuzzy numbers A and B by the relative position of their middle curves m_A and m_B.

Now, let $d_{BA}(\alpha)$ denote the signed horizontal distance between the middle lines m_B and m_A at level α. That is,

$$d_{BA}(\alpha) = m_B(\alpha) - m_A(\alpha) = \frac{a_B(\alpha) + b_B(\alpha)}{2} - \frac{a_A(\alpha) + b_A(\alpha)}{2}. \tag{22}$$

Then, by utilizing the preference measure $\mu^{(\lambda)}$, $\mu^{(\lambda)}(0, d_{BA}(\alpha))$ may be viewed as a quantity that characterizes the preference between A and B at α. That is, if $\mu^{(\lambda)}(0, d_{BA}(\alpha)) > \frac{1}{2}$, then A precedes B at α; otherwise A does not precede B at α. With this line of thinking, the mean of the $\mu^{(\lambda)}(0, d_{BA}(\alpha))$ preference measure values may be interpreted as a measure of the preference $A \prec B$ between the fuzzy numbers A and B. Based on the above considerations, we will define the preference implication-based preference measure for two fuzzy numbers as follows.

Definition 3.2. *Let $\mathbf{F}$ be a collection of fuzzy numbers. Let $A, B \in \mathbf{F}$, and let $\lambda > 0$. The preference implication-based measure $M^{(\lambda)}\colon \mathbf{F}^2 \to (0,1)$ of the preference $A \prec B$ (i.e. $M^{(\lambda)}(A,B)$) is given by*

$$M^{(\lambda)}(A,B) = \int_0^1 \mu^{(\lambda)}\left(0, d_{BA}(\alpha)\right) \mathrm{d}\alpha, \tag{23}$$

where

$$d_{BA}(\alpha) = \frac{l_B^{-1}(\alpha) + r_B^{-1}(\alpha)}{2} - \frac{l_A^{-1}(\alpha) + r_A^{-1}(\alpha)}{2}, \tag{24}$$

$\alpha \in [0,1]$.

It should be added that $M^{(\lambda)}(A,B)$ may be viewed as a quantity which measures how much the fuzzy number A precedes the fuzzy number B. Next, by utilizing Eq. (16), from Eq. (23) we have

$$M^{(\lambda)}(A,B) = \int_0^1 \frac{1}{1 + \mathrm{e}^{-\lambda d_{BA}(\alpha)}} \mathrm{d}\alpha. \tag{25}$$

Note that, depending on the functions l_A, r_A, l_B and r_B, the computation of the integral in Eq. (25) may be cumbersome. Alternatively, we can use the average $M^{(\lambda)}(A,B) \approx \frac{1}{n}\sum_{i=1}^{n} \frac{1}{1+\mathrm{e}^{-\lambda d_{BA}\left(\frac{i}{n}\right)}}$, which approximates the integral well, if n is sufficiently large (e.g. $n = 1000$).

3.1 Properties of the preference implication-based preference measure for fuzzy numbers

Now, we will show the key properties of the preference implication-based preference measure for fuzzy numbers. The following property is the reciprocity property of the preference measure $M^{(\lambda)}$.

Property 3.3. *Let A and B be two fuzzy numbers. Then,*

$$M^{(\lambda)}(A,B) + M^{(\lambda)}(B,A) = 1, \tag{26}$$

where $\lambda > 0$.

Proof. By using the signed distances $d_{BA}(\alpha)$ and $d_{AB}(\alpha)$ given by Eq. (24), where $\alpha \in [0,1]$, and noting Eq. (23), Eq. (17) and Eq. (18), we have

$$M^{(\lambda)}(A,B) + M^{(\lambda)}(B,A) = \int_0^1 \left(\mu^{(\lambda)}(0, d_{BA}(\alpha)) + \mu^{(\lambda)}(d_{BA}(\alpha), 0)\right) \mathrm{d}\alpha = \int_0^1 1 \mathrm{d}\alpha = 1.$$

□

Taking into account Property 3.3 with $B = A$, we immediately get Property 3.4.

Property 3.4. *Let A be a fuzzy number. Then,*

$$M^{(\lambda)}(A,A) = \frac{1}{2}, \tag{27}$$

where $\lambda > 0$.

3.2 The preference implication-based preference measure for trapezoidal fuzzy numbers

In this section, we will apply the preference implication-based measure $M^{(\lambda)}$ to fuzzy numbers that have trapezoidal membership functions. Namely, we will show how the measure $M^{(\lambda)}(A,B)$ can be computed for the spacial case where A and B are trapezoidal fuzzy numbers. Also, we will describe the key properties of $M^{(\lambda)}(A,B)$ for this special case.

Definition 3.5. *The trapezoidal fuzzy membership function $\mu_A\colon \mathbb{R} \to [0,1]$ with the parameters $\underline{x}_A^L < \overline{x}_A^L \leq \overline{x}_A^R < \underline{x}_A^R$ is given by*

$$\mu_A\left(x; \underline{x}_A^L, \overline{x}_A^L, \overline{x}_A^R, \underline{x}_A^R\right) = \begin{cases} 0, & \textit{if } x < \underline{x}_A^L \\ \frac{x-\underline{x}_A^L}{\overline{x}_A^L-\underline{x}_A^L}, & \textit{if } \underline{x}_A^L \leq x < \overline{x}_A^L \\ 1, & \textit{if } \overline{x}_A^L \leq x < \overline{x}_A^R \\ \frac{x-\underline{x}_A^R}{\overline{x}_A^R-\underline{x}_A^R}, & \textit{if } \overline{x}_A^R \leq x < \underline{x}_A^R \\ 0, & \textit{if } \underline{x}_A^R \geq x. \end{cases} \tag{28}$$

Suppose that A and B are two fuzzy numbers that have the trapezoidal membership functions μ_A and μ_B with the parameters $\underline{x}_A^L < \overline{x}_A^L \leq \overline{x}_A^R < \underline{x}_A^R$ and $\underline{x}_B^L < \overline{x}_B^L \leq \overline{x}_B^R < \underline{x}_B^R$, respectively. By noting Definition 3.1, the trapezoidal fuzzy numbers A and B are fuzzy numbers with the following left hand side and right hand side functions:

$$\begin{aligned} l_A(x) &= \frac{x-\underline{x}_A^L}{\overline{x}_A^L-\underline{x}_A^L}, \quad \underline{x}_A^L \leq x < \overline{x}_A^L, \qquad r_A(x) = \frac{x-\underline{x}_A^R}{\overline{x}_A^R-\underline{x}_A^R}, \quad \overline{x}_A^R \leq x < \underline{x}_A^R, \\ l_B(x) &= \frac{x-\underline{x}_B^L}{\overline{x}_B^L-\underline{x}_B^L}, \quad \underline{x}_B^L \leq x < \overline{x}_B^L, \qquad r_B(x) = \frac{x-\underline{x}_B^R}{\overline{x}_B^R-\underline{x}_B^R}, \quad \overline{x}_B^R \leq x < \underline{x}_B^R. \end{aligned} \tag{29}$$

Therefore, in this case, the middle curves m_A and m_B are straight lines. That is, m_A is the line that connects the midpoints of parallel sides of trapezoid A, and similarly, m_B is the line that connects the midpoints of parallel sides of trapezoid B. Here, utilizing Eq. (21) and Eq. (29), after direct calculation, we find that the α-cut intervals $I_A(\alpha)$ and $I_B(\alpha)$ in Eq. (20) are given by the points

$$\begin{aligned} a_A(\alpha) &= \alpha\overline{x}_A^L + (1-\alpha)\underline{x}_A^L, \quad b_A(\alpha) = \alpha\overline{x}_A^R + (1-\alpha)\underline{x}_A^R, \\ a_B(\alpha) &= \alpha\overline{x}_B^L + (1-\alpha)\underline{x}_B^L, \quad b_B(\alpha) = \alpha\overline{x}_B^R + (1-\alpha)\underline{x}_B^R, \end{aligned} \tag{30}$$

$\alpha \in [0,1]$.

3.2.1 Computing the preference implication-based preference measure for trapezoidal fuzzy numbers

Now, we will show how the preference implication-based preference measure given in Eq. (25) can be computed for two trapezoidal fuzzy numbers.

Proposition 3.6. *Let A and B be two fuzzy numbers that have the trapezoidal membership functions μ_A and μ_B with the parameters $\underline{x}_A^L < \overline{x}_A^L \leq \overline{x}_A^R < \underline{x}_A^R$ and $\underline{x}_B^L < \overline{x}_B^L \leq \overline{x}_B^R < \underline{x}_B^R$, respectively. Then, the preference implication-based preference measure $M^{(\lambda)}(A,B)$ for the fuzzy numbers A and B can be computed as*

$$M^{(\lambda)}(A,B) = \begin{cases} \frac{1}{\lambda p} \ln\left(\frac{\mathrm{e}^{\lambda(p+q)}+1}{\mathrm{e}^{\lambda q}+1}\right), & \text{if } p \neq 0 \quad (31) \\ \frac{1}{1+\mathrm{e}^{-\lambda q}}, & \text{if } p = 0, \quad (32) \end{cases}$$

where

$$p = \frac{1}{2}\left(\left(\overline{x}_B^R - \underline{x}_B^R\right) + \left(\overline{x}_B^L - \underline{x}_B^L\right) - \left(\overline{x}_A^L - \underline{x}_A^L\right) - \left(\overline{x}_A^R - \underline{x}_A^R\right) \right), \tag{33}$$

$$q = \frac{1}{2}\left(\underline{x}_B^L + \underline{x}_B^R - \underline{x}_A^L - \underline{x}_A^R\right), \tag{34}$$

and $\lambda > 0$.

Proof. Since A and B are trapezoidal fuzzy numbers, by noting Eq. (30), the signed distance $d_{BA}(\alpha)$ in Eq. (22) can be written as

$$d_{BA}(\alpha) = \frac{1}{2}\left(\left(\overline{x}_B^R - \underline{x}_B^R\right) + \left(\overline{x}_B^L - \underline{x}_B^L\right) - \left(\overline{x}_A^L - \underline{x}_A^L\right) - \left(\overline{x}_A^R - \underline{x}_A^R\right) \right)\alpha + \frac{1}{2}\left(\underline{x}_B^L + \underline{x}_B^R - \underline{x}_A^L - \underline{x}_A^R\right),$$

where $\alpha \in [0,1]$. That is,

$$d_{BA}(\alpha) = p\alpha + q, \tag{35}$$

where the constants p and q are given by Eq. (33) and Eq. (34), respectively. Therefore, $M^{(\lambda)}(A,B)$ in Eq. (25) can be written as

$$M^{(\lambda)}(A,B) = \int_0^1 \frac{1}{1+\mathrm{e}^{-\lambda(p\alpha+q)}} \mathrm{d}\alpha. \tag{36}$$

Let $p \neq 0$. Then, by using the substitution $u = \mathrm{e}^{\lambda p\alpha + \lambda q} + 1$, we can write

$$\int \frac{1}{1+\mathrm{e}^{-\lambda(p\alpha+q)}} \mathrm{d}\alpha = \frac{1}{\lambda p} \int \frac{1}{u} \mathrm{d}u = \frac{1}{\lambda p} \ln(u); \text{ and so } \int \frac{1}{1+\mathrm{e}^{-\lambda(p\alpha+q)}} \mathrm{d}\alpha = \frac{\ln\left(\mathrm{e}^{\lambda p\alpha+\lambda q}+1\right)}{\lambda p} + C,$$

where C is an arbitrary constant. Using this result and Eq. (36), we readily get Eq. (31).

Now, let $p = 0$. Then, Eq. (32) immediately follows from (36). □

3.2.2 Properties of the preference implication-based preference measure for trapezoidal fuzzy numbers

Now, we will show the key properties of the preference implication-based preference measure for fuzzy numbers that have trapezoidal membership functions.

Property 3.7. *By making use of L'Hospital's rule, we get*

$$\lim_{p\to 0}\left(\frac{1}{\lambda p} \ln\left(\frac{\mathrm{e}^{\lambda(p+q)}+1}{\mathrm{e}^{\lambda q}+1}\right)\right) = \frac{1}{1+\mathrm{e}^{-\lambda q}}.$$

This means that Eq. (32) is the limit case of Eq. (31) when $p \to 0$.

Later, we will make use of the following property of function M.

Property 3.8. *Let A and B be two fuzzy numbers that have the trapezoidal membership functions μ_A and μ_B with the parameters $\underline{x}_A^L < \overline{x}_A^L \leq \overline{x}_A^R < \underline{x}_A^R$ and $\underline{x}_B^L < \overline{x}_B^L \leq \overline{x}_B^R < \underline{x}_B^R$, respectively. Then*

1) $M^{(\lambda)}(A,B) > \frac{1}{2}$ *holds if and only if* $\int_0^1 d_{BA}(\alpha)\mathrm{d}\alpha > 0$

2) $M^{(\lambda)}(A,B) = \frac{1}{2}$ *holds if and only if* $\int_0^1 d_{BA}(\alpha)\mathrm{d}\alpha = 0$,

where the signed distance $d_{BA}(\alpha)$ is given by Eq. (22), $\alpha \in [0,1]$ and $\lambda > 0$.

Proof. In the proof of Proposition 3.6, we showed that the signed distance $d_{BA}(\alpha)$ can be written as $d_{BA}(\alpha) = p\alpha + q$, where the constants p and q are given by Eq. (33) and Eq. (34), respectively and $\alpha \in [0,1]$. Hence, we have

$$M^{(\lambda)}(A,B) = \int_0^1 \sigma_{p,q}^{(\lambda)}(\alpha)\mathrm{d}\alpha, \tag{37}$$

where the sigmoid function $\sigma_{p,q}^{(\lambda)} : \mathbb{R} \to (0,1)$ is given by

$$\sigma_{p,q}^{(\lambda)}(\alpha) = \begin{cases} \dfrac{1}{1+\mathrm{e}^{-\lambda p\left(\alpha+\frac{q}{p}\right)}}, & \text{if } p \neq 0 \quad (38) \\ \dfrac{1}{1+\mathrm{e}^{-\lambda q}}, & \text{if } p = 0\ . \quad (39) \end{cases}$$

Let I_l denote the integral of the linear function $p\alpha + q$ from 0 to 1; that is, $I_l = \int_0^1 d_{BA}(\alpha)\mathrm{d}\alpha = \int_0^1 (p\alpha + q)\,\mathrm{d}\alpha$. Here, let $\alpha \in \mathbb{R}$. First, we will prove 1). Now, let $p \neq 0$. Then, the line $p\alpha + q$ intersects the horizontal axis α at $\alpha = -\frac{q}{p}$; that is, $p\alpha + q = 0$ holds if and only if $\alpha = -\frac{q}{p}$. Next, based on Eq. (38), $\sigma_{p,q}^{(\lambda)}(\alpha) = \frac{1}{2}$ holds if and only if $\alpha = -\frac{q}{p}$. Here, we will distinguish two cases: (1a) $p > 0$ and (1b) $p < 0$.

(1a) If $p > 0$ then $p\alpha + q$ is an increasing function; and so, $I_l > 0$ holds if and only if $-\frac{q}{p} < \frac{1}{2}$. Also, due to the fact that $\lambda > 0$ and $p > 0$, the sigmoid function $\sigma_{p,q}^{(\lambda)}$ is increasing. Next, taking into account its symmetry, we get that the inequality $M^{(\lambda)}(A,B) > \frac{1}{2}$ holds if and only if $-\frac{q}{p} < \frac{1}{2}$. That is, $M^{(\lambda)}(A,B) > \frac{1}{2}$ is equivalent to $I_l > 0$.

(1b) If $p < 0$ then $p\alpha + q$ is a decreasing function; and so, $I_l > 0$ holds if and only if $-\frac{q}{p} > \frac{1}{2}$. Taking into consideration the fact that $\lambda > 0$ and $p < 0$, we get that the sigmoid function $\sigma_{p,q}^{(\lambda)}$ is decreasing. Also, noting its symmetry, we get that the inequality $M^{(\lambda)}(A,B) > \frac{1}{2}$ holds if and only if $-\frac{q}{p} > \frac{1}{2}$. Therefore, $M^{(\lambda)}(A,B) > \frac{1}{2}$ is equivalent to $I_l > 0$.

Now, let $p = 0$. Then, $d_{BA}(\alpha)$ has a constant value of q; and so, $I_l > 0$ holds if and only if $q > 0$. Also, based on Eq. (39), $\sigma_{p,q}^{(\lambda)}$ has a constant value of

$$\sigma_{p,q}^{(\lambda)}(\alpha)\big|_{p=0} = \frac{1}{1+\mathrm{e}^{-\lambda q}},$$

and so Eq. (37) gives us

$$M^{(\lambda)}(A,B) = \frac{1}{1+\mathrm{e}^{-\lambda q}}. \tag{40}$$

Since $\lambda > 0$, from the last equation we get that the inequality $M^{(\lambda)}(A,B) > \frac{1}{2}$ holds if and only if $q > 0$. That is, $M^{(\lambda)}(A,B) > \frac{1}{2}$ is equivalent to $I_l > 0$.

Now, we will prove 2). Here, we will distinguish two cases: (2a) $p \neq 0$ and (2b) $p = 0$.

(2a) If $p \neq 0$, then $p\alpha + q$ is either an increasing or a decreasing function; and so, $I_l = 0$ holds if and only if $-\frac{q}{p} = \frac{1}{2}$. Also, the sigmoid function $\sigma_{p,q}^{(\lambda)}$ is either increasing or decreasing, and by noting its symmetry, we get that $M^{(\lambda)}(A,B) = \frac{1}{2}$ holds if and only if $-\frac{q}{p} = \frac{1}{2}$. That is, $M^{(\lambda)}(A,B) = \frac{1}{2}$ is equivalent to $I_l = 0$.

(2b) If $p = 0$, then, $p\alpha + q$ has a constant value of q, which means that $I_l = 0$ holds if and only if $q = 0$. Here again, based on Eq. (37) and Eq. (39), we have Eq. (40). Therefore, $M^{(\lambda)}(A,B) = \frac{1}{2}$ holds if and only if $q = 0$. That is, $M^{(\lambda)}(A,B) = \frac{1}{2}$ is equivalent to $I_l = 0$. □

4 Ranking trapezoidal fuzzy numbers

Here, using the preference implication-based preference measure for trapezoidal fuzzy numbers, we will introduce crisp relations over a collection of trapezoidal fuzzy numbers, discuss their main algebraic properties and show how they can be utilized to rank such fuzzy numbers.

Definition 4.1. *Let* $\mathbf{F}$ *be a collection of fuzzy numbers with trapezoidal membership functions. The binary relation* $\prec_F$ *over the collection* $\mathbf{F}$ *is given by*

$$\prec_F := \left\{ (A,B) \in \mathbf{F} \times \mathbf{F} : M^{(\lambda)}(A,B) > \frac{1}{2} \right\},$$

where $\lambda > 0$.

In other words, based on Definition 4.1, the relation $A \prec_F B$ holds if and only if $M^{(\lambda)}(A,B) > \frac{1}{2}$, where A and B are two fuzzy numbers with trapezoidal membership functions. The following theorem states that the relation $\prec_F$ over a collection of trapezoidal fuzzy numbers is a strict order relation.

Theorem 4.2. *The relation $\prec_F$ given in Definition 4.1 is a strict order relation over a collection $\mathbf{F}$ of trapezoidal fuzzy numbers.*

Proof. Let $\mathbf{F}$ be a collection of fuzzy numbers with trapezoidal membership functions. In order to demonstrate that the relation $\prec_F$ is a strict order relation over the set $\mathbf{F}$, we need to show that $\prec_F$ is

1) irreflexive, i.e., for any $A \in \mathbf{F}$, $A \not\prec_F A$ holds

2) antisymmetric, i.e., for any $A, B \in \mathbf{F}$, if $A \prec_F B$, then $B \not\prec_F A$

3) transitive, i.e., for any $A, B, C \in \mathbf{F}$, if $A \prec_F B$ and $B \prec_F C$, then $A \prec_F C$.

1) (Irreflexivity.) Based on Property 3.4, $M^{(\lambda)}(A \prec A) = \frac{1}{2}$ holds for any $A \in \mathbf{F}$. Therefore, by noting Definition 4.1, we see that $A \prec_F A$ does not hold.
2) (Antisymmetry.) Let $A, B \in \mathbf{F}$ such that $A \prec_F B$. Then, based on Definition 4.1, $M^{(\lambda)}(A,B) > \frac{1}{2}$. Now, by noting the reciprocity property of function $M^{(\lambda)}$ (see Property 3.3), we have $M^{(\lambda)}(A,B) + M^{(\lambda)}(B,A) = 1$, from which we get that $M^{(\lambda)}(B,A) < \frac{1}{2}$. This means that $B \not\prec_F A$.
3) (Transitivity.) Let $A, B, C \in \mathbf{F}$ such that $A \prec_F B$ and $B \prec_F C$. Then, based on Definition 4.1, $A \prec_F B$ and $B \prec_F C$ means that $M^{(\lambda)}(A,B) > \frac{1}{2}$ and $M^{(\lambda)}(B,C) > \frac{1}{2}$. Now, by noting Property 3.8, $M^{(\lambda)}(A,B) > \frac{1}{2}$ is equivalent to

$$\int_0^1 d_{BA}(\alpha)\mathrm{d}\alpha > 0, \tag{41}$$

and $M^{(\lambda)}(B,C) > \frac{1}{2}$ is equivalent to

$$\int_0^1 d_{CB}(\alpha)\mathrm{d}\alpha > 0, \tag{42}$$

where the signed distances $d_{BA}(\alpha)$ and $d_{CB}(\alpha)$ are given by Eq. (24), and $\alpha \in [0,1]$. Next, by utilizing Eq. (24), the signed distance $d_{CA}(\alpha)$ can be written as

$$d_{CA}(\alpha) = \frac{a_C(\alpha) + b_C(\alpha)}{2} - \frac{a_A(\alpha) + b_A(\alpha)}{2} = d_{CB}(\alpha) + d_{BA}(\alpha). \tag{43}$$

Now, by noting Eq. (41) and Eq. (42) and Eq. (43), we can write

$$\int_0^1 d_{CA}(\alpha)\mathrm{d}\alpha = \int_0^1 \left(d_{CB}(\alpha) + d_{BA}(\alpha)\right)\mathrm{d}\alpha = \int_0^1 d_{CB}(\alpha)\mathrm{d}\alpha + \int_0^1 d_{BA}(\alpha)\mathrm{d}\alpha > 0, \tag{44}$$

which, based on Property 3.8, is equivalent to the inequality $M^{(\lambda)}(A,C) > \frac{1}{2}$. This means that $A \prec_F C$ holds. □

Example 4.3. *Let the trapezoidal fuzzy numbers A and B be given by the following parameters:*

$$\underline{x}_A^L = 14; \quad \overline{x}_A^L = 17; \quad \overline{x}_A^R = 22 \quad \underline{x}_A^R = 29$$
$$\underline{x}_B^L = 0; \quad \overline{x}_B^L = 24; \quad \overline{x}_B^R = 27 \quad \underline{x}_B^R = 31.$$

Figure 4 shows the plots of the membership functions of A and B. From the parameters of A and B, by using Eq. (33) *and Eq.* (34)*, we get $p = 12$ and $q = -6$. Next, by applying the formula for $M^{(\lambda)}(A,B)$ given in Eq.* (31)*, we get*

$$M^{(\lambda)}(A,B) = \frac{1}{\lambda p} \ln \frac{\mathrm{e}^{\lambda(p+q)} + 1}{\mathrm{e}^{\lambda q} + 1} = \frac{1}{12\lambda} \ln \frac{\mathrm{e}^{6\lambda} + 1}{\mathrm{e}^{-6\lambda} + 1} = \frac{1}{2}.$$

Now, by noting the reciprocity property of $M^{(\lambda)}$, we have $M^{(\lambda)}(B,A) = \frac{1}{2}$.

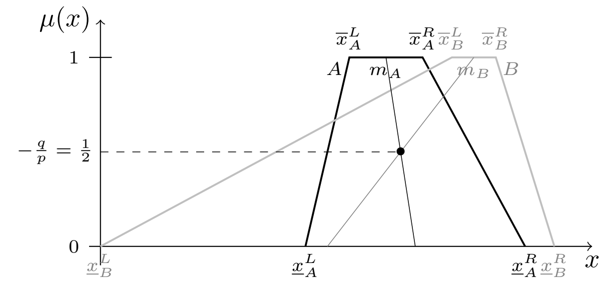

Figure 4: Two trapezoidal fuzzy numbers that are incomparable using the relation $\prec_F$.

Example 4.3 demonstrates a case where neither $A \prec_F B$, nor $B \prec_F A$ holds; that is, A and B are incomparable using the relation $\prec_F$. This also means that the relation $\prec_F$ is not total. That is, A and B are two trapezoidal fuzzy numbers for which neither of the following relations holds: $A \prec_F B$, $B \prec_F A$, $A = B$.

If the lines m_A and m_B are nonparallel, then the vertical coordinate of their intersection point is that α for which $d_{BA}(\alpha) = 0$. In the proof of Property 3.8, we showed that $d_{BA}(\alpha) = 0$ is equivalent to $\alpha = -\frac{q}{p}$. In the same proof, we also showed that $M^{(\lambda)}(A, B) = \frac{1}{2}$ holds if and only if $-\frac{q}{p} = \frac{1}{2}$. This means that if the trapezoidal fuzzy numbers A and B have such positions that the vertical coordinate of the intersection of the lines m_A and m_B is $\alpha = \frac{1}{2}$, then $M^{(\lambda)}(A, B) = \frac{1}{2}$.

It should be mentioned that the incomparability of the trapezoidal fuzzy numbers A and B using the relation $\prec_F$ in Example 4.3 is in line with the human perception that judging the order of these fuzzy numbers is difficult; and so, we tend to consider their order as being indifferent.

Now, we will introduce a crisp relation that may be interpreted as a relation that expresses the order indifference of trapezoidal fuzzy numbers. We will call this relation the order indifference relation.

Definition 4.4. *Let* $\mathbf{F}$ *be a collection of trapezoidal fuzzy numbers. The binary relation* $\equiv_F$ *(indifference relation) over the collection* $\mathbf{F}$ *is given by*

$$\equiv_F := \left\{ (A, B) \in \mathbf{F} \times \mathbf{F} : M^{(\lambda)}(A, B) = \frac{1}{2} \right\},$$

where $\lambda > 0$.

The following theorem states that the relation $\equiv_F$ over a collection of trapezoidal fuzzy numbers is an equivalence relation.

Theorem 4.5. *The relation* $\equiv_F$ *given in Definition 4.4 is an equivalence relation over a collection* $\mathbf{F}$ *of trapezoidal fuzzy numbers.*

Proof. Let $\mathbf{F}$ be a collection of fuzzy numbers with trapezoidal membership functions. In order to demonstrate that $\equiv_F$ is an equivalence relation, we need to show that $\equiv_F$ is

1) reflexive, i.e., for any $A \in \mathbf{F}$, $A \equiv_F A$ holds
2) symmetric, i.e., for any $A, B \in \mathbf{F}$, if $A \equiv_F B$, then $B \equiv_F A$
3) transitive, i.e., for any $A, B, C \in \mathbf{F}$, if $A \equiv_F B$ and $B \equiv_F C$, then $A \equiv_F C$.

1) (Irreflexivity.) Based on Property 3.4, $M^{(\lambda)}(A, A) = \frac{1}{2}$ holds for any $A \in \mathbf{F}$. Therefore, by noting Definition 4.4, we get that $A \equiv_F A$.
2) (Symmetry.) Let $A, B \in \mathbf{F}$ such that $A \equiv_F B$. Then, based on Definition 4.4, $M^{(\lambda)}(A, B) = \frac{1}{2}$. Now, by noting the reciprocity property of function $M^{(\lambda)}$ (see Property 3.3), we have $M^{(\lambda)}(A, B) + M^{(\lambda)}(B, A) = 1$, from which we get that $M^{(\lambda)}(B, A) = \frac{1}{2}$. This means that $B \equiv_F A$.
3) (Transitivity.) Let $A, B, C \in \mathbf{F}$ such that $A \equiv_F B$ and $B \equiv_F C$. Then, based on Definition 4.4, $A \equiv_F B$ and $B \equiv_F C$ means that $M^{(\lambda)}(A, B) = \frac{1}{2}$ and $M^{(\lambda)}(B, C) = \frac{1}{2}$. Now, by noting Property 3.8, $M^{(\lambda)}(A, B) = \frac{1}{2}$ is equivalent to

$$\int_0^1 d_{BA}(\alpha)\mathrm{d}\alpha = 0, \tag{45}$$

and $M^{(\lambda)}(B,C)=\frac{1}{2}$ is equivalent to

$$\int_0^1 d_{CB}(\alpha)\mathrm{d}\alpha = 0, \tag{46}$$

where the signed distances $d_{BA}(\alpha)$ and $d_{CB}(\alpha)$ are given by Eq. (24), and $\alpha \in [0,1]$. Now, by taking into account Eq. (43), Eq. (45) and Eq. (46), we have

$$\int_0^1 d_{CA}(\alpha)\mathrm{d}\alpha = \int_0^1 d_{BA}(\alpha)\mathrm{d}\alpha + \int_0^1 d_{CB}(\alpha)\mathrm{d}\alpha = 0,$$

which, based on Property 3.8, is equivalent to $M^{(\lambda)}(A,C)=\frac{1}{2}$. That is, $A \equiv_F C$ holds. □

4.1 The preference implication-based ranking method

Here, by the means of an example, we will present a method for ranking trapezoidal fuzzy numbers by using the preference implication-based preference measure and the crisp relations $\prec_F$ and $\equiv_F$. This method has been implemented in MATLAB 2019a computing environment (see https://github.com/dombijozsef).

Suppose that we wish to rank the elements of the collection $\mathbf{F} = \{A_1, A_2, \dots, A_n\}$ of fuzzy numbers that have trapezoidal membership functions. The parameter values of $A_1, A_2, \dots, A_n$ are shown in Table 1.

Table 1: Parameters of fuzzy numbers.

	$\underline{x}_A^L$	$\overline{x}_A^L$	$\overline{x}_A^R$	$\underline{x}_A^R$
A_1	43	65	68	74
A_2	33	65	68	74
A_3	55	58	62	73
A_4	0	24	27	31
A_5	42	71	87	111
A_6	57	72	77	85
A_7	55	60	65	70
A_8	14	17	22	29

Table 2: Matrix of the preference measure values.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
A_1	0.5000	0.2457	0.4623	0.0000	0.9993	0.9928	0.5000	0.0000
A_2	0.7543	0.5000	0.6138	0.0000	0.9996	0.9962	0.6745	0.0000
A_3	0.5377	0.3862	0.5000	0.0000	0.9994	0.9923	0.5575	0.0000
A_4	1.0000	1.0000	1.0000	0.5000	1.0000	1.0000	1.0000	0.5000
A_5	0.0007	0.0004	0.0006	0.0000	0.5000	0.0765	0.0005	0.0000
A_6	0.0072	0.0038	0.0077	0.0000	0.9235	0.5000	0.0067	0.0000
A_7	0.5000	0.3255	0.4425	0.0000	0.9995	0.9933	0.5000	0.0000
A_8	1.0000	1.0000	1.0000	0.5000	1.0000	1.0000	1.0000	0.5000

Table 3: The n_i, p_i, u_k and o_i values.

i	1	2	3	4	5	6	7	8
n_i	2	5	4	6	0	1	2	6
p_i	6	3	4	2	8	7	6	2
k	1	2	3	4	5	6		
u_k	2	3	4	6	7	8		
o_i	4	2	3	1	6	5	4	1

In this example $n = 8$. Here, using the formulas in Eq. (31) and Eq. (32) with a $\lambda = 0.5$ parameter value, the preference implication-based preference measure $M^{(\lambda)}(A_i, A_j)$ was computed for each (A_i, A_j) pair and stored in the preference measure matrix in Table 2, where $A_i, A_j \in \mathbf{F}$, $i, j = 1, 2, \dots, n$. In this matrix, we can see the reciprocity property of the index $M^{(\lambda)}$; that is, $M^{(\lambda)}(A_i, A_j) = 1 - M^{(\lambda)}(A_j, A_i)$ holds for any $i, j = 1, 2, \dots, n$. Note that the computed values in Table 2 have been rounded to four digits. Let n_i denote the cardinality of the set $\{j \in \{1, 2, \dots, n\} : A_i \prec_F A_j\}$; that is, n_i is the number of those A_j fuzzy sets for which $M^{(\lambda)}(A_i, A_j) > \frac{1}{2}$ holds, $i \in \{1, 2, \dots, n\}$, (see Table 3). Let p_i be given by $p_i = n - n_i$ for $i = 1, 2, \dots n$, and let the sequence $u_1 < u_2 < \cdots < u_m$ contain the unique values of $p_1, p_2, \dots, p_n$ in increasing order, $(m \le n)$. Now, for every $i \in \{1, 2, \dots, n\}$ let o_i be given by $o_i = k$ such that $u_k = p_i$, where $k \in \{1, 2, \dots, m\}$. Then, o_i may be interpreted as the order index of the fuzzy set A_i, where $i = 1, 2, \dots, n$. Table 3 summarizes the n_i, p_i, u_k and o_i values for our case.

Notice that the fuzzy numbers A_4 and A_8 have the same order index (1); that is, we consider the order of these two fuzzy numbers as being indifferent. From Table 2, we can see that $M^{(\lambda)}(A_4, A_8) = 0.5000$, which, after considering the definition for the relation $\equiv_F$ over the set $\mathbf{F}$, means that the relation $A_4 \equiv_F A_8$ holds. Also, the fuzzy numbers A_1 and A_7 have the same order index (4), which means that the relation $A_1 \equiv_F A_7$ holds as well. Hence, the fuzzy numbers with the same order index o_i form groups such that in each of these groups, we view the order of the fuzzy numbers as being indifferent. Using the relations $\prec_F$ and $\equiv_F$, the order of the fuzzy numbers in $\mathbf{F}$ can be represented by the following relation chain: $A_4 \equiv_F A_8 \prec_F A_2 \prec_F A_3 \prec_F A_1 \equiv_F A_7 \prec_F A_6 \prec_F A_5$.

Figure 5 shows a graph representation of this relation chain. In this figure, the black and gray-colored arrows represent the relations $\prec_F$ and $\equiv_F$, respectively. Figure 6 shows the fuzzy sets when they are ranked using the relations

$\prec_F$ and $\equiv_F$. In this figure, each of the gray-colored areas contain a group of fuzzy numbers whose order is viewed as being indifferent: group $\{A_4, A_8\}$ and group $\{A_1, A_7\}$.

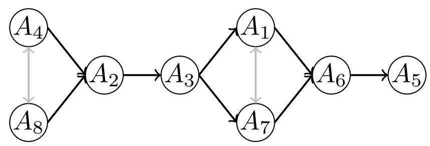

Figure 5: Graph representation of $\prec_F$ and $\equiv_F$.

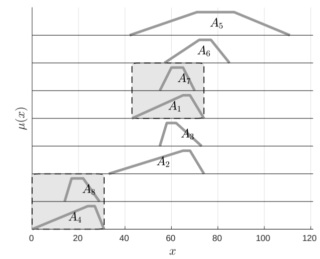

Figure 6: Ranked fuzzy numbers.

4.2 Comparisons with other ranking methods

We compared our fuzzy number ranking method with some well-known ones. The methods taken into account were the maximizing set and minimizing set method by Chen (1985) [10], the L-R spreads method by Abbasbandy and Hajjari (2009) [2], the sign distance method by Abbasbandy and Asady (2006) [1], the revised maximizing set and minimizing set method by Chou et al. (2011) [13], the distance method by Cheng (1998) [11], the inclusion index and bitset encoding method by Boulmakoul et al. (2017) [8], the maximizing barrier and minimizing barrier method by Choobineh and Li (1993) [12], the sign distance method by Abbasbandy and Asady (2006) [1], the distance minimization method by Asady and Zendehnam (2007) [6], the revised distance minimization method by Ezzati et al. (2012) [20], the centroid point and original point area by Chu and Tsao (2002) [14] and the decomposition and signed distance method by Yao and Wu (2000) [39].

Our comparison is based on eight samples of trapezoidal and triangular fuzzy numbers that were used as examples in the above-mentioned publications. Each fuzzy number is given by a quadruple according to the definition of the trapezoidal fuzzy number in Definition 3.5. Here, the triangular fuzzy numbers are also given by quadruples in which the two middle parameters are identical. The eight samples and results of the ranking methods are summarized in Table 4. The plots of the membership functions of the fuzzy numbers in the examined samples are shown in Figure 7. In Table 4, $\simeq$ stands for the indifference (equality) relation, while $\prec$ and $\succ$ denote preference relations. Note that the comparisons in Table 4 are based on the results available in the above-mentioned publications, and so not all the methods are compared for each sample (e.g., there are 3 methods for Sample 1, and there are 12 methods for Sample 7 in Table 4).

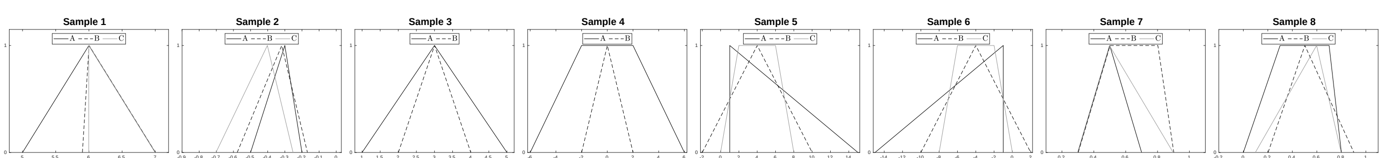

Figure 7: Samples of trapezoidal and triangular fuzzy numbers.

It should be noted that in line with the human intuition, if two fuzzy numbers have a common symmetry axis, then the outcome of our method is that the order of these two fuzzy numbers may be considered as being indifferent, see, e.g. Sample 3 and 4, or the fuzzy numbers B and C in Sample 5 and 6. We should also add that not all the examined methods own this property. In Sample 5 and 6, the fuzzy numbers denoted by the same letters are mirrored pairs. Therefore, we may expect that the order of $-A$, $-B$ and $-C$ is the reverse of the order of A, B and C. Our method and the methods of Abbasbandy and Hajjari (2009) and Boulmakoul et al. (2017) meet this expectation, but the method of Ezzati et al. (2012) does not. In sample 8, due to the overlapping of fuzzy numbers B and C, it is difficult to judge their order. The outcome of our method, and of some other methods as well, is that the order of B and C may be viewed as being indifferent. This outcome is in accordance with human intuition. These properties of our method suggest that it may be treated as a new alternative fuzzy number ranking method.

Table 4: Ranking results.

Sample ID	Sample	Method	Ranking
1	$A = (5, 6, 6, 7)$	Chen (1985)	$A \prec B \prec C$
	$B = (5.9, 6, 6, 7)$	Abbasbandy and Hajjari (2009)	$A \prec B \prec C$
	$C = (6, 6, 6, 7)$	Chou et al. (2011)	$C \prec B \prec A$
		Boulmakoul et al. (2017)	$A \prec B \prec C$
		The proposed method	$A \prec B \prec C$
2	$A = (0.5, 0.3, 0.2)$	Chen (1985)	$A \prec B \prec C$
	$B = (-0.58, -0.32, -0.32, -0.17)$	Cheng (1998)	$A \prec B \prec C$
	$C = (-0.7, -0.4, -0.4, -0.25)$	Choobineh and Li (1993)	$C \prec B \prec A$
		Boulmakoul et al. (2017)	$C \prec B \prec A$
		The proposed method	$C \prec B \prec A$
3	$A = (1, 3, 3, 5)$	Yao and Wu (2000)	$A \simeq B$
	$B = (2, 3, 3, 4)$	Abbasbandy and Hajjari (2009)	$A \simeq B$
		Ezzati et al. (2012)	$A \succ B$
		Boulmakoul et al. (2017)	$A \prec B$
		The proposed method	$A \simeq B$
4	$A = (-6, -2, 2, 6)$	Abbasbandy and Hajjari (2009)	$A \simeq B$
	$B = (-2, 0, 0, 2)$	Ezzati et al. (2012)	$A \succ B$
		Boulmakoul et al. (2017)	$A \prec B$
		The proposed method	$A \simeq B$
5	$A = (1, 1, 1, 15)$	Abbasbandy and Hajjari (2009)	$A \prec B \simeq C$
	$B = (-2, 4, 4, 10)$	Ezzati et al. (2012)	$A \prec C \prec B$
	$C = (0, 2, 6, 8)$	Boulmakoul et al. (2017)	$C \simeq B \prec A$
		The proposed method	$C \simeq B \prec A$
6	$-A = (-15, -1, -1, -1)$	Abbasbandy and Hajjari (2009)	$-C \simeq -B \prec -A$
	$-B = (-10, -4, -4, 2)$	Ezzati et al. (2012)	$-C \prec -B \prec -A$
	$-C = (-8, -6, -2, 0)$	Boulmakoul et al. (2017)	$-A \prec -B \simeq -C$
		The proposed method	$-A \prec -B \simeq -C$
7	$A = (0.3, 0.5, 0.5, 0.7)$	Boulmakoul et al. (2017)	$A \prec C \prec B$
	$B = (0.3, 0.5, 0.8, 0.9)$	Ezzati et al. (2012)	$A \prec C \prec B$
	$C = (0.3, 0.5, 0.5, 0.9)$	Abbasbandy and Hajjari (2009)	$A \prec C \prec B$
		Abbasbandy and Asady (2006) (p=1)	$A \prec C \prec B$
		Abbasbandy and Asady (2006) (p=2)	$A \prec C \prec B$
		Asady and Zendehnam (2007)	$A \prec C \prec B$
		Choobineh and Li (1993)	$A \prec B \prec C$
		Chen (1985)	$A \prec B \prec C$
		Chu and Tsao (2002)	$A \prec C \prec B$
		Yao and Wu (2000)	$A \prec C \prec B$
		Cheng (1998)	$A \prec C \prec B$
		The proposed method	$A \prec C \prec B$
8	$A = (0, 0.3, 0.7, 0.8)$	Boulmakoul et al. (2017)	$A \prec C \prec B$
	$B = (0.2, 0.5, 0.5, 0.9)$	Ezzati et al. (2012)	$B \prec A \prec C$
	$C = (0.1, 0.6, 0.6, 0.8)$	Abbasbandy and Hajjari (2009)	$B \prec A \prec C$
		Abbasbandy and Asady (2006) (p=1)	$A \prec B \simeq C$
		Abbasbandy and Asady (2006) (p=2)	$A \prec B \prec C$
		Asady and Zendehnam (2007)	$A \prec B \simeq C$
		Choobineh and Li (1993)	$A \prec B \prec C$
		Chen (1985)	$A \prec B \prec C$
		Chu and Tsao (2002)	$A \prec C \prec B$
		Yao and Wu (2000)	$A \prec B \simeq C$
		Cheng (1998)	$A \prec C \prec B$
		The proposed method	$A \prec B \simeq C$

5 Conclusions

The main findings of this study can be summarized as follows. Dombi and Baczyński first introduced the preference implication [18]. Now, in this study, we showed how the preference implication is connected with soft inequalities and explained why we modeled the latter ones by sigmoid functions. Utilizing this connection, we introduced the preference implication-based measure of preference for two real numbers and discussed its key properties. Next, we proposed the preference implication-based preference measure for two fuzzy numbers and highlighted some important properties of this new preference measure including its reciprocity. Furthermore, we derived the formulas using which the preference implication-based preference measure for two trapezoidal fuzzy numbers can be readily computed. The comparison of two fuzzy numbers should also be able to capture the situation where the order of the fuzzy numbers cannot be judged; and so, their order may be considered as being indifferent. Here, using the preference implication-based preference measure for two trapezoidal fuzzy numbers, we introduced two crisp relations over a collection of trapezoidal fuzzy numbers and discussed their main algebraic properties. We demonstrated that one of these two crisp relations is a strict, but not a total, order relation, and the other one is an equivalence relation. Here, we considered two fuzzy numbers as being comparable, when their order can be unambiguously determined. Next, we showed that our strict order relation can be used to rank comparable fuzzy numbers, while our equivalence relation, which we called the indifference relation, can be utilized to express the view that the order of some fuzzy numbers is indifferent. In an illustrative example, we showed how these two crisp relations can be used to rank a collection of trapezoidal fuzzy numbers. Based on the results of comparisons with other well-known fuzzy number ranking methods, our method may be viewed as a new alternative fuzzy number ranking method.

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