Iranian Journal of Fuzzy Systems

Volume 18, Number 3, (2021), pp. 1-11

Original paper

Fuzzy logic and enriched categories

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Abstract

We consider a category $\mathcal{C}$ enriched over the segment $[0, 1]$ whose hom-objects are real numbers from $[0, 1]$. For a suitably defined function $\hat{v}$ assigning to each formula φ some object of $\mathcal{C}$, the hom-object $\mathcal{C}(\hat{v}(\varphi), \hat{v}(\psi))$ represents the degree of derivability of ψ from φ. We reformulate completeness result for intuitionistic propositional logic, as well as Hájek's completeness results concerning the product, Gödel and Łukasiewicz fuzzy logic in the context of enriched category theory.

Keywords: Product fuzzy logic, Gödel fuzzy logic, Łukasiewicz fuzzy logic, t-norm, bicartesian closed $\mathcal{V}$-enriched category, self-enriched category.

1 Introduction

Between mathematical logic and category theory there are deep and rich connections, whose study has a long history (see e.g. classical books [7] and [10]). For instance, the basis for investigation of proof theory through category theory are *bicartesian categories*, i.e. categories in which all finite products and coproducts exist (including terminal and initial object). In such categories, we interpret product and coproduct as logical connectives $\wedge$ and $\vee$. One step further are *bicartesian closed categories*, which are the standard categorical models of intuitionistic propositional logic.

More precisely, let $\mathcal{C}(\times, +, \multimap, i, t)$ be an arbitrary bicartesian closed category where $\times$ is the product, $+$ is the coproduct, $\multimap$ is the right adjoint of $\times$, i is an initial object and t is a terminal object. Let P be the set of propositional letters, $\mathcal{F}$ be the set of formulae of intuitionistic propositional logic, and $v : P \to \mathrm{ob}(\mathcal{C})$ be an arbitrary function. We define $\hat{v} : \mathcal{F} \to \mathrm{ob}(\mathcal{C})$ by

$$\begin{aligned}
&\hat{v}(p) = v(p) \quad \text{for all } p \in P,\\
&\hat{v}(\varphi \wedge \psi) = \hat{v}(\varphi) \times \hat{v}(\psi),\\
&\hat{v}(\varphi \vee \psi) = \hat{v}(\varphi) + \hat{v}(\psi),\\
&\hat{v}(\varphi \to \psi) = \hat{v}(\varphi) \multimap \hat{v}(\psi),\\
&\hat{v}(\bot) = i, \quad \hat{v}(\top) = t.
\end{aligned}$$

Then we have that φ is a theorem of intuitionistic propositional logic if and only if for every bicartesian closed category $\mathcal{C}$ and for every $v : P \to \mathrm{ob}(\mathcal{C})$ we have $\mathcal{C}(t, \hat{v}(\varphi)) \neq \emptyset$. More generally, we have $\varphi \vdash \psi$ in intuitionistic propositional logic if and only if for every bicartesian closed category $\mathcal{C}$ and for every $v : P \to \mathrm{ob}(\mathcal{C})$ we have $\mathcal{C}(\hat{v}(\varphi), \hat{v}(\psi)) \neq \emptyset$.

Categories also arise in computer science, where the objects are data types and the arrows are programs. On the other side, *fuzzy logic* is an appropriate tool for reasoning in the presence of vagueness. It has various applications in natural language processing and artificial intelligence, where the intelligent behaviour is accomplished by creating fuzzy classes of certain parameters. Fuzzy logic also has a categorical semantics, which is studied by L. N. Stout in [11] and [12].

In this paper we present another categorical approach to intuitionistic and fuzzy logic, using *enriched category theory*. Namely, we consider a category $\mathcal{C}$ enriched over the segment $[0, 1]$ and instead of individual arrows, we deal

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Received: April 2020; Revised: October 2020; Accepted: December 2020.

with the *hom-objects* of the form $\mathcal{C}(a,b)$, which are real numbers from $[0,1]$. So, we replace the set $\mathcal{C}(\hat{v}(\varphi),\hat{v}(\psi))$ with a real number from $[0,1]$, which represents the *degree of derivability* of ψ from φ. Note that our category $\mathcal{C}$ is akin to Lawvere's metric spaces (cf. [8]), and enriched Lawvere theories that describe the *operational semantics* for programming languages (cf. [1]).

We have relied on the completeness results obtained by Hájek concerning the product, Gödel, and Łukasiewicz fuzzy logic, but we have moved to the level of derivations, while Hájeks results care about the level of formulae only.

2 Basic fuzzy logic and its extensions

In this section we define three important propositional fuzzy logics: product fuzzy logic, Gödel fuzzy logic and Łukasiewicz fuzzy logic (see [4]). All three of these logics are extensions of the basic fuzzy logic, which we introduce first.

Basic fuzzy logic BL has the primitive connectives &, $\rightarrow$ and $\bot$, while the derivable connectives are defined as follows:

$$\begin{aligned}
&\varphi \wedge \psi := \varphi \& (\varphi \rightarrow \psi),\\
&\varphi \vee \psi := ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi),\\
&\neg\varphi := \varphi \rightarrow \bot,\\
&\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \& (\psi \rightarrow \varphi),\\
&\top := \bot \rightarrow \bot.
\end{aligned}$$

The connective & is *strong conjunction*, while $\wedge$ is *weak conjunction*. The following formulae are axioms of the basic fuzzy logic BL:

$$\begin{aligned}
&(\text{A1}) && (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)),\\
&(\text{A2}) && (\varphi \& \psi) \rightarrow \varphi,\\
&(\text{A3}) && (\varphi \& \psi) \rightarrow (\psi \& \varphi),\\
&(\text{A4}) && (\varphi \& (\varphi \rightarrow \psi)) \rightarrow (\psi \& (\psi \rightarrow \varphi)),\\
&(\text{A5a}) && (\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \& \psi) \rightarrow \chi),\\
&(\text{A5b}) && ((\varphi \& \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi)),\\
&(\text{A6}) && ((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi),\\
&(\text{A7}) && \bot \rightarrow \varphi.
\end{aligned}$$

The *deduction rule* of BL is modus ponens. To give semantics for BL, we first introduce the following notions.

A *t-norm* (abbreviated from *triangular norm*) is a binary operation $*$ on $[0,1]$ satisfying the following conditions:

(1) $*$ is commutative and associative, i.e., for all $x, y, z \in [0,1]$,

$$x * y = y * x \quad \text{and} \quad x * (y * z) = (x * y) * z,$$

(2) $*$ is non-decreasing in both arguments, i.e.

$$x_1 \leq x_2 \Rightarrow x_1 * y \leq x_2 * y \quad \text{and} \quad y_1 \leq y_2 \Rightarrow x * y_1 \leq x * y_2,$$

(3) $1 * x = x$ and $0 * x = 0$ for all $x \in [0,1]$.

We say that a t-norm $*$ is *continuous* when it is a continuous mapping of $[0,1]^2$ into $[0,1]$ in both arguments (in the usual sense). The following are the most important examples of continuous t-norms:

- *Product t-norm*: $x * y := x \cdot y$ (product of real numbers),
- *Gödel t-norm*: $x * y := \min\{x, y\}$,
- *Łukasiewicz t-norm*: $x * y := \max\{x + y - 1, 0\}$.

We note that the three t-norms above are fundamental continuous t-norms, in the sense that, informally speaking, each continuous t-norm is a combination of the product, Łukasiewicz and Gödel t-norm (for more details see [4, Theorem 2.1.16]).

A continuous t-norm has a unique *residuum*, that is, a binary operation $\Rightarrow$ on $[0,1]$ such that for all x, y and z in $[0,1]$,

$$x * z \leq y \quad \text{if and only if} \quad z \leq (x \Rightarrow y), \tag{1}$$

see [4, Lemma 2.1.4]. A *valuation* of propositional variables is a function v assigning to each propositional variable p its truth value $v(p) \in [0,1]$. This extends uniquely to the valuation $\hat{v}$ of all formulae as follows:

$$\begin{aligned} &\hat{v}(\bot) = 0, \\ &\hat{v}(\varphi \& \psi) = \hat{v}(\varphi) * \hat{v}(\psi), \\ &\hat{v}(\varphi \to \psi) = \hat{v}(\varphi) \Rightarrow \hat{v}(\psi). \end{aligned}$$

For a fixed t-norm $*$, a formula φ is 1-*tautology* when $\hat{v}(\varphi) = 1$ for every v. We define now three important extensions of the basic fuzzy logic.

Product fuzzy logic Π is the extension of the basic fuzzy logic where strong conjunction is interpreted as product t-norm. Besides the axioms of the basic fuzzy logic, it has the following two axioms:

$$\begin{aligned} &(\Pi 1) \quad \neg\neg\chi \to ((\varphi \& \chi \to \psi \& \chi) \to (\varphi \to \psi)), \\ &(\Pi 2) \quad \varphi \wedge \neg\varphi \to \bot. \end{aligned}$$

Gödel fuzzy logic G is the extension of the basic fuzzy logic where strong conjunction is interpreted as Gödel t-norm. It has the following additional axiom:

$$(\mathrm{G}) \quad \varphi \to (\varphi \& \varphi).$$

In *Łukasiewicz fuzzy logic* Ł strong conjunction is interpreted as Łukasiewicz t-norm, and it is the extension of the basic fuzzy logic with the following axiom:

$$(\text{Ł}) \quad \neg\neg\varphi \to \varphi.$$

3 Category enriched over $[0,1]$

We recall that a *monoidal category* is a category $(\mathcal{V}, \otimes, I)$ equipped with a unit object I, a bifunctor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and natural isomorphisms $\alpha_{a,b,c} : a \otimes (b \otimes c) \to (a \otimes b) \otimes c$, $\lambda_a : I \otimes a \to a$ and $\rho_a : a \otimes I \to a$, subject to standard coherence conditions, [9, Section VII.1]. A monoidal category is *strict* when all the components of α, λ and ρ are identities. Explicitly, the following associativity and unit axioms hold

$$\begin{aligned} a \otimes (b \otimes c) &= (a \otimes b) \otimes c, & \qquad a \otimes I &= a = I \otimes a, \\ f \otimes (g \otimes h) &= (f \otimes g) \otimes h, & \qquad f \otimes \mathbb{1}_I &= f = \mathbb{1}_I \otimes f, \end{aligned}$$

for all objects a, b, c and arrows f, g, h of $\mathcal{V}$.

The partially ordered set $([0,1], \leq)$ can be regarded as a category whose objects are elements of $[0,1]$, and an arrow $a \to b$ exists if and only if $a \leq b$. So, for any pair of objects a and b in $[0,1]$ there is at most one arrow from a to b. Moreover, we can introduce a strict monoidal structure on the category $[0,1]$, e.g. when $\otimes$ is a t-norm, and I is the number 1.

We recall that a monoidal category $(\mathcal{V}, \otimes, I)$ is *symmetric* when there exists natural isomorphism $\sigma_{a,b} : a \otimes b \to b \otimes a$, which is self-inverse. Monoidal category $(\mathcal{V}, \otimes, I)$ is *closed* when there exists functor $a \multimap \_ : \mathcal{V} \to \mathcal{V}$, which is a right adjoint for $a \otimes \_ : \mathcal{V} \to \mathcal{V}$, for every object a of $\mathcal{V}$.

We say that a t-norm $*$ is *left-continuous* when it is a left-continuous mapping of $[0,1]^2$ into $[0,1]$ in both arguments (in the usual sense). The following proposition describes the connection between t-norms and symmetric monoidal structures on the segment $[0,1]$ envisaged as a category.

Proposition 3.1. *The function* $* : [0,1]^2 \to [0,1]$ *is a left-continuous t-norm if and only if* $([0,1], *, 1)$ *is a symmetric monoidal closed category.*

Proof. By [3, Proposition 5.4.2] we know that the residuum of a t-norm $*$ exists if and only if $*$ is left-continuous. Suppose now that $*$ is a left-continuous t-norm, and let $\Rightarrow$ be its residuum. As before, we have a strict monoidal structure on $[0,1]$, where tensor is a t-norm, and the unit object I is the number 1. Moreover, $* : [0,1]^2 \to [0,1]$ can be seen as a bifunctor, and $a \Rightarrow \_ : [0,1] \to [0,1]$ as a functor, where $a * \_$ is a left adjoint of $a \Rightarrow \_$. So, $([0,1], *, 1)$ is a symmetric monoidal closed category.

Conversely, suppose that $\mathcal{V} = ([0,1], *, 1)$ is a symmetric monoidal closed category. Let us prove that $a * 0 = 0$, for all $a \in [0,1]$, because $*$ immediately satisfies all other properties of a t-norm (commutativity of $*$ follows because $\mathcal{V}$ is symmetric, and monotonicity because $*$ is a bifunctor). We have $0 \leq a * 0$, because 0 is an initial object in $\mathcal{V}$, while the inequality $a * 0 \leq 0$ follows from the adjunction $\mathcal{V}(a * 0, 0) \cong \mathcal{V}(0, a \Rightarrow 0)$, where $a \Rightarrow \_$ is a right adjoint of $a * \_$. Also, this adjunction implies existence of a residuum for $*$, so again by [3, Proposition 5.4.2] we have that $*$ is left-continuous. □

In the rest of the paper, the monoidal category $([0,1], *, 1)$, where $*$ is a t-norm, is denoted by $\mathcal{V}$. The category $\mathcal{C}$ *enriched over the strict monoidal category* $\mathcal{V}$ (also called $\mathcal{V}$*-category*) consists of:

- a collection of objects, $\mathrm{ob}(\mathcal{C})$,
- a hom-object, $\mathcal{C}(a,b) \in [0,1] = \mathrm{ob}(\mathcal{V})$, for every pair of objects a, b of $\mathcal{C}$,
- composition arrows, $\circ : \mathcal{C}(a,b) * \mathcal{C}(b,c) \to \mathcal{C}(a,c)$ in the category $\mathcal{V}$, for all $a, b, c \in \mathrm{ob}(\mathcal{C})$, which give us inequality

$$\mathcal{C}(a,b) * \mathcal{C}(b,c) \leq \mathcal{C}(a,c), \tag{2}$$

- identity-assigning arrows, $\mathrm{id}_a : 1 \to \mathcal{C}(a,a)$ in the category $\mathcal{V}$, for all $a \in \mathrm{ob}(\mathcal{C})$, which give us equality

$$\mathcal{C}(a,a) = 1, \tag{3}$$

where composition is associative and unital (this instantly follows, because $\mathcal{V}$ is a preorder). So, the category $\mathcal{C}$ does not have any notion of individual arrow, and hom-*sets* are replaced by hom-*objects* (cf. Lawvere's metric spaces, [8]).

We can also generalise products and coproducts to the enriched context (see [1, Section 3] or [6]). The $\mathcal{V}$*-product* of objects a and b is an object $a \times b$ such that

$$\mathcal{C}(c,a) * \mathcal{C}(c,b) = \mathcal{C}(c, a \times b) \tag{4}$$

holds for every $c \in \mathrm{ob}(\mathcal{C})$. The $\mathcal{V}$*-terminal object* of $\mathcal{C}$ is $t \in \mathrm{ob}(\mathcal{C})$ such that

$$\mathcal{C}(a,t) = 1, \tag{5}$$

for all $a \in \mathrm{ob}(\mathcal{C})$. Similarly, the $\mathcal{V}$*-coproduct* of objects a and b is an object $a + b$ such that

$$\mathcal{C}(a,c) * \mathcal{C}(b,c) = \mathcal{C}(a+b, c) \tag{6}$$

for all $c \in \mathrm{ob}(\mathcal{C})$, and $\mathcal{V}$*-initial object* of $\mathcal{C}$ is $i \in \mathrm{ob}(\mathcal{C})$ such that

$$\mathcal{C}(i,a) = 1,$$

for every $a \in \mathrm{ob}(\mathcal{C})$.

Definition 3.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{V}$-enriched categories. A $\mathcal{V}$-*functor *$F : \mathcal{C} \to \mathcal{D}$ consists of a map which assigns to each object of $\mathcal{C}$ an object of $\mathcal{D}$ and for each pair of objects a and b in $\mathcal{C}$ this functor provides an arrow $F_{a,b} : \mathcal{C}(a,b) \to \mathcal{D}(Fa, Fb)$ in $\mathcal{V}$, such that F preserves composition and identity, i.e. the following diagrams commute:*

$$\begin{array}{ccc} \mathcal{C}(a,b) * \mathcal{C}(b,c) & \xrightarrow{\circ} & \mathcal{C}(a,c) \\ {\scriptstyle F_{a,b} * F_{b,c}} \downarrow & & \downarrow {\scriptstyle F_{a,c}} \\ \mathcal{D}(Fa,Fb) * \mathcal{D}(Fb,Fc) & \xrightarrow[\circ]{} & \mathcal{D}(Fa,Fc) \end{array}$$

$$\begin{array}{ccc} & 1 & \\ {\scriptstyle \mathrm{id}_a} \swarrow & & \searrow {\scriptstyle \mathrm{id}_{Fa}} \\ \mathcal{C}(a,a) & \xrightarrow[F_{a,a}]{} & \mathcal{D}(Fa,Fa) \end{array} .$$

The concept of adjoint functor extends straightforwardly to $\mathcal{V}$-categories. For example, the $\mathcal{V}$-functor $a \multimap \_ : \mathcal{C} \to \mathcal{C}$ is a $\mathcal{V}$-enriched right adjoint of the $\mathcal{V}$-functor $a \times \_ : \mathcal{C} \to \mathcal{C}$ when equality

$$\mathcal{C}(a \times b, c) = \mathcal{C}(b, a \multimap c), \tag{7}$$

holds for all objects $b, c \in \mathrm{ob}(\mathcal{C})$.

We say that $\mathcal{C}$ is a *bicartesian closed $\mathcal{V}$-enriched category* when $\mathcal{C}$ has all finite $\mathcal{V}$-products and $\mathcal{V}$-coproducts, a $\mathcal{V}$-initial object, a $\mathcal{V}$-terminal object and the $\mathcal{V}$-functor $a \times \_$ has a $\mathcal{V}$-enriched right adjoint. So, in a bicartesian closed $\mathcal{V}$-enriched category $\mathcal{C}$, inequality (2) and equalities (3)-(7) hold for all $a, b, c \in \mathrm{ob}(\mathcal{C})$.

Lemma 3.3. *For arbitrary bicartesian closed $\mathcal{V}$-enriched category $\mathcal{C}$ and for all $a, b, c, d \in \mathrm{ob}(\mathcal{C})$, we have:*

(1) $\mathcal{C}(t, a \multimap b) = \mathcal{C}(a, b)$,

(2) $\mathcal{C}(a \times b, c) = \mathcal{C}(b \times a, c)$ *and* $\mathcal{C}(a, b + c) = \mathcal{C}(a, c + b)$,

(3) $\mathcal{C}((a \times b) \times c, d) = \mathcal{C}(a \times (b \times c), d))$ *and* $\mathcal{C}(a, (b + c) + d) = \mathcal{C}(a, b + (c + d))$,

(4) $\mathcal{C}(a \times (a \multimap b), c) \geq \mathcal{C}(b, c)$.

Proof. (1) We know that $\mathcal{C}(t, a \multimap b) = \mathcal{C}(a \times t, b)$, so we want to show that $\mathcal{C}(a \times t, b) = \mathcal{C}(a, b)$. From $\mathcal{C}(a, a) = 1$ and $\mathcal{C}(a, t) = 1$ we obtain $\mathcal{C}(a, a \times t) = 1$. Thus, $\mathcal{C}(a, a \times t) * \mathcal{C}(a \times t, b) \leq \mathcal{C}(a, b)$ yields $\mathcal{C}(a \times t, b) \leq \mathcal{C}(a, b)$.

From $\mathcal{C}(a \times t, a) * \mathcal{C}(a \times t, t) = \mathcal{C}(a \times t, a \times t) = 1$ follows that $\mathcal{C}(a \times t, a) = 1$. Thus, $\mathcal{C}(a \times t, a) * \mathcal{C}(a, b) \leq \mathcal{C}(a \times t, b)$ yields $\mathcal{C}(a, b) \leq \mathcal{C}(a \times t, b)$. Therefore, it must be $\mathcal{C}(a, b) = \mathcal{C}(a \times t, b)$.

(2) Using commutativity for $*$, it is easy to see that $\mathcal{C}(a, b \times c) = \mathcal{C}(a, c \times b)$, so we have $\mathcal{C}(b \times a, a \times b) = \mathcal{C}(b \times a, b \times a) = 1$. Now, from $\mathcal{C}(b \times a, a \times b) * \mathcal{C}(a \times b, c) \leq \mathcal{C}(b \times a, c)$ and $\mathcal{C}(a \times b, b \times a) * \mathcal{C}(b \times a, c) \leq \mathcal{C}(a \times b, c)$, we conclude that $\mathcal{C}(a \times b, c) = \mathcal{C}(b \times a, c)$. We prove $\mathcal{C}(a, b + c) = \mathcal{C}(a, c + b)$ in a similar way.

(3) Similar to (2), save that we use associativity of $*$ instead of commutativity.

(4) Using that $\mathcal{C}(a \times (a \multimap b), b) = 1$, we have

$$\mathcal{C}(a \times (a \multimap b), b) * \mathcal{C}(b, c) \leq \mathcal{C}(a \times (a \multimap b), c),$$

and so $\mathcal{C}(a \times (a \multimap b), c) \geq \mathcal{C}(b, c)$. □

Lemma 3.4. *For every bicartesin closed $\mathcal{V}$-enriched category $\mathcal{C}$ and for all objects a and b, we have*

(1) $\mathcal{C}(a \times b, a) = \mathcal{C}(a \times b, b) = 1$,

(2) $\mathcal{C}(a, a + b) = \mathcal{C}(b, a + b) = 1$.

Proof. (1) We have $\mathcal{C}(a \times b, a) * \mathcal{C}(a \times b, b) = \mathcal{C}(a \times b, a \times b) = 1$. Since $\mathcal{C}(a \times b, b) \leq 1$, using monotonicity of $*$ we obtain

$$1 = \mathcal{C}(a \times b, a) * \mathcal{C}(a \times b, b) \leq \mathcal{C}(a \times b, a) * 1 = \mathcal{C}(a \times b, a),$$

i.e. $\mathcal{C}(a \times b, a) = 1$. Similarly, $\mathcal{C}(a \times b, b) = 1$.

(2) Using $\mathcal{C}(a, a + b) * \mathcal{C}(b, a + b) = 1$, $\mathcal{C}(b, a + b) \leq 1$ and monotonicity of $*$ we have

$$1 = \mathcal{C}(a, a + b) * \mathcal{C}(b, a + b) \leq \mathcal{C}(a, a + b) * 1 = \mathcal{C}(a, a + b),$$

i.e. $\mathcal{C}(a, a + b) = 1$. Similarly, $\mathcal{C}(b, a + b) = 1$. □

Corollary 3.5. *For every bicartesin closed $\mathcal{V}$-enriched category $\mathcal{C}$ and for all objects a, b and c, we have*

(1) $\mathcal{C}(a \times b, c) \geq \mathcal{C}(a, c)$ *and* $\mathcal{C}(a \times b, c) \geq \mathcal{C}(b, c)$,

(2) $\mathcal{C}(a, b + c) \geq \mathcal{C}(a, b)$ *and* $\mathcal{C}(a, b + c) \geq \mathcal{C}(a, c)$.

Proof. Using Lemma 3.4, we have $\mathcal{C}(a \times b, c) \geq \mathcal{C}(a \times b, a) * \mathcal{C}(a, c) = \mathcal{C}(a, c)$, and the first part of (1) follows. We prove remaining inequalities in a similar way. □

Lemma 3.6. *For every bicartesian closed $\mathcal{V}$-enriched category $\mathcal{C}$ and for all objects a_1, a_2, b_1, b_2, we have*

(1) $\mathcal{C}(a_1, a_2) * \mathcal{C}(b_1, b_2) \leq \mathcal{C}(a_1 \times b_1, a_2 \times b_2)$,

(2) $\mathcal{C}(a_1, a_2) * \mathcal{C}(b_1, b_2) \leq \mathcal{C}(a_1 + b_1, a_2 + b_2)$,

(3) $\mathcal{C}(a_1, a_2) * \mathcal{C}(b_1, b_2) \leq \mathcal{C}(a_2 \multimap b_1, a_1 \multimap b_2)$.

Proof. Using Corollary 3.5, we have

$$\mathcal{C}(a_1 \times b_1, a_2 \times b_2) = \mathcal{C}(a_1 \times b_1, a_2) * \mathcal{C}(a_1 \times b_1, b_2) \geq \mathcal{C}(a_1, a_2) * \mathcal{C}(b_1, b_2),$$

and (1) follows. In a similiar way we prove (2). To prove (3) first note that using Lemma 3.3 we have

$$\mathcal{C}(a_1 \multimap b_1, a_1 \multimap b_2) = \mathcal{C}(a_1 \times (a_1 \multimap b_1), b_2) \geq \mathcal{C}(b_1, b_2). \tag{8}$$

Using the inequality (8), the inequality (1) of this lemma and Lemma 3.3, we have

$$\begin{aligned}\mathcal{C}(a_2 \multimap b_1, a_1 \multimap b_1) &\geq \mathcal{C}(a_2 \multimap b_1, a_1 \multimap (a_1 \times (a_2 \multimap b_1))) \\ &\quad * \mathcal{C}(a_1 \multimap (a_1 \times (a_2 \multimap b_1)), a_1 \multimap (a_2 \times (a_2 \multimap b_1))) \\ &\quad * \mathcal{C}(a_1 \multimap (a_2 \times (a_2 \multimap b_1)), a_1 \multimap b_1) \\ &\geq 1 * \mathcal{C}((a_1 \times (a_2 \multimap b_1)), (a_2 \times (a_2 \multimap b_1))) \\ &\quad * \mathcal{C}((a_2 \times (a_2 \multimap b_1)), b_1) \\ &\geq \mathcal{C}(a_1, a_2) * \mathcal{C}(a_2 \multimap b_1, a_2 \multimap b_1) * 1,\end{aligned}$$

so $\mathcal{C}(a_2 \multimap b_1, a_1 \multimap b_1) \geq \mathcal{C}(a_1, a_2)$. Using this and inequality (8), we have

$$\mathcal{C}(a_2 \multimap b_1, a_1 \multimap b_2) \geq \mathcal{C}(a_2 \multimap b_1, a_1 \multimap b_1) * \mathcal{C}(a_1 \multimap b_1, a_1 \multimap b_2) \geq \mathcal{C}(a_1, a_2) * \mathcal{C}(b_1, b_2),$$

and the inequality (3) follows. □

Remark 3.7. *Note that the previous lemma says that $\times : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $+ : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and $\multimap: \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ are $\mathcal{V}$-bifunctors (cf. Definition 3.2).*

Remark 3.8. *Note that in the case when $*$ is the product t-norm, we have that $\mathcal{C}(a, b)$ is either 0 or 1, for any a and b. Indeed, from $\mathcal{C}(a, b \times b) = \mathcal{C}(a, b) \cdot \mathcal{C}(a, b)$ and $\mathcal{C}(a, b \times b) \geq \mathcal{C}(a, b) \cdot \mathcal{C}(b, b \times b)$ we obtain $\mathcal{C}(a, b) \cdot \mathcal{C}(a, b) \geq \mathcal{C}(a, b)$, which is only possible when $\mathcal{C}(a, b)$ is either 0 or 1.*

4 Bicartesian closed $\mathcal{V}$-enriched categories and intuitionistic propositional logic

We recall that a *Heyting algebra* is a partially ordered set $(H, \preccurlyeq)$ with a smallest element $\overline{0}$, a largest element $\overline{1}$, and three operations $\cap, \cup$ and $\Rightarrow$ satisfying the following conditions, for all $x, y, z \in H$

(1) $x \preccurlyeq \overline{1}$,

(2) $x \cap y \preccurlyeq x$,

(3) $x \cap y \preccurlyeq y$,

(4) $z \preccurlyeq x$ and $z \preccurlyeq y$ implies $z \preccurlyeq x \cap y$,

(5) $\overline{0} \preccurlyeq x$,

(6) $x \preccurlyeq x \cup y$,

(7) $y \preccurlyeq x \cup y$,

(8) $x \preccurlyeq z$ and $y \preccurlyeq z$ implies $x \cup y \preccurlyeq z$,

(9) $x \preccurlyeq (y \Rightarrow z)$ if and only if $x \cap y \preccurlyeq z$.

Note that we have used symbol $\preccurlyeq$ for the partial order on Heyting algebra, because symbol $\leq$ is reserved for the partial order on the segment $[0,1]$.

Definition 4.1. *Let $\mathcal{C}$ be a $\mathcal{V}$-enriched category. We say that objects a and b are* isomorphic *when $\mathcal{C}(a,b)=1$ and $\mathcal{C}(b,a)=1$.*

Proposition 4.2. *Every bicartesian closed $\mathcal{V}$-erniched category $(\mathcal{C},\times,+,\multimap,i,t)$ is a Heyting algebra, up to isomorphism of objects. Conversely, every Heyting algebra $(H,\cap,\cup,\Rightarrow,\overline{0},\overline{1})$ can be regarded as a bicartesian closed $\mathcal{V}$-enriched category.*

Proof. Let $\mathcal{C}$ be a $\mathcal{V}$-enriched bicartesian closed category. We define

$$a \precsim b \quad \text{if and only if} \quad \mathcal{C}(a,b)=1.$$

It is clear that $\precsim$ is a preorder. Consider the equivalence relation defined by

$$a \sim b \quad \text{if and only if} \quad \mathcal{C}(a,b)=1 \text{ and } \mathcal{C}(b,a)=1.$$

(It is easy to check that $\sim$ is indeed an equivalence relation). Let $H=\mathrm{ob}(\mathcal{C})/\sim$. This will be the desired Heyting algebra. We denote the equivalence class of a by $[a]$, and define

$$\begin{aligned}
&[a]\preccurlyeq[b] \quad \text{if and only if} \quad a\precsim b,\\
&[a]\cap[b]=[a\times b]\\
&[a]\cup[b]=[a+b]\\
&[a]\Rightarrow[b]=[a\multimap b]\\
&\mathcal{C}([a],[b])=\mathcal{C}(a,b)\\
&\overline{0}=[i] \quad \text{and} \quad \overline{1}=[t].
\end{aligned}$$

Let us show that this definition is correct. Suppose that $a_1\sim a_2$ and $b_1\sim b_2$. This means that $\mathcal{C}(a_1,a_2)=1$, $\mathcal{C}(a_2,a_1)=1$, $\mathcal{C}(b_1,b_2)=1$ and $\mathcal{C}(b_2,b_1)=1$. If $[a_1]\preccurlyeq[b_1]$, we have $\mathcal{C}(a_1,b_1)=1$, so

$$\mathcal{C}(a_2,b_2)\geq\mathcal{C}(a_2,a_1)*\mathcal{C}(a_1,b_1)*\mathcal{C}(b_1,b_2)=\mathcal{C}(a_1,b_1).$$

It follows that $\mathcal{C}(a_2,b_2)=1$, i.e. $[a_2]\preccurlyeq[b_2]$. Also, it is easy to see that $\preccurlyeq$ defined in this way is a partial order. Further, we want to show that $a_1\times b_1\sim a_2\times b_2$. Using Lemma 3.6, we have

$$\mathcal{C}(a_1\times b_1,a_2\times b_2)\geq\mathcal{C}(a_1,a_2)*\mathcal{C}(b_1,b_2)=1,$$

and similarly $\mathcal{C}(a_2\times b_2,a_1\times b_1)=1$, which exactly means that $a_1\times b_1\sim a_2\times b_2$. In a similar way we obtain $a_1+b_1\sim a_2+b_2$ and $a_1\multimap b_1\sim a_2\multimap b_2$. Finally, to prove that $\mathcal{C}(a_1,b_1)=\mathcal{C}(a_2,b_2)$, note that we have

$$\mathcal{C}(a_1,b_1)\geq\mathcal{C}(a_1,a_2)*\mathcal{C}(a_2,b_2)*\mathcal{C}(b_2,b_1)=\mathcal{C}(a_2,b_2),$$

and similarly $\mathcal{C}(a_2,b_2)\geq\mathcal{C}(a_1,b_1)$.

Now, it is routine to check that all conditions from the definition of Heyting algebra are fulfilled.

To prove the second part of the proposition, suppose that $(H,\cap,\cup,\Rightarrow,\overline{0},\overline{1})$ is a Heyting algebra and $\preccurlyeq$ partial order on it. If we define $\mathcal{C}$ such that the objects of $\mathcal{C}$ are elements of H, and

$$\mathcal{C}(a,b)=\begin{cases}1, & a\preccurlyeq b\\ 0, & \text{otherwise,}\end{cases}$$

while $\cap$ is $\times$, $\cup$ is $+$, $\Rightarrow$ is $\multimap$, $\overline{0}$ is i and $\overline{1}$ is t, then it is not hard to see that $\mathcal{C}$ is bicartesian closed $\mathcal{V}$-enriched category. For example, let us show that

$$\mathcal{C}(c,a)*\mathcal{C}(c,b)=\mathcal{C}(c,a\times b). \tag{9}$$

If $\mathcal{C}(c,a\times b)=1$, then $c\preccurlyeq a\cap b$. From the transitivity of $\preccurlyeq$ and the conditions (2) and (3) in the definition of Heyting algebra it follows that $c\preccurlyeq a$ and $c\preccurlyeq b$. Thus, $\mathcal{C}(c,a)=\mathcal{C}(c,b)=1$, and the equality (9) follows. If $\mathcal{C}(c,a\times b)=0$, then from the condition (4) in the definition of Heyting algebra follows that either $\mathcal{C}(c,a)=0$ or $\mathcal{C}(c,b)=0$, and the equality (9) holds again. □

For any bicartesian closed $\mathcal{V}$-enriched category $\mathcal{C}$ let us denote by $\Phi(\mathcal{C})$ the Heyting algebra obtained from $\mathcal{C}$ as described in the proof of the previous proposition. Similarly, for a Heyting alegebra H, we denote by $\Psi(H)$ the bicartesian closed category obtained from H in the way described in the previous proof. Then it is not hard to see that $\Phi(\Psi(H)) = H$. In order to formulate corollary concerning $\Psi(\Phi(\mathcal{C}))$, we first need the following definition.

Definition 4.3. *Let $\mathcal{C}$ be a $\mathcal{V}$-enriched category. Then a* skeleton *of $\mathcal{C}$ is a $\mathcal{V}$-enriched category $\mathcal{D}$ such that*

- $\mathrm{ob}(\mathcal{D}) \subseteq \mathrm{ob}(\mathcal{C})$,
- *for every pair of objects a and b of $\mathcal{D}$ we have $\mathcal{D}(a, b) = \mathcal{C}(a, b)$,*
- *each object of $\mathcal{C}$ is isomorphic (in $\mathcal{C}$) to exactly one object of $\mathcal{D}$.*

Note that a category $\mathcal{C}$ is $\mathcal{V}$-*equivalent* to its skeleton $\mathcal{D}$ in the sense that there exist $\mathcal{V}$-enriched functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that for every $c \in \mathrm{ob}(\mathcal{C})$ and $d \in \mathrm{ob}(\mathcal{D})$ we have $FGd \cong d$ and $GFc \cong c$.

Corollary 4.4. *Let $\mathcal{C}$ be a bicartesian closed category enriched over $\mathcal{V}$ with product t-norm. Then $\Psi(\Phi(\mathcal{C}))$ is a skeleton of $\mathcal{C}$ (and consequently is $\mathcal{V}$-equivalent to $\mathcal{C}$).*

Proof. Follows directly from Remark 3.8 and the definitions od Φ and Ψ. □

Let $\mathcal{C}$ be a bicartesian closed $\mathcal{V}$-enriched category, P be a set of propositional letters and let $v : P \to \mathrm{ob}(\mathcal{C})$ be an arbitrary function. Denote by $\mathcal{F}$ the set of formulae built out of the set P, constant $\bot$ and binary connectives $\wedge, \vee$ and $\to$. Also, we define $\top := \bot \to \bot$.

Let $\hat{v} : \mathcal{F} \to \mathrm{ob}(\mathcal{C})$ be defined so that

$$\begin{aligned}
&\hat{v}(p) = v(p) \quad \text{for all } p \in P,\\
&\hat{v}(\varphi \wedge \psi) = \hat{v}(\varphi) \times \hat{v}(\psi),\\
&\hat{v}(\varphi \vee \psi) = \hat{v}(\varphi) + \hat{v}(\psi),\\
&\hat{v}(\varphi \to \psi) = \hat{v}(\varphi) \multimap \hat{v}(\psi),\\
&\hat{v}(\bot) = i, \quad \hat{v}(\top) = t.
\end{aligned}$$

Definition 4.5. *Let $\mathcal{C}$ be a bicartesian closed $\mathcal{V}$-enriched category. We say that a formula φ is $\mathcal{C}$-*valid *if for every function $v : P \to \mathrm{ob}(\mathcal{C})$ we have $\mathcal{C}(t, \hat{v}(\varphi)) = 1$.*

In a similar way, for a Heyting algebra H and for arbitrary function $w : P \to H$ we define $\hat{w} : \mathcal{F} \to H$.

Definition 4.6. *Let H be a Heyting algebra. We say that a formula φ is H-*valid *if for every function $w : P \to H$ we have $\hat{w}(\varphi) = \overline{1}$.*

Lemma 4.7. *A formula φ is $\mathcal{C}$-valid for every bicartesian closed $\mathcal{V}$-enriched category $\mathcal{C}$ if and only if it is H-valid for every Heyting algebra H.*

Proof. Suppose that φ is $\mathcal{C}$-valid for every bicartesian closed $\mathcal{V}$-enriched category $\mathcal{C}$. Let H be an arbitrary Heyting algebra, and $w : P \to H$ be an arbitrary function. We need to show that $\hat{w}(\varphi) = \overline{1}$. Let us denote $\Psi(H)$ by $\mathcal{C}$. Since φ is $\mathcal{C}$-valid, we have that $\mathcal{C}(t, \hat{w}(\varphi)) = 1$. This implies $\overline{1} \preccurlyeq \hat{w}(\varphi)$, which together with the condition (1) in the definition of a Heyting algebra yields $\hat{w}(\varphi) = \overline{1}$.

Conversely, suppose that φ is H-valid for every Heyting algebra H. Let $\mathcal{C}$ be an arbitrary bicartesian closed $\mathcal{V}$-enriched category, and $v : P \to \mathrm{ob}(\mathcal{C})$ be an arbitrary function. We need to show that $\mathcal{C}(t, \hat{v}(\varphi)) = 1$. Let us denote $\Phi(\mathcal{C})$ by H and consider the function $w : P \to H$ defined by $w(p) = [v(p)]$, where $[v(p)]$ is the equivalence class of $v(p)$, with respect to the equivalence relation defined in the proof of Proposition 4.2. It is not hard to see that $\hat{w}(\varphi) = [\hat{v}(p)]$. Since φ is H-valid, we have that $\hat{w}(\varphi) = \overline{1}$, and in particular $\overline{1} \preccurlyeq \hat{w}(\varphi)$. This means that $[t] \preccurlyeq [\hat{v}(\varphi)]$, which implies $t \precsim \hat{v}(\varphi)$, and consequently $\mathcal{C}(t, \hat{v}(\varphi)) = 1$. □

Theorem 4.8. *A formula φ is a theorem of intuitionistic propositional logic if and only if it is $\mathcal{C}$-valid for every bicartesian closed $\mathcal{V}$-enriched category $\mathcal{C}$.*

Proof. Intuitionistic propositional logic is complete with respect to Heyting algebras, i.e. the formula φ is a theorem of intuitionistic propositional logic if and only if it is H-valid for every Heyting algebra H. Now theorem follows directly from Lemma 4.7. □

5 Self-enrichment and fuzzy logic

By [2, Proposition 6.2.6], if $\mathcal{V}$ is a symmetric monoidal closed category, $\mathcal{V}$ itself can be provided with the structure of a $\mathcal{V}$-category. Precisely, there is a $\mathcal{V}$-category denoted $\widehat{\mathcal{V}}$ with the same objects as $\mathcal{V}$ and the hom-objects given by the internal homs. We say that $\widehat{\mathcal{V}}$ is *self-enriched*.

Let now, as before, $\mathcal{V}$ be the category $([0,1],*,1)$ where $*$ is a t-norm. Suppose that $*$ is left-continuous. Then by Proposition 3.1, the category $\mathcal{V}$ is symmetric monoidal closed, so we can define the category $\widehat{\mathcal{V}}$. Hence, the objects of $\widehat{\mathcal{V}}$ are the real numbers from the segment $[0,1]$, while the hom-objects are defined by

$$\widehat{\mathcal{V}}(a,b) = (a \Rightarrow b) \quad \text{for } a,b \in [0,1].$$

When we want to emphasize a t-norm $*$, we shall denote self-enriched category with respect to $*$ by $\widehat{\mathcal{V}}_*$. Since the product t-norm, Gödel t-norm and Łukasiewicz t-norm are left-continuous (in fact, they are continuous), they have the residuums, so the category $\mathcal{V}$ with respect to any of these three t-norms is symmetric monoidal closed. Therefore, we can define self-enriched categories with respect to the product t-norm, Gödel t-norm and Łukasiewicz t-norm, and we denote them by $\widehat{\mathcal{V}}_\Pi$, $\widehat{\mathcal{V}}_{\mathrm{G}}$ and $\widehat{\mathcal{V}}_{\mathrm{L}}$, respectively.

Also, $\widehat{\mathcal{V}}$ is a symmetric monoidal closed $\mathcal{V}$-enriched category, because (besides other straightforward properties) we have

$$\widehat{\mathcal{V}}(a*b,c) = \widehat{\mathcal{V}}(b, a \Rightarrow c) \quad \text{for } a,b \in [0,1].$$

To show this, we actually need to show that $((a*b) \Rightarrow c) = (b \Rightarrow (a \Rightarrow c))$ holds in $\mathcal{V}$. Using the equivalence (1) and associativity of a t-norm, we have

$$\begin{aligned} ((a*b) \Rightarrow c) \leq (b \Rightarrow (a \Rightarrow c)) \quad &\text{iff} \quad b*((a*b) \Rightarrow c) \leq (a \Rightarrow c) \\ &\text{iff} \quad a*(b*((a*b) \Rightarrow c)) \leq c \\ &\text{iff} \quad (a*b)*((a*b) \Rightarrow c) \leq c \\ &\text{iff} \quad ((a*b) \Rightarrow c) \leq ((a*b) \Rightarrow c). \end{aligned}$$

Hence, inequality $((a*b) \Rightarrow c) \leq (b \Rightarrow (a \Rightarrow c))$ follows. In a similar way we show $(b \Rightarrow (a \Rightarrow c)) \leq ((a*b) \Rightarrow c)$.

Hájek proved the following completeness results for the product, Gödel and Łukasiewicz fuzzy logic (see [4, Theorems 3.2.13, 4.1.13 and 4.2.17]).

Theorem 5.1. *(1) A formula φ is provable in the product logic Π if and only if it is a 1-tautology of the product logic.*
(2) A formula φ is provable in Gödel logic G if and only if it is a 1-tautology of Gödel logic.
(3) A formula φ is provable in Łukasiewicz logic Ł if and only if it is a 1-tautology of Łukasiewicz logic.

Let P be a set of propositional letters and let $v : P \to \mathrm{ob}(\widehat{\mathcal{V}})$ be an arbitrary function. Denote by $\mathcal{F}$ the set of formulae of the basic fuzzy logic BL.

Let $\hat{v} : \mathcal{F} \to \mathrm{ob}(\widehat{\mathcal{V}})$ be defined so that

$$\begin{aligned} &\hat{v}(p) = v(p) \quad \text{for all } p \in P, \\ &\hat{v}(\varphi \& \psi) = \hat{v}(\varphi) * \hat{v}(\psi), \\ &\hat{v}(\varphi \to \psi) = \hat{v}(\varphi) \Rightarrow \hat{v}(\psi), \\ &\hat{v}(\bot) = 0, \quad \hat{v}(\top) = 1. \end{aligned}$$

We say that a formula φ is $\widehat{\mathcal{V}}_*$*-valid* if for every function $v : P \to [0,1]$ we have $\widehat{\mathcal{V}}_*(1, \hat{v}(\varphi)) = 1$. In the case of product t-norm, Gödel t-norm or Łukasiewicz t-norm, we get the notions of $\widehat{\mathcal{V}}_\Pi$-valid, $\widehat{\mathcal{V}}_{\mathrm{G}}$-valid and $\widehat{\mathcal{V}}_{\mathrm{L}}$-valid formulae.

Lemma 5.2. *Let $*$ be a left-continuous t-norm. A formula φ is 1-tautology with respect to $*$ if and only if it is $\widehat{\mathcal{V}}_*$-valid.*

Proof. Suppose that φ is 1-tautology with respect to $*$. This means that for every $v : P \to [0,1]$ we have $\hat{v}(\varphi) = 1$. So, $\widehat{V}_*(1, \hat{v}(\varphi)) = \widehat{V}_*(1,1) = 1$, i.e. φ is $\widehat{\mathcal{V}}_*$-valid.

Conversely, let φ be $\widehat{\mathcal{V}}_*$-valid, i.e. for every function $v : P \to [0,1]$ we have $\widehat{\mathcal{V}}_*(1, \hat{v}(\varphi)) = 1$. This means that in category $\mathcal{V}$ we have $(1 \Rightarrow \hat{v}(\varphi)) = 1$, and in particular $1 \leq (1 \Rightarrow \hat{v}(\varphi))$. Now from the equivalence (1) we have $1 * 1 \leq \hat{v}(\varphi)$, i.e. $\hat{v}(\varphi) = 1$. Hence, φ is 1-tautology with respect to $*$. □

Theorem 5.3. *(1) A formula φ is provable in the product logic Π if and only if it is $\widehat{\mathcal{V}}_{\Pi}$-valid.*

(2) A formula φ is provable in Gödel logic G if and only if it is $\widehat{\mathcal{V}}_{\mathrm{G}}$-valid.

(3) A formula φ is provable in Łukasiewicz logic $\mathrm{Ł}$ if and only if it is $\widehat{\mathcal{V}}_{\mathrm{Ł}}$-valid.

Proof. Follows directly from Theorem 5.1 and Lemma 5.2. □

6 Conclusions and final remarks

The standard notion of the existence of a derivation between formulae φ and ψ, i.e. $\mathrm{Hom}(\varphi, \psi) \neq \emptyset$, can be expressed by the fact that in the category **Set** there exists an arrow from a terminal object I (the unit of the monoidal category **Set**, which is a singleton in that category) to $\mathrm{Hom}(\varphi, \psi)$, i.e. there exists an arrow $f : I \to \mathrm{Hom}(\varphi, \psi)$.

Analogously, in the context of the categories enriched over $[0, 1]$ the existence of a derivation between formulae φ and ψ can be expressed by the fact that there exists an arrow $f : 1 \to \mathrm{Hom}(\varphi, \psi)$, i.e. $1 \leq \mathrm{Hom}(\varphi, \psi)$. Since $\mathrm{Hom}(\varphi, \psi) \leq 1$, we have that

$$\text{there exists a derivation between } \varphi \text{ and } \psi \quad \text{iff} \quad \mathrm{Hom}(\varphi, \psi) = 1. \tag{10}$$

It follows from (10) that in the context of the categories enriched over $[0, 1]$ the notion of the isomorphism of two objects coincides with the notion of "mutual derivability" (there exists a derivation from φ to ψ and a derivation from ψ to φ).

Another consequence of (10) is that we entirely destroy the spectrum of the particular importance to "classical" categorial proof theory, i.e. the distinction between derivations from φ to ψ. However, this enriched approach opens the spectrum which was hidden in the "classical" categorial proof theory under the non-existence of a derivation between φ and ψ, and that is now covered by $\mathrm{Hom}(\varphi, \psi) \in [0, 1)$. In that sense, this enriched approach allows us to develop a theory of "non-proofs", and that is the point where this direction of research can be continued.

We note that the notion of "non-proofs" or "wrong proofs" appears in [5], where the author discusses their significance in light of the recent transformations in proof theory, focused not so much on proofs themselves as on their dynamic. In *loc. cit.* it is argued that reasoning in its general sense includes "wrong reasoning" (mistakes, inconsistencies, interruption, etc.) and that from the proofs-as-programs viewpoint, the non-proofs are also worth studying. We quote ([5], p. 59)

> But overall, because of the status of proofs having changed, that of non-proofs, as a result, also has changed. (...) This becomes clear once one tackles proofs in computational terms: then a proof is a typed program, whose typability ensures that certain computational properties are satisfied... Nevertheless, untypable programs – those processes that do not correspond to proofs and that Logic fails to civilize – share with proofs their life as dynamic entities, their computational nature.

We believe that some concrete need for fuzzy logic (e.g. in computer science) could be connected with some categories enriched over $[0, 1]$ and one could give some technical results which are equally interesting to both proof theory and computer science.

Acknowledgement

We would like to thank Zoran Petrić for suggesting the problem studied here, and for very helpful discussions and detailed comments during the course of research and writing this paper. Also, we would like to thank Dragan Doder for useful suggestions.

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