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On the fuzzy solutions of time-fractional problems

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Abstract

The main purpose of this paper is to obtain an analytical solution for the time-fractional fuzzy equation. To do this, the time-fractional equation is transformed into an algebraic equation using the fuzzy Laplace and Fourier transforms. The fractional derivatives are described in the Caputo gH-differentiability. In addition, to demonstrate the efficiency of the method some various examples are solved.

Keywords: Fuzzy time-fractional differential equations, fuzzy Laplace transform, fuzzy Fourier transform, Caputo gH-differentiability.

1 Introduction

During the last decade, the interest of mathematicians in the theory of fractional calculus has been steadily increasing. The main reason for this development is the application of fractional differential equations to modeling phenomena in different branches of science and engineering. But mathematical modeling of this phenomenon requires gathering information and data from various sources. This type of data is often uncertain due to inaccurate measurements. Fuzzy set theory is a well-known way for modeling systems with uncertainties and over the past decades, with the development of the fuzzy concept, the proposed models are more consistent with reality and can express with a more comprehensive view. Allahviranloo has displayed several types of uncertainties including fuzzy in [3].

The topic of differentiability of a fuzzy function based on Hukuhara difference [24] has been discussed in [15, 25, 28]. Due to some shortcomings of this method, the weakly generalized differentiability and strongly generalized differentiability for a fuzzy function are introduced by Bede and Gal [15]. But the fuzzy differential equations expressed by the strongly generalized derivative didn't have a unique solution. So, Stefanini and Bede defined generalized Hukuhara difference and derivative for interval valued functions differentiability and all the conditions for existence of this kind of difference are examined in [16, 32]. The results obtained in [14] show that the generalized Hukuhara difference always exists for triangular fuzzy numbers.

Along with the development in fuzzy differential equations, fuzzy fractional differential equations have also made great progress. Agarwal et al. [1] introduced the concept fuzzy fractional differential equations. The existence and uniqueness of fuzzy solution for fractional differential equations are proved in [13]. Some concepts of the fractional theory such as fractional Riemaann-Liouville and caputo H-differentiability for fuzzy functions are introduced by Allahviranloo et al. [4, 7, 30, 31]. The authors of articles [4, 30, 31] have described the Laplace transform for a fuzzy function. Most of these articles used the concept of Hukuhrara differentiability or the strongly generalized differentiability to define the fractional derivative of a fuzzy function. As stated, the fuzzy differential equations expressed by these concepts of differentiability do not have a unique solution. In order to solve this problem, Allahviranloo et al. [5] have proposed the generalized Hukuhara fractional Caputo derivative of a fuzzy function and they prove the existence and uniqueness of the solution for the fractional differential equations with a fuzzy initial value under the Krasnoselskii-Krein condition. The existence and uniqueness of the fuzzy solution for delay fractional differential equation studied in [22, 34]. In

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recent years, many methods have been utilized for finding an analytical or numerical fuzzy solution for fuzzy fractional differential equations and many others [10, 9, 12, 11].

For the first time, H. Viet Long et al. examined the fuzzy fractional partial differential equations [35]. They introduced the concepts of fuzzy fractional integral and Caputo partial differentiability based on generalized Hukurara differentiability for the fuzzy multivariable functions. The fuzzy Caputo-Katugampola fractional differential equations in fuzzy space are considered in [23] and under generalized Lipschitz condition, the existence and uniqueness of the solution are proved.

The main objective of this study is to obtain the explicit solution of the time-fractional fuzzy problems. For this purpose, we use fuzzy Laplace and Fourier transforms and the concept of fuzzy generalized Hukuhara differentiability.

A brief outline of the contents is now given as follows. Some concepts associated with fuzzy numbers and fuzzy generalized Hukuhara differentiability and etc are expressed and some new theorems and lemmas to be used in the main part of the paper are proved in Section2. In Section 3 we develop some new properties for the fuzzy Laplace and fuzzy Fourier transforms. In Section 4 we introduce a fuzzy time-fractional equation and obtain the fuzzy explicit solution of the mentioned equation and it is followed up by solving several examples in Section 5. Finally, conclusions are drawn in Section 6.

2 Mathematical preliminaries

Let $\mathbb{E}_{\mathcal{T}}$ be the set of all triangular fuzzy numbers in the apace of fuzzy sets ($\mathbb{E}$). The generalized Hukuhara difference of two fuzzy numbers $a, b \in \mathbb{E}$ is the fuzzy number c, (if it exists), such that

$$a \ominus_{gH} b = c \Longleftrightarrow (i).\ a = b \oplus c, \quad or \quad (ii).\ b = a \oplus (-1)c. \tag{1}$$

Now, we consider $a, b \in \mathbb{E}_{\mathcal{T}}$, then

$$a \ominus_{gH} b = c \Longleftrightarrow \begin{cases} (i).\ c = (a_1 - b_1, a_2 - b_2, a_3 - b_3); \\ or\ (ii).\ c = (a_3 - b_3, a_2 - b_2, a_1 - b_1). \end{cases}$$

provided that c is a triangular fuzzy number [2, 14]. The results obtained in [14] show that if $a, b \in \mathbb{E}_{\mathcal{T}}$, then $a \ominus_{gH} b$ always exists in $\mathbb{E}_{\mathcal{T}}$.

Let us consider $\mathbb{D} \subseteq \mathbb{R}^2$ and $\mathbb{J} = [a, b]$. Moreover, suppose that the space of all continuous fuzzy functions on domain $\mathbb{D}$ is denoted by $C^F(\mathbb{D})$ and also $L^F(\mathbb{D})$ denotes the space of all Lebesgue integrable fuzzy functions. Throughout the rest of this paper, we assume that $f(t) \in C^F(\mathbb{J}) \cap L^F(\mathbb{J})$ and $u(t, x) \in C^F(\mathbb{D}) \cap L^F(\mathbb{D})$ are two triangular fuzzy functions.

Definition 2.1. [14] *The fuzzy function $f(t)$ is called the generalized Hukuhara differentiable ($gH-$differentiable) at $t_0 \in \mathbb{J}$ if*

$$f'_{gH}(t_0) = \lim_{h \to 0} \frac{f(t_0 + h) \ominus_{gH} f(t_0)}{h},$$

belongs to $\mathbb{E}_{\mathcal{T}}$. In addition, we can say that $f(t)$ is

- $[(i) - gH]-$*differentiable function if and only if for all $t \in \mathbb{J}$*

$$f'_{i.gH}(t) = \Big(f'_1(t), f'_2(t), f'_3(t)\Big),$$

defines a triangular fuzzy number.

- $[(ii) - gH]-$*differentiable function if and only if for all $t \in \mathbb{J}$*

$$f'_{ii.gH}(t) = \Big(f'_3(t), f'_2(t), f'_1(t)\Big),$$

is a triangular fuzzy number.

Definition 2.2. *A fuzzy function f is piecewise continuous on the interval $[0, \infty)$ if*

1. $\lim_{t \to 0^+} f(t) = f(0^+)$.

2. *f is continuous on every finite interval $(0, b)$ except possibly at a finite number of points $\tau_1, \tau_2, ..., \tau_n$ in $(0, b)$ at which f has jump discontinuity.*

Definition 2.3. [14] *Let $f : \mathbb{J} \to \mathbb{E}_{\mathcal{T}}$ be a triangular fuzzy function, then*

$$\int_a^b f(t)dt = \Big(\int_a^b f_1(t)dt, \int_a^b f_2(t)dt, \int_a^b f_3(t)dt\Big).$$

Definition 2.4. [21] *Let $f : \mathbb{R} \to \mathbb{E}$. We say that $f(t)$ is fuzzy absolutely integrable if*

$$\int_{-\infty}^{\infty} D(f(t), 0) < \infty,$$

where D is the Hausdorff distance.

Lemma 2.5. [35] *If $f : \mathbb{J} \to \mathbb{E}_{\mathcal{T}}$ be a triangular fuzzy function with no switching point in interval $\mathbb{J}$, then we have*

1. *If $f(t)$ is $[(i) - gH]-$differentiable, then*

$$\int_a^b f'_{i.gH}(t)dt = f(b) \ominus f(a).$$

2. *If $f(t)$ is $[(ii) - gH]-$differentiable, then*

$$\int_a^b f'_{ii.gH}(t)dt = (-1)f(a) \ominus (-1)f(b).$$

According to the results obtained in [19], the gH-derivative of product of a differentiable real-valued function and a gH-differentiable interval function depends on the sign of the real-valued function and its first derivative. So in the following, we extend the results for a triangular fuzzy function.

Theorem 2.6. *Let $f : \mathbb{J} \to \mathbb{E}_{\mathcal{T}}$ be generalized Hukuhara differentiable on $\mathbb{J}$ and type of $gH-$differentiability doesn't change on $\mathbb{J}$ and $q(t)$ is a real-valued continuous function. Then we have Table 1 for the $gH-$derivative of $q(t) \odot f(t)$ and the type of $gH-$differentiability of $q(t) \odot f(t)$.*

	Type of diff of f	Sign $q(t)$	Sign $q'(t)$	$\Big(q(t) \odot f(t)\Big)'_{gH}$	Type of $\Big(q(t) \odot f(t)\Big)'_{gH}$
1	(i)	> 0	> 0	$q'(t) \odot f(t) \oplus q(t) \odot f'_{gH}(t)$	(i)
2	(i)	> 0	< 0	$q(t) \odot f'_{gH}(t) \ominus (-q'(t)) \odot f(t)$	(i)
3	(i)	< 0	> 0	$q(t) \odot f'_{gH}(t) \ominus (-q'(t)) \odot f(t)$	(ii)
4	(i)	< 0	< 0	$q'(t) \odot f(t) \oplus q(t) \odot f'_{gH}(t)$	(ii)
5	(ii)	> 0	> 0	$q(t) \odot f'_{gH}(t) \ominus (-q'(t)) \odot f(t)$	(ii)
6	(ii)	> 0	< 0	$q(t) \odot f'_{gH}(t) \oplus q'(t) \odot f(t)$	(ii)
7	(ii)	< 0	> 0	$q'(t) \odot f(t) \oplus q(t) \odot f'_{gH}(t)$	(i)
8	(ii)	< 0	< 0	$q(t) \odot f'_{gH}(t) \ominus (-q'(t)) \odot f(t)$	(i)

Table 1: The type of $gH-$differentiability of $q(t) \odot f(t)$

Provided the Hukuhara differences exists.

Proof. We only give a proof of Case 3. The rest of cases are proven in the same way and we do not go into details. According to the assumptions given in Table 1, $f(t)$ is a $[(i) - gH]-$differentiable function and $q(t) < 0$ and $q'(t) > 0$

$$\begin{aligned}
q(t) \odot f'_{gH}(t) \ominus (-q'(t)) \odot f(t) &= q(t) \odot \Big(f'_1(t), f'_2(t), f'_3(t)\Big) \ominus (-q'(t)) \odot \Big(f_1(t), f_2(t), f_3(t)\Big) \\
&= \Big(q(t)f'_3(t), q(t)f'_2(t), q(t)f'_1(t)\Big) \ominus (-1)\Big(q'(t)f_1(t), q'(t)f_2(t), q'(t)f_3(t)\Big) \\
&= \Big(q(t)f'_3(t), q(t)f'_2(t), q(t)f'_1(t)\Big) \ominus \Big(-q'(t)f_3(t), -q'(t)f_2(t), -q'(t)f_1(t)\Big) \\
&= \Big(q(t)f'_3(t) + q'(t)f_3(t), q(t)f'_2(t) + q'(t)f_2(t), q(t)f'_1(t) + q'(t)f_1(t)\Big) \\
&= \Big(q(t) \odot f(t)\Big)'_{ii.gH}.
\end{aligned}$$

□

Example 2.7. *Consider* $f(t) = \Big(3.3e^{-t}, 5.9e^{-t}, 7.2e^{-t}\Big)$ *and* $q(t) = -t^2$ *are defined for* $t > 2$. *We have*

$$q(t) \odot f(t) = \Big(-7.2t^2e^{-t}, -5.9t^2e^{-t}, -3.3t^2e^{-t}\Big),$$

and

$$\Big(q(t) \odot f(t)\Big)'_{gH} = \Big(3.3e^{-t}(t^2-2t), 5.9e^{-t}(t^2-2t), 7.2e^{-t}(t^2-2t)\Big).$$

As you can see $f(t)$ *is a* $[(ii)-gH]-$*differentiable function,* $q(t) \odot f(t)$ *is a* $[(i)-gH]-$*differentiable function, and* $\Big(q(t) \odot f(t)\Big)'_{gH} = q(t) \odot f'_{gH}(t) \ominus (-q'(t)) \odot f(t)$.
Now, let the $[(i)-gH]-$*differentiable fuzzy function* $f(t) = \Big(2.5t^2, 6t^2, 8.6t^2\Big)$.

$$q(t) \odot f(t) = \Big(-8.6t^4, -6t^4, -2.5t^4\Big),$$

and

$$\Big(q(t) \odot f(t)\Big)'_{gH} = \Big(-34.4t^3, -24t^3, -10t^3\Big).$$

So, Case 4 in Theorem 2.6 is established.

Theorem 2.8. *(Integration by part) Consider* $f : \mathbb{J} \to \mathbb{E}_{\mathcal{T}}$ *and* $f(t)$ *is* $gH-$*differentiable such that the type of differentiability does not change in* $\mathbb{J}$. *If* $q(t)$ *is a differentiable real-valued function, then*

1 . *If* $f(t)$ *is a* $[(i)-gH]-$*differentiable function and*

1-1 . *If* $q(t) > 0$ *and* $q'(t) > 0$, *then*

$$\int_a^b q(t) \odot f'_{gH}(t)dt = \Big(q(b) \odot f(b) \ominus q(a) \odot f(a)\Big) \ominus \int_a^b q'(t) \odot f(t)dt.$$

1-2 . *If* $q(t) > 0$ *and* $q'(t) < 0$, *then*

$$\int_a^b q(t) \odot f'_{gH}(t)dt = q(b) \odot f(b) \ominus q(a) \odot f(a) \oplus \int_a^b (-q'(t)) \odot f(t)dt.$$

1-3 . *If* $q(t) < 0$ *and* $q'(t) > 0$, *then*

$$\int_a^b q(t) \odot f'_{gH}(t)dt = (-q(a)) \odot f(a) \ominus (-q(b)) \odot f(b) \oplus \int_a^b (-q'(t)) \odot f(t)dt.$$

1-4 . *If* $q(t) < 0$ *and* $q'(t) < 0$, *then*

$$\int_a^b q(t) \odot f'_{gH}(t)dt = \Big((-q(a)) \odot f(a) \ominus (-q(b)) \odot f(b)\Big) \ominus \int_a^b q'(t) \odot f(t)dt.$$

2 . *If* $f(t)$ *is a* $[(ii)-gH]-$*differentiable function, then*

2-1 . *If* $q(t) > 0$ *and* $q'(t) > 0$, *then*

$$\int_a^b q(t) \odot f'_{gH}(t)dt = (-q(a)) \odot f(a) \ominus (-q(b)) \odot f(b) \oplus \int_a^b (-q'(t)) \odot f(t)dt.$$

2-2 . *If* $q(t) > 0$ *and* $q'(t) < 0$, *then*

$$\int_a^b q(t) \odot f'_{gH}(t)dt = \Big((-q(a)) \odot f(a) \ominus (-q(b)) \odot f(b)\Big) \ominus \int_a^b q'(t) \odot f(t)dt.$$

2-3 . *If $q(t) < 0$ and $q'(t) > 0$, then*

$$\int_a^b q(t) \odot f'_{gH}(t)dt \quad = \quad \Big(q(b) \odot f(b) \ominus q(a) \odot f(a)\Big) \ominus \int_a^b q'(t) \odot f(t)dt.$$

2-4 . *If $q(t) < 0$ and $q'(t) < 0$, then*

$$\int_a^b q(t) \odot f'_{gH}(t)dt \quad = \quad q(b) \odot f(b) \ominus q(a) \odot f(a) \oplus \int_a^b (-q'(t)) \odot f(t)dt.$$

Proof. This can be proved by Lemma 2.5 and the equations given in Table 1. For instant, we want to prove Case 1-4. Therefore by Case 4 in Table 1, we have

$$\Big(q(t) \odot f(t)\Big)'_{ii.gH} = q'(t) \odot f(t) \oplus q(t) \odot f'_{gH}(t).$$

We integrated the above equation, hence using Lemma 2.5

$$\int_a^b \Big(q(t) \odot f(t)\Big)'_{ii.gH} dt = \int_a^b q'(t) \odot f(t)dt \oplus \int_a^b q(t) \odot f'_{gH}(t)dt.$$

So

$$\Big((-q(a)) \odot f(a) \ominus (-q(b)) \odot f(b)\Big) \ominus \int_a^b q'(t) \odot f(t)dt = \int_a^b q(t) \odot f'_{gH}(t)dt. \tag{2}$$

Similarly, the rest of the cases can be proved. □

Definition 2.9. [21] *A triangular fuzzy function $u(t, x)$, without any switching point on $\mathbb{D}$ is called*

- *$[(i) - p]-$differentiable w.r.t. x at $(t, x) \in \mathbb{D}$ if*

$$\frac{\partial u(t,x)}{\partial x} = \Big(\frac{\partial u_1(t,x)}{\partial x}, \frac{\partial u_2(t,x)}{\partial x}, \frac{\partial u_3(t,x)}{\partial x}\Big).$$

- *$[(ii) - p]-$differentiable w.r.t. x at $(t, x) \in \mathbb{D}$ if*

$$\frac{\partial u(t,x)}{\partial x} = \Big(\frac{\partial u_3(t,x)}{\partial x}, \frac{\partial u_2(t,x)}{\partial x}, \frac{\partial u_1(t,x)}{\partial x}\Big).$$

Corollary 2.10. *Let $f(x)$ be a triangular gH-differentiable without any switching points and $u(t, x) = e^{-at}f(x)$, for $a > 0$. We say that*

1. *$u(t, x)$ is $[i - p]-$differentiable with respect to t if $\frac{\partial u(t,x)}{\partial t} = \ominus a e^{-at} f(x)$.*

1. *$u(t, x)$ is $[ii - p]-$differentiable with respect to t if $\frac{\partial u(t,x)}{\partial t} = (-1)a e^{-at} f(x)$.*

Definition 2.11. [21] *Let $u(t, x)$ be a fuzzy continuous function on $[a, b] \times [0, \infty)$. Suppose $\int_c^\infty u(t, x)dt$ converges for each $t \in [a, b]$. We say that the integral $F(x) = \int_c^\infty u(t, x)dt$ converges uniformly on $x \in [a, b]$ if for any $\epsilon > 0$ there is a number N depending only on ϵ such that*

$$D\Big(F(x), \int_c^d u(t, x)dt\Big) < \epsilon,$$

whenever $d \geq N$ for all $x \in [a, b]$, that is

$$\sup_{x\in[a,b]} D\Big(\int_c^d u(t, x)dt, 0\Big) \to 0 \quad as \quad d \to \infty.$$

Theorem 2.12. [21]*(Differentiation under the integral sign) Suppose both, $u(t, x)$ and $\frac{\partial u(t,x)}{\partial x}$, are fuzzy continuous in $[a, b] \times [c, \infty)$. Suppose also that the integral*

$$F(x) = \int_c^\infty u(t, x)dt, \tag{3}$$

converges for $x \in \mathbb{R}$, and the integral $\int_c^\infty \frac{\partial u(t,x)}{\partial x} dt$ converges uniformly on $[a, b]$. Then F is gH-differentiable on $[a, b]$ and

$$F'_{gH}(x) = \int_c^\infty \frac{\partial u(t,x)}{\partial x} dt.$$

3 Fuzzy integral transforms

The integral transform technique is interpreted as powerful tools for solving different types of mathematical equations. This technique is a method to remove partial derivative from considered equations. So the partial differential equation is transformed into the algebraic equation in a transform domain. This section briefly examines the fuzzy integral transforms which are used in this paper to reduce the differential operators to an algebraic form.

3.1 The fuzzy Laplace transform

Definition 3.1. [30] *A fuzzy function $f(t)$ has exponential order a on $0 \leq t < \infty$ if there exist positive constants K and a such that for some $t_0 \geq 0$*

$$D(f(t), 0) \leq Ke^{at}, \quad t \geq t_0.$$

Definition 3.2. [4, 30] *The fuzzy Laplace transform of fuzzy function $f(t)$ is defined as following*

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} e^{-s\tau} \odot f(\tau)d\tau = \lim_{\mathcal{R}\to\infty} \int_0^{\mathcal{R}} e^{-s\tau} \odot f(\tau)d\tau,$$

whenever the limit exists.
Similarly, $\mathcal{L}^{-1}$ is used to denote the inverse fuzzy Laplace transform of $F(s)$ and we have

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} \odot F(s)ds,$$

with fixed $\gamma \in \mathbb{R}$.

Remark 3.3. *Consider the fuzzy function $f(t) = \Big(f_1(t), f_2(t), f_3(t)\Big)$, then by attention to Definition 2.3, the fuzzy Laplace transform of this function is denoted by*

$$F(s) = \mathcal{L}[f(t)] \quad = \quad \Big(L[f_1(t)],\ L[f_2(t)], L[f_3(t)]\Big),$$

where L is the definition of classical Laplace transform, and

$$L[f_i(t)] \quad = \quad \int_0^{\infty} e^{-s\tau} f_i(\tau)d\tau = \lim_{\mathcal{R}\to\infty} \int_0^{\mathcal{R}} e^{-s\tau} f_i(\tau)d\tau, \quad i = 1, 2, 3.$$

Theorem 3.4. [30] *If fuzzy function $f(t)$ be bounded piecewise continuous on $[0, \infty)$, and of exponential order a, then the fuzzy Laplace transform of $f(t)$ exists for all s provided $Re(s) > a$.*

Theorem 3.5. [4, 30] *Let us consider*

1. *f is a fuzzy continuous function for $t \geq 0$ and of exponential order a.*

2. *$f'_{gH}(t)$ is a piecewise continuous in every finite closed interval $\mathbb{J}$.*

Moreover, assume that $f(t)$ is $gH-$differentiable in $\mathbb{J}$ provided that the type of $gH-$differentiability doesn't change in interval $\mathbb{J}$. Then for $Re(s) > \alpha$ we have

$$\mathcal{L}[f'_{i.gH}(t)] = sF(s) \ominus f(0), \qquad \mathcal{L}[f'_{ii.gH}(t)] = (-1)f(0) \ominus (-1)sF(s),$$

where $F(s) = \mathcal{L}[f(t)]$.

Example 3.6. *Consider the $[(i)-gH]-$differential fuzzy function $f(t) = \Big(2t+1, 5t+4, 6t+8\Big)$. Using Remark 3.3, we will take the fuzzy Laplace transform of $f(t)$. We obtain $F(s) = \Big(\frac{2}{s^2} + \frac{1}{s}, \frac{5}{s^2} + \frac{4}{s}, \frac{6}{s^2} + \frac{8}{s}\Big)$. Given that $f(0) = \Big(1, 4, 8\Big)$, we conclude that*

$$sF(s) \ominus f(0) = \Big(\frac{2}{s}, \frac{5}{s}, \frac{6}{s}\Big).$$

Which is equal to $\mathcal{L}[f'_{i.gH}(t)]$. So the relation stated in Theorem 3.5 is established.

Now, let $f(t) = \Big(3.2e^{-2t}, 5.7e^{-2t}, 7.9e^{-2t}\Big)$. We have $\mathcal{L}[f(t)] = \Big(\frac{3.2}{s+2}, \frac{5.7}{s+2}, \frac{7.9}{s+2}\Big)$ and $(-1)f(0) \ominus (-1)sF(s) = \Big(\frac{-15.8}{s+2}, \frac{-11.4}{s+2}, \frac{-6.4}{s+2}\Big)$ which is equal $\mathcal{L}[f'_{ii.gH}(t)]$.

Definition 3.7. *The convolution of two functions $q(t)$ and $f(t)$ for $t > 0$ is given by*

$$(f * q)(t) = \int_0^t f(\tau) \odot q(t-\tau)d\tau,$$

where $q(t)$ is a real-valued positive piecewise continuous function and $f(t)$ is a triangular fuzzy piecewise continuous function. Substituting $u = t - \tau$ gives

$$(f * q)(t) = \int_0^t q(u) \odot f(t-u)du = (q * f)(t),$$

so, the convolution is commutative.

Theorem 3.8. *(Convolution Theorem) If $f(t)$ is a triangular fuzzy piecewise continuous function on $[0, \infty)$ and of exponential order a.*

$$\mathcal{L}[(f * q)(t)] = \mathcal{L}[f(t)] \odot \mathcal{L}[q(t)], \qquad (Re(s) > a),$$

where $q(t)$ is a positive real-valued piecewise continuous function on $[0, \infty)$.

Proof. We start with product

$$\begin{aligned}\mathcal{L}[f(t)] \odot \mathcal{L}[q(t)] &= \Big(\int_0^\infty e^{-s\tau} \odot f(\tau)d\tau\Big) \odot \Big(\int_0^\infty e^{-s\sigma} q(\sigma)d\sigma\Big)\\ &= \int_0^\infty \Big(\int_0^\infty e^{-s(\tau+\sigma)} \odot f(\tau)d\tau\Big) \odot q(\sigma)d\sigma.\end{aligned}$$

Let us hold τ fixed in the interior integral, substituting $t = \tau + \sigma$ and $d\sigma = dt$, we obtain

$$\begin{aligned}&\mathcal{L}[f(t)] \odot \mathcal{L}[q(t)] = \int_0^\infty \Big(\int_\tau^\infty e^{-st} \odot f(\tau) \odot q(t-\tau)dt\Big)d\tau\\ &= \Big(\int_0^\infty \int_\tau^\infty e^{-st} f_1(\tau)q(t-\tau)dtd\tau, \int_0^\infty \int_\tau^\infty e^{-st} f_2(\tau)q(t-\tau)dtd\tau, \int_0^\infty \int_\tau^\infty e^{-st} f_3(\tau)q(t-\tau)dtd\tau\Big).\end{aligned}$$

Due to the assumptions on f and q, the improper integrals of f and q converge absolutely, so by Theorem 3.2 in [21] we can reverse the order of integration

$$\begin{aligned}&\mathcal{L}[f(t)] \odot \mathcal{L}[q(t)] =\\ &= \Big(\int_0^\infty \int_0^t e^{-st} f_1(\tau)q(t-\tau)d\tau dt, \int_0^\infty \int_0^t e^{-st} f_2(\tau)q(t-\tau)d\tau dt, \int_0^\infty \int_0^t e^{-st} f_3(\tau)q(t-\tau)d\tau dt\Big)\\ &= \Big(\int_0^\infty e^{-st}\Big(\int_0^t f_1(\tau)q(t-\tau)d\tau\Big)dt, \int_0^\infty e^{-st}\Big(\int_0^t f_2(\tau)q(t-\tau)d\tau\Big)dt, \int_0^\infty e^{-st}\Big(\int_0^t f_3(\tau)q(t-\tau)d\tau\Big)dt\Big).\end{aligned}$$

So

$$\mathcal{L}[f(t)] \odot \mathcal{L}[q(t)] = \int_0^\infty e^{-st} \odot \Big(\int_0^t f(\tau) \odot q(t-\tau)d\tau\Big)dt = \mathcal{L}[(f * q)(t)].$$

□

Example 3.9. *Let $f(t) = \Big(2t+1, 5t+4, 6t+8\Big)$ and $q(t) = t^2$. Using Theorem 3.8*

$$\mathcal{L}[(f * q)(t)] = \mathcal{L}[f(t)] \odot \mathcal{L}[q(t)].$$

Recall $\mathcal{L}[q(t)] = \frac{1}{s^3}$ and $\mathcal{L}[f(t)] = \Big(\frac{2}{s^2} + \frac{1}{s}, \frac{5}{s^2} + \frac{4}{s}, \frac{6}{s^2} + \frac{8}{s}\Big)$. We conclude that

$$\mathcal{L}[(f * q)(t)] = \Big(\frac{2}{s^5} + \frac{1}{s^4}, \frac{5}{s^5} + \frac{4}{s^4}, \frac{6}{s^5} + \frac{8}{s^4}\Big).$$

Next, we investigate the fuzzy Laplace transform for the generalized Hukuhara Caputo derivative of a fuzzy function. The generalized Hukuhara fractional Caputo derivative of a fuzzy function $f(t)$ of order α is defined as follows [5]

$$ {}^{C}_{gH}\mathcal{D}^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'_{gH}(\tau)}{(t-\tau)^{\alpha}} d\tau, $$

where $0 < \alpha \leq 1$. Let $f(t)$ is ${}^{CF}[gH]-$differentiable, then $f(t)$ is [34].

- ${}^{CF}[(i) - gH]$-differentiable if for all $t \in [a, b]$

$$ {}^{C}_{i.gH}\mathcal{D}^{\alpha}_{a^+} f(t) = \Big({}^{C}D^{\alpha}_{a^+} f_1(t),\ {}^{C}D^{\alpha}_{a^+} f_2(t),\ {}^{C}D^{\alpha}_{a^+} f_3(t) \Big). $$

- ${}^{CF}[(ii) - gH]$-differentiable if for all $t \in [a, b]$

$$ {}^{C}_{ii.gH}\mathcal{D}^{\alpha}_{a^+} f(t) = \Big({}^{C}D^{\alpha}_{a^+} f_3(t),\ {}^{C}D^{\alpha}_{a^+} f_2(t),\ {}^{C}D^{\alpha}_{a^+} f_1(t) \Big). $$

Theorem 3.10. *Let $f(t)$ is a fuzzy continuous function on $[0, \infty)$ and α be a real number, $0 < \alpha \leq 1$. Then*

1. *If $f(t)$ is ${}^{CF}[(i) - gH]-$differentiable on $[0, \infty)$, then*

$$ \mathcal{L}[\ {}^{C}_{i.gH}\mathcal{D}^{\alpha}_{0} f(t)] = s^{\alpha}\mathcal{L}[f(t)] \ominus s^{\alpha-1} f(0). $$

2. *If $f(t)$ is ${}^{CF}[(ii) - gH]-$differentiable on $[0, \infty)$, then*

$$ \mathcal{L}[\ {}^{C}_{ii.gH}\mathcal{D}^{\alpha}_{0} f(t)] = (-1)s^{\alpha-1} f(0) \ominus (-1)s^{\alpha}\mathcal{L}[f(t)]. $$

Proof. We have

$$ {}^{C}_{i.gH}\mathcal{D}^{\alpha}_{0} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \odot f'_{i.gH}(\tau) d\tau = \frac{1}{\Gamma(1-\alpha)} \Big(t^{-\alpha} * f'_{i.gH}(t) \Big). $$

Under the assumptions expressed in Theorem 3.5 and Theorem 3.8

$$ \mathcal{L}[{}^{C}_{i.gH}\mathcal{D}^{\alpha}_{0} f(t)] = \frac{1}{\Gamma(1-\alpha)} \mathcal{L}[t^{-\alpha}] \odot \mathcal{L}[f'_{i.gH}(t)] = s^{\alpha}\mathcal{L}[f(t)] \ominus s^{\alpha-1} f(0). $$

And a similar approach can be applied to ${}^{CF}[(ii) - gH]-$differentiable function. □

3.2 The fuzzy Fourier transform

The fuzzy Fourier transform based on the generalized Hukuhara derivative is investigated in [21]. Here, we introduce a different definition for the fuzzy Fourier transform, so it is necessary to prove some properties for the fuzzy Fourier transform based on this new definition.

Definition 3.11. *Consider $f : \mathbb{R} \to \mathbb{E}_{\mathcal{T}}$ be a fuzzy function. The fuzzy Fourier transform of $f(t)$ is given by the integral*

$$ \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t) \odot e^{ikt} dt = F(k), $$

then the fuzzy inverse Fourier transform of $F(k)$ is

$$ \mathcal{F}^{-1}[F(k)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) \odot e^{-ikt} dk = f(t). $$

According to the Theorem 4.2 in [21], we can say that the fuzzy inverse Fourier transform exists if $F(k)$ is fuzzy absolutely integrable.

Remark 3.12. *Let $f(t;r) = [f^-(t;r) \quad , \quad f^+(t;r)]$ is the r-cut form of fuzzy function $f(t)$. It follows by using the definition of classical Fourier transform*

$$\mathcal{F}[f(t;r)] = \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f^-(t;r)e^{-iwx}dx \quad , \quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f^+(t;r)e^{-iwx}dx\right].$$

Theorem 3.13. *Let $f(t)$ and $f'_{gH}(t)$ are fuzzy continuous and absolutely integrable function on $(-\infty,\infty)$. Then*

1. *If $f(t)$ is $[(i)-gH]-$differentiable, then $\mathcal{F}[f'_{i.gH}(t)] = \ominus ik\mathcal{F}[f(t)]$.*

2. *If $f(t)$ is $[(ii)-gH]-$differentiable, then $\mathcal{F}[f'_{ii.gH}(t)] = (-1)ik\mathcal{F}[f(t)]$.*

Proof. Let us to consider $f(t)$ is a $[(i)-gH]-$differentiable triangular fuzzy function, then from the Definition 3.11,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\odot e^{-ikt}dk.$$

Using Definition 2.11, we can show that $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\odot \frac{\partial}{\partial t}\left(e^{-ikt}\right)dk$ is converges uniformly. Hence, Theorem 2.12 yields,

$$f'_{i.gH}(t) = \frac{\partial}{\partial t}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\odot e^{-ikt}dk\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\odot \frac{\partial}{\partial t}\left(e^{-ikt}\right)dk.$$

$f(t)$ is a $[i-gH]-$differentiable function then using Corollary 2.10,

$$f'_{i.gH}(t) = \frac{\ominus ik}{2\pi}\int_{-\infty}^{\infty} F(k)\odot e^{-ikt}dk = \ominus ikf(t).$$

By using the fuzzy Fourier transform, we can obtain the desired result. And a similar approach can be applied to $[(ii)-gH]-$differentiable function. □

4 Fuzzy explicit solution of the time-fractional problem

This section examines the explicit solution of the following fuzzy time-fractional problem

$$ {}_{gH}^{C}\mathcal{D}_t^{\alpha}u(t,x) = \lambda \odot \frac{\partial u(t,x)}{\partial x}, \qquad t>0, \quad -\infty < x < \infty, \tag{4}$$

with $0 < \alpha \leq 1$ and λ is a real positive constant. Here ${}_{gH}^{C}\mathcal{D}_t^{\alpha}u(t,x)$ is the Caputo generalized Hukuhara partial derivative (${}^{CF}[gH-p]$-differentiability for short) of order α with respect to t defined by

$$ {}_{gH}^{C}\mathcal{D}_t^{\alpha}u(t,x) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(\tau,x)}{\partial\tau}\frac{1}{(t-\tau)^{\alpha}}d\tau, \tag{5}$$

for $0 < \alpha \leq 1$ [35]. In equation (5) $\frac{\partial u(\tau,x)}{\partial\tau}$ is the generalized Hukuhara partial derivative ($[gH-p]$-differentiability for short) of $u(\tau,x)$ with respect to τ [6], and in equation (4), $\frac{\partial u(t,x)}{\partial x}$ is the generalized Hukuhara partial derivative of $u(t,x)$ with respect to x defined by

$$\frac{\partial u(t,x)}{\partial x} = \lim_{h\to 0}\frac{u(t,x+h)\ominus_{gH} u(t,x)}{h}.$$

Now, we investigate the fuzzy explicit solution of the fuzzy linear partial fractional differential equation (4), with the following fuzzy boundary condition

$$u(0+,x) = g(x). \tag{6}$$

where $g(x)$ is a triangular fuzzy continuous function.

Suppose that $u(t,x)$ is the fuzzy explicit solution of problem (4) provided that the types of ${}^{CF}[gH-p]$-differentiability with respect to t and $[gH-p]-$differentiability with respect to x are the same.

The explicit fuzzy solution of problem (4) with fuzzy boundary conditions (6) is derived by the fuzzy Laplace transform of a fuzzy function $u(t,x)$ with respect to t

$$\mathcal{L}_t[u(t,x)] = \int_0^\infty e^{-st} \odot u(t,x)dt = \mathrm{U}_t(s,x), \tag{7}$$

for any fixed $x \in \mathbb{R}$, and the Fuzzy Fourier transform with respect to x

$$\mathcal{F}_x[u(t,x)] = \int_{-\infty}^\infty e^{ikx} \odot u(t,x)dx = \mathcal{U}_x(t,k), \tag{8}$$

for any fixed $t > 0$. Using the definitions given in Subsections 3.1 and 3.2, the fuzzy inverse Laplace transform and the fuzzy inverse Fourier transform can be expressed as follows

$$\mathcal{L}_s^{-1}[\mathrm{U}_t(s,x)] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} \odot \mathrm{U}_t(s,x)ds, \qquad \mathcal{F}_k^{-1}[\mathcal{U}_x(t,k)] = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-ikx} \odot \mathcal{U}_x(t,k)dk.$$

Such that the following equations exist for them

$$\mathcal{L}_s^{-1}\mathcal{L}_t[u(t,x)] = u(t,x), \qquad \mathcal{F}_k^{-1}\mathcal{F}_x[u(t,x)] = u(t,x).$$

Let $\mathcal{LF}$ denotes the space of all fuzzy functions $u(t,x)$ such that the fuzzy Laplace transform and the fuzzy Fourier transform exist for them and we introduce the following notation

$$\tilde{u}(s,k) := \mathcal{F}_x\mathcal{L}_t[u(t,x)] = \int_{-\infty}^\infty \int_0^\infty e^{-st}e^{ikx} \odot u(t,x)dtdx, \qquad t > 0. \tag{9}$$

Lemma 4.1. *Let $g(x)$ be a fuzzy function such that there exists the fuzzy Fourier transform $G(k)$. Then the solution $u(t,x) \in \mathcal{LF}$ of the problem (4) with fuzzy boundary values (6) is given by the*

$$u(t,x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}s^{\alpha-1} \odot \Big(\frac{1}{2\pi}\int_{-\infty}^\infty \frac{G(K)}{s^\alpha + i\lambda k}e^{-ikx}dk\Big)ds, \tag{10}$$

provided that the integral on the right-hand side of equation (10) exists.

Proof. Let $u(t,x)$ is ${}^{CF}[i-gH]-$ differentiable w.r.t t and $[(i)-p]-$differentiable w.r.t x.

$${}^{C}_{i.gH}\mathcal{D}_t^\alpha u(t,x) = \lambda \odot \frac{\partial u(t,x)}{\partial x}. \tag{11}$$

The fuzzy Laplace transform with respect to t is applied to equation (11). According to Theorem 3.10 and boundary condition (6), we obtain

$$s^\alpha \mathcal{L}_t[u(t,x)] \ominus s^{\alpha-1}g(x) = \lambda \odot \frac{\partial}{\partial x}\Big(\mathcal{L}_t[u(t,x)]\Big). \tag{12}$$

Applying the fuzzy Fourier transform (8) and using Theorem 3.13 we obtain

$$s^\alpha\tilde{u}(s,k) \ominus s^{\alpha-1}G(k) = \ominus_{gH} i\lambda \odot k\tilde{u}(s,k), \tag{13}$$

so

$$\tilde{u}(s,k) = \frac{s^{\alpha-1} \odot G(k)}{s^\alpha + i\lambda k}. \tag{14}$$

The fuzzy inverse Fourier and fuzzy inverse Laplace transform are applied to equation (14), then the solution (10) will be obtained.

Let $u(t,x)$ is ${}^{CF}[ii-gH]-$ differentiable w.r.t t and $[ii-p]-$differentiable w.r.t x.

$${}^{C}_{ii.gH}\mathcal{D}_t^\alpha u(t,x) = \lambda \odot \frac{\partial u(t,x)}{\partial x}, \tag{15}$$

The fuzzy solution for this type of differentiability can be obtained by using a similar procedure. Applying the fuzzy Laplace transform with respect to t to equation (15), then by using Theorem 3.10

$$(-1)s^{\alpha-1}g(x)\ominus_{gH}(-1)s^{\alpha}\mathcal{L}_t[u(t,x)]=\lambda\odot\frac{\partial}{\partial x}\Big(\mathcal{L}_t[u(t,x)]\Big).$$

So the fuzzy Fourier transform is applied to above equation and using Theorem 3.13 we obtain

$$(-1)s^{\alpha-1}G(k)\ominus_{gH}(-1)s^{\alpha}\odot\tilde{u}(s,k)=(-1)i\lambda k\tilde{u}(s,k).$$

Now multiply both sides of the above equation by (-1), therefore

$$\tilde{u}(s,k)=\frac{s^{\alpha-1}\odot G(k)}{i\lambda k+s^{\alpha}}. \tag{16}$$

The fuzzy inverse Fourier and fuzzy inverse Laplace transform are applied to equation (16), then the solution (10) will be obtained. □

In the following Lemmas, we intend to simplify the fuzzy solution given in Lemma 4.1. For this purpose, we will use the Mittag-Leffler function [20]

$$E_{\alpha}(z)=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\alpha j+1)}. \tag{17}$$

Lemma 4.2. *Let $g(x)$ be a fuzzy function such that its fuzzy Fourier transform $G(k)$ exists and $E_{\alpha}(\mu t^{\alpha})$ is the Mittag-Leffler function. Then the solution $u(t,x)\in\mathcal{LF}$ of the problem (4) with the fuzzy boundary values (6) is*

$$u(t,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}E_{\alpha}(-i\lambda kt^{\alpha})e^{-ikx}\odot G(k)dk, \tag{18}$$

provided that the integral on the right-hand side of this equation exists.

Proof. Let $u(t,x)$ is ${}^{CF}[i-gH]-$differentiable w.r.t t and $[(i)-p]-$differentiable w.r.t x. By applying the fuzzy inverse Laplace transform to equation (14) we have

$$\mathcal{F}_x[u(t,x)] \quad = \quad \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{st}s^{\alpha-1}\odot G(k)}{s^{\alpha}+i\lambda k}ds.$$

On the other hand, according to the Laplace transform of the Mittag-Leffler function $E_{\alpha}(\mu t^{\alpha})$ [20]

$$\mathcal{L}[E_{\alpha}(\mu t^{\alpha})]=\frac{s^{\alpha-1}}{s^{\alpha}-\mu},\qquad (|\mu s^{-\alpha}|<1).$$

Assuming $\mu=-i\lambda k$, we can conclude that

$$\mathcal{F}_x[u(t,x)] \quad = \quad E_{\alpha}(-i\lambda kt^{\alpha})\odot G(k).$$

Applying the inverse Fourier transform to above equation and so we have equation (18). In addition, note that this solution holds in the initial condition $u(0+,x)=g(x)$ because

$$u(0,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}E_{\alpha}(0)e^{-ikx}\odot G(k)dk,$$

and $E_{\alpha}(0)=1$.

By applying the same method to equation (16), the same solution will be obtained when $u(t,x)$ is ${}^{CF}[ii-gH]-$differentiable w.r.t t and $[ii-p]-$differentiable w.r.t x.

□

Lemma 4.3. *Let $g(x)$ be a fuzzy function such that*

$$\lim_{|x|\to\infty} g^{(j)}(x) = 0, \qquad (j = 0, 1, 2, ...).$$

Then the solution of the problem (4) *and* (6) *is given by*

$$u(t,x) = \sum_{j=0}^{\infty} \frac{(\lambda t^{\alpha})^j}{\Gamma(\alpha j + 1)} \odot g_{gH}^{(j)}(x), \tag{19}$$

provided that the series in (19) *converges for any $x \in \mathbb{R}$ and any $t > 0$.*

Proof. According to Lemma 4.2, the following fuzzy solution was obtained for problem (4)

$$u(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{\alpha}(-i\lambda k t^{\alpha}) G(k) \odot e^{-ikx} dk, \tag{20}$$

Substitute equation (17) into equation (20), so

$$\begin{aligned} u(t,x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Big[\sum_{j=0}^{\infty} \frac{(-i\lambda k t^{\alpha})^j}{\Gamma(\alpha j + 1)} \Big] G(k) \odot e^{-ikx} dk \\ &= \sum_{j=0}^{\infty} \frac{(\lambda t^{\alpha})^j}{\Gamma(\alpha j + 1)} \Big(\frac{1}{2\pi} \int_{-\infty}^{+\infty} (-ik)^j G(k) \odot e^{-ikx} dk \Big). \end{aligned} \tag{21}$$

Because of the power series representing the Mittag-Leffler function is uniform convergence for all $t \in \mathbb{C}$ [20], we can interchange the order of integration and series. Now, using the Theorem 3.3 in [21], we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} (-ik)^j G(k) \odot e^{-ikx} dk = \frac{\partial^j}{\partial x^j} \Big(\mathcal{F}_k^{-1}[G(x)] \Big) = g_{gH}^{(j)}(x). \tag{22}$$

Then by comparing equation (21) and equation (22), the solution given in equation (19) will be obtained. □

5 Examples

To illustrate the efficiency and accuracy of the method for solving the fuzzy time-fractional problems, some different examples will be solved in this section. All calculations were performed on a PC running Mathematica software.

Example 5.1. *Consider the following fuzzy fractional partial differential equation*

$$\begin{cases} {}_{gH}^{C}\mathcal{D}_t^{\frac{1}{2}} u(t,x) = \frac{\partial u(t,x)}{\partial x}, \qquad (t > 0, \ x \in \mathbb{R}), \qquad 0 < \alpha \leq 1, \\ u(0+, x) = \Big(2x^2, 4.2x^2, 6.6x^2\Big). \end{cases} \tag{23}$$

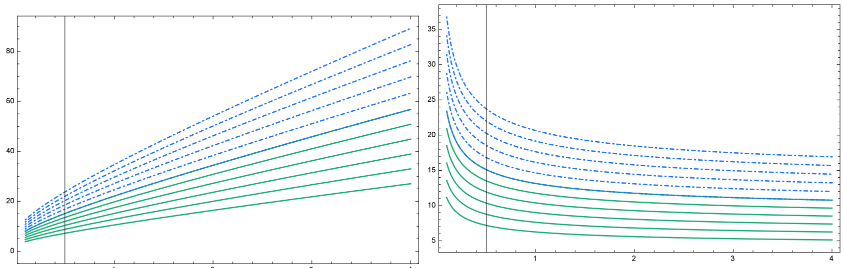
By attention to Lemma 4.2, equation (18) *and Definition 2.3, problem* (23) *has the following fuzzy solution*

$$u(t,x) = \tfrac{1}{2\pi} \Big(\int_{-\infty}^{\infty} E_{\frac{1}{2}}(-ikt^{\frac{1}{2}}) \mathcal{F}[2x^2] \odot e^{-ikx} dk, \int_{-\infty}^{\infty} E_{\frac{1}{2}}(-ikt^{\frac{1}{2}}) \mathcal{F}[4.2x^2] \odot e^{-ikx} dk, \int_{-\infty}^{\infty} E_{\frac{1}{2}}(-ikt^{\frac{1}{2}}) \mathcal{F}[6.6x^2] \odot e^{-ikx} dk \Big),$$

therefore

$$u(t,x) = \Big(4t + \frac{(8x\sqrt{t})}{\sqrt{\pi}} + 2x^2, 8.4t + 9.47839x\sqrt{t} + 4.2x^2, 13.2t + 14.8946x\sqrt{t} + 6.6x^2\Big).$$

To illustrate the behavior of the gH-differentiability of solution, $u(t,x;r) = \Big[4t + \frac{(8x\sqrt{t})}{\sqrt{\pi}} + 2x^2 + r(4.4t + 4.96487\sqrt{t}x + 2.2x^2), \ 13.2t + 14.8946x\sqrt{t} + 6.6x^2 - r(4.8t + 5.41621\sqrt{t}x + 2.4x^2\Big]$. *and* ${}_{gH}^{C}\mathcal{D}_t^{\frac{1}{2}} u(t,x;r) = \Big[4 + \frac{2.25676x}{\sqrt{t}} + r\Big(4.4 + \frac{2.48244x}{\sqrt{t}}\Big), \ 13.2 + \frac{7.4473x}{\sqrt{t}} + r\Big(-4.8 - \frac{2.70811x}{\sqrt{t}}\Big)\Big]$ *are presented in Figures 1 (a) and (b) when $x = 1$. In Figure 2(b), the position of lower cut(Green) and upper cut(Blue) for* $C_{gH}\mathcal{D}_t^{\frac{1}{2}} u(t,x;r)$ *is changed. It shows that $u(t,x)$ is* ${}^{CF}[ii - gH]-$*differentiable w.r.t t.*



(a). $u(t,x)$. (b). ${}^{C}_{gH}\mathcal{D}_t^{\frac{1}{2}}u(t,x)$.

Figure 1: Plots of $u(t,x)$, ${}^{C}_{gH}\mathcal{D}_t^{\frac{1}{2}}u(t,x)$ in $x=1$ for $\alpha\in[0,1]$ of Example 5.1.

Example 5.2. *Consider the following fuzzy fractional partial differential equation*

$$\begin{cases} {}^{C}_{gH}\mathcal{D}_t^{\frac{1}{2}}u(t,x) = \frac{\partial u(t,x)}{\partial x}, \qquad (t>0,\ x\in\mathbb{R}), \\ u(0+,x) = \Big(x^3, \sqrt{3}x^3, 2\sqrt{2}x^3\Big). \end{cases} \tag{24}$$

By attention to Lemma 4.2, equation (18) *and Definition 2.3, problem* (23) *has the following fuzzy solution*

$$u(t,x) = \Bigg(6tx + x^3 + \frac{(2\sqrt{t}(4t+3x^2)}{\sqrt{\pi}}, \sqrt{3}\Big(6tx + x^3 + \frac{(2\sqrt{t}(4t+3x^2)}{\sqrt{\pi}}\Big), 2\sqrt{2}\Big(6tx + x^3 + \frac{(2\sqrt{t}(4t+3x^2)}{\sqrt{\pi}}\Big)\Bigg).$$

To determine the type of gH-differentiability of the above fuzzy solution, we take the Caputo generalized Hukuhara derivative of order α *with respect to* t *and generalized Hukuhara partial derivative with respect to* x

$$\begin{aligned} \frac{\partial u(t,x)}{\partial x} &= {}^{C}_{gH}\mathcal{D}_t^{\frac{1}{2}}u(t,x) \\ &= \Bigg(3(2t + \frac{(4\sqrt{t}x)}{\sqrt{\pi}} + x^2), 3\sqrt{\frac{3}{\pi}}(4\sqrt{t}x + \sqrt{\pi}(2t+x^2), 6\sqrt{\frac{2}{\pi}}(4\sqrt{t}x + \sqrt{\pi}(2t+x^2)\Bigg). \end{aligned}$$

This solution is ${}^{CF}[i-gH]-$ *differentiable w.r.t* t *and* $[i-p]-$*differentiable w.r.t* x.

Example 5.3. *Consider the following fuzzy fractional partial differential equation*

$$\begin{aligned} {}^{C}_{gH}\mathcal{D}_t^{\alpha}u(t,x) &= \lambda\frac{\partial u(t,x)}{\partial x}, \quad t>0,\ x\in\mathbb{R}, \\ u(0+,x) &= \Big(e^{-\mu x}, 2.2e^{-\mu x}, 5.3e^{-\mu x}\Big), \quad (\mu>0). \end{aligned}$$

All the conditions in Lemma 4.3 hold for this problem. So according to the results in Lemma 4.3 the solution of the given problem is

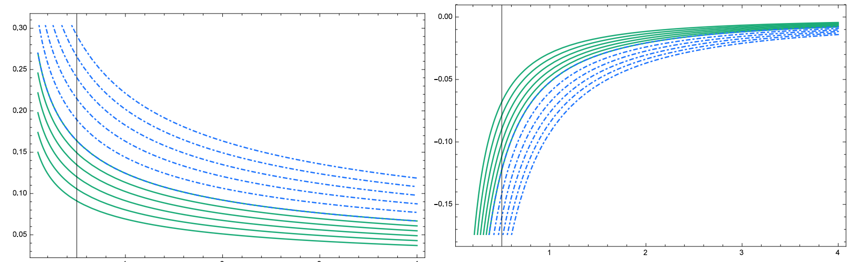
$$u(t,x) = \Big(e^{-\mu x}E_\alpha(-\mu\lambda t^\alpha), 2.2e^{-\mu x}E_\alpha(-\mu\lambda t^\alpha), 5.3e^{-\mu x}E_\alpha(-\mu\lambda t^\alpha)\Big).$$

In the particular case, consider the following fuzzy fractional differential of problem

$$\begin{aligned} {}^{C}_{gH}\mathcal{D}_t^{\frac{1}{2}}u(t,x) &= \frac{\partial u(t,x)}{\partial x}, \quad t>0, x\in\mathbb{R}. \\ u(0+,x) &= \Big(2e^{-2x}, 3.6e^{-2x}, 6.4e^{-2x}\Big), \end{aligned}$$

where $\alpha=\frac{1}{2}$, $\mu=2$ *and* $\lambda=1$. *Then we conclude that*

$$u(t,x) = \Big(2e^{4t-2x}Erfc[2\sqrt{t}], 3.6e^{4t-2x}Erfc[2\sqrt{t}], 6.4e^{4t-2x}Erfc[2\sqrt{t}]\Big). \tag{25}$$



(a). $u(t,x)$. (b). ${}^{C}_{gH}\mathcal{D}_t^{\frac{1}{2}}u(t,x)$.

Figure 2: Plots of $u(t,x)$, ${}^{C}_{gH}\mathcal{D}_t^{\frac{1}{2}}u(t,x)$ in $x=1$ for $\alpha \in [0,1]$ of Example 5.3.

Where $Erfc[z] = \frac{2}{\sqrt{\pi}}\int_z^{\infty} e^{-t^2}dt$.

Now, we want to determine the type of gH-differentiability of fuzzy solution (25)*. For this solution we have*

$$\frac{\partial u(t,x)}{\partial x} ={}^{C}_{gH}\mathcal{D}_t^{\frac{1}{2}}u(t,x) = -\Big(4e^{(4t-2x)}Erfc[2\sqrt{t}], 7.2e^{(4t-2x)}Erfc[2\sqrt{t}], 12.8e^{(4t-2x)}Erfc[2\sqrt{t}]\Big).$$

As you can see, the solution (25) *is* ${}^{CF}[ii-gH]-$ *differentiable w.r.t* t *and* $[ii-p]-$*differentiable w.r.t* x.

We plot the solution $u(t,x;r) = \Big[e^{4t-2x}(2+1.6r)Erfc[2\sqrt{t}],\ e^{4t-2x}(6.4-2.8r)Erfc[2\sqrt{t}]\Big]$ *and it's fuzzy Caputo gH-derivative* ${}^{C}_{gH}\mathcal{D}_t^{\frac{1}{2}}u(t,x;r)$ *in Figures 2 (a) and (b) when* $x=1$*. In Figure 1(b), the position of lower cut(Green) and upper cut(Blue) for* $C_{gH}\mathcal{D}_t^{\frac{1}{2}}u(t,x;r)$ *doesn't changed. It shows that* $u(t,x)$ *is* ${}^{CF}[i-gH]-$ *differentiable w.r.t* t.

6 Conclusion

In this paper, the triangular fuzzy explicit solution of the time-fractional differential equation based on the type of ${}^{CF}[gH-p]-$ differentiability w.r.t t and $[gH-p]-$ differentiability w.r.t x, were examined. We studied the triangular fuzzy fractional partial differential equation based on generalized Hukuhara differentiability and this kind of differentiability always exists for triangular fuzzy functions. To obtain an analytical solution for the time-fractional differential equation, the fuzzy Laplace transform and the fuzzy Fourier transform were introduced for the triangular fuzzy functions and some new properties have been proved for these fuzzy transforms. This technique is very powerful and efficient for obtaining triangular fuzzy analytical solution of the fuzzy differential equations without embedding them to crisp equations. For this reason, it will be necessary to investigate this method for solving different kinds of triangular fuzzy differential equations and we hope that this work be a step in this direction.

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References

[1] R. P. Agarwal, S. Arshad, D. ÓRegan, V. Lupulescu, *Fuzzy fractional integral equations under compactness type condition*, Fractional Calculus and Applied Analysis, **15** (2012), 572-590.

[2] R. Alikhani, F. Bahrami, *Fuzzy partial differential equations under the cross product of fuzzy numbers*, Information Sciences, **494** (2019), 80-99.

[3] T. Allahviranloo, *Uncertain information and linear systems, in: Studies in systems*, Decision and Control, Springer International Publishing, **254** (2020), 109-119.

[4] T. Allahviranloo, M. B. Ahmadi, *Fuzzy Laplace transforms*, Soft Computing, **14** (2010), 235-243.

[5] T. Allahviranloo, A. Armand, Z. Gouyandeh, *Fuzzy fractional differential equations under generalized fuzzy Caputo derivative*, Journal of Intelligent and Fuzzy Systems, **26** (2014), 1481-1490.

[6] T. Allahviranloo, Z. Gouyandeh, A. Armand, A. Hasanoglu, *On fuzzy solutions for heat equation based on generalized hukuhara differentiability*, Fuzzy Sets and Systems, **265** (2015), 1-23.

[7] T. Allahviranloo, S. Salahshour, S. Abbasbandy, *Explicit solutions of fractional differential equations with uncertainty*, Soft Computing, **16** (2012), 297-302.

[8] G. A. Anastassiou, *Fuzzy mathematics: Approximation theory. Studies in fuzziness and soft computing*, Springer, Berlin Heidelberg, 2010.

[9] O. A. Arqub, *Adaptation of reproducing kernel algorithm for solving fuzzy Fredholm-Volterra integro-differential equations*, Neural Computing and Applications, **28** (2017), 1591-1610.

[10] O. A. Arqub, M. Al-Smadi, *Fuzzy conformable fractional differential equations: Novel extended approach and new numerical solutions*, Soft Computing, **24** (2020), 12501-12522.

[11] O. A. Arqub, M. AL-Smadi, S. Momani, T. Hayat, *Numerical solutions of fuzzy differential equations using reproducing kernel Hilbert space method*, Soft Computing, **20** (2016), 3283-3302.

[12] O. A. Arqub, M. Al-Smadi, S. Momani, T. Hayat, *Application of reproducing kernel algorithm for solving second-order, two-point fuzzy boundary value problems*, Soft Computing, **21** (2017), 7191-7206.

[13] S. Arshad, V. Lupulescu, *On the fractional differential equations with uncertainty*, Nonlinear Analysis, **75** (2011), 3685-3693.

[14] B. Bede, *Mathematics of fuzzy sets and fuzzy logic*, Springer, London, 2013.

[15] B. Bede, S. G. Gal, *Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations*, Fuzzy Sets and Systems, **151**(3) (2005), 581-599.

[16] B. Bede, L. Stefanini, *Generalized differentiability of fuzzy-valued functions*, Fuzzy Sets and Systems, **230** (2013), 119-141.

[17] A. M. Bertone, R. M. Jafelice, L. C. De Barros, R. C. Bassanezi, *On fuzzy solutions for partial differential equations*, Fuzzy Sets and Systems, **219** (2013), 68-80.

[18] J. J. Buckley, T. Feuring, *Introduction to fuzzy partial differential equations*, Fuzzy Sets and Systems, **105** (1999), 241-248.

[19] Y. Chalco-Canoa, G. Maqui-Huamán, G. N. Silva, M. D. Jiménez-Gamero, *Algebra of generalized Hukuhara differentiable interval-valued functions: Review and new properties*, Fuzzy Sets and Systems, **375**(15) (2019), 53-69.

[20] F. Ge, Y. Chen, C. Kou, *Regional analysis of time-fractional diffusion processes*, Springer International Publishing, 2018.

[21] Z. Gouyandeh, T. Allahviranloo, S. Abbasbandy, A. Armand, *A fuzzy solution of heat equation under generalized Hukuhara differentiability by fuzzy Fourier transform*, Fuzzy Sets and Systems, **309** (2017), 81-97.

[22] N. V. Hoa, *Fuzzy fractional functional integral and differential equations*, Fuzzy Sets and Systems, **280** (2015), 58-90.

[23] N. V. Hoa, H. Vu, T. Minh Duc, *Fuzzy fractional differential equations under Caputo-Katugampola fractional derivative approach*, Fuzzy Sets and Systems, **375** (2019), 70-99.

[24] M. Hukuhara, *Integration des applications mesurables dont lavaleur est un compact convex*, Funkcialaj Ekvacioj, **10** (1967), 205-223.

[25] O. Kaleva, *Fuzzy differential equations*, Fuzzy Sets and Systems, **24**(3) (1987), 301-317.

[26] A. Kaufmann, M. M. Gupta, *Introduction fuzzy arithmetic*, Van Nostrand Reinhold, New York, 1985.

[27] V. Lakshmikantham, T. Bhaskar, J. Devi, *Theory of set differential equations in metric spaces*, Cambridge Scientific Publishers, 2006.

[28] M. L. Puri, D. A. Ralescu, *Differentials of fuzzy functions*, Journal of Mathematical Analysis and Applications, **91**(2) (1983), 552-558.

[29] M. Rahman, *Applications of Fourier transforms to generalized functions*, WIT Press, 2011.

[30] S. Salahshour, T. Allahviranloo, *Applications of fuzzy Laplace transforms*, Soft Computing, **17** (2013), 145-158.

[31] S. Salahshour, T. Allahviranloo, S. Abbasbandy, *Solving fuzzy fractional differential equations by fuzzy Laplace transforms*, Communications in Nonlinear Science and Numerical Simulation, **17** (2012), 1372-1381.

[32] L. Stefanini, *A generalization of hukuhara difference and division for interval and fuzzy arithmetic*, Fuzzy Sets and Systems, **161**(11) (2010), 1564-1584.

[33] N. Thi Kim Son, *A foundation on semigroups of operators defined on the set of triangular fuzzy numbers and its application to fuzzy fractional evolution equations*, Fuzzy Sets and Systems, **347** (2018), 1-28.

[34] N. Van Hoa, *Fuzzy fractional functional differential equations under Caputo gH-differentiability*, Communications in Nonlinear Science and Numerical Simulation, **22** (2015), 1134-1157.

[35] H. Viet Long, N. Thi Kim Son, H. Thi Thanh Tam, *The solvability of fuzzy fractional partial differential equations under Caputo gH-differentiability*, Fuzzy Sets and Systems, **309** (2017), 35-63.