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Original paper

# Jensen's inequalities for pseudo-integrals

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**Abstract**

In this paper, we introduce a general $(\oplus,\otimes)$-convex function based on semirings $([a,b],\oplus,\otimes)$ with pseudo-addition $\oplus$ and pseudo-multiplication $\otimes$. The generalization of the finite Jensen's inequality, as well as pseudo-integral with respect to $(\oplus,\otimes)$-convex functions, is obtained. This also generalizes Jensen's inequalities for Lebesgue integral and the results of Pap and Štrboja [12]. Meanwhile, we also prove Jensen's inequalities for pseudo-integrals on semirings $([a,b],\sup,\otimes)$ with respect to nondecreasing functions and present corresponding results for generalized fuzzy integrals.

*Keywords:* Jensen's inequality, semiring, pseudo-integral, pseudo-operation, $(\oplus,\otimes)$-convex function.

## 1 Introduction

It is well-known that Jensen's inequality is a part of the classical mathematical analysis [15], and it has played an important role in other fields of mathematics, such as probability theory, optimization, control theory, etc. The problem of the generalization of Jensen's inequalities is a contemporary issue. Recently, Jensen's inequalities are obtained on the basis of various kinds of non-additive integrals, such as Sugeno integral [1, 14], generalized Sugeno integral [5], extremal universal integrals [13], Choquet integral [7, 21], and integrals for fuzzy-interval-valued functions [3].

It is worth emphasizing that pseudo-analysis [2, 6, 8, 9, 10, 11], as a generalization of the classical analysis, is commonly used to solve nonlinear and uncertainty problems in many different fields, such as systems, optimization, decision making, control theory, differential equations, etc. The pseudo-integral [4, 8, 10, 17, 18, 19, 20, 22, 23] is the main branch in pseudo-analysis and it is a useful tool in optimization, decision making, control theory, etc. Recently, a generalization of Jensen's inequality for pseudo-integral was obtained [12]. The main result is as follow:

**Theorem 1.1** (Theorem 4, Pap and Štrboja [12])**.** *Let $\varphi:[a,b]\to[a,b]$ be a convex and non-increasing function. If a generator $g:[a,b]\to[0,\infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\otimes$ is a convex and increasing function, then for any measurable function $f:[0,1]\to[a,b]$, we have*

$$\varphi\left(\int_{[0,1]}^{\oplus} f(x)dx\right) \leqslant \int_{[0,1]}^{\oplus} \varphi(f(x))dx. \tag{1}$$

Jensen's inequality is interesting and valuable, but the inequality (1) is not the complete generalization of the classical Jensen's inequality, while the pseudo-integral is an extension of Lebesgue integrals. Further, as a sufficient condition, the convex function $\varphi$ is supposed to be non-increasing, which is not necessary in the classical one. Hence Jensen's inequality for pseudo-integral should be revisited. In this paper, we introduce a general $(\oplus,\otimes)$-convex function related to real semirings $([a,b],\oplus,\otimes)$ with pseudo-addition $\oplus$ and pseudo-multiplication $\otimes$. We prove Jensen's inequalities for a finite case and pseudo-integrals. Then Jensen's inequalities are discussed in contexts of some important cases of pseudo-integrals. These generalize not only the classical Jensen's inequalities but also the theorem 1.1. Meanwhile, Jensen's other inequalities different from the theorem 1.1 are also addressed.

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The rest of the paper is divided into six parts: in Section 2, as a preparation, we remind the real semiring on intervals with three basic cases and corresponding pseudo-integrals. Section 3 is devoted to the new notion of $(\oplus,\otimes)$-convex function and related finite version of Jensen's inequality. In Section 4, we investigate two special case of $(\oplus,\otimes)$-convex functions with the corresponding interpretations: max-plus case and $g$-semiring. In Section 5, we prove a generalized Jensen's inequality for pseudo-integrals with $(\oplus,\otimes)$-convex functions. Section 6 is devoted to special cases. In the concluding part, we remark that the obtained results can be applied to Jensen's inequalities for pseudo-integrals of set-valued functions from an earlier study [17].

## 2 Pseudo-integral

In this section, as a foreground, we remind some notions related to semirings and pseudo-integrals, adopted from earlier works [10, 11, 12]. Let $[a,b]$ be a closed (in some cases can be considered semiclosed) subinterval of $[-\infty,\infty]$. The full order on $[a,b]$ will be denoted by $\preceq$. A binary operation $\oplus$ on $[a,b]$ is pseudo-addition, if it is commutative, non-decreasing (with respect to $\preceq$), associative and with a zero (neutral) element denoted by $\mathbf{0}$. Let $[a,b]_+\subseteq[a,b]$ with $\mathbf{0}\preceq x$. A binary operation $\otimes$ on $[a,b]$ is pseudo-multiplication, if it is commutative, positively non-decreasing, i.e., $x\preceq y$ implies $x\otimes z\preceq y\otimes z$ for all $z\in[a,b]_+$, associative and with unit element $\mathbf{1}\in[a,b]$, i.e., for each $x\in[a,b]$, $\mathbf{1}\otimes x=x$. We assume also $\mathbf{0}\otimes x=\mathbf{0}$ and $\otimes$ is distributive over $\oplus$, i.e.,

$$x\otimes(y\oplus z)=(x\otimes y)\oplus(x\otimes z).$$

$([a,b],\oplus,\otimes)$ is a (real) semiring. In this paper, all pseudo-operations are supposed to be continuous. There are three canonical cases of these semirings [10, 11] (proved for a pair of pan-operations [9]).

**Case I:** Idempotent $\oplus$ and not non-idempotent $\otimes$:

$$x\oplus y=\sup(x,y), x\otimes y=g^{-1}(g(x)\cdot g(y)),$$

where $g:[a,b]\to[0,\infty]$ is an increasing bijection.

A full order is induced as follows: $x\preceq y\Leftrightarrow\sup(x,y)=y,\quad \mathbf{0}=a$. Special important case is $\mathbb{R}_{\max}$, i.e., $[-\infty,\infty[$ endowed with pseudo-addition max and pseudo-multiplication $+$, i.e., $g(x)=e^x$, see [2, 6, 10, 11].

**Case II:** $g$-semiring, i.e., pseudo-operations are given by:

$$x\oplus y=g^{-1}(g(x)+g(y))\quad\text{and}\quad x\otimes y=g^{-1}(g(x)\cdot g(y)),$$

where $g:[a,b]\to[0,\infty]$ is continuous and strictly monotone function.

(a) If $g$ is a strictly increasing generator, then $\mathbf{0}=a$, the usual order induced by $\oplus$ as follows:

$$x\preceq y\Leftrightarrow g(x)\leqslant g(y).$$

(b) If $g$ is a strictly decreasing generator, then $\mathbf{0}=b$, the usual order induced by $\oplus$ as follows:

$$x\preceq y\Leftrightarrow g(x)\geqslant g(y).$$

**Case III:** Idempotent $\oplus$ and $\otimes$: $x\vee y=\sup(x,y),\ x\wedge y=\inf(x,y)$.

A full order is induced as follows:

$$x\preceq y\Leftrightarrow\sup(x,y)=y\quad\text{and}\quad\mathbf{0}=a,\mathbf{1}=b.$$

We suppose that $([a,b],\oplus)$ and $([a,b],\otimes)$ are the complete lattice ordered semigroups, and $[a,b]$ is endowed with a metric $d$ compatible with sup and inf, i.e., $\liminf\limits_{n\to\infty}x_n=x$ and $\limsup\limits_{n\to\infty}x_n=x$ imply $\lim\limits_{n\to\infty}d(x_n,x)=0$, and which satisfies at least one of the following conditions:

(a) $d(x\oplus y,x'\oplus y')\leqslant d(x,x')+d(y,y')$;

(b) $d(x\oplus y,x'\oplus y')\leqslant\max\{d(x,x'),d(y,y')\}$.

Both conditions (a) and (b) imply:

$$d(x_n,y_n)\to0\Rightarrow d(x_n\oplus z,y_n\oplus z)\to0.$$

Metric $d$ is also monotone, i.e.,

$$x\preceq z\preceq y\Rightarrow d(x,y)\geqslant\sup\{d(y,z),d(x,z)\}.$$

A semiring $([a,b],\oplus,\otimes)$ is said to be having pseudo-inverses if for each non-zero element $x\in[a,b]$, there exists a $x^{(-1)}$, such that $x\otimes x^{(-1)}=\mathbf{1}$, and $x^{(-1)}$ is called a pseudo-inverse of $x$.

It is easy to see that semirings for Case I and Case II are all having pseudo-inverses. Since for each $x\neq\mathbf{0}$, there is $x^{(-1)}=g^{-1}\left(\frac{1}{g(x)}\right)$ such that

$$x\otimes x^{(-1)}=g^{-1}(g(x)\cdot g(x^{(-1)}))=g^{-1}\left(g(x)\cdot\frac{1}{g(x)}\right)=g^{-1}(1)=\mathbf{1}.$$

Let $X$ be a non-empty set and $\mathcal{A}$ be a $\sigma-$algebra of subsets of $X$, i.e., $(X,\mathcal{A})$ is a measurable space.

**Definition 2.1.** *A set-function $m:\mathcal{A}\to[a,b]_+$ is a $\oplus$-measure if it satisfies the following conditions*

*(i) $m(\emptyset)=\mathbf{0}$ (if $\oplus$ is not idempotent, i.e., $x\oplus x=x$ for all $x\in[a,b]$).*

*(ii) For any sequence $\{A_i(i\geq1)\}$ of pairwise disjoint sets from $\mathcal{A}$ we have*

$$m\left(\bigcup_{i=1}^{\infty}A_i\right)=\bigoplus_{i=1}^{\infty}m(A_i)=\lim_{n\to\infty}\bigoplus_{i=1}^{n}m(A_i).$$

Let $f,h:X\to[a,b]$ be two functions. Then for any $x\in X$ and $\lambda\in[a,b]$ we define

$$(f\oplus g)(x)=f(x)\oplus g(x),\ (f\otimes g)(x)=f(x)\otimes g(x),\ (\lambda\otimes f)(x)=\lambda\otimes f(x).$$

The pseudo-characteristic function on $A\subseteq X$ with values in a semiring $([a,b],\oplus,\otimes)$ is defined by $\chi_A(x)=\mathbf{1}$ with $x\in A$ and $\mathbf{0}$ with $x\notin A$. A step (measurable) function is a mapping $e:X\to[a,b]$ that has the following representation

$$e=\oplus_{i=1}^{n}a_i\otimes\chi_{A_i},$$

for $a_i\in[a,b]$ and sets $A_i\in\mathcal{A}$ are pairwise disjoint if $\oplus$ is nonidempotent. Let $\varepsilon$ be a positive real number and $B\subset[a,b]$. A subset $\{l_i^{\varepsilon}(i\geq1)\}$ of set $B$ is a $\varepsilon$-net on $B$ if for each $x\in B$ there exists $l_i^{\varepsilon}$ such that $d(l_i^{\varepsilon},x)\leqslant\varepsilon$. If we, also, have $l_i^{\varepsilon}\preceq x$ , then we call $\{l_i^{\varepsilon}(i\geqslant1)\}$ a lower $\varepsilon$-net. If $l_i^{\varepsilon}\preceq l_{i+1}^{\varepsilon}$ holds, then $\{l_i^{\varepsilon}(i\geq1)\}$ is monotone.

**Definition 2.2.** *Let $m:\mathcal{A}\to[a,b]_+$ be a $\oplus$-measure.*
*(i) The pseudo-integral of a step function $e:X\to[a,b]$ is defined by*

$$\int_X^{\oplus}e\otimes dm=\mathop{\oplus}_{i=1}^{n}a_i\otimes m(A_i).$$

*(ii) The pseudo-integral of a bounded measurable function $f:X\to[a,b]$, (if $\oplus$ is nonidempotent we suppose that for $\varepsilon>0$ there exists a monotone $\varepsilon$-net in $f(X)$) and is defined by*

$$\int_X^{\oplus}f\otimes dm=\lim_{n\to\infty}\int_X^{\oplus}e_n(x)\otimes dm,$$

*where $\{e_n\}$ is a sequence of step functions such that $d(e_n(x),f(x))\to0$ uniformly as $n\to\infty$.*

**Example 2.3.** *Let $([a,b],\oplus,\otimes)$ be a g-semiring. Then the pseudo-integral for a function $f:X\to[a,b]$ reduce to the g-integral,*

$$\int_X^{\oplus}f\otimes dm=g^{-1}\left(\int_X(g\circ f)dg\circ m\right).$$

**Example 2.4.** *Let $([a,b],\sup,\otimes)$ be a semiring. Then the pseudo-integral for a function $f:X\to[a,b]$ is given by*

$$\int_X^{\sup}f\otimes dm=\sup_{x\in X}(f(x)\otimes\psi(x))=\sup_{t\in[a,b]}(t\otimes m(t\preceq f)),$$

*where the function $\psi:X\to[a,b]$ defines a $\sigma$-sup-measure $m$ by $m(A)=\sup_{x\in A}\psi(x)$ and $(t\preceq f)=\{x\in X\mid t\preceq f(x)\}$.*

# 3 Jensen's inequality for finite case

In this section, we shall first introduce the notion of $(\oplus,\otimes)$-convex sets (pseudo-convex set for Case II in [4]). Then we introduce $(\oplus,\otimes)$-convex functions with respect to pseudo-operations on a $(\oplus,\otimes)$-convex sets. A generalized finite Jensen's inequality for $(\oplus,\otimes)$-convex functions is obtained.

**Definition 3.1.** *Let $([a,b],\oplus,\otimes)$ be a semiring. A subset $C$ of $[a,b]$ is said to be $(\oplus,\otimes)$-convex (pseudo-convex for Case II [4]) if and only if for all $x,y\in C$ and $\lambda,\mu\in[a,b]_+$ with $\lambda\oplus\mu=\mathbf{1}$ implies $\lambda\otimes x\oplus\mu\otimes y\in C$.*

**Definition 3.2.** *Let $([a,b],\oplus,\otimes)$ be a semiring and $C\subseteq[a,b]$ be a $(\oplus,\otimes)$-convex set. A function $\varphi: C\to[a,b]$ is said to be $(\oplus,\otimes)$-convex (resp., $(\oplus,\otimes)$-concave) if there holds:*

$$\varphi(\lambda\otimes x\oplus\mu\otimes y)\preceq\lambda\otimes\varphi(x)\oplus\mu\otimes\varphi(y), \tag{2}$$

$$(resp., \varphi(\lambda\otimes x\oplus\mu\otimes y)\succeq\lambda\otimes\varphi(x)\oplus\mu\otimes\varphi(y)),$$

*for all $x,y\in C$ and $\lambda,\mu\in[a,b]_+$ with $\lambda\oplus\mu=\mathbf{1}$.*

By the preceding definition, we can obtain the following property:

**Proposition 3.3.** *(Jensen's inequality for finite case) Let $([a,b],\oplus,\otimes)$ be a semiring having pseudo-inverses and $C\subseteq[a,b]$ be a $(\oplus,\otimes)$-convex set. Then a function $\varphi: C\to[a,b]$ is $(\oplus,\otimes)$-convex (resp., $(\oplus,\otimes)$-concave) if and only if*

$$\varphi\left(\bigoplus_{i=1}^{n}\lambda_i\otimes x_i\right)\preceq\bigoplus_{i=1}^{n}\lambda_i\otimes\varphi(x_i), \tag{3}$$

$$(resp., \varphi\left(\bigoplus_{i=1}^{n}\lambda_i\otimes x_i\right)\succeq\bigoplus_{i=1}^{n}\lambda_i\otimes\varphi(x_i)),$$

*for all $x_i\in C$, $\lambda_i\in[a,b]_+$, $1\leqslant i\leqslant n$ with $\bigoplus_{i=1}^{n}\lambda_i=\mathbf{1}$.*

*Proof.* It is sufficient to prove the case of $(\oplus,\otimes)$-convexity. The "if" part is obvious. Next, for the "only if" part, we prove it by induction on $n$. The result is trivially true when $n=1$ , and it is true for $n=2$ , as this reduces to (2). Next, for all $x_i\in C$, $\lambda_i\in[a,b]_+$, $1\leqslant i\leqslant n$ with $\bigoplus_{i=1}^{n}\lambda_i=\mathbf{1}$, take

$$y=(\lambda_{n-1}\oplus\lambda_n)^{(-1)}\otimes\lambda_{n-1}\otimes x_{n-1}\oplus(\lambda_{n-1}\oplus\lambda_n)^{(-1)}\otimes\lambda_n\otimes x_n.$$

Then we obtain, first applying inequality for $n-1$ and then the inequality for $n-2$,

$$\begin{aligned}
\varphi(\bigoplus_{i=1}^{n}\lambda_i\otimes x_i) &= \varphi(\bigoplus_{i=1}^{n-2}\lambda_i\otimes x_i\oplus(\lambda_{n-1}\oplus\lambda_n)\otimes y)\\
&\preceq \bigoplus_{i=1}^{n-2}\lambda_i\otimes\varphi(x_i)\oplus(\lambda_{n-1}\oplus\lambda_n)\otimes\varphi(y)\\
&\preceq \bigoplus_{i=1}^{n-2}\lambda_i\otimes\varphi(x_i)\oplus(\lambda_{n-1}\oplus\lambda_n)\\
&\quad\otimes[(\lambda_{n-1}\oplus\lambda_n)^{(-1)}\otimes\lambda_{n-1}\otimes\varphi(x_{n-1})\oplus(\lambda_{n-1}\oplus\lambda_n)^{(-1)}\otimes\lambda_n\otimes\varphi(x_n)]\\
&= \bigoplus_{i=1}^{n-2}\lambda_i\otimes\varphi(x_i)\oplus\lambda_{n-1}\otimes\varphi(x_{n-1})\oplus\lambda_n\otimes\varphi(x_n)\\
&= \bigoplus_{i=1}^{n}\lambda_i\otimes\varphi(x_i).
\end{aligned}$$

This gives (3). □

**Example 3.4.** *Let $C=[c,d]\subset\mathbb{R}$ and let the pseudo-addition $\oplus$ is the usual addition "+", pseudo-multiplication $\otimes$ is the usual multiplication "·" . If $\varphi:[c,d]\to\mathbb{R}$ is a usual convex function, then it is (+,·)-convex.*

# 4 Special cases of $(\oplus,\otimes)$-convex functions

In this section, we deal with two kinds of special $(\oplus,\otimes)$-convex functions based on semirings Case I for $\oplus=\max$, $\otimes=+$ and $g$-semirings, and their interpretations are shown.

### 4.1 Semiring $\mathbb{R}_{\max}$

In the special Case I for $\oplus = \max$, $\otimes = +$ , the $(\max, +)$-convex function has been introduced by Zimmermann [24]. Let $\mathbb{R}_{\max} = [-\infty, \infty[$ be the semifield endowed with pseudo-addition max and usually addition + as pseudo-multiplication.

**Definition 4.1.** *A* $(\max, +)$ *convex set is a subset* $C \subset \mathbb{R}_{\max}$ *which satisfies that* $x, y \in C$, $\alpha, \beta \in \mathbb{R}_{\max}$ *with* $\max(\alpha, \beta) = 0$ *imply*

$$\max(\alpha + x, \beta + y) \in C.$$

Let $C \subset \mathbb{R}_{\max}$ be a $(\max, +)$-convex set. Then by Definition 3.2 we say that $\varphi$ is $(\max, +)$-convex on the set $C$ (see also [2, 24]) , i.e.,

$$\varphi(\max(\alpha + x, \beta + y)) \leqslant \max(\alpha + \varphi(x), \beta + \varphi(y)).$$

We have the following property by [24].

**Proposition 4.2.** *A function* $\varphi : \mathbb{R}_{\max} \to [-\infty, \infty]$ *is* ( $max, +$)*-convex if its lower level sets*

$$S_t = \{x \in X \mid t \geqslant \varphi(x)\} \quad (t \in \mathbb{R}_{\max}),$$

*are* $(\max, +)$*-convex subsets of* $\mathbb{R}_{\max}$.

The epigraph (supergraph, upper hull) of a function $f : \mathbb{R} \to \mathbb{R}$ is the set of points lying on or above its graph:

$$\mathrm{epi} f = \{(x, t) \mid x \in \mathbb{R},\, t \in \mathbb{R},\, t \geqslant f(x)\} \subseteq \mathbb{R}^2.$$

One of the fundamental results of the convex analysis is that classical convex lower semi-continuous functions are upper hulls of affine maps, see [16]. We cannot obtain the $(\max, +)$-convex function as the sup of affine functions, where the affine function has the following form

$$\max(\max(\alpha + x), \beta),$$

for $x, y \in \mathbb{R}_{\max}$ and $\alpha, \beta \in \mathbb{R}_{\max}$. However, the following characterization was obtained, see [2] (Theorem 4.8). Before giving the cited theorem, we need the notion of difference $\div$ given for $\alpha, \beta \in \mathbb{R}_{\max}$ by

$$\alpha \div \beta = \begin{cases} \alpha & \text{if } \alpha > \beta, \\ -\infty & \text{otherwise.} \end{cases}$$

**Theorem 4.3.** *A* $\varphi$ *is* $(\max, +)$*-convex and lower semi-continuous if and only if it is* sup *of differences* $\div$ *of affine functions, where the difference of affine functions* $\max(\max(\alpha + x), \beta)$ *and* $\max(\max(\gamma + x), \delta)$ *is given by*

$$y = \max(\max(\alpha + x), \beta)) \div \max(\max(\gamma + x), \delta)).$$

We remark that every difference of the affine functions is $(\max, +)$-convex and lower semi-continuous (Proposition 4.6 from [2]). The four generic differences of affine functions over $\mathbb{R}_{\max}$ are:

(i) For $\alpha \leqslant \gamma$ and $\beta \leqslant \delta$ we have $y = \mathbf{0}$;

(ii) For $\alpha \leqslant \gamma$ and $\beta > \delta$ we have $y = \begin{cases} \beta & \text{if } x < \beta \lozenge \gamma, \\ \mathbf{0} & \text{otherwise} \end{cases}$ ;

(iii) For $\alpha > \gamma$ and $\beta \leqslant \delta$ we have $y = \begin{cases} \mathbf{0} & \text{if } x \leqslant \alpha \lozenge \delta, \\ \alpha + x & \text{otherwise} \end{cases}$ ;

(iv) For $\alpha > \gamma$ and $\beta > \delta$ we have $y = \max(\alpha + x, \beta)$,

where the operation $\lozenge$ on $\overline{\mathbb{R}}_{\max} = [-\infty, \infty]$, is given by $(-\infty)\lozenge(-\infty) = \infty\lozenge\infty = \infty$ and $x \lozenge y = y - x$ for other values of $x$ and $y$.

An example of a $(\max, +)$-convex function over $\mathbb{R}_{\max}$ is given in Figure 1 of an earlier article [2], together with epigraphs of differences of affine functions.

## 4.2 $g$-semiring

It was shown that for Case II any sub-interval of $[a,b]$ is $(\oplus,\otimes)$-convex [4].

**Lemma 4.4.** *Given a $g$-semiring $([a,b],\oplus,\otimes)$ with a generator $g$. Then we have that a subset $C\subseteq[a,b]$ is $(\oplus,\otimes)$-convex if and only if $g(C)$ is convex, i.e., it is an subinterval of $[0,\infty]$.*

**Theorem 4.5.** *Let $C$ be a $(\oplus,\otimes)$-convex set. Then we have*
*(i) if $g$ is a strictly increasing generator, then $\varphi$ is $(\oplus,\otimes)$-convex (resp., $(\oplus,\otimes)$-concave) if and only if $g\circ\varphi\circ g^{-1}:g(C)\to[0,\infty]$ is convex(resp., concave);*
*(ii) if $g$ is a strictly decreasing generator, then $\varphi$ is $(\oplus,\otimes)$-convex (resp., $(\oplus,\otimes)$-concave) if and only if $g\circ\varphi\circ g^{-1}:g(C)\to[0,\infty]$ is concave(resp., convex).*

*Proof.* It is sufficient to prove (i), since the proof of (ii) is analogous. Let $\varphi$ be $(\oplus,\otimes)$-convex function. For all $x,y\in C$ and $\lambda,\mu\in[a,b]_+$ with $\lambda\oplus\mu=\mathbf{1}$, we have

$$\begin{aligned}
&\varphi \text{ is } (\oplus,\otimes) \text{ -convex}\\
&\Leftrightarrow \varphi(\lambda\otimes x\oplus\mu\otimes y)\preceq\lambda\otimes\varphi(x)\oplus\mu\otimes\varphi(y)\\
&\Leftrightarrow g(\varphi(\lambda\otimes x\oplus\mu\otimes y))\leqslant g(\lambda\otimes\varphi(x)\oplus\mu\otimes\varphi(y))\\
&= g(g^{-1}(g(\lambda\otimes\varphi(x))+g(\mu\otimes\varphi(y))))=g(\lambda\otimes\varphi(x))+g(\mu\otimes\varphi(y))\\
&= g(\lambda)\cdot g(\varphi(x))+g(\mu)\cdot g(\varphi(y))\\
&\Leftrightarrow g\cdot\varphi\cdot g^{-1}(g(\lambda)g(x)+g(\mu)g(y))\leqslant g(\lambda)\cdot g(\varphi(g^{-1}(g(x))))+g(\mu)\cdot g(\varphi(g^{-1}(g(y))))\\
&\Leftrightarrow (g\circ\varphi\circ g^{-1})(g(\lambda)g(x)+g(\mu)g(y))\\
&\leqslant g(\lambda)\cdot(g\circ\varphi\circ g^{-1})(g(x))+g(\mu)\cdot(g\circ\varphi\circ g^{-1})(g(y))\\
&\Leftrightarrow g\circ\varphi\circ g^{-1} \text{ is convex function on } g(C),
\end{aligned}$$

for all $g(x),g(y)\in g(C)$ and $g(\lambda),g(\mu)\in[g(a),g(b)]$ with $g(\lambda)+g(\mu)=1$. □

**Remark 4.6.** *Interpretation of $(\oplus,\otimes)$-convex functions on $g$-semirings:*

*(i) A notion of g-convex-concave (resp., g-concave-convex) was introduced by Todorov et al. (see[20], Definition 17). If it is restricted on a g-semiring, then we can obtain the definition of g-convex (resp., g-concave ) function, i.e., a function $\varphi:C\to[a,b]$ is said to be g-convex (resp., g-concave) if $g\circ\varphi\circ g^{-1}$ is convex (resp., concave). In this case, the definitions of $(\oplus,\otimes)$-convex functions and g-convex functions coincide.*

*(ii) If $\varphi:C\to[a,b]$ is $(\oplus,\otimes)$-convex function, then $\psi=g\circ\varphi\circ g^{-1}:g(C)\to[0,\infty]$ is a convex function. Further we have $\varphi=g^{-1}\circ\psi\circ g$. Thus $\varphi$ is obtained by a transformation $g^{-1}\circ()\circ g$ from the convex function $\psi$. This transformation can be viewed as a "distortion", hence "$(\oplus,\otimes)$-convex" can also be called "pseudo-convex". In order to distinguish with the classical "quasi-convex, pseudo-convex", we use the name "$(\oplus,\otimes)$-convex".*

*(iii) In the classical analysis [15], we have that if $f:[c,d]\to[0,\infty]$ is a convex function, then $f$ is continuous at each point $t\in]c,d[$. Therefore for Case II a $(\oplus,\otimes)$-convex function $\varphi:[a,b]\to[a,b]$, is continuous at each point $t\in]a,b[$.*

**Example 4.7.** *(i) Let $g(x)=x^\alpha$ for some $\alpha\in[1,\infty[$ and $x\in[0,\infty[$. The corresponding pseudo-operations are*

$$x\oplus y=\sqrt[\alpha]{x^\alpha+y^\alpha},\quad x\otimes y=xy.$$

*We construct a $(\oplus,\otimes)$-convex function $\varphi$ by the convex function $f(x)=(x-1)^2$ in the following way*

$$\varphi(x)=(g^{-1}\circ f\circ g)(x)=\sqrt[\alpha]{x}\circ(x-1)^2\circ x^\alpha=\sqrt[\alpha]{x}\circ(x^\alpha-1)^2=\sqrt[\alpha]{(x^\alpha-1)^2},$$

*for $x\in[0,\infty[$.*

*(ii) Let $g(x)=e^x$. The corresponding pseudo-operations are*

$$x\oplus y=\ln(e^x+e^y),\quad x\otimes y=x+y.$$

*We construct a $(\oplus,\otimes)$-convex function by the convex function $f(x)=x^2$ in the following way*

$$\varphi(x)=\ln x\circ x^2\circ e^x=\ln x\circ(e^x)^2=\ln e^{2x}=2x,$$

*for $x\in[0,\infty[$.*

# 5 Jensen's inequality for pseudo-integral

In this section, we give Jensen's inequality for pseudo-integral on general semirings $([a,b],\oplus,\otimes)$.

**Theorem 5.1.** *Let $([a,b],\oplus,\otimes)$ be a semiring having pseudo-inverses and $m(X)=\mathbf{1}$. Then for any measurable function $f:X\to[a,b]$, we have the following assertions:*
*(i) (Jensen's inequality) If $\varphi:[a,b]\to[a,b]$ is a continuous $(\oplus,\otimes)$-convex function, then*

$$\varphi\left(\int_X^{\oplus} f\otimes dm\right)\preceq\int_X^{\oplus}(\varphi\circ f)\otimes dm. \tag{4}$$

*(ii) (Reverse Jensen's inequality). If $\varphi:[a,b]\to[a,b]$ is a continuous $(\oplus,\otimes)$-concave function, then*

$$\varphi\left(\int_X^{\oplus} f\otimes dm\right)\succeq\int_X^{\oplus}(\varphi\circ f)\otimes dm.$$

*Proof.* It is sufficient to prove (i). Step 1: For step functions. Let a step function $e:X\to[a,b]$ be given by $e=\oplus_{i=1}^{n}a_i\otimes\chi_{A_i}$ for $a_i\in[a,b]$ and sets $A_i\in\mathcal{A}$ are pairwise disjoint if $\oplus$ is nonidempotent. Then we have

$$\varphi\left(\int_X^{\otimes} e\otimes dm\right)=\varphi\left(\overset{n}{\underset{i=1}{\oplus}}a_i\otimes m(A_i)\right).$$

Since

$$m\left(\bigcup_{i=1}^{n}A_i\right)=\overset{n}{\underset{i=1}{\oplus}}m(A_i)=\mathbf{1},$$

and $\varphi$ is $(\oplus,\otimes)$-convex, by Proposition 3.3 we obtain

$$\varphi\left(\overset{n}{\underset{i=1}{\oplus}}a_i\otimes m(A_i)\right)\preceq\overset{n}{\underset{i=1}{\oplus}}\varphi(a_i)\otimes m(A_i)=\int_X^{\oplus}\varphi(e)\otimes dm.$$

That is

$$\varphi\left(\int_X^{\oplus} e\otimes dm\right)\preceq\int_X^{\oplus}(\varphi\circ e)dm.$$

Step 2: For any measurable function $f:X\to[a,b]$, then there exists a sequence of step functions $\{e_n\}$ such that $d(e_n(x),f(x))\to 0$ uniformly as $n\to\infty$, and

$$\int_X^{\oplus} f\otimes dm=\lim_{n\to\infty}\int_X^{\oplus}e_n(x)\otimes dm.$$

Then by the continuity of $\varphi$ and $\int_X^{\oplus}f\otimes dm$, by Step1 we obtain

$$\varphi\left(\int_X^{\oplus} f\otimes dm\right)=\lim_{n\to\infty}\varphi\left(\int_X^{\oplus}e_n(x)\otimes dm\right)\preceq\lim_{n\to\infty}\int_X^{\oplus}(\varphi\circ e_n)\otimes dm.$$

By the classical analysis, it is easy to see that $d((\varphi\circ e_n)(x),f(x))\to 0$ uniformly as $n\to\infty$, since $\varphi$ is continuous and $d(e_n(x),f(x))\to 0$ uniformly as $n\to\infty$,

Hence

$$\lim_{n\to\infty}\int_X^{\oplus}(\varphi\circ e_n)\otimes dm=\int_X^{\oplus}(\varphi\circ f)\otimes dm.$$

That gives (4).

□

# 6 Special cases of Jensen's inequality for pseudo-integral

In this section, we shall first give Jensen's inequality for pseudo-integral on $g$-semirings, then prove Jensen's inequality for pseudo-integral on semirings $([a,b],\sup,\otimes)$.

## 6.1 $g$-semiring

Further, two assertions were reported in earlier work [12]. It is easy to see that they can be obtained directly by Theorem 5.1.

**Corollary 6.1** (Theorem 4. [12]). *Let $f : X \to [a, b]$ be a measurable function and $\int_X^{\oplus} f \otimes dm \in\, ]a, b[$ and let $m(X) = \mathbf{1}$. If $\varphi : [a, b] \to [a, b]$ is a convex and non-increasing function and the generator $g$ is a convex and strictly increasing function, then we have*

$$\varphi\left(\int_X^{\oplus} f \otimes dm\right) \preceq \int_X^{\oplus} (\varphi \circ f) \otimes dm. \tag{5}$$

*Proof.* It was shown in the proof of Theorem 4 [12], that the composition $g \circ \varphi \circ g^{-1}$ is a convex function. As the generator $g$ is strictly increasing, we know that $\varphi$ is $(\oplus, \otimes)$-convex by Theorem 4.5. By the classical analysis, we have that $\varphi$ is convex and implies that $\varphi$ is continuous, then conditions in Theorem 5.1 (i) are satisfied, and inequality (5) holds. The proof is complete. □

**Corollary 6.2** (Theorem 5. [12]). *Let $f : X \to [a, b]$ be a measurable function and $\int_X^{\oplus} f \otimes dm \in\, ]a, b[$ and let $m(X) = \mathbf{1}$. If $\varphi : [a, b] \to [a, b]$ is a concave and non-decreasing function and the generator $g$ is a convex and strictly decreasing function, then we have*

$$\varphi\left(\int_X^{\oplus} f \otimes dm\right) \succeq \int_X^{\oplus} (\varphi \circ f) \otimes dm. \tag{6}$$

*Proof.* It has been shown in the proof of Theorem 5 [12], that the composition $g \circ \varphi \circ g^{-1}$ is a convex function. As the generator $g$ is strictly decreasing, we know that $\varphi$ is $((\oplus, \otimes)$-concave by Theorem 4.5. By the classical analysis, we have $\varphi$ is concave implies $\varphi$ is continuous, then conditions in Theorem 5.1(ii) are satisfied, and inequality (6) holds. The proof is complete. □

In a similar manner, we can obtain the next two corollaries.

**Corollary 6.3.** *Let $f : X \to [a, b]$ be a measurable function and $\int_X^{\oplus} f \otimes dm \in\, ]a, b[$ and let $m(X) = \mathbf{1}$. If $\varphi : [a, b] \to [a, b]$ is a convex and non-decreasing function and the generator $g$ is a concave and strictly decreasing function, then we have*

$$\varphi\left(\int_X^{\oplus} f \otimes dm\right) \preceq \int_X^{\oplus} (\varphi \circ f) \otimes dm.$$

**Corollary 6.4.** *Let $f : X \to [a, b]$ be a measurable function and $\int_X^{\oplus} f \otimes dm \in\, ]a, b[$ and let $m(X) = \mathbf{1}$. If $\varphi : [a, b] \to [a, b]$ is a concave and non-decreasing function and the generator $g$ is a concave and strictly increasing function, then we have*

$$\varphi\left(\int_X^{\oplus} f \otimes dm\right) \succeq \int_X^{\oplus} (\varphi \circ f) \otimes dm.$$

For a $g$-semiring and a monotone function $\varphi : [a, b] \to [a, b]$. Based on the monotonicity and convexity or concavity on $\varphi$ and $g$, there exist 16 kinds of compositions $g \circ \varphi \circ g^{-1}$, then we have a question: "Is there any other case when the composition is convex or concave except for the above four corollaries?" The answer is no. For the details, see the following table:

| $\varphi$ / $g \circ \varphi \circ g^{-1}$ / $g$ | | | $g$ | | | |
|---|---|---|---|---|---|---|
| | | | increasing | | decreasing | |
| | | | convex | concave | convex | concave |
| $\varphi$ | non-decreasing | convex | uncertain | uncertain | uncertain | **concave** |
| | | concave | uncertain | **concave** | **convex** | uncertain |
| | non-increasing | convex | **convex** | uncertain | uncertain | uncertain |
| | | concave | uncertain | uncertain | uncertain | uncertain |

We explain the table by an example: if $\varphi$ is non-decreasing and convex, and $g$ is increasing and convex, then convexity of $g \circ \varphi \circ g^{-1}$ cannot be judged, this means "uncertain" (other situations are similar).

**Example 6.5.** *(i) Let $g(x) = x^\alpha$ for some $\alpha \in [1,\infty[$ for $x \in [0,\infty[$. The corresponding pseudo-operations are $x \oplus y = \sqrt[\alpha]{x^\alpha + y^\alpha}$ and $x \otimes y = xy$. By Example 4.7 (i), we have known that $\varphi(x) = \sqrt[\alpha]{(x^\alpha - 1)^2}$ is a $(\oplus,\otimes)$-convex function. Then we have*

$$\sqrt[\alpha]{((\int_{[0,1]} f(x)dx)^\alpha - 1)^2} \preceq \sqrt[\alpha]{\int_{[0,1]} (f(x)^\alpha - 1)^2 dx},$$

*i.e.,*

$$\left(\left(\int_{[0,1]} f(x)dx\right)^\alpha - 1\right)^2 \leqslant \int_{[0,1]} (f(x)^\alpha - 1)^2 dx.$$

*(ii) Let $g(x) = e^x$. The corresponding pseudo-operations are $x \oplus y = \ln(e^x + e^y)$ and $x \otimes y = x + y$. By Example 4.7 (ii), we have known that $\varphi(x) = 2x$ is a $(\oplus,\otimes)$-convex function. Then we have*

$$2\left(\ln \int_{[0,1]} e^{f(x)}dx\right) \preceq \ln\left(\int_{[0,1]} e^{2f(x)}dx\right),$$

*i.e.,*

$$\left(\int_{[0,1]} e^{f(x)}dx\right)^2 \leqslant \int_{[0,1]} e^{2f(x)}dx.$$

## 6.2 Semiring $([a,b], \sup, \otimes)$

Jensen's inequalities for generalized Sugeno integrals have been studied [5]. We have an analogous inequality for pseudo-integral on a semiring $([a,b], \sup, \otimes)$. The special important case is $\mathbb{R}_{\max}$, see subsection 4.1. We write $m(\mathcal{A}) = \{m(A) \mid A \in \mathcal{A}\}$.

**Theorem 6.6.** *(Reverse Jensen's inequality) Let $\varphi : [a,b] \to [a,b]$ be a nondecreasing function with $\varphi(\mathbf{0}) = \mathbf{0}$. Then the inequality*

$$\varphi\left(\int_X^{\sup} f \otimes dm\right) \succeq \int_X^{\sup} \varphi \circ f \otimes dm,$$

*holds for any measurable functions $f : X \to [a,b]$ if and only if $\varphi(t \otimes v) \succeq \varphi(t) \otimes v$ for any $t \in [a,b], v \in m(\mathcal{A})$ .*

*Proof.* Sufficiency: Since $\varphi : [a,b] \to [a,b]$ is nondecreasing, by Example 2.4 and the given condition, we have

$$\varphi\left(\int_X^{\sup} f \otimes dm\right) = \varphi(\sup_{x\in X} f(x) \otimes \psi(x)) \succeq \sup_{x\in X} \varphi(f(x) \otimes \psi(x)) \succeq \sup_{x\in X} \varphi(f(x)) \otimes \psi(x) = \int_X^{\sup} \varphi \circ f \otimes dm.$$

Necessity: Taking $f = t \otimes \chi_A$, where $t \in [a,b]$, $A \in \mathcal{A}$, we have $\varphi \circ f = \varphi(t) \otimes \chi_A$. Then we obtain

$$\varphi\left(\int_X^{\sup} f \otimes dm\right) = \varphi(t \otimes m(A)),$$

$$\int_X^{\sup} \varphi \circ f \otimes dm = \varphi(t) \otimes m(A).$$

Therefore we have $\varphi(t \otimes v) \succeq \varphi(t) \otimes v$ for any $t \in [a,b]$, $v \in m(\mathcal{A})$. □

It is easy to obtain the following equivalent form of the above theorem.

**Theorem 6.7.** *(Jensen's inequality) Let $\varphi : [a,b] \to [a,b]$ be a nondecreasing function with $\varphi(\mathbf{0}) = \mathbf{0}$. Then the inequality*

$$\varphi\left(\int_X^{\sup} f \otimes dm\right) \prec \int_X^{\sup} \varphi \circ f \otimes dm,$$

*holds for any measurable functions $f : X \to [a,b]$ if and only if $\varphi(t \otimes v) \prec \varphi(t) \otimes v$ for any $t \in [a,b]$, $v \in m(\mathcal{A})$ .*

**Theorem 6.8.** *(Jensen's inequality) Let $\varphi : [a, b] \to [a, b]$ be a left-continuous and nondecreasing function with $\varphi(\mathbf{0}) = \mathbf{0}$. Then the inequality*

$$\varphi\left(\int_X^{\sup} f \otimes dm\right) \preceq \int_X^{\sup} \varphi \circ f \otimes dm,$$

*holds for any measurable functions $f : X \to [a, b]$ if and only if $\varphi(t \otimes v) \preceq \varphi(t) \otimes v$ for any $t \in [a, b], v \in m(\mathcal{A})$ .*

*Proof.* It is similar to the proof of Theorem 2.1 in [5]. □

**Theorem 6.9.** *(Reverse Jensen's inequality) Let $\varphi : [a, b] \to [a, b]$ be a left-continuous and nondecreasing function with $\varphi(\mathbf{0}) = \mathbf{0}$. Then the inequality*

$$\varphi\left(\int_X^{\sup} f \otimes dm\right) \succ \int_X^{\sup} \varphi \circ f \otimes dm,$$

*holds for any measurable functions $f : X \to [a, b]$ if and only if $\varphi(t \otimes v) \succ \varphi(t) \otimes v$ for any $t \in [a, b], v \in m(\mathcal{A})$.*

**Corollary 6.10.** *Let $\varphi : [a, b] \to [a, b]$ be a left-continuous and nondecreasing function with $\varphi(\mathbf{0}) = \mathbf{0}$. Then the equality*

$$\varphi\left(\int_X^{\sup} f \otimes dm\right) = \int_X^{\sup} \varphi \circ f \otimes dm,$$

*holds for any measurable functions $f : X \to [a, b]$ if and only if $\varphi(t \otimes v) = \varphi(t) \otimes v$ for any $t \in [a, b]$ , $v \in m(\mathcal{A})$.*

## 7 Concluding remarks

We have introduced a general notion of $(\oplus, \otimes)$-convex function. We prove a general Jensen's inequality for the finite case and for the pseudo-integral. Then we discuss Jensen's inequalities for special cases of semirings. The first one is on $g$-semiring, where these inequalities cover not only the classical Jensen's inequalities, but also those of Pap and Štrboja [12]. The second type of Jensen's inequalities are obtained on semiring $([a, b], \sup, \otimes)$. In the future, it will be investigated how to extend the obtained results on functions with many variables. Further, Jensen's inequalities for pseudo-integrals from this paper can be extended on Jensen's inequalities for pseudo-integrals of set-valued functions obtained in an earlier study [17].

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## References

[1] S. Abbaszadeh, M. E. Gordji, E. Pap, A. Szakál, *Jensen-type inequalities for Sugeno integral*, Information Sciences, **376** (2017), 148-157.

[2] G. Cohen, S. Gaubert, J. P. Quadrat, I. Singer, *Max-plus convex sets and functions*, Idempotent Mathematics and Mathematical Physics (Eds. G. L. Litvinov, V. P. Maslov), Contemporary Mathematics, American Mathematical Society, **377** (2005), 105-129.

[3] T. M. Costa, *Jensen's inequality type integral for fuzzy-interval-valued functions*, Fuzzy Sets and Systems, **327** (2017), 31-47.

[4] T. Grbić, I. Štajner-Papuga, M. Štrboja, *An approach to pseudo-integration of set-valued functions*, Information Sciences, **181** (2011), 2278-2292.

[5] M. Kaluszka, A. Okolewski, M. Boczek, *On the Jensen type inequality for generalized Sugeno integral*, Information Sciences, **266** (2014), 140-147.

[6] G. L. Litvinov, V. P. Maslov (Eds), *Idempotent mathematics and mathematical physics*, Contemporary Mathematics, American Mathematical Society, **377**, 2005.

[7] R. Mesiar, J. Li, E. Pap, *The Choquet integral as Lebesgue integral and related inequalities*, Kybernetika, **46** (2010), 1098-1107.

[8] R. Mesiar, E. Pap, *Idempotent integral as limit of g-integrals*, Fuzzy Sets and Systems, **102** (1999), 385-392.

[9] R. Mesiar, J. Rybarik, *Pan-operations structure*, Fuzzy Sets and Systems, **74** (1995), 365-369.

[10] E. Pap, *Null-additive set functions*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1995.

[11] E. Pap, *Generalized real analysis and its application*, International Journal of Approximate Reasoning, **47** (2008), 368-386.

[12] E. Pap, M. Štrboja, *Generalization of the Jensen inequality for pseudo-integral*, Information Sciences, **180** (2010), 543-548.

[13] E. Pap, M. Štrboja, *Jensen type inequality for extremal universal integrals*, in: Proceedings of 2012 IEEE 10th Jubilee International Symposium on Intelligent Systems and Informatics (SISY), IEEE (2012), 525-529.

[14] H. Román-Flores, A. Flores-Franulic, Y. Chalco-Cano, *A Jensen type inequality for fuzzy integrals*, Information Sciences, **177** (2007), 3192-3201.

[15] H. L. Royden, *Real analysis*, Macmillan, New York, 1988.

[16] I. Singer, *Abstract convex analysis*, Wiley-Interscience and Canadian Mathematics, 1997.

[17] M. Štrboja, T. Grbić, I. Štajner-Papuga, G. Grujić, S. Medić, *Jensen and Chebyshev inequalities for pseudo-integrals of set-valued functions*, Fuzzy Sets and Systems, **222** (2013), 18-32.

[18] M. Štrboja, E. Pap, B. Mihailović, *Transformation of the pseudo-integral and related convergence theorems*, Fuzzy Sets and Systems, **355** (2019), 67-82.

[19] M. Sugeno, T. Murofushi, *Pseudo-additive measures and integrals*, Journal of Mathematical Analysis and Applications, **122** (1987), 197-222.

[20] M. Todorov, M. Štrboja, E. Pap, B. Mihailović, *Jensen type inequality for the bipolar pseudo-integrals*, Fuzzy Sets and Systems, **379** (2020), 82-101.

[21] R. Wang, *Some inequalities and convergence theorems for Choquet integrals*, Journal of Applied Mathematics and Computing, **35** (2011), 305-321.

[22] D. Zhang, R. Mesiar, E. Pap, *Pseudo-integral and generalized Choquet integral*, Fuzzy Sets and Systems, (2020), doi.org/10.1016/j.fss.2020.12.005.

[23] D. Zhang, E. Pap, *Fubini theorem and generalized Minkowski inequality for the pseudo-integral*, International Journal of Approximate Reasoning, **122** (2020), 9-23.

[24] K. Zimmermann, *Extremely convex functions*, Z. Pad. Hochschule, N. K. Krupskaya, **1** (1979), 3-7.