Iranian Journal of Fuzzy Systems

Volume 18, Number 3, (2021), pp. 143-160

Original paper

Hyers-Ulam-Rassias stability of quaternion multidimensional fuzzy nonlinear difference equations with impulses

Z. Li[1], C. Wang[2], R. P. Agarwal[3] and R. Sakthivel[4]

[1,2] *Department of Mathematics, Yunnan University, Kunming, Yunnan, China*
[3] *Department of Mathematics, Texas A&M University-Kingsville, TX, USA*
[3] *Distinguished University Professor of Mathematics, Florida Institute of Technology, USA*
[4] *Department of Applied Mathematics, Bharathiar University, Coimbatore, India*

lizhienyunnan@163.com, chaowang@ynu.edu.cn, ravi.agarwal@tamuk.edu, krsakthivel0209@gmail.com

Abstract

In this paper, we consider the Hyers-Ulam-Rassias stability of the impulsive fuzzy nonlinear difference equations in the multidimensional fuzzy quaternion space. Some sufficient conditions of the Hyers-Ulam-Rassias stability are established for the quaternion multidimensional fuzzy nonlinear difference equations with impulses and several examples are demonstrated to show the feasibility of our obtained results.

Keywords: Fuzzy nonlinear difference equations, quaternion space, Hyers-Ulam-Rassias stability, impulses.

1 Introduction

Fuzzy dynamic equations are a hot research field, due to their broad applications on the analysis of natural phenomena with inherent inaccuracy (see [6]). It is well-known that there are two main types of fuzzy dynamic equations (i.e., fuzzy differential equations in the continuous time situation and fuzzy difference equations in the discrete time situation) that can accurately describe the evolution behavior of the phenomena in the imprecise patterns (see [10, 15]). For the continuous time case, a notion of strongly generalized differentiability was introduced in the literature [2] and it can be widely applied to study fuzzy differential equations (see [3]). This related topic is extensively studied in many literatures including initial and boundary value problems (see [19, 20]), periodic problems (see [12]), almost periodic problems (see [32]), and fuzzy fractional differential equations (see [1, 16, 30]) etc. Moreover, a generalized Hukuhara difference and division for interval were proposed in fuzzy arithmetic direction and the unidimensional and multidimensional boxes were initiated to deal with high-dimensional problems (see [26]). For the discrete time case, some new fuzzy difference equations were proposed and analyzed including stability and existence analysis and these discrete results were proved to be very effective in the fuzzy control applications (see [5, 11, 14]). For the unification of both differential equations and difference equations, we refer the readers to the book [33] for more details.

On the other hand, the evolution process of phenomena does not always keep in constant status, it may be perturbed by a sudden change and the original trajectory will be broken in this case. Such a sudden change is described by "impulses" in the dynamic equations (see [13, 31]). The introduction of impulses in the fuzzy dynamic equations is the critical point in our discussion. This effective approach will get rid of the limitations of non-impulsive dynamic equations in the field of applications, and is apparently suitable to analyze the evolution of real phenomena which are perturbed by the external factors and bring perfect adaptiveness for control processes in many application areas (see [23]).

In 1989, Buckley first introduced the notion of fuzzy complex number (see [4]) which was used to study complex fuzzy sets and complex membership grade (see [21, 28]). As demonstrated in these literatures, the superiority of this augmented definition of complex fuzzy sets is its capacity to accommodate fuzzy cycles. To expand fuzzy complex

Corresponding Author: C. Wang
Received: May 2020; Revised: October 2020; Accepted: January 2021.

number, Moura et al. (see [18]) proposed the concept of the fuzzy quaternion number and studied some main concepts such as distance, supremum, infimum and arithmetic properties, etc. In addition, for quaternions have a powerful modeling of rotation and orientation, many advantages over real-valued vectors in physics and engineering applications were demonstrated under the fuzzy background (see [35]).

In 1940, a new stability of the linear functional equation was proposed by Hyers (see [7]) and was formed as a series of mathematical problems in the book by Ulam (see [29]) and discussed by Rassias (see [22]). Since then many literatures focused on this topic and formed a new generalized stability theory of various types of differential equations (see [8, 9] for functional equations, [17, 27] for differential operators, [24] for ordinary differential equations, [25] for fuzzy differential equations, [34] for impulsive ordinary differential equations).

However, there is no work on the Hyers-Ulam-Rassias stability for quaternion multidimensional fuzzy difference equations with impulses. Motivated by the above, in the multidimensional fuzzy quaternion space, we will consider the Hyers-Ulam-Rassias stability of three types of impulsive fuzzy nonlinear difference equations in Section 3 and establish some sufficient conditions to guarantee the Hyers-Ulam-Rassias stability of their fuzzy iteration solutions. The paper is organized as four sections. In Section 2, some fundamental results of fuzzy quaternion vector are established in the multidimensional fuzzy quaternion space. In Section 3, we obtain the sufficient conditions of the Hyers-Ulam-Rassias stability for the quaternion multidimensional fuzzy nonlinear difference equations with impulses and provide several examples to illustrate the effectiveness of the obtained results. In Section 5, a conclusion is presented to end this paper.

2 Fuzzy quaternion vector in the multidimensional fuzzy quaternion space

Throughout the paper, we denote the set of quaternion by $\mathbb{Q}$, the natural numbers set by $\mathbb{N}$ and $\dot{n}$ a natural number, the $\dot{n}$-dimension quaternion vectors space by $\mathbb{Q}^{\dot{n}}$, the $\dot{n}\times\dot{n}$-quaternion-matrices set by $\mathbb{Q}_{\dot{n}\times\dot{n}}$, the real numbers set by $\mathbb{R}$, the $\dot{n}$-dimension real vectors space by $\mathbb{R}^{\dot{n}}$ and the integers set by $\mathbb{Z}$. In addition, i, j, k are the quaternion unites and satisfy the following multiplication

$$i^2 = j^2 = k^2 = -1, jk = -kj = i, ki = -ik = j, ij = -ji = k.$$

Let $x_{\dot{n}} \in \mathbb{Q}^{\dot{n}}$, $x_{\dot{n}} = (x_1, x_2, \ldots, x_{\dot{n}})^T = (x_{10}+x_{11}i+x_{12}j+x_{13}k, x_{20}+x_{21}i+x_{22}j+x_{23}k, \ldots, x_{\dot{n}0}+x_{\dot{n}1}i+x_{\dot{n}2}j+x_{\dot{n}3}k)^T$, $x_{hm} \in \mathbb{R}$ and $0 \le m \le 3$ and $1 \le h \le \dot{n}$, we define $\|x_{\dot{n}}\| = \sum_{m=1}^{\dot{n}} \sum_{h=0}^{3} |x_{hm}|$. For $A = [a_{h,m}]_{\dot{n}\times\dot{n}} = [a_{h,m,0} + a_{h,m,1}i + a_{h,m,2}j + a_{h,m,3}k]_{\dot{n}\times\dot{n}} \in \mathbb{Q}_{\dot{n}\times\dot{n}}$, define $\|A\| = \sum_{m=1}^{\dot{n}} \sum_{h=1}^{\dot{n}} \sum_{v=0}^{3} |x_{h,m,v}|$.

Definition 2.1. [18] *A fuzzy real set is a function $\overline{A} : \mathbb{R} \to [0, 1]$ and $\overline{A}$ is a fuzzy real number if and only if: (i) $\overline{A}$ is a normal, i.e., there exists $x \in \mathbb{R}$ such that $\overline{A}(x) = 1$; (ii) for all $\alpha \in (0, 1]$, the set $[\overline{A}]^\alpha = \{x \in \mathbb{R} : \overline{A}(x) \ge \alpha\}$ is limited set. The set of all fuzzy real numbers is denoted by $\mathbb{R}_F$. Similarly, a fuzzy quaternion numbers is given by $h' : \mathbb{Q} \to [0, 1]$ such that $h'(a + bi + cj + dk) = \min\{\overline{A}(a), \overline{B}(b), \overline{C}(c), \overline{G}(d)\}$ for some $\overline{A}, \overline{B}, \overline{C}, \overline{G} \in \mathbb{R}_F$. The set of all fuzzy quaternion number is denoted by $\mathbb{Q}_F$.*

Lemma 2.2. [18] *For every $h' \in \mathbb{Q}_F$ and $\alpha \in (0, 1]$, $[h']^\alpha = [\overline{A}]^\alpha \times [\overline{B}]^\alpha \times [\overline{C}]^\alpha \times [\overline{G}]^\alpha$, i.e., $[h']^\alpha$ is a hiper-cube in $\mathbb{R}^4$.*

Example 2.3. *Let*

$$A(x) = \begin{cases} 2x, \ x \in [0, \frac{1}{2}], \\ 0, \ otherwise, \end{cases} \quad B(x) = \begin{cases} 1 - \cos x\pi, \ x \in [0, \frac{1}{2}], \\ 0, \ otherwise, \end{cases} \quad C(x) = \begin{cases} \sin x\pi, \ x \in [0, \frac{1}{2}], \\ 0, \ otherwise, \end{cases} \quad G(x) = \begin{cases} \frac{1-\cos x\pi+\sin x\pi}{2}, \ x \in [0, \frac{1}{2}], \\ 0, \ otherwise. \end{cases}$$

Then $h'(a + bi + cj + dk) = \max\{2a, 1 - \cos b\pi, \sin c\pi, \frac{1-\cos d\pi+\sin d\pi}{2}\}$ for $a, b, c, d \in [0, \frac{1}{2}]$, $h'(a + bi + cj + dk) = 0$ otherwise. $h'[\frac{1}{2} + \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k] = 1$, $[h]^0 = [0, \frac{1}{2}] \times [0, \frac{1}{2}] \times [0, \frac{1}{2}] \times [0, \frac{1}{2}]$, $[h']^{\frac{1}{2}} = [\frac{1}{4}, \frac{1}{2}] \times [\frac{1}{3}, \frac{1}{2}] \times [\frac{1}{6}, \frac{1}{2}] \times [\frac{1}{4}, \frac{1}{2}]$. In fact $[h']^{\frac{1}{2}} = [\overline{A}]^{\frac{1}{2}} \times [\overline{B}]^{\frac{1}{2}} \times [\overline{C}]^{\frac{1}{2}} \times [\overline{G}]^{\frac{1}{2}}$, where $[A]^{\frac{1}{2}} = [\frac{1}{4}, \frac{1}{2}]$, $[B]^{\frac{1}{2}} = \{x : 1 \ge 1 - \cos x\pi \ge \frac{1}{2}\} \cap [0, \frac{1}{2}] = [\frac{1}{3}, \frac{1}{2}]$, $[C]^{\frac{1}{2}} = \{x : 1 \ge \sin x\pi \ge \frac{1}{2}\} \cap [0, \frac{1}{2}] = [\frac{1}{6}, \frac{1}{2}]$, $[G]^{\frac{1}{2}} = \{x : 1 \ge \frac{1-\cos x\pi+\sin x\pi}{2} \ge \frac{1}{2}\} \cap [0, \frac{1}{2}] = [\frac{1}{4}, \frac{1}{2}]$.

Definition 2.4. [6] *Let $T = [a, b] \subset \mathbb{R}$ be a compact interval and denote $E^{\dot{n}} = \{u : \mathbb{R}^{\dot{n}} \to [0, 1] | u \text{ satisfies } (i) - (iv) \text{ below}\}$, where (i) u is a normal, i.e., there exists an $x_0 \in \mathbb{R}^{\dot{n}}$ such that $u(x_0) = 1$; (ii) u is fuzzy convex; (iii) u is upper semi-continuous; (iv) $[u]^0 = \overline{\{x \in \mathbb{R}^{\dot{n}} : u(x) > 0\}}$ is compact. Moreover denote $[u]^\alpha = \{x \in \mathbb{R}^{\dot{n}} : u(x) \ge \alpha\}$ for $\alpha \in (0, 1]$, $[u]^\alpha$ is a nonempty compact convex subset of $\mathbb{R}^{\dot{n}}$, $[u + v]^\alpha = [u]^\alpha + [u]^\alpha$ and $[cu]^\alpha = c[u]^\alpha$ for any $u, v \in E^{\dot{n}}$ and $c \in \mathbb{R}^+$.*

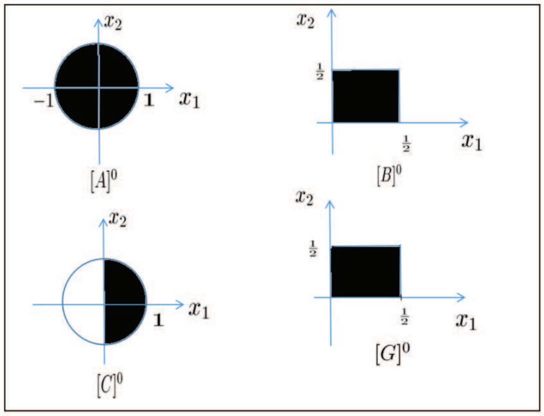

Figure 1: Schematic diagram of the "0"-level set

Example 2.5. *Let* $\dot{n} = 3$, $x_{\dot{n}} = (x_1, x_2, x_3)^T \in \mathbb{R}^3$, $u(x_{\dot{n}}) = \max_{1\leq n\leq 3}\{x_n\}$ *for* $\sum_{n=1}^{3} x_n^2 \leq 1$ *and* $x_n \in [0,1]$, $u(x_{\dot{n}}) = 0$ *otherwise. Then* $u \in E^3$. *In fact, (i)* $u(x_{\dot{n}}) = 1$ *if* $x_{\dot{n}} = (1,0,0)^T$*; (ii) let* $x_{\dot{n}1} = (x_{11}, x_{21}, x_{31})^T$ *and* $x_{\dot{n}2} = (x_{12}, x_{22}, x_{32})^T$, *if* $\sum_{n=1}^{3} x_{nl}^2 \leq 1$ *for* $l = 1,2$, *then* $\sum_{n=1}^{3}[\lambda x_{n1} + (1-\lambda)x_{n2}]^2 \leq 1$. *Hence for* $x_{hm} > 0$, $h = 1,2,3$, $m = 1,2$, *there exists* $n_0 \in \{1,2,3\}$ *such that* $u\big(\lambda x_{\dot{n}1} + (1-\lambda)x_{\dot{n}2}\big) = \lambda x_{n_01} + (1-\lambda)x_{n_02} \geq \min\{x_{n_01}, x_{n_02}\}$, *which implies* $u(\lambda x_{\dot{n}1} + (1-\lambda)x_{\dot{n}2}) \geq \min\{u(x_{\dot{n}1}), u(x_{\dot{n}2})\}$*; (iii) immediately; (iv)* $[u]^0 = \{(x_1,x_2,x_3)^T : x_1^2 + x_2^2 + x_3^2 \leq 1, x_n \in [0,1], n = 1,2,3\}$ *is compact.*

Now, we introduce the notion of fuzzy quaternion vector in the multidimensional quaternion space.

Definition 2.6. *If* $h' : \mathbb{Q}^{\dot{n}} \to [0,1]$ *and satisfies* $h'(a_{\dot{n}} + b_{\dot{n}}i + c_{\dot{n}}j + d_{\dot{n}}k) = \min\{A(a_{\dot{n}}), B(b_{\dot{n}}), C(c_{\dot{n}}), G(d_{\dot{n}})\}$, *then* h' *is called a fuzzy quaternion vector, where* $a_{\dot{n}}, b_{\dot{n}}, c_{\dot{n}}, d_{\dot{n}} \in \mathbb{R}^{\dot{n}}$, $A, B, C, G \in E^{\dot{n}}$. *The set of all fuzzy quaternion vector is denoted by* $\mathbb{Q}_F^{\dot{n}}$.

Example 2.7. *Let* $\dot{n} = 2$, $x_{\dot{n}} = (x_1, x_2)$,

$$A(x_{\dot{n}}) = \begin{cases} |x_1|, & x_1^2 + x_2^2 \leq 1, \\ 0, & \textit{otherwise}; \end{cases} \qquad B(x_{\dot{n}}) = \begin{cases} \frac{\sin x_1\pi + \cos x_2\pi}{2}, & x_1, x_2 \in [0, \frac{1}{2}], \\ 0, & \textit{otherwise}; \end{cases}$$

$$C(x_{\dot{n}}) = \begin{cases} x_1, & x_1^2 + x_2^2 \leq 1, \ x_1 \geq 0, \\ 0, & \textit{otherwise}; \end{cases} \qquad G(x_{\dot{n}}) = \begin{cases} \frac{2-\sin x_1\pi - \cos x_2\pi}{2}, & x_1, x_2 \in [0, \frac{1}{2}], \\ 0, & \textit{otherwise}. \end{cases}$$

Then $h' \in \mathbb{Q}_F^2$, *i.e.,* $h'[a_1 + b_1i + c_1j + d_1k, a_2 + b_2i + c_2j + d_2k]^T := L = \max\{|a_1|, \frac{\sin b_1\pi + \cos b_2\pi}{2}, c_1, \frac{2-\sin d_1\pi - \cos d_2\pi}{2}\}$ *for* $a_1^2 + a_2^2 \leq 1$, $b_1, b_2, d_1, d_2 \in [0, \frac{1}{2}]$, $c_1^2 + c_2^2 \leq 1$ *and* $c_1 \geq 0$*;* $L = 0$ *otherwise,* $[A]^0 = \{(x_1,x_2)^T : x_1^2 + x_2^2 \leq 1\}$, $[B]^0 = [G]^0 = [0, \frac{1}{2}] \times [0, \frac{1}{2}]$, $[C]^0 = \{(x_1,x_2)^T : x_1^2 + x_2^2 \leq 1, x_1 \geq 0\}$ *(see Figure 1),* $h'[1 + \frac{1}{2}i + j, \frac{1}{2}k]^T = 1$, *i.e.,* $a_{\dot{n}} = c_{\dot{n}} = [1,0]^T$, $b_{\dot{n}} = [\frac{1}{2}, 0]^T$, $d_{\dot{n}} = [0, \frac{1}{2}]^T$. *In fact,* $A(a_{\dot{n}}) = C(c_{\dot{n}}) = 1$, $B(b_{\dot{n}}) = \frac{\sin\frac{\pi}{2} + \cos 0}{2} = 1$, $G(d_{\dot{n}}) = \frac{2 - \sin 0 - \cos\frac{\pi}{2}}{2} = 1$, $h'(a_{\dot{n}} + b_{\dot{n}}i + c_{\dot{n}}j + d_{\dot{n}}k) = \min\{A(a_{\dot{n}}), B(b_{\dot{n}}), C(c_{\dot{n}}), G(d_{\dot{n}})\} = 1$.

Theorem 2.8. *Let* $s', h' \in \mathbb{Q}_F^{\dot{n}}$, $A, B, C, G \in E^{\dot{n}}$, $\alpha \in (0,1]$, *then*

(i) $[h']^\alpha = [A]^\alpha \times [B]^\alpha \times [C]^\alpha \times [G]^\alpha$, *where* $[A]^\alpha = \{x_{\dot{n}} \in \mathbb{R}^{\dot{n}} : A(x_{\dot{n}}) \in [\alpha, 1]\}$, $[B]^\alpha = \{x_{\dot{n}} \in \mathbb{R}^{\dot{n}} : B(x_{\dot{n}}) \in [\alpha, 1]\}$, $[C]^\alpha = \{x_{\dot{n}} \in \mathbb{R}^{\dot{n}} : C(x_{\dot{n}}) \in [\alpha, 1]\}$, $[G]^\alpha = \{x_{\dot{n}} \in \mathbb{R}^{\dot{n}} : G(x_{\dot{n}}) \in [\alpha, 1]\}$*;*

(ii) h' *is a fuzzy convex set for any* $h' \in \mathbb{Q}_F^{\dot{n}}$*;*

(iii) $[h']^{\alpha_1} \supseteq [h']^{\alpha_2}$ *if* $0 < \alpha_1 \leq \alpha_2 \leq 1$*;*

(iv) $[h']^0 = [A]^0 \times [B]^0 \times [C]^0 \times [G]^0$ *is compact.*

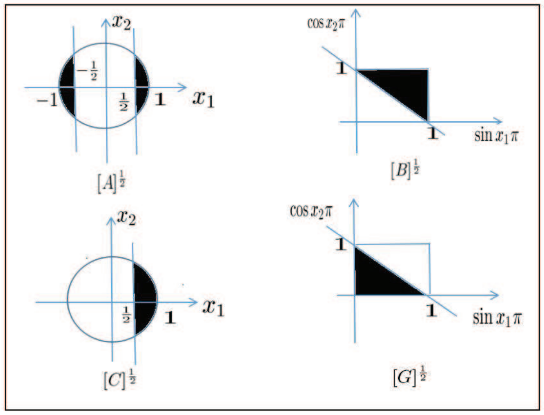

Figure 2: Schematic diagram of the "$\frac{1}{2}$"-level set

Proof. **Case** (i). Let $z = a_{\dot n}+b_{\dot n}i+c_{\dot n}j+d_{\dot n}k \in [h']^\alpha$, i.e., $h'(z) = \min\{A(a_{\dot n}), B(b_{\dot n}), C(c_{\dot n}), G(d_{\dot n}\} \geq \alpha$, then $A(a_{\dot n}) \geq \alpha$, $B(b_{\dot n}) \geq \alpha$, $C(c_{\dot n}) \geq \alpha$, $G(d_{\dot n}) \geq \alpha$, i.e., $a_{\dot n} \in [A]^\alpha$, $b_{\dot n} \in [B]^\alpha$, $c_{\dot n} \in [C]^\alpha$, $d_{\dot n} \in [G]^\alpha$. Hence $z \in [A]^\alpha\times[B]^\alpha\times[C]^\alpha\times[G]^\alpha$. On the other hand, if $z = a_{\dot n}+b_{\dot n}i+c_{\dot n}j+d_{\dot n}k \in [A]^\alpha\times[B]^\alpha\times[C]^\alpha\times[G]^\alpha$, i.e., $a_{\dot n} \in [A]^\alpha$, $b_{\dot n} \in [B]^\alpha$, $c_{\dot n} \in [C]^\alpha$ and $d_{\dot n} \in [G]^\alpha$, then $h'(z) = \min\{A(a_{\dot n}), B(b_{\dot n}), C(c_{\dot n}), G(d_{\dot n}\} \geq \alpha$, which implies $z \in [h']^\alpha$. Thus $[h']^\alpha = [A]^\alpha\times[B]^\alpha\times[C]^\alpha\times[G]^\alpha$.

Case (ii). Let $x_1 = a_{10} + a_{11}i + a_{12}j + a_{13}k$ and $x_2 = a_{20} + a_{21}i + a_{22}j + a_{23}k$, $a_{hm} \in \mathbb{R}^{\dot n}$. By Definition 2.4 (ii), we have $A\big(\lambda x_{10}+(1-\lambda)x_{20}\big) \geq \min\{A(x_{10}), A(x_{20})\}$, $B\big(\lambda x_{11}+(1-\lambda)x_{21}\big) \geq \min\{B(x_{11}), B(x_{21})\}$, $C\big(\lambda x_{12}+(1-\lambda)x_{22}\big) \geq \min\{C(x_{12}), C(x_{22})\}$, $G\big(\lambda x_{13} + (1-\lambda)x_{23}\big) \geq \min\{G(x_{13}), G(x_{23})\}$. Hence

$$\begin{aligned} h'\big(\lambda x_1 + (1-\lambda)x_2\big) &= h'\big(\lambda x_{10} + (1-\lambda)x_{20} + \lambda x_{11}i + (1-\lambda)x_{21}i + \lambda x_{12}j + (1-\lambda)x_{22}j + \lambda x_{13}k \\ &\quad +(1-\lambda)x_{23}k\big) = \min\big\{A\big(\lambda x_{10} + (1-\lambda)x_{20}\big), B\big(\lambda x_{11} + (1-\lambda)x_{21}\big), C\big(\lambda x_{12} \\ &\quad +(1-\lambda)x_{22}\big), G(\lambda x_{13} + (1-\lambda)x_{23})\big\} \\ &\geq \min\{A(x_{10}), A(x_{20}), B(x_{11}), B(x_{21}), C(x_{12}), C(x_{22}), G(x_{13}), G(x_{23})\} \\ &= \min\{h'(x_1), h'(x_2)\}. \end{aligned}$$

Case (iii). For $0 < \alpha_1 \leq \alpha_2 \leq 1$, if $x \in [h']^{\alpha_2}$, then $h'(x) \geq \alpha_2 \geq \alpha_1$, i.e., $x \in [h']^{\alpha_1}$. Hence $[h']^{\alpha_1} \supseteq [h']^{\alpha_2}$.

Case (iv). Let $h'_m = A_m \times B_m \times C_m \times G_m$ be a sequence on $\mathbb{Q}_F^{\dot n}$. Since the $[u]^0$ is compact in $\mathbb{R}^{\dot n}$ for any $u \in E^{\dot n}$ and $A_m \in E^{\dot n}$, then there exists a subsequence $\{A_{m_{l1}}\}$ of the sequence $\{A_m\}$ such that $\{A_{m_{l1}}\}$ is convergence on $[A]^0$. It is easy to see that the sequence $\{A_{m_{l1}} \times B_{m_{l1}} \times C_{m_{l1}} \times G_{m_{l1}}\}$ is a subsequence of $\{A_m \times B_m \times C_m \times G_m\}$. Similarly, there exists a subsequence $\{B_{m_{l2}}\}$ of the sequence $\{B_{m_{l1}}\}$ such that $\{B_{m_{l2}}\}$ is convergent on $[B]^0$, i.e., $\{A_{m_{l2}}, B_{m_{l2}}\}$ is convergent on $[A]^0 \times [B]^0$. Similarly, we can obtain a subsequence $\{A_{m_{l4}} \times B_{m_{l4}} \times C_{m_{l4}} \times G_{m_{l4}}\}$ of $\{h'_m\}$ such that $\{A_{m_{l4}} \times B_{m_{l4}} \times C_{m_{l4}} \times G_{m_{l4}}\}$ convergent, i.e., $\{h'_{m_{l4}}\}$ is convergence, which implies that for any sequence $\{h'_m\}$ there exist a subsequence $\{h'_{m_{l4}}\}$ such that $\{h'_{m_{l4}}\}$ is convergent on $[h']^\alpha$. Hence $[h']^0 = [A]^0 \times [B]^0 \times [C]^0 \times [G]^0$ is compact. The proof is completed. □

Example 2.9. *Let* $\dot n = 2$, $h' = A \times B \times C \times G$, $h'[a_1 + b_1i + c_1j + d_1k, a_2 + b_2i + c_2j + d_2k]^T := H$, $H = \max\{|a_1|, \frac{\sin b_1\pi+\cos b_2\pi}{2}, c_1, \frac{2-\sin d_1\pi-\cos d_2\pi}{2}\}$ *for* $a_1^2 + a_2^2 \leq 1$, $b_1, b_2, d_1, d_2 \in [0, \frac{1}{2}]$, $c_1^2 + c_2^2 \leq 1$ *and* $c_1 \geq 0$*;* $H = 0$ *otherwise. Then* $[h']^{\frac{1}{2}} = [A]^{\frac{1}{2}} \times [B]^{\frac{1}{2}} \times [C]^{\frac{1}{2}} \times [G]^{\frac{1}{2}}$*, where* $[A]^{\frac{1}{2}} = \{(x_1, x_2)^T : x_1^2 + x_2^2 \leq 1, x_1 \in [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]\}$, $[B]^{\frac{1}{2}} = \{(x_1, x_2)^T, \sin x_1\pi + \cos x_2\pi \geq 1,\ x_1, x_2 \in [0, \frac{1}{2}]\}$, $[C]^{\frac{1}{2}} = \{(x_1, x_2)^T : x_1^2 + x_2^2 \leq 1, x_1 \in [\frac{1}{2}, 1]\}$, $[G]^{\frac{1}{2}} = \{(x_1, x_2)^T : \sin x_1\pi + \cos x_2\pi \leq 1,\ x_1, x_2 \in [0, \frac{1}{2}]\}$ *(see Figure 2).*

Definition 2.10. *Let* $s', h' \in \mathbb{Q}_F^{\dot n}$*. If there exists* $q' \in \mathbb{Q}_F^{\dot n}$ *such that* $s' = h' \oplus q'$*, then* q' *is called the* H*-difference of* s', h' *and it can be expressed by* $s' \ominus h'$*. Moreover, we define* $0'$ *by*

$$0'(x_{\dot n}) = \begin{cases} 1, \textit{ for } x_{\dot n} = (x_{\dot n1}, x_{\dot n2}, \ldots, x_{\dot n\dot n})^T,\ \ x_{\dot n1} = \ldots = x_{\dot n\dot n} = 0, \\ 0,\ \ \textit{otherwise}. \end{cases}$$

For $h' \in \mathbb{Q}_F^{\dot n}$*, if there exists* $h'' \in \mathbb{Q}_F^{\dot n}$ *such that* $h' \oplus h'' = 0'$*, then we denote* h'' *by* $\ominus h'$*.*

Example 2.11. *Let $\dot{n}=2$, for the hiper-cubes $Q_1=[q_1,q_2]\times[q_3,q_4]$ and $P_1=[p_1,p_2]\times[p_3,p_4]$, where $q_m,p_m\in\mathbb{Q}$, $m=1,2,3,4$, i.e., $Q_1=[q_{10},q_{20}]\times[q_{11},q_{21}]\times[q_{12},q_{22}]\times[q_{13},q_{23}]\times[q_{30},q_{40}]\times[q_{31},q_{41}]\times[q_{32},q_{42}]\times[q_{33},q_{43}]$, define the addition between Q_1 and P_1 by $Q_1+P_1=\prod_{m=0}^{3}[q_{1m}+p_{2m},q_{2m}+p_{1m}]\prod_{m=0}^{3}[q_{3m}+p_{4m},q_{4m}+p_{3m}]$,*

$$h'\begin{bmatrix}a_1+b_1i+c_1j+d_1k\\ a_2+b_2i+c_2j+d_2k\end{bmatrix}=\begin{cases}\max\{\frac{a_1+a_2}{2},\frac{\sin b_1\pi+\cos b_2\pi}{2},c_1,\frac{2-\sin d_1\pi-\cos d_2\pi}{2}\}, & a_1,a_2,c_1,c_2\in[0,1],\\ b_1,b_2,d_1,d_2\in[0,\frac{1}{2}], & \\ 0, & \text{otherwise}\end{cases}$$

$$0'\begin{bmatrix}a_1+b_1i+c_1j+d_1k\\ a_2+b_2i+c_2j+d_2k\end{bmatrix}=\begin{cases}1, & a_1=a_2=b_1=b_2=c_1=0,\\ c_2=d_1=d_2=0, & \\ 0, & \text{otherwise.}\end{cases}$$

Then $\ominus h'[a_1+b_1i+c_1j+d_1k,a_2+b_2i+c_2j+d_2k]^T:=R=\max\{\frac{-a_1-a_2}{2},\frac{-\sin b_1\pi+\cos b_2\pi}{2},-c_1,\frac{2+\sin d_1\pi-\cos d_2\pi}{2}\}$ for $a_1,a_2,c_1,c_2\in[-1,0]$ and $b_1,b_2,d_1,d_2\in[\frac{-1}{2},0]$; $R=0$ otherwise. In fact, $[h']^0=[0,1]\times[0,\frac{1}{2}]\times[0,1]\times[0,\frac{1}{2}]\times[0,1]\times[0,\frac{1}{2}]\times[0,1]\times[0,\frac{1}{2}]$, $[\ominus h']^0=[-1,0]\times[\frac{-1}{2},0]\times[-1,0]\times[\frac{-1}{2},0]\times[-1,0]\times[\frac{-1}{2},0]\times[-1,0]\times[\frac{-1}{2},0]$, $[h']^0+[\ominus h']^0=\{0\}\times\{0\}\times\{0\}\times\{0\}\times\{0\}\times\{0\}\times\{0\}\times\{0\}=[0']^0$.

Lemma 2.12. [6] *Let $u,v,w\in E^{\dot{n}}$ and $\lambda\in\mathbb{R}$. Then $D_{\mathbb{R}^{\dot{n}}}(u\oplus w,v\oplus w)=D_{\mathbb{R}^{\dot{n}}}(u,v)$ and $D_{\mathbb{R}^{\dot{n}}}(\lambda u,\lambda v)=|\lambda|D_{\mathbb{R}^{\dot{n}}}(u,v)$, recall that the Huasdorff metric is defined as $d(A,B)=\inf\{\epsilon:A\subset N(B,\epsilon),B\subset N(A,\epsilon)\}$, where $A,B\in\mathscr{P}_\kappa(\mathbb{R}^{\dot{n}})$ and $N(A,\epsilon)=\{x\in\mathbb{R}^{\dot{n}}:\|x-y\|<\epsilon$ for some $y\in A\}$, the symbol $\mathscr{P}_\kappa(\mathbb{R}^{\dot{n}})$ denoted the family of all nonempty compact convex subsets of $\mathbb{R}^{\dot{n}}$.*

Now, based on the properties of fuzzy quaternion vectors, we introduce some fuzzy arithmetics in the multidimensional fuzzy quaternion space which will be used later.

Definition 2.13. *Let $s',h'\in\mathbb{Q}_F^{\dot{n}}$, $s'=(X,Y,Z,W)$ and $h'=(A,B,C,G)$. Then*

(i) the addition and multiplication is defined by $s'\oplus h'=(X\oplus A,Y\oplus B,Z\oplus C,W\oplus G)$ and $s'h'=(XA\ominus YB\ominus ZC\ominus WG,XB\oplus YA\oplus ZG\ominus WC,XC\ominus YG\oplus ZA\oplus WB,XG\ominus YC\ominus ZB\oplus WA)$;

(ii) the scalar multiplication is defined by $\omega h'=(xA\ominus yB\ominus zC\ominus wG,xB\oplus yA\oplus zG\ominus wC,xC\ominus yG\oplus zA\oplus wB,xG\ominus yC\ominus zB\oplus wA)$ for $\omega=x+yi+zj+wk$, $[\|\omega\|h']^\alpha=\|\omega\|[h']^\alpha$;

(iii) let $Q\in\mathbb{Q}_{\dot{n}\times\dot{n}}$, define $[Qh']^\alpha$ by $[Qh]^\alpha=Q[h']^\alpha$ for any $\alpha\in[0,1]$;

(iv) the metric between s' and h' is defined by $D(s',h')=\sup_{\alpha\in[0,1]}\{d([X]^\alpha,[A]^\alpha)+d([Y]^\alpha,[B]^\alpha)+d([Z]^\alpha,[C]^\alpha)+d([W]^\alpha,[G]^\alpha)\}$, where $d([X]^\alpha,[A]^\alpha)=\inf\{r:[X]^\alpha\subset N([A]^\alpha,r),[A]^\alpha\subset N([X]^\alpha,r)\}$, $N([A]^\alpha,r)=\{x_{\dot{n}}\in\mathbb{R}^{\dot{n}}:\|x_{\dot{n}}-y_{\dot{n}}\|<r$ for some $y_{\dot{n}}\in[A]^\alpha\}$, similar to $d([Y]^\alpha,[B]^\alpha)$, $d([Z]^\alpha,[C]^\alpha)$ and $d([W]^\alpha,[G]^\alpha)$.

Based on fuzzy arithmetics of fuzzy quaternion vectors, the following theorem can be obtained.

Theorem 2.14. *Let $x,y,z\in\mathbb{Q}_F^{\dot{n}}$, $Q\in\mathbb{Q}_{\dot{n}\times\dot{n}}$. Then (i) $D(x,y)\le D(x,z)+D(z,y)$; (ii) $D(x\oplus z,y\oplus z)=D(x,y)$; (iii) $D(Qx,Qy)\le\|Q\|D(x,y)$; (iv) $x\oplus(\ominus y)=x\ominus y$ and $x\ominus(\ominus y)=x\oplus y$.*

Proof. Let $x=x_0+x_1i+x_2j+x_3k$, $y=y_0+y_1i+y_2j+y_3k$, $z=z_0+z_1i+z_2j+z_3k$, $\alpha\in[0,1]$, where $x_m,y_m,z_m\in E^{\dot{n}}$, $m=0,1,2,3$.

(*i*) Since $\|x_{\dot{n}}-y_{\dot{n}}\|\le\|x_{\dot{n}}-z_{\dot{n}}\|+\|z_{\dot{n}}-y_{\dot{n}}\|$ for $x_{\dot{n}},y_{\dot{n}},z_{\dot{n}}\in\mathbb{R}^{\dot{n}}$. Then $d([x_0]^\alpha,[y_0]^\alpha)\le d([x_0]^\alpha,[z_0]^\alpha)+d([x_0]^\alpha,[y_0]^\alpha)$ and by $\sup_{\alpha\in[0,1]}\{f(\alpha)+g(\alpha)\}\le\sup_{\alpha\in[0,1]}f(\alpha)+\sup_{\alpha\in[0,1]}g(\alpha)$, we have

$$\begin{aligned}D(x,y)&=\sup_{\alpha\in[0,1]}\{d([x_0]^\alpha,[y_0]^\alpha)+d([x_1]^\alpha,[y_1]^\alpha)+d([x_2]^\alpha,[y_2]^\alpha)+d([x_3]^\alpha,[y_3]^\alpha)\}\\&\le\sup_{\alpha\in[0,1]}\{d([x_0]^\alpha,[z_0]^\alpha)+d([x_1]^\alpha,[z_1]^\alpha)+d([x_2]^\alpha,[z_2]^\alpha)+d([x_3]^\alpha,[z_3]^\alpha)\\&\quad+d([z_0]^\alpha,[y_0]^\alpha)+d([z_1]^\alpha,[y_1]^\alpha)+d([z_2]^\alpha,[y_2]^\alpha)+d([z_3]^\alpha,[y_3]^\alpha)\}\\&\le\sup_{\alpha\in[0,1]}\{d([x_0]^\alpha,[z_0]^\alpha)+d([x_1]^\alpha,[z_1]^\alpha)+d([x_2]^\alpha,[z_2]^\alpha)+d([x_3]^\alpha,[z_3]^\alpha)\}\\&\quad+\sup_{\alpha\in[0,1]}\{d([z_0]^\alpha,[y_0]^\alpha)+d([z_1]^\alpha,[y_1]^\alpha)+d([z_2]^\alpha,[y_2]^\alpha)+d([z_3]^\alpha,[y_3]^\alpha)\}\\&=D(x,z)+D(z,y).\end{aligned}$$

(ii) By Lemma 2.12 and Definition 2.13, we have

$$\begin{aligned} D(x\oplus z, y\oplus z) &= D_{\mathbb{R}^{\dot{n}}}(x_0\oplus z_0, y_0\oplus z_0)+D_{\mathbb{R}^{\dot{n}}}(x_1\oplus z_1, y_1\oplus z_1)+D_{\mathbb{R}^{\dot{n}}}(x_2\oplus z_2, y_2\oplus z_2)\\ &\quad +D_{\mathbb{R}^{\dot{n}}}(x_3\oplus z_3, y_3\oplus z_3)=D_{\mathbb{R}^{\dot{n}}}(x_0,y_0)+D_{\mathbb{R}^{\dot{n}}}(x_1,y_1)+D_{\mathbb{R}^{\dot{n}}}(x_2,y_2)+D_{\mathbb{R}^{\dot{n}}}(x_3,y_3)\\ &= D(x,y). \end{aligned}$$

(iii) By Lemma 2.12 and Definition 2.13, we have

$$\begin{aligned} D(Qx,Qy) &= \sup_{\alpha\in[0,1]}\{d([Qx_0]^\alpha,[Qy_0]^\alpha)+d([Qx_1]^\alpha,[Qy_1]^\alpha)+d([Qx_2]^\alpha,[Qy_2]^\alpha)+d([Qx_3]^\alpha,[Qy_3]^\alpha)\}\\ &= \sup_{\alpha\in[0,1]}\{d(Q[x_0]^\alpha,Q[y_0]^\alpha)+d(Q[x_1]^\alpha,Q[y_1]^\alpha)+d(Q[x_2]^\alpha,Q[y_2]^\alpha)+d(Q[x_3]^\alpha,Q[y_3]^\alpha)\}. \end{aligned}$$

Moerover, we have $d(Q[x_0]^\alpha, Q[y_0]^\alpha)=\inf\{r: Q[x_0]^\alpha\subset N(Q[y_0]^\alpha,r), Q[y_0]^\alpha\subset N(Q[x_0]^\alpha,r)\}$, $N(Q[x_0]^\alpha,r)=\{Qx_{\dot{n}}\in\mathbb{R}^{\dot{n}}: \|Qx_{\dot{n}}-Qy_{\dot{n}}\|<r \text{ for some } y_{\dot{n}}\in[x_0]^\alpha\}$. By $\|Qx_{\dot{n}}-Qy_{\dot{n}}\|\le\|Q\|\|x_{\dot{n}}-y_{\dot{n}}\|$, then we have $d(Q[x_0]^\alpha,Q[y_0]^\alpha)\le\|Q\|d([x_0]^\alpha,[y_0]^\alpha)$. Hence $D(Qx,Qy)\le\|Q\|D(x,y)$.

(iv) Let $w=x\ominus y$, by Definition 2.4 and Theorem 2.8 (i), we have $x=w\oplus y$, i.e., $[x]^\alpha=[w]^\alpha+[y]^\alpha$ and $[w]^\alpha=[x]^\alpha-[y]^\alpha$. On the other hand, $[x\oplus(\ominus y)]^\alpha=[x]^\alpha+[\ominus y]^\alpha$ and $[y]^\alpha+[\ominus y]^\alpha=[0']^\alpha$. Hence $[x\oplus(\ominus y)]^\alpha=[x]^\alpha-[y]^\alpha$, i.e., $[x\oplus(\ominus y)]^\alpha=[x\ominus y]^\alpha$. Hence $x\oplus(\ominus y)=x\ominus y$. Next, we will prove $x\ominus(\ominus y)=x\oplus y$. Let $z=x\ominus(\ominus y)$, then $x=z\oplus(\ominus y)=z\ominus y$, i.e., $z=x\oplus y$. Hence $x\ominus(\ominus y)=x\oplus y$. The proof is completed. □

For more convenient discussion of the Hyers-Ulam-Rassias stability of the fuzzy iteration solutions for the quaternion multidimensional fuzzy difference equations with impulses, we introduce the following definition.

Definition 2.15. *Let $y:\mathbb{Z}\to\mathbb{Q}_F^{\dot{n}}$, $n_0\in\mathbb{Z}$. $\Delta y(n_0)$ is said to be the fuzzy difference between $y(n_0)$ and $y(n_0+1)$ if $\Delta y(n_0)$ exists and is one of the following:*

(H_1) *$y(n_0+1)\ominus y(n_0)=\Delta y(n_0)$ (y is called admitting (H_1)-fuzzy difference at n_0) or*

(H_2) *$y(n_0)\ominus y(n_0+1)=\Delta y(n_0)$ (y is called admitting (H_2)-fuzzy difference at n_0).*

Remark 2.16. *In Definition 2.15, let $n_0\in\mathbb{Z}$, $k\in\mathbb{N}^+$, $y:\mathbb{Z}\to\mathbb{Q}_F^{\dot{n}}$, then (i) $y(n_0+k)=y(n_0)\oplus\Delta y(n_0)\oplus\Delta y(n_0+1)\oplus\ldots\oplus\Delta y(n_0+k-1)$ if y is (H_1)-admitting fuzzy difference on $[n_0,n_0+k]$; (ii) $y(n_0+k)=y(n_0)\ominus\Delta y(n_0)\ominus\Delta y(n_0+1)\ominus\ldots\ominus\Delta y(n_0+k-1)$ if y is (H_2)-admitting fuzzy difference on $[n_0,n_0+k]$.*

3 Hyers-Ulam-Rassias stability analysis

In this section, we will consider the Hyers-Ulam-Rassias stability of the following three types of multidimensional impulsive fuzzy nonlinear difference equations:

$$\begin{cases} y(n+1)=A(n)y(n)\oplus f(n,y(n)), & n\in\dot{N}\backslash\{n_m\},\\ \Delta y(n_m)=I_m\big(y(n_m)\big), & m\in\mathbb{M}; \end{cases}\tag{1}$$

$$\begin{cases} y(n+1)\oplus A(n)y(n)=f(n,y(n)), & n\in\dot{N}\backslash\{n_m\},\\ \Delta y(n_m)=I_m\big(y(n_m)\big), & m\in\mathbb{M}; \end{cases}\tag{2}$$

$$\begin{cases} y(n+1)\oplus f(n,y(n))=A(n)y(n), & n\in\dot{N}\backslash\{n_m\},\\ \Delta y(n_m)=I_m\big(y(n_m)\big), & m\in\mathbb{M}, \end{cases}\tag{3}$$

where $A:\mathbb{Z}\to\mathbb{Q}^{\dot{n}\times\dot{n}}$, $f:\mathbb{Z}\times\mathbb{Q}_F^{\dot{n}}\to\mathbb{Q}_F^{\dot{n}}$, $I_m:\mathbb{Q}_F^{\dot{n}}\to\mathbb{Q}_F^{\dot{n}}$, $\dot{N}=\{n\in\mathbb{Z}: n_0\le n<N\}$ for some finite number $N\in\mathbb{Z}$, $n_m\in\dot{N}$, $n_m<n_{m+1}$, $m\in\mathbb{M}=\{1,2,\ldots,k\}$, where $n_k<N$ and $k\in\mathbb{N}$ is a finite number.

Remark 3.1. *Let $n, s \in \mathbb{Z}$, if y is (H_1)-admitting fuzzy difference on $\dot{N}$, through the iteration methods of solutions, then the Cauchy formulas (without impulsive points) of* (1), (2) *and* (3) *can be given by:*

$$\begin{aligned} w_1(n+1,s,y(s)) &= \prod_{v=0}^{n-s} A(n-v)y(s) \oplus \sum_{l=1}^{n-s} \prod_{v=0}^{n-(s+l)} A(n-v)f(s+l-1,y(s+l-1)) \oplus f(n,y(n)), \\ w_2(n+1,s,y(s)) &= f(n,y(n)) \oplus \prod_{v=0}^{n-s} \ominus A(n-v)y(s) \oplus \sum_{l=1}^{n-s} \prod_{v=0}^{n-(s+l)} \ominus A(n-v)f(s+l-1,y(s+l-1)), \\ w_3(n+1,s,y(s)) &= \prod_{v=0}^{n-s} A(n-v)y(s) \ominus \sum_{l=1}^{n-s} \prod_{v=0}^{n-(s+l)} A(n-v)f(s+l-1,y(s+l-1)) \ominus f(n,y(n)). \end{aligned}$$

Repeating the iteration steps with impulsive terms, the solutions of (1), (2) *and* (3) *can be given by:*

$$\begin{aligned} y_1(n,n_0,y(n_0)) &= \begin{cases} w_1(n,n_0,y(n_0)), & n \in [n_0,n_1], \\ y_1(n_m) \oplus I_m(y_1(n_m)), & n = n_m+1, \\ w_1(n,n_m+1,y_1(n_m) \oplus I_m(y_1(n_m))), & n_m+1 < n \le n_{m+1}; \end{cases} \\ y_2(n,n_0,y(n_0)) &= \begin{cases} w_2(n,n_0,y(n_0)), & n \in [n_0,n_1], \\ y_2(n_m) \oplus I_m(y_2(n_m)), & n = n_m+1, \\ w_2(n,n_m+1,y_2(n_m) \oplus I_m(y_2(n_m))), & n_m+1 < n \le n_{m+1}; \end{cases} \\ y_3(n,n_0,y(n_0)) &= \begin{cases} w_3(n,n_0,y(n_0)), & n \in [n_0,n_1], \\ y_3(n_m) \oplus I_m(y_3(n_m)), & n = n_m+1, \\ w_3(n,n_m+1,y_3(n_m) \oplus I_m(y_3(n_m))), & n_m+1 < n \le n_{m+1}, \end{cases} \end{aligned}$$

i.e.,

$$\begin{aligned} y_1(n_m+1,n_0,y(n_0)) &= \prod_{h=m-1}^{1} \prod_{l=0}^{n_{h+1}-(n_h+1)} A(n_{h+1}-l) \prod_{v=0}^{n_1-n_0} A(n_1-v)y(n_0) \oplus \sum_{l=0}^{m-2} \prod_{v=m-1}^{m-l-1} \prod_{h=0}^{n_{v+1}-(n_v+1)} A(n_{v+1}-h)(M_{1,m-l-2} \\ &\quad \oplus I_{m-l-1}(y(n_{m-l-1}))) \oplus M_{1,m-1} \oplus I_m(y(n_m)), \\ y_2(n_m+1,n_0,y(n_0)) &= \prod_{h=m-1}^{1} \prod_{l=0}^{n_{h+1}-(n_h+1)} \ominus A(n_{h+1}-l) \prod_{v=0}^{n_1-n_0} \ominus A(n_1-v)y(n_0) \\ &\quad \oplus \sum_{l=0}^{m-2} \prod_{v=m-1}^{m-l-1} \prod_{h=0}^{n_{v+1}-(n_v+1)} \ominus A(n_{v+1}-h)(M_{2,m-l-2} \oplus I_{m-l-1}(y(n_{m-l-1}))) \oplus M_{2,m-1} \oplus I_m(y(n_m)), \\ y_3(n_m+1,n_0,y(n_0)) &= \prod_{h=m-1}^{1} \prod_{l=0}^{n_{h+1}-(n_h+1)} A(n_{h+1}-l) \prod_{v=0}^{n_1-n_0} A(n_1-v)y(n_0) \oplus \sum_{l=0}^{m-2} \prod_{v=m-1}^{m-l-1} \prod_{h=0}^{n_{v+1}-(n_v+1)} A(n_{v+1}-h)(M_{3,m-l-2} \\ &\quad \oplus I_{m-l-1}(y(n_{m-l-1}))) \oplus M_{3,m-1} \oplus I_m(y(n_m)), \end{aligned}$$

where

$$\begin{aligned} M_{1,0} &= \sum_{l=1}^{n_1-n_0} \prod_{v=0}^{n_1-(n_0+l)} A(n_1-v)f(n_0+l-1,y(n_0+l-1)) \oplus f(n_1,y(n_1)), \\ M_{1,m} &= \sum_{l=1}^{n_{m+1}-n_m-1} \prod_{v=0}^{n_{m+1}-(n_m+l+1)} A(n_{m+1}-v)f(n_m+l,y(n_m+l)) \oplus f(n_{m+1},y(n_{m+1})), \\ M_{2,0} &= \sum_{l=1}^{n_1-n_0} \prod_{v=0}^{n_1-(n_0+l)} \ominus A(n_1-v)f(n_0+l-1,y(n_0+l-1)) \oplus f(n_1,y(n_1)), \\ M_{2,m} &= \sum_{l=1}^{n_{m+1}-n_m-1} \prod_{v=0}^{n_{m+1}-(n_m+l+1)} \ominus A(n_{m+1}-v)f(n_m+l,y(n_m+l)) \oplus f(n_{m+1},y(n_{m+1})), \end{aligned}$$

$$M_{3,0}=\ominus\sum_{l=1}^{n_1-n_0}\prod_{v=0}^{n_1-(n_0+l)}A(n_1-v)f(n_0+l-1,y(n_0+l-1))\ominus f(n_1,y(n_1)),$$
$$M_{3,m}=\ominus\sum_{l=1}^{n_{m+1}-n_m-1}\prod_{v=0}^{n_{m+1}-(n_m+l+1)}A(n_{m+1}-v)f(n_m+l,y(n_m+l))\ominus f(n_{m+1},y(n_{m+1})).$$

Remark 3.2. *Let $n,s\in\mathbb{Z}$, if y is (H_2)-admitting fuzzy difference on $\dot{N}$, through iteration method, then the Cauchy formulas (without impulsive points) of* (1), (2) *and* (3) *are given by:*

$$\begin{aligned}
w_1(n+1,s,y(s)) &= \prod_{v=0}^{n-s}A(n-v)y(s)\oplus\sum_{l=1}^{n-s}\prod_{v=0}^{n-(s+l)}A(n-v)f(s+l-1,y(s+l-1))\oplus f(n,y(n)),\\
w_2(n+1,s,y(s)) &= f(n,y(n))\oplus\prod_{v=0}^{n-s}\ominus A(n-v)y(s)\oplus\sum_{l=1}^{n-s}\prod_{v=0}^{n-(s+l)}\ominus A(n-v)f(s+l-1,y(s+l-1)),\\
w_3(n+1,s,y(s)) &= \prod_{v=0}^{n-s}A(n-v)y(s)\ominus\sum_{l=1}^{n-s}\prod_{v=0}^{n-(s+l)}A(n-v)f(s+l-1,y(s+l-1))\ominus f(n,y(n)).
\end{aligned}$$

Moreover, the solutions of (1), (2) *and* (3) *are given by:*

$$\begin{aligned}
y_1(n,n_0,y(n_0)) &= \begin{cases} w_1(n,n_0,y(n_0)), & n\in[n_0,n_1],\\ y_1(n_m)\ominus I_m(y_1(n_m)), & n=n_m+1,\\ w_1(n,n_m+1,y_1(n_m)\ominus I_m(y_1(n_m))), & n_m+1<n\le n_{m+1};\end{cases}\\
y_2(n,n_0,y(n_0)) &= \begin{cases} w_2(n,n_0,y(n_0)), & n\in[n_0,n_1],\\ y_2(n_m)\ominus I_m(y_2(n_m)), & n=n_m+1,\\ w_2(n,n_m+1,y_2(n_m)\ominus I_m(y_2(n_m))), & n_m+1<n\le n_{m+1};\end{cases}\\
y_3(n,n_0,y(n_0)) &= \begin{cases} w_3(n,n_0,y(n_0)), & n\in[n_0,n_1],\\ y_3(n_m)\ominus I_m(y_3(n_m)), & n=n_m+1,\\ w_3(n,n_m+1,y_3(n_m)\ominus I_m(y_3(n_m))), & n_m+1<n\le n_{m+1},\end{cases}
\end{aligned}$$

i.e.,

$$\begin{aligned}
y_1(n_m+1,n_0,y(n_0)) &= \prod_{h=m-1}^{1}\prod_{l=0}^{n_{h+1}-(n_h+1)}A(n_{h+1}-l)\prod_{v=0}^{n_1-n_0}A(n_1-v)y(n_0)\oplus\sum_{l=0}^{m-2}\prod_{v=m-1}^{m-l-1}\prod_{h=0}^{n_{v+1}-(n_v+1)}A(n_{v+1}-h)(M_{1,m-l-2}\\
&\quad\ominus I_{m-l-1}(y(n_{m-l-1})))\oplus M_{1,m-1}\ominus I_m(y(n_m)),\\
y_2(n_m+1,n_0,y(n_0)) &= \prod_{h=m-1}^{1}\prod_{l=0}^{n_{h+1}-(n_h+1)}\ominus A(n_{h+1}-l)\prod_{v=0}^{n_1-n_0}\ominus A(n_1-v)y(n_0)\\
&\quad\oplus\sum_{l=0}^{m-2}\prod_{v=m-1}^{m-l-1}\prod_{h=0}^{n_{v+1}-(n_v+1)}\ominus A(n_{v+1}-h)(M_{2,m-l-2}\ominus I_{m-l-1}(y(n_{m-l-1})))\oplus M_{2,m-1}\ominus I_m(y(n_m)),\\
y_3(n_m+1,n_0,y(n_0)) &= \prod_{h=m-1}^{1}\prod_{l=0}^{n_{h+1}-(n_h+1)}A(n_{h+1}-l)\prod_{v=0}^{n_1-n_0}A(n_1-v)y(n_0)\oplus\sum_{l=0}^{m-2}\prod_{v=m-1}^{m-l-1}\prod_{h=0}^{n_{v+1}-(n_v+1)}A(n_{v+1}-h)(M_{3,m-l-2}\\
&\quad\ominus I_{m-l-1}(y(n_{m-l-1})))\oplus M_{3,m-1}\ominus I_m(y(n_m)).
\end{aligned}$$

For $\forall\epsilon>0$, $\phi,\varphi:\mathbb{Z}\to[0,+\infty)$, consider the mapping $y:\dot{N}\to\mathbb{Q}_{\dot{n}\times\dot{n}}$ with the condition (H_1) or (H_2) satisfying one of the following:

$$\begin{cases} D(y(n+1),A(n)y(n)\oplus f(n,y(n)))\le\phi(n)\epsilon, & n\in\dot{N}\backslash\{n_m\},\\ D(\Delta y(n_m),I_m(y(n_m)))\le\varphi(m)\epsilon, & m\in\mathbb{M},\\ D(I_m(y),I_m(z))\le LD(y,z), & \text{for } y,z\in\mathbb{Q}_F^{\dot{n}} \text{ and } L>0;\end{cases}\tag{4}$$

$$\begin{cases} D(y(n+1)\oplus A(n)y(n),f(n,y(n)))\le\phi(n)\epsilon, & n\in\dot{N}\backslash\{n_m\},\\ D(\Delta y(n_m),I_m(y(n_m)))\le\varphi(m)\epsilon, & m\in\mathbb{M},\\ D(I_m(y),I_m(z))\le LD(y,z), & \text{for } y,z\in\mathbb{Q}_F^{\dot{n}} \text{ and } L>0;\end{cases}\tag{5}$$

$$\begin{cases} D(y(n+1)\oplus f(n,y(n)),A(n)y(n))\leq \phi(n)\epsilon, \;\; n\in \dot{N}\backslash\{n_m\}, \\ D(\Delta y(n_m),I_m(y(n_m)))\leq \varphi(m)\epsilon, \;\; m\in\mathbb{M}, \\ D(I_m(y),I_m(z))\leq LD(y,z), \text{ for } y,z\in\mathbb{Q}_F^{\dot{n}} \text{ and } L>0. \end{cases} \tag{6}$$

Definition 3.3. *We say* (1) ((2) *or* (3), *respectively) is Hyers-Ulam-Rassias stable under* (H_1) (*or* (H_2)) *if for each* $\epsilon>0$ *and each solution of* (4) ((5) *or* (6), *respectively) there exists a* $\tilde{y}\in\mathbb{Q}_F^{\dot{n}}$ *satisfying* (H_1) (*or* (H_2)) *and* $\tilde{y}$ *is a solution of* (1) ((2) *or* (3), *respectively) with* $D(y(n),\tilde{y}(n))\leq \hat{Q}(\varphi(n)+\phi(n))\epsilon$ *for* $\hat{Q}>0$ *and* $n\in\mathbb{Z}$.

Next, we will consider the stability of (1) under the condition (H_1). Notice that we will give the proof process of Theorem 3.4 in full, and through using the similar proof method, one can obtain Theorems 3.5-3.9 immediately and we will not repeat these similar proofs here.

Theorem 3.4. *Let* $\omega\in\mathbb{Q}$, $f:\mathbb{Z}\times\mathbb{Q}_F^{\dot{n}}\to\mathbb{Q}_F^{\dot{n}}$, y *be a solution of* (4) *and* $\tilde{y}$ *be a solution of* (1). *If*

$$\prod_{t=n-(n_m+1)}^{1} A^{-1}(n-t)\big[y(n)\ominus \sum_{l=1}^{n-(n_m+2)}\prod_{v=1}^{n-(n_m+1+l)} A(n-v)f(n_m+l,y(n_m+l))\ominus f(n-1,y(n-1))\big]$$
$$\ominus\sum_{l=0}^{m-2}\prod_{v=m-1}^{m-l-1}\prod_{h=0}^{n_{v+1}-(n_v+1)} A(n_{v+1}-h)(M_{1,m-l-2}\oplus I_{m-l-1}(y(n_{m-l-1})))\ominus M_{1,m-1}\ominus I_m(y(n_m))$$

exists for all $n\in\dot{N}\backslash\{n_m\}$. *Then there exists an unique* $y_0\in\mathbb{Q}_F^{\dot{n}}$ *with*

$$D(y(n),\tilde{y}(n))\leq\epsilon\prod_{v=1}^{n-(n_m+1)}\|A(n-v)\|\sum_{v=1}^{n-s}\prod_{t=n-(n_m+1)}^{v}\|A^{-1}(n-t)\|\phi(n-v)$$

for $n\in[n_m+1,n_{m+1}]$ *and* $s\in[n_m+1,n]$, $m\in\mathbb{M}$ *and*

$$\begin{aligned} D(y(n_{m+1}+1),\tilde{y}(n_{m+1}+1)) \quad &\leq \quad \epsilon\bigg(\varphi(m+1)+(1+L)\prod_{v=1}^{n_{m+1}-(n_m+1)}\|A(n_{m+1}-v)\| \\ &\times\sum_{v=1}^{n_{m+1}-s}\prod_{t=n_{m+1}-(n_m+1)}^{v}\|A^{-1}(n_{m+1}-t)\|\phi(n_{m+1}-v)\bigg). \end{aligned}$$

Proof. By Remark 3.1, for $n\in[n_m+1,n_m]$, we have

$$\begin{aligned} \tilde{y}(n) \quad = \quad & \prod_{v=1}^{n-(n_m+1)} A(n-v)\Big[\prod_{h=m-1}^{1}\prod_{l=0}^{n_{h+1}-(n_h+1)} A(n_{h+1}-l)\prod_{v=0}^{n_1-n_0} A(n_1-v)y(n_0) \\ & \oplus\sum_{l=0}^{m-2}\prod_{v=m-1}^{m-l-1}\prod_{h=0}^{n_{v+1}-(n_v+1)} A(n_{v+1}-h)(M_{1,m-l-2}\oplus I_{m-l-1}(y(n_{m-l-1})))\oplus M_{1,m-1} \\ & \oplus I_m(y(n_m))\Big]\oplus\sum_{l=1}^{n-(n_m+2)}\prod_{v=1}^{n-(n_m+1+l)} A(n-v)f(n_m+l,y(n_m+l))\oplus f(n-1,y(n-1)). \end{aligned}$$

$$\begin{aligned} \text{Let}\, q(n) \quad = \quad & \prod_{t=n-(n_m+1)}^{1} A^{-1}(n-t)\big[y(n)\ominus\sum_{l=1}^{n-(n_m+2)}\prod_{v=1}^{n-(n_m+1+l)} A(n-v)f(n_m+l,y(n_m+l)) \\ & \ominus f(n-1,y(n-1))\big]\ominus\sum_{l=0}^{m-2}\prod_{v=m-1}^{m-l-1}\prod_{h=0}^{n_{v+1}-(n_v+1)} A(n_{v+1}-h)(M_{1,m-l-2}\oplus I_{m-l-1}(y(n_{m-l-1}))) \\ & \ominus M_{1,m-1}\ominus I_m(y(n_m)). \end{aligned}$$

For all $n, s \in [n_m+1, n_{m+1}]$ and $n > s$, we have

$$\begin{aligned}
& D(q(n), q(s)) \\
= \; & D\Big(\prod_{t=n-(n_m+1)}^{1} A^{-1}(n-t)\big[y(n) \ominus \sum_{l=1}^{n-(n_m+2)} \prod_{v=1}^{n-(n_m+1+l)} A(n-v) f(n_m+l, y(n_m+l)) \\
& \ominus f(n-1, y(n-1))\big] \ominus \sum_{l=0}^{m-2} \prod_{v=m-1}^{m-l-1} \prod_{h=0}^{n_{v+1}-(n_v+1)} A(n_{v+1}-h)(M_{1,m-l-2} \oplus I_{m-l-1}(y(n_{m-l-1}))) \\
& \ominus M_{1,m-1} \ominus I_m(y(n_m)), \prod_{t=s-(n_m+1)}^{1} A^{-1}(s-t)\big[y(s) \ominus \sum_{l=1}^{s-(n_m+2)} \prod_{v=1}^{s-(n_m+1+l)} A(s-v) \\
& \times f(n_m+l, y(n_m+l)) \ominus f(s-1, y(s-1))\big] \ominus \sum_{l=0}^{m-2} \prod_{v=m-1}^{m-l-1} \prod_{h=0}^{n_{v+1}-(n_v+1)} A(n_{v+1}-h) \\
& \times (M_{1,m-l-2} \oplus I_{m-l-1}(y(n_{m-l-1}))) \ominus M_{1,m-1} \ominus I_m(y(n_m))\Big) \\
= \; & D\Big(\prod_{t=n-(n_m+1)}^{1} A^{-1}(n-t)\big[y(n) \ominus \sum_{l=1}^{n-(n_m+2)} \prod_{v=1}^{n-(n_m+1+l)} A(n-v) f(n_m+l, y(n_m+l)) \\
& \ominus f(n-1, y(n-1))\big], \prod_{t=s-(n_m+1)}^{1} A^{-1}(s-t)\big[y(s) \ominus \sum_{l=1}^{s-(n_m+2)} \prod_{v=1}^{s-(n_m+1+l)} A(s-v) \\
& \times f(n_m+l, y(n_m+l)) \ominus f(s-1, y(s-1))\big]\Big) \\
= \; & D\Big(\prod_{t=n-(n_m+1)}^{1} A^{-1}(n-t) y(n) \ominus \sum_{l=1}^{n-(n_m+2)} \prod_{t=1}^{l} A^{-1}(n_m+t) f(n_m+l, y(n_m+l)) \\
& \ominus \prod_{t=n-(n_m+1)}^{1} A^{-1}(n-t) f(n-1, y(n-1))\big], \prod_{t=s-(n_m+1)}^{1} A^{-1}(s-t) y(s) \\
& \ominus \sum_{l=1}^{s-(n_m+2)} \prod_{t=1}^{l} A^{-1}(n_m+l) f(n_m+l, y(n_m+l)) \ominus \prod_{t=s-(n_m+1)}^{1} A^{-1}(s-t) f(s-1, y(s-1))\Big) \\
= \; & D\Big(\prod_{t=n-(n_m+1)}^{1} A^{-1}(n-t) y(n) \ominus \sum_{l=s-n_m}^{n-(n_m+2)} \prod_{t=1}^{l} A^{-1}(n_m+t) f(n_m+l, y(n_m+l)) \\
& \ominus \prod_{t=n-(n_m+1)}^{1} A^{-1}(n-t) f(n-1, y(n-1)), \prod_{t=s-(n_m+1)}^{1} A^{-1}(s-t) y(s)\Big) \\
= \; & D\Big(\prod_{t=n-(n_m+1)}^{1} A^{-1}(n-t) y(n), \prod_{t=s-(n_m+1)}^{1} A^{-1}(s-t) y(s) \oplus \sum_{l=s-n_m}^{n-(n_m+2)} \prod_{t=1}^{l} A^{-1}(n_m+t) \\
& \times f(n_m+l, y(n_m+l)) \oplus \prod_{t=n-(n_m+1)}^{1} A^{-1}(n-t) f(n-1, y(n-1))\Big) \\
= \; & D\Big(\prod_{t=n-(n_m+1)}^{1} A^{-1}(n-t) y(n), \prod_{t=n-(n_m+1)}^{1} A^{-1}(n-t)\Big[\prod_{t=1}^{n-s} A(n-t) y(s) \\
& \oplus \sum_{l=s-n_m}^{n-(n_m+2)} \prod_{v=1}^{n-(n_m+l+1)} A(n-v) f(n_m+l, y(n_m+l)) \oplus f(n-1, y(n-1))\Big]\Big) \\
\leq \; & D\Big(\prod_{t=n-(n_m+1)}^{1} A^{-1}(n-t) y(n), \prod_{t=n-(n_m+1)}^{1} A^{-1}(n-t)[A(n-1) y(n-1)
\end{aligned}$$

$$
\begin{aligned}
&\oplus f(n-1,y(n-1))]\Big)+D\Big(\prod_{t=n-(n_m+1)}^{1}A^{-1}(n-t)[A(n-1)y(n-1)\\
&\oplus f(n-1,y(n-1))],\prod_{t=n-(n_m+1)}^{1}A^{-1}(n-t)[A(n-1)A(n-2)y(n-2)\\
&\oplus A(n-1)f(n-2,y(n-2))\oplus f(n-1,y(n-1))]\Big)+\ldots+\\
&+D\Big(\prod_{t=n-(n_m+1)}^{1}A^{-1}(n-t)\Big[\prod_{t=1}^{n-(s+1)}A(n-t)y(s+1)\oplus\sum_{l=s+1-n_m}^{n-(n_m+2)}\prod_{v=1}^{n-(n_m+l+1)}A(n-v)\\
&\times f(n_m+l,y(n_m+l))\oplus f(n-1,y(n-1))\Big]\Big),\prod_{t=n-(n_m+1)}^{1}A^{-1}(n-t)\Big[\prod_{t=1}^{n-s}A(n-t)y(s)\\
&\oplus\sum_{l=s-n_m}^{n-(n_m+2)}\prod_{v=1}^{n-(n_m+l+1)}A(n-v)f(n_m+l,y(n_m+l))\oplus f(n-1,y(n-1))\Big]\Big)\\
=\;&D\Big(\prod_{t=n-(n_m+1)}^{1}A^{-1}(n-t)y(n),\prod_{t=n-(n_m+1)}^{1}A^{-1}(n-t)[A(n-1)y(n-1)\\
&\oplus f(n-1,y(n-1))]\Big)+D\Big(\prod_{t=n-(n_m+1)}^{2}A^{-1}(n-t)y(n-1),\prod_{t=n-(n_m+1)}^{2}A^{-1}(n-t)\\
&\times[A(n-2)y(n-2)\oplus f(n-2,y(n-2))]\Big)+\ldots+\\
&+D\Big(\prod_{t=n-(n_m+1)}^{n-s}A^{-1}(n-t)y(s+1),\prod_{t=n-(n_m+1)}^{n-s}A^{-1}(n-t)[A(s)y(s)\oplus f(s,y(s))]\Big)\\
\le\;&\prod_{t=n-(n_m+1)}^{1}\|A^{-1}(n-t)\|\phi(n-1)\epsilon+\prod_{t=n-(n_m+1)}^{2}\|A^{-1}(n-t)\|\phi(n-2)\epsilon+\ldots+\\
&+\prod_{t=n-(n_m+1)}^{n-s}\|A^{-1}(n-t)\|\phi(s)\epsilon=\epsilon\sum_{v=1}^{n-s}\prod_{t=n-(n_m+1)}^{v}\|A^{-1}(n-t)\|\phi(n-v).
\end{aligned}
$$

Since the space $\mathbb{Q}_F^{\dot{n}}$ is complete, then there exists a $y_0=y(n_0)\in\mathbb{Q}_F^{\dot{n}}$ such that $q(s)$ converges to

$$
\prod_{h=m-1}^{1}\prod_{l=0}^{n_{h+1}-(n_h+1)}A(n_{h+1}-l)\prod_{v=0}^{n_1-n_0}A(n_1-v)y(n_0).
$$

Hence

$$
\begin{aligned}
&D\big(y(n),\tilde{y}(n)\big)\\
=\;&D\Big(y(n),\prod_{v=1}^{n-(n_m+1)}A(n-v)\Big[\prod_{h=m-1}^{1}\prod_{l=0}^{n_{h+1}-(n_h+1)}A(n_{h+1}-l)\prod_{v=0}^{n_1-n_0}A(n_1-v)y(n_0)\\
&\oplus\sum_{l=0}^{m-2}\prod_{v=m-1}^{m-l-1}\prod_{h=0}^{n_{v+1}-(n_v+1)}A(n_{v+1}-h)(M_{1,m-l-2}\oplus I_{m-l-1}(y(n_{m-l-1})))\oplus M_{1,m-1}\\
&\oplus I_m(y(n_m))\Big]\oplus\sum_{l=1}^{n-(n_m+2)}\prod_{v=1}^{n-(n_m+1+l)}A(n-v)f(n_m+l,y(n_m+l))\oplus f(n-1,y(n-1))\Big)\\
=\;&D\Big(y(n),\prod_{v=1}^{n-(n_m+1)}A(n-v)\Big[q(s)\oplus\sum_{l=0}^{m-2}\prod_{v=m-1}^{m-l-1}\prod_{h=0}^{n_{v+1}-(n_v+1)}A(n_{v+1}-h)(M_{1,m-l-2}
\end{aligned}
$$

$$
\begin{aligned}
&\oplus I_{m-l-1}(y(n_{m-l-1}))) \oplus M_{1,m-1} \oplus I_m(y(n_m))\Big] \oplus \sum_{l=1}^{n-(n_m+2)} \prod_{v=1}^{n-(n_m+1+l)} A(n-v) \\
&\times f(n_m+l, y(n_m+l)) \oplus f(n-1, y(n-1))\Big) + D\Big(\prod_{v=1}^{n-(n_m+1)} A(n-v)\Big[q(s) \\
&\oplus \sum_{l=0}^{m-2} \prod_{v=m-1}^{m-l-1} \prod_{h=0}^{n_{v+1}-(n_v+1)} A(n_{v+1}-h)(M_{1,m-l-2} \oplus I_{m-l-1}(y(n_{m-l-1}))) \oplus M_{1,m-1} \\
&\oplus I_m(y(n_m))\Big] \oplus \sum_{l=1}^{n-(n_m+2)} \prod_{v=1}^{n-(n_m+1+l)} A(n-v) f(n_m+l, y(n_m+l)) \oplus f(n-1, y(n-1)), \\
&\prod_{v=1}^{n-(n_m+1)} A(n-v)\Big[\prod_{h=m-1}^{1} \prod_{l=0}^{n_{h+1}-(n_h+1)} A(n_{h+1}-l) \prod_{v=0}^{n_1-n_0} A(n_1-v) y(n_0) \\
&\oplus \sum_{l=0}^{m-2} \prod_{v=m-1}^{m-l-1} \prod_{h=0}^{n_{v+1}-(n_v+1)} A(n_{v+1}-h)(M_{1,m-l-2} \oplus I_{m-l-1}(y(n_{m-l-1}))) \oplus M_{1,m-1} \\
&\oplus I_m(y(n_m))\Big] \oplus \sum_{l=1}^{n-(n_m+2)} \prod_{v=1}^{n-(n_m+1+l)} A(n-v) f(n_m+l, y(n_m+l)) \oplus f(n-1, y(n-1))\Big) \\
= \;& D\Big(\prod_{v=1}^{n-(n_m+1)} A(n-v) \prod_{v=n-(n_m+1)}^{1} A^{-1}(n-v)\big[y(n) \ominus \sum_{l=1}^{n-(n_m+2)} \prod_{v=1}^{n-(n_m+1+l)} A(n-v) \\
&\times f(n_m+l, y(n_m+l)) \ominus f(n-1, y(n-1))\big], \prod_{v=1}^{n-(n_m+1)} A(n-v)\Big[q(s) \\
&\oplus \sum_{l=0}^{m-2} \prod_{v=m-1}^{m-l-1} \prod_{h=0}^{n_{v+1}-(n_v+1)} A(n_{v+1}-h)(M_{1,m-l-2} \oplus I_{m-l-1}(y(n_{m-l-1}))) \oplus M_{1,m-1} \\
&\oplus I_m(y(n_m))\Big]\Big) + D\Big(\prod_{v=1}^{n-(n_m+1)} A(n-v) q(s), \prod_{v=1}^{n-(n_m+1)} A(n-v) \\
&\times \prod_{h=m-1}^{1} \prod_{l=0}^{n_{h+1}-(n_h+1)} A(n_{h+1}-l) \prod_{v=0}^{n_1-n_0} A(n_1-v) y(n_0)\Big) \\
= \;& D\Big(\prod_{v=1}^{n-(n_m+1)} A(n-v) q(n), \prod_{v=1}^{n-(n_m+1)} A(n-v) q(s)\Big) + D\Big(\prod_{v=1}^{n-(n_m+1)} A(n-v) q(s), \\
&\prod_{v=1}^{n-(n_m+1)} A(n-v) \prod_{h=m-1}^{1} \prod_{l=0}^{n_{h+1}-(n_h+1)} A(n_{h+1}-l) \prod_{v=0}^{n_1-n_0} A(n_1-v) y(n_0)\Big) \\
\le \;& \epsilon \prod_{v=1}^{n-(n_m+1)} \|A(n-v)\| \sum_{v=1}^{n-s} \prod_{t=n-(n_m+1)}^{v} \|A^{-1}(n-t)\| \phi(n-v).
\end{aligned}
$$

For $n = n_m + 1$ and $s \in [n_m + 1, n_{m+1}]$, we have

$$
\begin{aligned}
&D\big(y(n_{m+1}+1), \tilde{y}(n_{m+1}+1)\big) = D\big(y(n_{m+1}+1) \ominus y(n_{m+1}) \oplus y(n_{m+1}), \tilde{y}(n_{m+1}) \oplus I_{m+1}(\tilde{y}(n_{m+1}))\big) \\
\le \;& D\big(\Delta y(n_{m+1}) \oplus y(n_{m+1}), y(n_{m+1}) \oplus I_{m+1}(y(n_{m+1}))\big) + D\big(y(n_{m+1}) \oplus I_{m+1}(y(n_{m+1})), \tilde{y}(n_{m+1}) \oplus I_{m+1}(y(n_{m+1}))\big) \\
&+ D\big(\tilde{y}(n_{m+1}) \oplus I_{m+1}(y(n_{m+1})), \tilde{y}(n_{m+1}) \oplus I_{m+1}(\tilde{y}(n_{m+1}))\big) \\
= \;& D\big(\Delta y(n_{m+1}), I_{m+1}(y(n_{m+1}))\big) + D\big(y(n_{m+1}), \tilde{y}(n_{m+1})\big) + D\big(I_{m+1}(y(n_{m+1})), I_{m+1}(\tilde{y}(n_{m+1}))\big) \\
\le \;& \varphi(m+1)\epsilon + (1+L) D\big(y(n_{m+1}), \tilde{y}(n_{m+1})\big)
\end{aligned}
$$

$$\leq \epsilon\big(\varphi(m+1)+(1+L)\prod_{v=1}^{n_{m+1}-(n_m+1)}\|A(n_{m+1}-v)\|\times\sum_{v=1}^{n_{m+1}-s}\prod_{t=n_{m+1}-(n_m+1)}^{v}\|A^{-1}(n_{m+1}-t)\|\phi(n_{m+1}-v)\big).$$

Now we prove the uniqueness of y_0. Assume that there exists a $y_1\in\mathbb{Q}_F^{\dot{n}}$ such that

$$D\big(y(n),\tilde{y}(n)\big)\leq\epsilon\prod_{v=1}^{n-(n_m+1)}\|A(n-v)\|\sum_{v=1}^{n-s}\prod_{t=n-(n_m+1)}^{v}\|A^{-1}(n-t)\|\phi(n-v)$$

for $n\in[n_m+1,n_{m+1}]$. Then

$$\begin{aligned}
D(y_0,y_1) &= D\Big(\prod_{v=0}^{n_1-n_0}A^{-1}(n_0+v)\prod_{h=1}^{m-1}\prod_{l=1}^{n_{h+1}-n_h}A^{-1}(n_h+l)\prod_{h=m-1}^{1}\prod_{l=0}^{n_{h+1}-(n_h+1)}A(n_{h+1}-l)\\
&\times\prod_{v=0}^{n_1-n_0}A(n_1-v)y_0,\ \prod_{v=0}^{n_1-n_0}A^{-1}(n_0+v)\prod_{h=1}^{m-1}\prod_{l=1}^{n_{h+1}-n_h}A^{-1}(n_h+l)\\
&\times\prod_{h=m-1}^{1}\prod_{l=0}^{n_{h+1}-(n_h+1)}A(n_{h+1}-l)\prod_{v=0}^{n_1-n_0}A(n_1-v)y_1\Big)\\
&= D\Big(\prod_{v=0}^{n_1-n_0}A^{-1}(n_0+v)\prod_{h=1}^{m-1}\prod_{l=1}^{n_{h+1}-n_h}A^{-1}(n_h+l)\prod_{v=n-(n_m+1)}^{1}A^{-1}(n-v)\\
&\times\Big\{\prod_{v=1}^{n-(n_m+1)}A(n-v)\Big[\prod_{h=m-1}^{1}\prod_{l=0}^{n_{h+1}-(n_h+1)}A(n_{h+1}-l)\prod_{v=0}^{n_1-n_0}A(n_1-v)y_0\\
&\oplus\sum_{l=0}^{m-2}\prod_{v=m-1}^{m-l-1}\prod_{h=0}^{n_{v+1}-(n_v+1)}A(n_{v+1}-h)(M_{1,m-l-2}\oplus I_{m-l-1}(y(n_{m-l-1})))\\
&\oplus M_{1,m-1}\oplus I_m(y(n_m))\Big]\oplus\sum_{l=1}^{n-(n_m+2)}\prod_{v=1}^{n-(n_m+1+l)}A(n-v)f(n_m+l,y(n_m+l))\\
&\oplus f(n-1,y(n-1))\Big\},\ \prod_{v=0}^{n_1-n_0}A^{-1}(n_0+v)\prod_{h=1}^{m-1}\prod_{l=1}^{n_{h+1}-n_h}A^{-1}(n_h+l)\prod_{v=n-(n_m+1)}^{1}A^{-1}(n-v)\\
&\times\Big\{\prod_{v=1}^{n-(n_m+1)}A(n-v)\Big[\prod_{h=m-1}^{1}\prod_{l=0}^{n_{h+1}-(n_h+1)}A(n_{h+1}-l)\prod_{v=0}^{n_1-n_0}A(n_1-v)y_1\\
&\oplus\sum_{l=0}^{m-2}\prod_{v=m-1}^{m-l-1}\prod_{h=0}^{n_{v+1}-(n_v+1)}A(n_{v+1}-h)(M_{1,m-l-2}\oplus I_{m-l-1}(y(n_{m-l-1})))\oplus M_{1,m-1}\\
&\oplus I_m(y(n_m))\Big]\oplus\sum_{l=1}^{n-(n_m+2)}\prod_{v=1}^{n-(n_m+1+l)}A(n-v)f(n_m+l,y(n_m+l))\oplus f(n-1,y(n-1))\Big\}\Big)\\
&\leq D\Big(\prod_{v=0}^{n_1-n_0}A^{-1}(n_0+v)\prod_{h=1}^{m-1}\prod_{l=1}^{n_{h+1}-n_h}A^{-1}(n_h+l)\prod_{v=n-(n_m+1)}^{1}A^{-1}(n-v)\\
&\times\Big\{\prod_{v=1}^{n-(n_m+1)}A(n-v)\Big[\prod_{h=m-1}^{1}\prod_{l=0}^{n_{h+1}-(n_h+1)}A(n_{h+1}-l)\prod_{v=0}^{n_1-n_0}A(n_1-v)y_0\\
&\oplus\sum_{l=0}^{m-2}\prod_{v=m-1}^{m-l-1}\prod_{h=0}^{n_{v+1}-(n_v+1)}A(n_{v+1}-h)(M_{1,m-l-2}\oplus I_{m-l-1}(y(n_{m-l-1})))\\
&\oplus M_{1,m-1}\oplus I_m(y(n_m))\Big]\oplus\sum_{l=1}^{n-(n_m+2)}\prod_{v=1}^{n-(n_m+1+l)}A(n-v)f(n_m+l,y(n_m+l))
\end{aligned}$$

$$\oplus f(n-1,y(n-1))\Big\}, \prod_{v=0}^{n_1-n_0} A^{-1}(n_0+v)\prod_{h=1}^{m-1}\prod_{l=1}^{n_{h+1}-n_h} A^{-1}(n_h+l)\prod_{v=n-(n_m+1)}^{1} A^{-1}(n-v)$$
$$\times y(n)\Big) + D\Big(\prod_{v=0}^{n_1-n_0} A^{-1}(n_0+v)\prod_{h=1}^{m-1}\prod_{l=1}^{n_{h+1}-n_h} A^{-1}(n_h+l)\prod_{v=n-(n_m+1)}^{1} A^{-1}(n-v)$$
$$\times y(n), \prod_{v=0}^{n_1-n_0} A^{-1}(n_0+v)\prod_{h=1}^{m-1}\prod_{l=1}^{n_{h+1}-n_h} A^{-1}(n_h+l)\prod_{v=n-(n_m+1)}^{1} A^{-1}(n-v)$$
$$\times\Big\{\prod_{v=1}^{n-(n_m+1)} A(n-v)\Big[\prod_{h=m-1}^{1}\prod_{l=0}^{n_{h+1}-(n_h+1)} A(n_{h+1}-l)\prod_{v=0}^{n_1-n_0} A(n_1-v)y_1$$
$$\oplus\sum_{l=0}^{m-2}\prod_{v=m-1}^{m-l-1}\prod_{h=0}^{n_{v+1}-(n_v+1)} A(n_{v+1}-h)(M_{1,m-l-2}\oplus I_{m-l-1}(y(n_{m-l-1})))$$
$$\oplus M_{1,m-1}\oplus I_m(y(n_m))\Big]\oplus\sum_{l=1}^{n-(n_m+2)}\prod_{v=1}^{n-(n_m+1+l)} A(n-v)f(n_m+l,y(n_m+l))$$
$$\oplus f(n-1,y(n-1))\Big\}\Big)\le 2\epsilon\prod_{v=0}^{n_1-n_0}\|A^{-1}(n_0+v)\|\prod_{h=1}^{m-1}\prod_{l=1}^{n_{h+1}-n_h}\|A^{-1}(n_h+l)\|$$
$$\times\prod_{v=n-(n_m+1)}^{1}\|A^{-1}(n-v)\|\prod_{v=1}^{n-(n_m+1)}\|A(n-v)\|\sum_{v=1}^{n-s}\prod_{t=n-(n_m+1)}^{v}\|A^{-1}(n-t)\|\phi(n-v).$$

Hence $y_0=y_1$. The proof is completed. □

Theorem 3.5. *Let $\omega\in\mathbb{Q}$, $f:\mathbb{Z}\times\mathbb{Q}_F^{\dot n}\to\mathbb{Q}_F^{\dot n}$, y be a solution of (4) and $\tilde y$ be a solution of (1). If*

$$\prod_{t=n-(n_m+1)}^{1} A^{-1}(n-t)\big[y(n)\ominus\sum_{l=1}^{n-(n_m+2)}\prod_{v=1}^{n-(n_m+1+l)} A(n-v)f(n_m+l,y(n_m+l))\ominus f(n-1,y(n-1))\big]$$
$$\ominus\sum_{l=0}^{m-2}\prod_{v=m-1}^{m-l-1}\prod_{h=0}^{n_{v+1}-(n_v+1)} A(n_{v+1}-h)(M_{1,m-l-2}\ominus I_{m-l-1}(y(n_{m-l-1})))\ominus M_{1,m-1}\oplus I_m(y(n_m))$$

exists for all $n\in\dot N\backslash\{n_m\}$. Then there exists a unique $y_0\in\mathbb{Q}_F^{\dot n}$ with

$$D(y(n),\tilde y(n))\le\epsilon\prod_{v=1}^{n-(n_m+1)}\|A(n-v)\|\sum_{v=1}^{n-s}\prod_{t=n-(n_m+1)}^{v}\|A^{-1}(n-t)\|\phi(n-v)$$

for $n\in[n_m+1,n_{m+1}]$ and $s\in[n_m+1,n]$, $m\in\mathbb{M}$ and

$$\begin{aligned} D(y(n_{m+1}+1),\tilde y(n_{m+1}+1)) \quad\le\quad & \epsilon\Big(\varphi(m+1)+(1+L)\prod_{v=1}^{n_{m+1}-(n_m+1)}\|A(n_{m+1}-v)\| \\ & \times\sum_{v=1}^{n_{m+1}-s}\prod_{t=n_{m+1}-(n_m+1)}^{v}\|A^{-1}(n_{m+1}-t)\|\phi(n_{m+1}-v)\Big). \end{aligned}$$

Theorem 3.6. *Let $\omega\in\mathbb{Q}$, $f:\mathbb{Z}\times\mathbb{Q}_F^{\dot n}\to\mathbb{Q}_F^{\dot n}$, y be a solution of (5) and $\tilde y$ be a solution of (2). If*

$$\prod_{t=n-(n_m+1)}^{1}\ominus A^{-1}(n-t)\big[y(n)\ominus\sum_{l=1}^{n-(n_m+2)}\prod_{v=1}^{n-(n_m+1+l)}\ominus A(n-v)f(n_m+l,y(n_m+l))\ominus f(n-1,y(n-1))\big]$$
$$\ominus\sum_{l=0}^{m-2}\prod_{v=m-1}^{m-l-1}\prod_{h=0}^{n_{v+1}-(n_v+1)}\ominus A(n_{v+1}-h)(M_{2,m-l-2}\oplus I_{m-l-1}(y(n_{m-l-1})))\ominus M_{2,m-1}\ominus I_m(y(n_m))$$

exists for all $n \in \dot{N}\backslash\{n_m\}$. *Then there exists a unique* $y_0 \in \mathbb{Q}_F^{\dot{n}}$ *with*

$$D(y(n),\tilde{y}(n)) \leq \epsilon \prod_{v=1}^{n-(n_m+1)} \|\ominus A(n-v)\| \sum_{v=1}^{n-s} \prod_{t=n-(n_m+1)}^{v} \|\ominus A^{-1}(n-t)\|\phi(n-v)$$

for $n \in [n_m+1, n_{m+1}]$ *and* $s \in [n_m+1, n]$, $m \in \mathbb{M}$ *and*

$$\begin{aligned} D(y(n_{m+1}+1),\tilde{y}(n_{m+1}+1)) \quad &\leq \quad \epsilon\Big(\varphi(m+1) + (1+L) \prod_{v=1}^{n_{m+1}-(n_m+1)} \|\ominus A(n_{m+1}-v)\| \\ &\times \sum_{v=1}^{n_{m+1}-s} \prod_{t=n_{m+1}-(n_m+1)}^{v} \|\ominus A^{-1}(n_{m+1}-t)\|\phi(n_{m+1}-v)\Big). \end{aligned}$$

Theorem 3.7. *Let* $\omega \in \mathbb{Q}$, $f : \mathbb{Z} \times \mathbb{Q}_F^{\dot{n}} \to \mathbb{Q}_F^{\dot{n}}$, y *be a solution of* (5) *and* $\tilde{y}$ *be a solution of* (2). *If*

$$\begin{aligned} &\prod_{t=n-(n_m+1)}^{1} \ominus A^{-1}(n-t)\big[y(n) \ominus \sum_{l=1}^{n-(n_m+2)} \prod_{v=1}^{n-(n_m+1+l)} \ominus A(n-v)f(n_m+l, y(n_m+l)) \ominus f(n-1, y(n-1))\big] \\ &\ominus \sum_{l=0}^{m-2} \prod_{v=m-1}^{m-l-1} \prod_{h=0}^{n_{v+1}-(n_v+1)} \ominus A(n_{v+1}-h)(M_{2,m-l-2} \ominus I_{m-l-1}(y(n_{m-l-1}))) \ominus M_{2,m-1} \oplus I_m(y(n_m)) \end{aligned}$$

exists for all $n \in \dot{N}\backslash\{n_m\}$. *Then there exists a unique* $y_0 \in \mathbb{Q}_F^{\dot{n}}$ *with*

$$D(y(n),\tilde{y}(n)) \leq \epsilon \prod_{v=1}^{n-(n_m+1)} \|\ominus A(n-v)\| \sum_{v=1}^{n-s} \prod_{t=n-(n_m+1)}^{v} \|\ominus A^{-1}(n-t)\|\phi(n-v)$$

for $n \in [n_m+1, n_{m+1}]$ *and* $s \in [n_m+1, n]$, $m \in \mathbb{M}$ *and*

$$\begin{aligned} D(y(n_{m+1}+1),\tilde{y}(n_{m+1}+1)) \quad &\leq \quad \epsilon\Big(\varphi(m+1) + (1+L) \prod_{v=1}^{n_{m+1}-(n_m+1)} \|\ominus A(n_{m+1}-v)\| \\ &\times \sum_{v=1}^{n_{m+1}-s} \prod_{t=n_{m+1}-(n_m+1)}^{v} \|\ominus A^{-1}(n_{m+1}-t)\|\phi(n_{m+1}-v)\Big). \end{aligned}$$

Theorem 3.8. *Let* $\omega \in \mathbb{Q}$, $f : \mathbb{Z} \times \mathbb{Q}_F^{\dot{n}} \to \mathbb{Q}_F^{\dot{n}}$, y *be a solution of* (6) *and* $\tilde{y}$ *be a solution of* (3). *If*

$$\begin{aligned} &\prod_{t=n-(n_m+1)}^{1} A^{-1}(n-t)\big[y(n) \oplus \sum_{l=1}^{n-(n_m+2)} \prod_{v=1}^{n-(n_m+1+l)} A(n-v)f(n_m+l, y(n_m+l)) \oplus f(n-1, y(n-1))\big] \\ &\ominus \sum_{l=0}^{m-2} \prod_{v=m-1}^{m-l-1} \prod_{h=0}^{n_{v+1}-(n_v+1)} A(n_{v+1}-h)(M_{3,m-l-2} \oplus I_{m-l-1}(y(n_{m-l-1}))) \ominus M_{3,m-1} \ominus I_m(y(n_m)) \end{aligned}$$

exists for all $n \in \dot{N}\backslash\{n_m\}$. *Then there exists a unique* $y_0 \in \mathbb{Q}_F^{\dot{n}}$ *with*

$$D(y(n),\tilde{y}(n)) \leq \epsilon \prod_{v=1}^{n-(n_m+1)} \|A(n-v)\| \sum_{v=1}^{n-s} \prod_{t=n-(n_m+1)}^{v} \|A^{-1}(n-t)\|\phi(n-v)$$

for $n \in [n_m+1, n_{m+1}]$ *and* $s \in [n_m+1, n]$, $m \in \mathbb{M}$ *and*

$$\begin{aligned} D(y(n_{m+1}+1),\tilde{y}(n_{m+1}+1)) \quad &\leq \quad \epsilon\Big(\varphi(m+1) + (1+L) \prod_{v=1}^{n_{m+1}-(n_m+1)} \|A(n_{m+1}-v)\| \\ &\times \sum_{v=1}^{n_{m+1}-s} \prod_{t=n_{m+1}-(n_m+1)}^{v} \|A^{-1}(n_{m+1}-t)\|\phi(n_{m+1}-v)\Big). \end{aligned}$$

Theorem 3.9. *Let $\omega \in \mathbb{Q}$, $f : \mathbb{Z} \times \mathbb{Q}_F^{\dot{n}} \to \mathbb{Q}_F^{\dot{n}}$, y be a solution of (6) and $\tilde{y}$ be a solution of (3). If*

$$\prod_{t=n-(n_m+1)}^{1} A^{-1}(n-t)\big[y(n) \oplus \sum_{l=1}^{n-(n_m+2)} \prod_{v=1}^{n-(n_m+1+l)} A(n-v)f(n_m+l, y(n_m+l)) \oplus f(n-1, y(n-1))\big]$$
$$\ominus \sum_{l=0}^{m-2} \prod_{v=m-1}^{m-l-1} \prod_{h=0}^{n_{v+1}-(n_v+1)} A(n_{v+1}-h)(M_{3,m-l-2} \ominus I_{m-l-1}(y(n_{m-l-1}))) \ominus M_{3,m-1} \oplus I_m(y(n_m))$$

exists for all $n \in \dot{N}\backslash\{n_m\}$. Then there exists a unique $y_0 \in \mathbb{Q}_F^{\dot{n}}$ with

$$D(y(n), \tilde{y}(n)) \leq \epsilon \prod_{v=1}^{n-(n_m+1)} \|A(n-v)\| \sum_{v=1}^{n-s} \prod_{t=n-(n_m+1)}^{v} \|A^{-1}(n-t)\|\phi(n-v)$$

for $n \in [n_m+1, n_{m+1}]$ and $s \in [n_m+1, n]$, $m \in \mathbb{M}$ and

$$\begin{aligned} D(y(n_{m+1}+1), \tilde{y}(n_{m+1}+1)) \quad \leq \quad & \epsilon\bigg(\varphi(m+1) + (1+L) \prod_{v=1}^{n_{m+1}-(n_m+1)} \|A(n_{m+1}-v)\| \\ & \times \sum_{v=1}^{n_{m+1}-s} \prod_{t=n_{m+1}-(n_m+1)}^{v} \|A^{-1}(n_{m+1}-t)\|\phi(n_{m+1}-v)\bigg). \end{aligned}$$

4 Example

Let $\dot{n} = 2$, $n_0 = 0$, $n \in [0, 5] \cap \mathbb{Z}$, $n_1 = 2$, $n_2 = 4$, $f(n, y(n)) = ny(n)$,

$$A(n) = \begin{bmatrix} \frac{1+(-1)^{n+1}i+\cos(n+1)\pi j+\sin\frac{(2n-1)\pi}{2}}{4} & 0 \\ 0 & \frac{1+(-1)^n i+\cos n\pi j+\sin\frac{(2n+1)\pi}{2}}{4} \end{bmatrix},$$

and the inequalities

$$\begin{cases} D(y(n+1), A(n)y(n) \oplus f(n, y(n))) \leq \phi(n)\epsilon, \;\; n \in \dot{N}\backslash\{n_m\}, \\ D(\Delta y(n_m), I_m(y(n_m))) \leq \varphi(m)\epsilon, \;\; m = 1, 2, \\ D(I_m(y), I_m(z)) \leq LD(y, z), \text{ for } y, z \in \mathbb{Q}_F^{\dot{n}} \text{ and } L > 0; \end{cases}$$

where $\phi(n) = n + 1$, $\varphi(m) = m$.

Case (I)**.** $\Delta y(n_m) = y(n_m+1) \ominus y(n_m) = I_m(y(n_m)) = \frac{1}{m}y(n_m)$. By Remark 3.1, the solution of (1) is given as: $\hat{y}(1) = A(0)\hat{y}(0)$, $\hat{y}(2) = [I + A(0)]\hat{y}(0)$, $\hat{y}(3) = 2[I + A(0)]\hat{y}(0)$, $\hat{y}(4) = [9I + 4A(0)]\hat{y}(0)$, $\hat{y}(5) = [\frac{27}{2}I + 6A(0)]\hat{y}(0)$ and the solution of (4) is given as: $y(0) = \hat{y}(0) \oplus g_0'$, $y(1) = A(0)\hat{y}(0) \oplus g_1$, $y(2) = [I + A(0)]\hat{y}(0) \oplus g_2$, $y(3) = 2[I + A(0)]\hat{y}(0) \oplus g_3$, $y(4) = [9I + 4A(0)]\hat{y}(0) \oplus g_4$, $y(5) = [\frac{27}{2}I + 6A(0)]\hat{y}(0) \oplus g_5$, where $D(0', g_1) < \epsilon$, $D(0', g_2) < 4\epsilon$, $D(0', g_3) < (5 + 4L)\epsilon$, $D(0', g_4) < (60 + 48L)\epsilon$, $D(0', g_4) < (62 + 108L + 48L^2)\epsilon$. In fact, $\hat{y}(1) = A(0)\hat{y}(0) \oplus 0\hat{y}(0) = A(0)\hat{y}(0)$, $\hat{y}(2) = A(1)[A(0)\hat{y}(0)] \oplus A(0)\hat{y}(0) = \hat{y}(0) \oplus A(0)\hat{y}(0) = [I + A(0)]\hat{y}(0)$, $\hat{y}(3) = \hat{y}(2) \oplus \frac{1}{1}\hat{y}(2) = 2[I + A(0)]\hat{y}(0)$, $\hat{y}(4) = 2A(3)[I + A(0)]\hat{y}(0) \oplus 6[I + A(0)]\hat{y}(0) = [9I + 4A(0)]\hat{y}(0)$, $\hat{y}(5) = \hat{y}(4) \oplus \frac{1}{2}\hat{y}(4) = [\frac{27}{2}I + 6A(0)]\hat{y}(0)$.

Case (II)**.** $\Delta y(n_m) = y(n_m) \ominus y(n_m+1) = I_m(y(n_m)) = \frac{1}{n_m}y(n_m)$. By Remark 3.2, the solution of (1) is given by: $\hat{y}(1) = A(0)\hat{y}(0)$, $\hat{y}(2) = [I + A(0)]\hat{y}(0)$, $\hat{y}(3) = \frac{1}{2}[I + A(0)]\hat{y}(0)$, $\hat{y}(4) = [\frac{9}{4}I + A(0)]\hat{y}(0)$, $\hat{y}(5) = [\frac{27}{16}I + \frac{3}{4}A(0)]\hat{y}(0)$ and the solution of (4) can be given as: $y(0) = \hat{y}(0) \oplus g_0'$, $y(1) = A(0)\hat{y}(0) \oplus g_1'$, $y(2) = [I + A(0)]\hat{y}(0) \oplus g_2'$, $y(3) = \frac{1}{2}[I + A(0)]\hat{y}(0) \oplus g_3'$, $y(4) = [\frac{9}{4}I + A(0)]\hat{y}(0) \oplus g_4'$, $y(5) = [\frac{27}{16}I + \frac{3}{4}A(0)]\hat{y}(0) \oplus g_5'$, where $D(0', g_1') < \epsilon$, $D(0', g_2') < 4\epsilon$, $D(0', g_3') < (5 + 4L)\epsilon$, $D(0', g_4') < (60 + 48L)\epsilon$, $D(0', g_4') < (62 + 108L + 48L^2)\epsilon$. In fact, $\hat{y}(3) = \hat{y}(2) \ominus \frac{1}{2}\hat{y}(2) = \frac{1}{2}[I+A(0)]\hat{y}(0)$, $\hat{y}(4) = \frac{1}{2}A(3)[I+A(0)]\hat{y}(0) \oplus \frac{3}{2}[I+A(0)]\hat{y}(0) = [\frac{9}{4}I+A(0)]\hat{y}(0)$, $\hat{y}(5) = \hat{y}(4) \ominus \frac{1}{4}\hat{y}(4) = [\frac{27}{16}I+\frac{3}{4}A(0)]\hat{y}(0)$.

5 Conclusions

In this paper, we conduct the first discussion of the Hyers-Ulam-Rassias stability for the quaternion multidimensional fuzzy difference equations with impulses including three main nonlinear types. Through establishing some basic results of fuzzy quaternion vectors in the multidimensional fuzzy quaternion space, some sufficient conditions to guarantee the Hyers-Ulam-Rassias stability of the fuzzy iteration solutions for these fuzzy difference equations are obtained. The results established in this paper fill the gap of Hyers-Ulam-Rassias stability theory of the quaternion multidimensional fuzzy difference equations in the research field.

Based on this research, the Hyers-Ulam-Rassias stability of the fuzzy iteration solutions for the quaternion multidimensional fuzzy dynamic equations with impulses including three main nonlinear types on discrete hybrid domains can be established in our future work.

Acknowledgement

This work is supported by NSFC (No. 11961077, No. 11601470), IRTSTYN and Joint Key Project of Yunnan Provincial Science and Technology Department of Yunnan University (No. 2018FY001(-014)).

References

[1] R. P. Agarwal, S. Arshad, D. O'Regan, V. Lupulescu, *Fuzzy fractional integral equations under compactness type condition*, Fractional Calculus and Applied Analysis, **15** (2012), 572-590.

[2] B. Bede, S. G. Gal, *Almost periodic fuzzy-number-valued functions*, Fuzzy Sets and Systems, **147** (2004), 385-403.

[3] B. Bede, L. Stefanini, *Generalized differentiability of fuzzy-valued functions*, Fuzzy Sets and Systems, **230** (2013), 119-141.

[4] J. J. Buckley, *Fuzzy complex numbers*, Fuzzy Sets and Systems, **33** (1989), 333-345.

[5] E. Y. Deeba, A. D. Korvin, E. L. Koh, *A fuzzy difference equation with an application*, Journal of Difference Equations and Applications, **2** (1996), 365-374.

[6] P. Diamond, P. Kloeden, *Metric spaces of fuzzy sets*, World Scientific, Singapore, 1994.

[7] D. H. Hyers, *On the stability of the linear functional equation*, Proceedings of the National Academy of Sciences, **27** (1941), 222-224.

[8] D. H. Hyers, G. Isac, T. M. Rassias, *Stability of functional equations in several variables*, Birkhäuser, 1998.

[9] S. M. Jung, *Hyers-Ulam-Rassias stability of functional equations in mathematical analysis*, Hadronic Press, Palm Harbor, 2001.

[10] O. Kaleva, *A note on fuzzy differential equations*, Nonlinear Analysis, **64** (2006), 895-900.

[11] A. Khastan, *New solutions for first order linear fuzzy difference equations*, Journal of Computational and Applied Mathematics, **312** (2017), 156-166.

[12] A. Khastan, R. Rodríguez-López, *On periodic solutions to first order linear fuzzy differential equations under differential inclusions' approach*, Information Sciences, **322** (2015), 31-50.

[13] V. Lakshmikantham, D. D. Bainov, P. S. Simeonov, *Theory of impulsive differential equations*, World Scientific, Singapore, 1989.

[14] V. Lakshmikantham, A. S. Vatsala, *Basic theory of fuzzy difference equations*, Journal of Difference Equations and Applications, **8** (2002), 957-968.

[15] R. Liu, J. Wang, D. O'Regan, *Ulam type stability of first-order linear impulsive fuzzy differential equations*, Fuzzy Sets and Systems, **400** (2020), 34-89.

[16] V. Lupulescu, *Fractional calculus for interval-valued functions*, Fuzzy Sets and Systems, **265** (2015), 63-85.

[17] T. Miura, S. Miyajima, S. E. Takahasi, *Hyers-Ulam stability of linear differential operator with constant coefficients*, Mathematische Nachrichten, **258** (2003), 90-96.

[18] R. P. A. Moura, F. B. Bergamaschi, R. H. N. Santiago, B. R. C. Bedregal, *Fuzzy quaternion numbers*, IEEE International Conference on Fuzzy Systems, (2013), 1-6.

[19] J. J. Nieto, *The Cauchy problem for continuous fuzzy differential equations*, Fuzzy Sets and Systems, **102** (1999), 259-262.

[20] J. J. Nieto, R. Rodríguez-López, *Bounded solutions for fuzzy differential and integral equations*, Chaos, Solitons and Fractals, **27** (2006), 1376-1386.

[21] D. Ramot, R. Milo, M. Friedman, A. Kandel, *Complex fuzzy sets*, IEEE Transactions on Fuzzy Systems, **10** (2002), 171-186.

[22] T. M. Rassias, *On the stability of linear mappings in Banach spaces*, Proceedings of the American Mathematical Society, **72** (1978), 297-300.

[23] R. Rodríguez-López, *Periodic boundary value problems for impulsive fuzzy differential equations*, Fuzzy Sets and Systems, **159** (2008), 1384-1409.

[24] I. A. Rus, *Ulam stability of ordinary differential equations*, Studia Univ. "Babeş Bolyai" Mathematica, **54** (2009), 125-133.

[25] Y. Shen, *On the Ulam stability of first order linear fuzzy differential equations under generalized differentiability*, Fuzzy Sets and Systems, **280** (2015), 27-57.

[26] L. Stefanini, *A generalization of Hukuhara difference and division for interval and fuzzy arithmetic*, Fuzzy Sets and Systems, **161** (2010), 1564-1584.

[27] H. Takagi, T. Miura, S. E. Takahasi, *Essential norms and stability constants of weighted composition operator on* $C(X)$, Bulletin of the Korean Mathematical Society, **40** (2003), 583-591.

[28] D. E. Tamir, L. Jin, A. Kandel, *A new interpretation of complex membership grade*, Journal of Intelligent and Fuzzy Systems, **26** (2011), 285-312.

[29] S. M. Ulam, *A collection of mathematical problems*, Interscience Publishers, New York, 1968.

[30] N. Van Hoa, V. Lupulescu, D. O'Regan, *Solving interval-valued fractional initial value problems under Caputo gH-fractional differentiability*, Fuzzy Sets and Systems, **309** (2017), 1-34.

[31] C. Wang, R. P. Agarwal, *Almost periodic solution for a new type of neutral impulsive stochastic Lasota-Wazewska timescale model*, Applied Mathematics Letters, **70** (2017), 58-65.

[32] C. Wang, R. P. Agarwal, D. O'Regan, *Calculus of fuzzy vector-valued functions and almost periodic fuzzy vector-valued functions on time scales*, Fuzzy Sets and Systems, **375** (2019), 1-52.

[33] C. Wang, R. P. Agarwal, D. O'Regan, R. Sakthivel, *Theory of translation closedness for time scales*, Springer, Developments in Mathematics Series, **62**, Switzerland, 2020.

[34] J. Wang, M. Fěckan, Y. Zhou, *Ulam's type stability of impulsive ordinary differential equations*, Journal of Mathematical Analysis and Applications, **395** (2012), 258-264.

[35] Z. P. Yang, T. Z. Xu, M. Qi. *The Cauchy problem for quaternion fuzzy fractional differential equations*, Journal of Intelligent and Fuzzy Systems, **29** (2015), 451-461.