AGE REPLACEMENT POLICY IN UNCERTAIN ENVIRONMENT

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Abstract. Age replacement policy is concerned with finding an optional time to minimize the cost, at which time the unit is replaced even if it does not fail. So far, age replacement policy involving random age has been proposed. This paper will assume the age of the unit is an uncertain variable, and find the optimal time to replace the unit.

1. **Introduction**

Maintenance is usually carried out to ensure the reliability of a system. Generally, there are two types of maintenance: preventive maintenance and corrective maintenance depending on whether the system is maintained before or after it breaks down. Corrective maintenance usually is costly, so it is necessary to carry out preventive maintenance. However, it is unwise to preventively maintain the system too frequently. From this point, many policies integrating preventive and corrective maintenance have been proposed such as age replacement policy, block replacement policy, and periodic replacement policy.

Originally, the age of a system was considered to be a random variable. In 1965, Barlow and Proschan [1] studied the basic replacement policies. After that, age replacement policy with discounting was studied by Fox [7] in 1966. Later age replacement policy with random charges was studied by Cleroux *et al* [5] in 1979. Multistage block replacement policy was introduced by Marathe [22] in 1966. Tilquin and Cleroux [26] considered the random cost in the block replacement policy in 1975. Periodic replacement policy with adjustment costs are first studied by Tilquin and Cleroux [27] in 1975. Then Boland and Proschan [2] studied the case when the repair cost increases with age. For more development of replacement policies, readers may refer to Nakagawa [23].

As we know, a premise to apply probability is that the estimated probability distribution is close enough to the real frequency. In our daily life, we often meet with the situation where there are few or even no observed data. In that case, we have to invite some domain experts to evaluate their degree of belief that each event will occur. However, human tends to overweight unlikely events (Kahneman and Tversky [11]), thus the degree of belief may have a much larger range than the

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real frequency. In this situation, if we insist to deal with the degree of belief using probability theory, some counterintuitive results will be obtained (Liu [19]).

In order to deal with the experts' degree of belief, an uncertainty theory was founded by Liu [12] in 2007 and refined by Liu [17] in 2011. Then Yao [32] assumed the age of the system to be an uncertain variable, proposed uncertain block replacement policy, and found the optimal replacement time. In this paper, we will study age replacement policy in uncertain environment. The remainder of this paper is organized as follows: In Section 2, we review some concepts and properties of uncertainty theory. In Section 3, some uncertain renewal processes are introduced for better modeling the age replacement policy. Age replacement policy in uncertain environment is presented in Section 4. Finally, some remarks are made in Section 5.

2. Preliminaries

Uncertainty theory was found by Liu [12] in 2007 and refined by Liu [17] in 2011 to model human uncertainty. Many researchers have contributed to this area such as Gao [8], You [34], Liu and Ha [21], Peng and Iwamura [25], Chen and Dai [4], and Dai and Chen [6]. Uncertainty theory has become a branch of axiomatic mathematics, and has been applied to uncertain programming (Liu [14]), uncertain risk analysis (Liu [16]), uncertain game (Gao *et al* [9, 10]), uncertain statistics (Wang *et al* [28]), uncertain inference (Liu [15]), uncertain logic ([18]), uncertain finance (Peng and Yao [24], Chen [3], Yao [31]), and uncertain optimal control (Zhu [35]). In this section, we will introduce some useful definitions and theorems needed throughout this paper.

Definition 2.1. [12] An uncertain variable ξ is a measurable function from the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$$

is an event.

For a sequence of uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ and a measurable function f, Liu [12] proved that $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ defined as

$$\xi(\gamma) = f(\xi_1(\gamma), \xi_2(\gamma), \cdots, \xi_n(\gamma)), \quad \forall \gamma \in \Gamma$$

is also an uncertain variable. In order to describe an uncertain variable, a concept of uncertainty distribution is introduced as follows.

Definition 2.2. [12] The uncertainty distribution of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}$$

for any real number x.

Peng and Iwamura [25] proved that a function $\Phi : \Re \to [0, 1]$ is an uncertainty distribution if and only if it is a monotone increasing function unless $\Phi(x) \equiv 0$ or $\Phi(x) \equiv 1$. The inverse function Φ^{-1} is called the inverse uncertainty distribution of ξ. Inverse uncertainty distribution is an important tool in the operation of uncertain variables.

Theorem 2.3. [17] *Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(x_1, x_2, \cdots, x_n)$ is increasing with respect to $x_1, x_2, \cdots, x_m$ and decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$$

is an uncertain variable with an inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)\right).$$

Expected value is the average of an uncertain variable in the sense of uncertain measure. It is an important index to rank uncertain variables.

Definition 2.4. [12] Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^0 \mathcal{M}\{\xi \leq r\}\mathrm{d}r$$

provided that at least one of the two integrals is finite.

In order to calculate the expected value via inverse uncertainty distribution, Liu and Ha [21] proved that

$$E[\xi] = \int_0^1 f\left(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)\right)\mathrm{d}\alpha$$

under the condition described in Theorem 2.3. Generally, the expected value operator E has no linearity property for arbitrary uncertain variables. But, for independent uncertain variables ξ and η, Liu [17] proved that

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta]$$

for any real numbers a and b.

3. Uncertain Renewal Process

An uncertain process (Liu [13]) is essentially a sequence of uncertain variables indexed by time or space. Renewal process is one of the most important uncertain processes.

Definition 3.1. [13] Let $\xi_1, \xi_2, \cdots$ be iid positive uncertain variables. Define $S_0 = 0$ and $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ for $n \geq 1$. Then the uncertain process

$$N_t = \max_{n \geq 0}\{n | S_n \leq t\}$$

is called an uncertain renewal process.

An uncertain renewal process can only take nonnegative integer values, and each sample path is a right-continuous function. Liu [17] proved that N_t/t converges in distribution to $1/\xi_1$ which is an uncertain variable. Based on this, Liu [17] proved the elementary renewal theorem, i.e.,

$$\lim_{t \to \infty} E\left[\frac{N_t}{t}\right] = E\left[\frac{1}{\xi_1}\right]$$

provided that $E[1/\xi_1]$ exists. Following that, Yao [29] presented uncertain calculus with respect to renewal process.

Definition 3.2. [17] Let $\xi_1, \xi_2, \cdots$ be iid positive uncertain interarrival times, and let $\eta_1, \eta_2, \cdots$ be iid uncertain rewards. It is also assumed that $\xi_1, \eta_1, \xi_2, \eta_2, \cdots$ are independent. Then

$$R_t = \sum_{i=1}^{N_t} \eta_i$$

is called an uncertain renewal reward process, where N_t is the renewal process with uncertain interarrival times $\xi_1, \xi_2, \cdots$

Liu [17] proved that the reward rate R_t/t converges in distribution to η_1/ξ_1, based on which he proved the renewal reward theorem, i.e.,

$$\lim_{t\to\infty} E\left[\frac{R_t}{t}\right] = E\left[\frac{\eta_1}{\xi_1}\right]$$

provided that $E[\eta_1/\xi_1]$ exists. Besides, Liu [20] applied uncertain renewal reward process to insurance model.

Definition 3.3. [30] Let $\xi_1, \xi_2, \cdots$ be iid uncertain on-times, and $\eta_1, \eta_2, \cdots$ be iid uncertain off-times. Assume $\xi_1, \eta_1, \xi_2, \eta_2, \cdots$ are also independent uncertain variables. Then

$$A_t = \begin{cases} t - \sum_{i=1}^{N_t} \eta_i, & \text{if } \sum_{i=1}^{N_t} (\xi_i + \eta_i) \le t < \sum_{i=1}^{N_t} (\xi_i + \eta_i) + \xi_{N_t+1} \\ \sum_{i=1}^{N_t+1} \xi_i, & \text{if } \sum_{i=1}^{N_t} (\xi_i + \eta_i) + \xi_{N_t+1} \le t < \sum_{i=1}^{N_t+1} (\xi_i + \eta_i) \end{cases}$$

is called an uncertain alternating renewal process, where N_t is the renewal process with uncertain interarrival times $\xi_1 + \eta_1, \xi_2 + \eta_2, \cdots$

Yao and Li [30] proved that the availability rate R_t/t converges in distribution to $\eta_1/(\xi_1 + \eta_1)$, based on which they proved the alternating renewal theorem, i.e.,

$$\lim_{t\to\infty} E\left[\frac{A_t}{t}\right] = E\left[\frac{\xi_1}{\xi_1 + \eta_1}\right]$$

provided that $E\left[\xi_1/(\xi_1 + \eta_1)\right]$ exists.

4. Age Replacement Policy

Age replacement means that a unit is always replaced at failure or at an age T, whichever occurs first. Assume that the ages of the units are iid uncertain variables $\xi_1, \xi_2, \cdots$ Then the actual ages of the units are iid uncertain variables $\xi_1 \wedge T, \xi_2 \wedge T, \cdots$ which generate an uncertain renewal process

$$N_t = \max\left\{ n \;\middle|\; \sum_{i=1}^{n} \xi_i \wedge T \le t \right\}.$$

Let b denote the cost of replacing a unit when it fails, and let a denote the cost of replacing a unit at age T. Usually, we have $a < b$. For simplicity, we introduce

$$f(x) = \begin{cases} b & \text{if } x < T \\ a & \text{if } x = T. \end{cases}$$

Then $f(\xi_i \wedge T)$ denotes the cost to replace the ith unit. The expect total replacement cost before time t is

$$E\left[\sum_{i=1}^{N_t} f(\xi_i \wedge T)\right],$$

and the average cost is

$$\lim_{t\to\infty} E\left[\sum_{i=1}^{N_t} f(\xi_i \wedge T)/t\right].$$

Age replacement policy aims at finding an optimal time T to minimize the average replacement cost, i.e.,

$$\min_T \lim_{t\to\infty} E\left[\sum_{i=1}^{N_t} f(\xi_i \wedge T)/t\right].$$

Theorem 4.1. *Let ξ be a positive uncertain variable with an uncertainty distribution Φ. Given that*

$$f(x) = \begin{cases} b & \text{if } x < T \\ a & \text{if } x = T \end{cases}$$

with $0 < a < b$, the uncertain variable

$$\frac{f(\xi \wedge T)}{\xi \wedge T}$$

has an uncertainty distribution

$$\Psi(x) = \begin{cases} 0 & \text{if } x < a/T \\ 1 - \Phi(T) & \text{if } a/T \le x \le b/T \\ 1 - \Phi(b/x) & \text{if } x > b/T. \end{cases}$$

Proof. Since $f(\xi \wedge T) \ge a$ and $\xi \wedge T \le T$, we have

$$\frac{f(\xi \wedge T)}{\xi \wedge T} \ge \frac{a}{T}, \quad a.s.,$$

thus

$$\Psi(x) = \mathcal{M}\left\{\frac{f(\xi \wedge T)}{\xi \wedge T} \le x\right\} = 0$$

for any $x \in (-\infty, a/T)$. If $x \in [a/T, b/T]$, then

$$\Psi(x) = \mathcal{M}\left\{\frac{f(\xi \wedge T)}{\xi \wedge T} \le x\right\} = \mathcal{M}\{\xi \ge T\} = 1 - \Phi(T).$$

If $x > b/T$, then

$$\Psi(x) = \mathcal{M}\left\{\frac{f(\xi \wedge T)}{\xi \wedge T} \le x\right\} = \mathcal{M}\left\{\frac{b}{\xi} \le x\right\} = \mathcal{M}\{\xi \ge \frac{b}{x}\} = 1 - \Phi\left(\frac{b}{x}\right).$$

The theorem is proved. □

Lemma 4.2. [33] *Let N_t be an uncertain renewal process, $\xi_1, \xi_2, \cdots$ be iid positive uncertain variables, and f be a positive function. Then*

$$\sum_{i=1}^{N_t} f(\xi_i) \Bigg/ \sum_{i=1}^{N_t} g(\xi_i) \text{ and } f(\xi_1)\,/g(\xi_1)$$

have a common uncertainty distribution.

Theorem 4.3. *Let $\xi_1, \xi_2, \cdots$ be a sequence of iid positive uncertain variables with a common uncertainty distribution Φ, and N_t be an uncertain renewal process with uncertain interarrivals $\xi_1 \wedge T, \xi_2 \wedge T, \cdots$ Given that*

$$f(x) = \begin{cases} b & \text{if } x < T \\ a & \text{if } x = T \end{cases}$$

with $0 < a < b$, the uncertainty distribution $\Psi_t(x)$ of the uncertain variable

$$\sum_{i=1}^{N_t} f(\xi_i \wedge T)\,/t$$

satisfies

$$\Psi_t(x) \geq \begin{cases} 0 & \text{if } x < a/T \\ 1 - \Phi(T) & \text{if } a/T \leq x \leq b/T \\ 1 - \Phi(b/x) & \text{if } x > b/T. \end{cases}$$

Proof. Since

$$\sum_{i=1}^{N_t} (\xi_i \wedge T) \leq t,$$

we have

$$\begin{aligned}
\Psi_t(x) =& \mathcal{M}\left\{\sum_{i=1}^{N_t} f(\xi_i \wedge T)\,/t \leq x\right\} \\
=& \mathcal{M}\left\{\left(\sum_{i=1}^{N_t} f(\xi_i \wedge T) \Bigg/ \sum_{i=1}^{N_t} \xi_i \wedge T\right)\left(\sum_{i=1}^{N_t} \xi_i \wedge T/t\right) \leq x\right\} \\
\geq& \mathcal{M}\left\{\sum_{i=1}^{N_t} f(\xi_i \wedge T) \Bigg/ \sum_{i=1}^{N_t} (\xi_i \wedge T) \leq x\right\} \\
=& \mathcal{M}\left\{\frac{f(\xi_1 \wedge T)}{\xi_1 \wedge T} \leq x\right\} \quad \text{(Lemma 4.2)} \\
=& \begin{cases} 0 & \text{if } x < a/T \\ 1 - \Phi(T) & \text{if } a/T \leq x \leq b/T \\ 1 - \Phi(b/x) & \text{if } x > b/T. \end{cases}
\end{aligned}$$

The theorem is thus proved. □

Theorem 4.4. *Let $\xi_1, \xi_2, \cdots$ be a sequence of iid positive uncertain variables with a common uncertainty distribution Φ, and N_t be an uncertain renewal process with* *uncertain interarrivals* $\xi_1 \wedge T, \xi_2 \wedge T, \cdots$ *Given that*

$$f(x) = \begin{cases} b & \text{if } x < T \\ a & \text{if } x = T \end{cases}$$

with $0 < a < b$, *the uncertain variable*

$$\sum_{i=1}^{N_t} f(\xi_i \wedge T)/t$$

converges in distribution to $f(\xi_1 \wedge T)/(\xi_1 \wedge T)$.

Proof. **Case I:** Assume that $x < a/T$. When $t > Ta/(a - Tx)$, we have

$$\begin{aligned}\sum_{i=1}^{N_t} f(\xi_i \wedge T)/t \geq \frac{aN_t}{t} &= \frac{a(t/T - 1)}{t} = \frac{a}{T}\left(1 - \frac{T}{t}\right) \\ &> \frac{a}{T}\left(1 - \frac{T}{Ta/(a - Tx)}\right) = x.\end{aligned}$$

So we have

$$\lim_{t\to\infty} \Psi_t(x) = \mathcal{M}\left\{\sum_{i=1}^{N_t} f(\xi_i \wedge T)/t \leq x\right\} = 0.$$

Case II: Assume that $x = b/T$. We will first prove that

$$\left\{\sum_{i=1}^{N_t} f(\xi_i \wedge T)/t \leq \frac{b}{T}\right\} \subset \bigcup_{i=1}^{\infty} \{\xi_i > T - \delta\}$$

for any $\delta > 0$ provided that t is large enough. Write

$$A = \left\{\sum_{i=1}^{N_t} f(\xi_i \wedge T)/t \leq \frac{b}{T}\right\}, \quad B = \bigcup_{i=1}^{\infty} \{\xi_i > T - \delta\}.$$

Fix

$$\gamma \in B^c = \bigcap_{i=1}^{\infty} \{\xi_i \leq T - \delta\}.$$

When $t \geq T(T - \delta)/\delta$, we have

$$\begin{aligned}\sum_{i=1}^{N_t(\gamma)} f(\xi_i(\gamma) \wedge T)/t = \frac{bN_t(\gamma)}{t} &\geq \frac{b(t/(T - \delta) - 1)}{t} = b\left(\frac{1}{T - \delta} - \frac{1}{t}\right) \\ &> b\left(\frac{1}{T - \delta} - \frac{1}{T(T - \delta)/\delta}\right) = \frac{b}{T}.\end{aligned}$$

Thus

$$B^c = \bigcap_{i=1}^{\infty} \{\xi_i \leq T - \delta\} \subset \left\{\sum_{i=1}^{N_t} f(\xi_i \wedge T)/t > \frac{b}{T}\right\} = A^c.$$

That is

$$A = \left\{ \sum_{i=1}^{N_t} f(\xi_i \wedge T)/t \leq \frac{b}{T} \right\} \subset \bigcup_{i=1}^{\infty} \{\xi_i > T - \delta\} = B$$

provided that $t \geq T(T-\delta)/\delta$. Hence, we obtain that

$$\lim_{t\to\infty} \Psi_t\left(\frac{b}{T}\right) = \mathcal{M}\left\{ \sum_{i=1}^{N_t} f(\xi_i \wedge T)/t \leq \frac{b}{T} \right\} \leq \mathcal{M}\left\{ \bigcup_{i=1}^{\infty} (\xi_i > T - \delta) \right\} = 1 - \Phi\left(T - \delta\right).$$

Letting $\delta \to 0$, we have

$$\lim_{t\to\infty} \Psi_t\left(b/T\right) \leq 1 - \Phi(T).$$

By Theorem 4.3, we have

$$\Psi_t\left(x\right) \geq 1 - \Phi(T)$$

for any $x \in [a/T, b/T]$. According to the monotonicity of uncertainty distribution, we have

$$\lim_{t\to\infty} \Psi_t\left(x\right) = 1 - \Phi(T)$$

for $x \in [a/T, b/T]$.

Case III: Assume that $x \geq b/T$. We will first prove that

$$\left\{ \sum_{i=1}^{N_t} f(\xi_i \wedge T)/t \leq x \right\} \subset \bigcup_{i=1}^{\infty} \left\{ \xi_i > \frac{b}{x} - \delta \right\}$$

for any $\delta > 0$ provided that t is large enough. Write

$$A = \left\{ \sum_{i=1}^{N_t} f(\xi_i \wedge T)/t \leq x \right\}, \quad B = \bigcup_{i=1}^{\infty} \left\{ \xi_i > \frac{b}{x} - \delta \right\}.$$

Fix

$$\gamma \in B^c = \bigcap_{i=1}^{\infty} \left\{ \xi_i \leq \frac{b}{x} - \delta \right\}.$$

When $t \geq b(b-\delta x)/(\delta x^2)$, we have

$$\begin{aligned}\sum_{i=1}^{N_t(\gamma)} f(\xi_i(\gamma) \wedge T)/t = \frac{bN_t(\gamma)}{t} &\geq \frac{b}{t}\left(\frac{t}{b/x-\delta} - 1\right) = b\left(\frac{x}{b-\delta x} - \frac{1}{t}\right) \\ &> b\left(\frac{x}{b-\delta x} - \frac{\delta x^2}{b(b-\delta x)}\right) = x.\end{aligned}$$

Thus

$$B^c = \bigcap_{i=1}^{\infty} \left\{ \xi_i \leq \frac{b}{x} - \delta \right\} \subset \left\{ \sum_{i=1}^{N_t} f(\xi_i \wedge T)/t > x \right\} = A^c.$$

That is

$$A = \left\{ \sum_{i=1}^{N_t} f(\xi_i \wedge T)/t \leq x \right\} \subset \bigcup_{i=1}^{\infty} \left\{ \xi_i > \frac{b}{x} - \delta \right\} = B$$

provided that $t \geq b(b-\delta x)/(\delta x^2)$, Hence, we obtain that

$$\lim_{t\to\infty} \Psi_t(x) = \mathcal{M}\left\{\sum_{i=1}^{N_t} f(\xi_i \wedge T)/t \leq x\right\} \leq \mathcal{M}\left\{\bigcup_{i=1}^{\infty}\left(\xi_i > \frac{b}{x} - \delta\right)\right\} = 1 - \Phi\left(\frac{b}{x} - \delta\right).$$

Letting $\delta \to 0$, we have

$$\lim_{t\to\infty} \Psi_t(x) \leq 1 - \Phi(b/x).$$

By Theorem 4.3, we have

$$\Psi_t(x) \geq 1 - \Phi(b/x)$$

for any $x > b/T$. Hence, we get

$$\Psi_t(x) = 1 - \Phi(b/x).$$

The theorem is proved. □

Theorem 4.5. *Let $\xi_1, \xi_2, \cdots$ be a sequence of iid positive uncertain variables with a common uncertainty distribution Φ, and N_t be an uncertain renewal process with uncertain interarrivals $\xi_1 \wedge T, \xi_2 \wedge T, \cdots$ Assume that*

$$f(x) = \begin{cases} b & \text{if } x < T \\ a & \text{if } x = T \end{cases}$$

with $0 < a < b$. Then

$$\lim_{t\to\infty} E\left[\sum_{i=1}^{N_t} f(\xi_i \wedge T)/t\right] = \frac{a}{T} + \frac{b-a}{T}\Phi(T) + b\int_0^T \frac{\Phi(x)}{x^2}\mathrm{d}x.$$

Proof. Let Ψ denote the uncertainty distribution of $f(\xi_1 \wedge T)/(\xi_1 \wedge T)$. It is easy to verify that

$$\begin{aligned} E\left[\frac{f(\xi_1 \wedge T)}{\xi_1 \wedge T}\right] &= \int_0^{+\infty} 1 - \Psi(x)\mathrm{d}x \\ &= \frac{a}{T} + \frac{b-a}{T}\Phi(T) + \int_{b/T}^{+\infty} \Phi\left(\frac{b}{x}\right)\mathrm{d}x \\ &= \frac{a}{T} + \frac{b-a}{T}\Phi(T) + b\int_0^T \frac{\Phi(x)}{x^2}\mathrm{d}x. \end{aligned}$$

Since $1 - \Psi_t(x) \leq 1 - \Psi(x)$ by Theorem 4.3, according to the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} \lim_{t\to+\infty} E\left[\sum_{i=1}^{N_t} f(\xi_i \wedge T)/t\right] &= \lim_{t\to+\infty}\int_0^{+\infty} 1 - \Psi_t(x)\mathrm{d}x = \int_0^{+\infty} 1 - \Psi(x)\mathrm{d}x \\ &= \frac{a}{T} + \frac{b-a}{T}\Phi(T) + b\int_0^T \frac{\Phi(x)}{x^2}\mathrm{d}x. \end{aligned}$$

□

From Theorem 4.5, in order to find the optimal replacement time, we just need to find T^* which solves the problem

$$\min_{T} \frac{a}{T} + \frac{b-a}{T}\Phi(T) + b\int_0^T \frac{\Phi(x)}{x^2}\mathrm{d}x.$$

5. Conclusions

This paper first introduced uncertainty theory to age replacement policy, and treated the age of the system as uncertain variable instead of random variable. The optimal time to replace the unit was also obtained.

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