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Calculation of centroid of high dimensional fuzzy number and application

G. Wang[1] and Y. Xu[2]

[1,2] *Institute of Operational Research and Cybernetics, Hangzhou Dianzi University, Hangzhou, China*

g.x.wang@hdu.edu.cn, yfxu@hdu.edu.cn

Abstract

In this paper, the conception of centroid of n-dimensional fuzzy number is introduced via regarding its membership function as the density function on its support set, and some properties of it are obtained. Compared with the mean of the multi dimensional fuzzy number, the centroid takes into account the overall relationship between the edge membership functions of the membership function of the multi dimensional fuzzy number. Therefore, it can approximate (characterize) the fuzzy number more objectively and reasonably than using the mean of the multi dimensional fuzzy number. The most important work of this paper is that for two special kinds of multi dimensional fuzzy numbers (fuzzy n-cell numbers and fuzzy n-ellipsoid numbers), we respectively give calculation formulas, which can be used conveniently in application since the formulas are based on a definite integral of the level set functions of the multi dimensional fuzzy number on the unit interval $[0, 1]$, rather than the multiple integral of the membership function of the multi dimensional fuzzy number itself on its support set. Then, by using the calculation formulas, we obtain another special property of the centroid for fuzzy n-cell number and fuzzy n-ellipsoid number. Finally, as an example of application, by using the centroid of multi dimensional fuzzy number, we define a fuzzy order on n-dimensional fuzzy number space, which can be used to rank uncertain or imprecise multichannel digital information.

Keywords: Fuzzy number, fuzzy $n-$cell number, fuzzy $n-$ellipsoid number, centroid of high dimensional fuzzy number, fuzzy order.

1 Introduction

The concept of fuzzy numbers was introduced by Chang and Zadeh [6] in 1972 with the consideration of the properties of probability functions. Since then both the numbers and the problems in relation to them have been widely studied, see for example, [2, 10, 12, 31, 35] and the references therein. With the development of theories and applications of fuzzy numbers, this concept becomes more and more important [3, 5, 7, 11, 15, 16].

Recently, there are still a lot of work in this area or related to this area. For example, in 2014, Arotaritei and Ionescu introduced "fuzzy Voronoi" diagrams for fuzzy numbers of dimension two by extension of Voronoi diagrams for fuzzy numbers in [1]; Coroianu, Gagolewski and Grzegorzewski studied the problem of the nearest approximation of fuzzy numbers by piecewise linear 1-knot fuzzy numbers in [8]; Haji, Zare, Eslamipoor and Sepehriar presented a new method for ranking fuzzy numbers based on the left and right using distance method and level set in [19]; Yazdi, GhasemiGol and Effati et al presented a new hierarchical tree approach to clustering fuzzy numbers in [33]. In 2015, Gong, Zhang and Zhu studied the problem of statistical convergence for sequences of fuzzy-number-valued functions in[17]. In 2016, Coroianu and Stefanini studied the problem of general approximation of fuzzy numbers by F-transform in [9]; Ban and Coroianu studied the problem of symmetric triangular approximations of fuzzy numbers under a general condition and properties in [4]; Gong, Hai and Li studied the convexity and differentiability of $n-$dimensional fuzzy number-valued functions and their applications in [13, 18]. In 2018, Yeh also studied the problem of symmetric triangular approximations of fuzzy numbers under a general condition and properties in [34]. This year, Gong and Hao studied fuzzy Laplace transform based on the Henstock integral for fuzzy-number-valued functions in [14]; Yang and

Corresponding Author: G. Wang

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Gong studied fuzzy Fredholm integral equations of the first kind and regularization methods for fuzzy-number-valued functions in [32].

One of the reasons why fuzzy numbers have become the research object of many researchers is that they have strong application background. For example, fuzzy numbers can be used to express uncertain or imprecise digital information [24, 26]. Thus, the method of ranking, classifying and recognizing the uncertain or imprecise digital information can be established [24, 27, 28] by studying the methods of ranking, classifying and recognizing the fuzzy numbers. In the research of ranking, classification and recognition of fuzzy numbers, an important numerical characteristics called mean of fuzzy numbers is often used. If the uncertain or imprecise digital information to be ranked, classified and recognized is multi channel, then we need to establish methods of ranking, classifying and recognizing multi dimensional fuzzy numbers. However, for general multi dimensional fuzzy number, the concept of mean can not be introduced as $1-$dimensional fuzzy number because of its complex structure (the boundary of its level set can not be expressed by a real functions which can directly obtain from its membership function). Even though we can define the concept of mean for some special multi dimensional fuzzy numbers (such as fuzzy cell numbers and fuzzy ellipsoid numbers), there are some shortcomings in their application (see the beginning of Section 3 Centroid of n-dimensional fuzzy number). In this paper, we are going to introduce the concept of the centroid for fuzzy numbers directly using the fuzzy membership function itself, and investigate its properties, especially its calculations so that it can be easily used in applications. The introduced centroid can not only be defined for general multi-dimensional fuzzy numbers, but also can characterize the fuzzy number more reasonably than the mean for special multi dimensional fuzzy numbers such as fuzzy cell numbers or fuzzy ellipsoid numbers (also see the beginning of Section 3 Centroid of n-dimensional fuzzy number), so that the processing method of multi channel uncertain or imprecise information such as ranking, classifying and recognizing based on the concept of the centroid will be more reasonable than that based on the concept of the mean.

This paper is arranged as follows: In Section 2, we review some basic definitions and notations and results which will be used in this paper. In Section 3, we give the definition of centroid of general n-dimensional fuzzy number, investigate its properties. In Section 4, for fuzzy n-cell numbers, we obtain a calculation formula of the centroid, which can be used conveniently in application since the formulas are based on a definite integral of the level set functions of the multi dimensional fuzzy number on the unit interval [0,1], rather than the multiple integral of the membership function of the multi dimensional fuzzy number itself on its support set. And approve a property for the centroid of fuzzy cell number by using the obtained calculation formula. In Section 5, we also discuss the calculation of the centroid for fuzzy n-ellipsoid numbers and give a calculation formula that can be used conveniently in application and a property. In Section 6, as an example of application, by using the centroid of multi dimensional fuzzy number, we define a fuzzy order on n-dimensional fuzzy number space, which can be used to rank uncertain or imprecise multichannel digital information. Finally, in Section 7, we make a conclusion to this paper.

2 Basic definitions and notations

Let n be a natural number, R be the real number set, and R^n be the $n-$dimensional Euclidean space. A fuzzy subset (for short, a fuzzy set) of R^n is a function $u : R^n \to [0,1]$. For each such fuzzy set u, we denote the $r-$level set of u by $[u]^r$, i.e., $[u]^r = \{\mathbf{x} \in R^n : u(\mathbf{x}) \geq r\}$ for any $r \in (0,1]$. By supp$u$ we denote the support of u, i.e., the set $\{\mathbf{x} \in R^n : u(\mathbf{x}) > 0\}$. By $[u]^0$ we denote the closure of the set supp$u$, i.e., $[u]^0 = \overline{\{\mathbf{x} \in R^n : u(\mathbf{x}) > 0\}}$.

If u is a normal and fuzzy convex fuzzy set of R^n, $u(\mathbf{x})$ is upper semi-continuous, and $[u]^0$ is compact, then u is called a $n-$dimensional fuzzy number, and the collection of all $n-$dimensional fuzzy numbers is denoted by E^n.

Let $K(R^n)$ denote the collection of non-empty compact subsets of R^n. The space $K(R^n)$ has a linear structure induced by the addition and scalar multiplication $A + B = \{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in A, \mathbf{b} \in B\}$ and $\lambda A = \{\lambda\mathbf{a} \mid \mathbf{a} \in A\}$ for any $A, B \in K(R^n),\ \lambda \in R$.

For any $\mathbf{a} \in R^n$, define an $n-$dimensional fuzzy number $\hat{\mathbf{a}}$ by

$$\hat{\mathbf{a}}(\mathbf{x}) = \begin{cases} 1 & \text{if} \quad \mathbf{x} = \mathbf{a} \\ 0 & \text{if} \quad \mathbf{x} \neq \mathbf{a} \end{cases}$$

for any $\mathbf{x} \in R^n$.

The addition, scalar multiplication on E^n are defined by

$$(u + v)(\mathbf{x}) = \sup_{\mathbf{y}+\mathbf{z}=\mathbf{x}} \min(u(\mathbf{y}), v(\mathbf{z})),$$

$$(\lambda u)(\mathbf{x}) = \begin{cases} u(\lambda^{-1}\mathbf{x}) & \text{if} \quad \lambda \neq 0 \\ \hat{0}(\mathbf{x}) & \text{if} \quad \lambda = 0 \end{cases}$$

for $u, v \in E^n$ and $\lambda \in R$.

It is known that if $u \in E^n$, then for each $r \in [0,1]$, $[u]^r$ is a non-empty compact convex set of R^n.

Let $a_i, b_i \in R$ with $a_i \le b_i$ $(i = 1, 2, \cdots, n)$. We denote $\mathbf{CE}\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{bmatrix}$ (in short, $\mathbf{CE}_{i=1}^n[a_i, b_i]) = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$, and $\mathbf{EL}\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{bmatrix}$ (in short, $\mathbf{EL}_{i=1}^n[a_i,\ b_i]) = \{(x_1, x_2, \cdots, x_n) \in R^n | \sum_{i=1}^n \frac{(x_i - \frac{b_i + a_i}{2})^2}{(\frac{a_i - b_i}{2})} \le 1\}$.

Let $u \in E^n$. If for each $r \in [0,1]$, exist $\underline{u_i}(r)$, $\overline{u_i}(r) \in R$ with $\underline{u_i}(r) \le \overline{u_i}(r)$, $i = 1, 2, \cdots, n$ such that $[u]^r = \mathbf{CE}_{i=1}^n[\underline{u_i}(r), \overline{u_i}(r)]$, then we call u a fuzzy n-cell number [29]. And, we denote the collection of all fuzzy n-cell numbers by $\mathbf{C}(E^n)$.

Let $u \in E^n$. If for each $r \in [0,1]$, exist $\underline{u_i}(r)$, $\overline{u_i}(r) \in R$ with $\underline{u_i}(r) \le \overline{u_i}(r)$, $i = 1, 2, \cdots, n$ such that $[u]^r = \mathbf{EL}_{i=1}^n[\underline{u_i}(r), \overline{u_i}(r)]$, then we call u a fuzzy n-ellipsoid number [26]. And, we denote the collection of all fuzzy n-ellipsoid numbers by $\mathbf{E}(E^n)$.

Let $u_i \in E$ $(= E^1)$, $i = 1, 2, \cdots, n$. We call the ordered 1-dimensional fuzzy number class $u_1, u_2, \cdots, u_n$ (i.e. the Cartesian product of 1-dimensional fuzzy numbers $u_1, u_2, \cdots, u_n$) a $n-$dimensional fuzzy vector, denote it as $(u_1, u_2, \cdots, u_n)$, and call the collection of all n-dimensional fuzzy vectors (i.e. the Cartesian product $\overbrace{E \times E \times \cdots \times E}^{n}$) n-dimensional fuzzy vector space, and denote it as $(E)^n$.

By Theorem 3.1 in [21], we see fuzzy n-cell number and $n-$dimensional fuzzy vector can represent each other, and the representation is unique, so $\mathbf{C}(E^n)$ and $(E)^n$ may be regarded as identity. If $u \in \mathbf{C}(E^n)$ and $(u_1, u_2, \cdots, u_n) \in (E)^n$ express each other, then we can denote as $u = \mathbf{F}_c(u_1, u_2, \cdots, u_n)$.

By Theorem 3.1 in [23], we see fuzzy n-ellipsoid number and $n-$dimensional fuzzy vector can represent each other, and the representation is unique, so $\mathbf{E}(E^n)$ and $(E)^n$ may be regarded as identity. If $u \in \mathbf{E}(E^n)$ and $(u_1, u_2, \cdots, u_n) \in (E)^n$ express each other, then we can denote as $u = \mathbf{F}_e(u_1, u_2, \cdots, u_n)$.

For $\mathbf{a}, \mathbf{b} \in R^n$ with $\mathbf{a} = (a_1, a_2, \cdots, a_n)$ and $\mathbf{b} = (b_1, b_2, \cdots, b_n)$, we define $\mathbf{a} \le \mathbf{b}$ if and only if $a_i \le b_i$ for any $i = 1, 2, \cdots, n$.

For $u, v \in \mathbf{C}(E^n)$ (resp. $u, v \in \mathbf{E}(E^n)$) with $u = \mathbf{F}_c(u_1, u_2, \cdots, u_n)$ and $v = \mathbf{F}_c(v_1, v_2, \cdots, v_n)$ (resp. $u = \mathbf{F}_e(u_1, u_2, \cdots, u_n)$ and $v = \mathbf{F}_e(v_1, v_2, \cdots, v_n)$), we define $u \le v$ if and only if $\underline{u_i}(r) \le \underline{v_i}(r)$ and $\overline{u_i}(r) \le \overline{v_i}(r)$ for any $r \in [0,1]$ and $i = 1, 2, \cdots, n$, i.e., $u_i \le v_i, i = 1, 2, \cdots, n$.

3 Centroid of n-dimensional fuzzy number

As trying to characterize fuzzy numbers with their numerical characteristics, the means of fuzzy numbers are often used. For some n-dimensional fuzzy numbers, such as a fuzzy n-cell fuzzy number $u = \mathbf{F}_c(u_1, u_2, \cdots, u_n)$ or a fuzzy n-ellipsoid fuzzy number $u = \mathbf{F}_e(u_1, u_2, \cdots, u_n)$, its mean [24, 27, 28] is a n-dimensional vector of which components are all real numbers:

$$\mathbf{M}(u) = (M(u_1), M(u_2), \cdots, M(u_n)),$$

where $M(u_i) = \int_0^1 r[\underline{u_i}(r) + \overline{u_i}(r)]dr$.

For general high-dimensional fuzzy number, because there is no the expression of edge membership functions like cell or ellipsoid fuzzy number, the concept of such mean can not be introduced. In addition, for high-dimensional fuzzy cell or ellipsoid number u, although the concept of the mean can be defined, from the definition of mean $\mathbf{M}(u)$, we can easily see that each component $M(u_i)$ of the mean $\mathbf{M}(u)$ depends only on its own corresponding component u_i of the fuzzy n-cell fuzzy number $u = \mathbf{F}_c(u_1, u_2, \cdots, u_n)$ or the fuzzy n-ellipsoid fuzzy number $u = \mathbf{F}_e(u_1, u_2, \cdots, u_n)$, and has nothing to do with other components, i.e., the mean $\mathbf{M}(u)$ does not well reflect the overall structure of the high-dimensional fuzzy number u (the membership function of u, i.e., the joint membership function of membership functions of all components u_i $(i = 1, 2, \cdots, n)$ of u, see [21, 23]) of the fuzzy n-cell number or the fuzzy n-ellipsoid number. Therefore, using the means of components of the fuzzy n-cell number or the fuzzy n-ellipsoid number to approximate the fuzzy n-cell number or the fuzzy n-ellipsoid number has some shortcomings.

On the other hand, it is known that for a fuzzy number, the greater the degree of membership of a point in the real field R, the greater the contribution of the point to the fuzzy number. In other words, the greater the degree of a point in the real field R belonging to a fuzzy number, the greater the importance of the point to the fuzzy number. Therefore, we can regard a multi dimensional fuzzy number u as an object which is the closure $[u]^0$ of the support of u taking the membership function $u(x) = u(x_1, x_2, \cdots, x_n)$ as the density function. Therefore, we have the reason of using the centroid (we also call it the centroid of the multi dimensional fuzzy number, and denote it as $\mathbf{C}(u)$) of the object to approximate the multi dimensional fuzzy number u. And we will see (see the Theorem 4.2) that each component $\mathbf{C}(u_i)$ of the centroid $\mathbf{C}(u)$ of the multi dimensional fuzzy number u depends not only on its own corresponding component u_i, but also on all other components of the multi dimensional fuzzy number u. In other words, the components $C_1(u), C_2(u), \cdots, C_n(u)$ of the centroid $\mathbf{C}(u)$ are all related to the overall structure of multi dimensional fuzzy number u (this avoids the shortcomings of the mean $\mathbf{M}(u)$). From a structural perspective, using centroid $\mathbf{C}(u)$ of multi dimensional fuzzy number u is more objective to approximate the multi dimensional fuzzy number u than using its mean $\mathbf{M}(u)$.

In the following, we give the definition of centroid of multi dimensional fuzzy number:

Definition 3.1. *Let $u \in E^n$. For any $i = 1, 2, \cdots, n$, we define $C_i(u)$ as following:*

$$C_i(u) = \frac{\overbrace{\int \cdots \int}^{n}_{[u]^0} x_i u(x_1, x_2, \cdots, x_n) dx_1 dx_2 \cdots dx_n}{\overbrace{\int \cdots \int}^{n}_{[u]^0} u(x_1, x_2, \cdots, x_n) dx_1 dx_2 \cdots dx_n}, \tag{1}$$

then we call the $n-$dimensional real number vector $(C_1(u), C_2(u), \cdots, C_n(u))$ the centroid of multi dimensional fuzzy number u, and denote it by $\mathbf{C}(u)$, i.e., $\mathbf{C}(u) = (C_1(u), C_2(u), \cdots, C_n(u))$.

Although the addition and scalar multiplication of fuzzy numbers are different from the usual addition and scalar multiplication of real functions, we can obtain the following results for the concept of the centroid of $n-$dimensional fuzzy number:

Property 3.2. *Let $u \in E^n$. We have that*

1) $\mathbf{C}(u + \hat{\mathbf{a}}) = \mathbf{C}(u) + \mathbf{a}$ *(i.e., $\mathbf{C}(u + \hat{\mathbf{a}}) = (C_1(u) + a_1, C_2(u) + a_2, \cdots, C_n(u) + a_n)$) for any $\mathbf{a} = (a_1, a_2, \cdots, a_n) \in R^n$;*

2) $\mathbf{C}(\alpha u) = \alpha \mathbf{C}(u)$ *(i.e., $\mathbf{C}(\alpha u) = (\alpha C_1(u), \alpha C_2(u), \cdots, \alpha C_n(u))$) for any $\alpha \in R$.*

Proof. By the definition of addition on E^n, we have that

$$\begin{aligned}(u + \hat{\mathbf{a}})(\mathbf{x}) &= \sup_{\mathbf{y}+\mathbf{z}=\mathbf{x}} \min\{u(\mathbf{y}), \hat{\mathbf{a}}(\mathbf{z})\} = \sup_{\mathbf{y}+\mathbf{a}=\mathbf{x}} \min\{u(\mathbf{y}), \hat{\mathbf{a}}(\mathbf{a})\} \\ &= \sup_{\mathbf{y}+\mathbf{a}=\mathbf{x}} \min\{u(\mathbf{y}), 1\} = \sup_{\mathbf{y}+\mathbf{a}=\mathbf{x}} u(\mathbf{y}) = u(\mathbf{x} - \mathbf{a}),\end{aligned}$$

Therefore, for any $i = 1, 2, \cdots, n$, by Equation (1), we have that

$$\begin{aligned}C_i(u + \hat{\mathbf{a}}) &= \frac{\overbrace{\int \cdots \int}^{n}_{[u+\hat{\mathbf{a}}]^0} x_i \cdot (u + \hat{\mathbf{a}})(x_1, x_2, \cdots, x_n) dx_1 dx_2 \cdots dx_n}{\overbrace{\int \cdots \int}^{n}_{[u+\hat{\mathbf{a}}]^0} (u + \hat{\mathbf{a}})(x_1, x_2, \cdots, x_n) dx_1 dx_2 \cdots dx_n} \\ &= \frac{\overbrace{\int \cdots \int}^{n}_{[u+\hat{\mathbf{a}}]^0} x_i \cdot u(x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n) dx_1 dx_2 \cdots dx_n}{\overbrace{\int \cdots \int}^{n}_{[u+\hat{\mathbf{a}}]^0} u(x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n) dx_1 dx_2 \cdots dx_n}\end{aligned}$$

$$= \frac{\overbrace{\int \cdots \int}^{n}_{[u]^0+\{\hat{\mathbf{a}}\}} x_i \cdot u(x_1-a_1, x_2-a_2, \cdots, x_n-a_n) d(x_1-a_1) d(x_2-a_2) \cdots d(x_n-a_n)}{\overbrace{\int \cdots \int}^{n}_{[u]^0+\{\hat{\mathbf{a}}\}} u(x_1-a_1, x_2-a_2, \cdots, x_n-a_n) d(x_1-a_1) d(x_2-a_2) \cdots d(x_n-a_n)}$$

$$= \frac{\overbrace{\int \cdots \int}^{n}_{[u]^0+\{\hat{\mathbf{a}}\}} (y_i+a_i) \cdot u(y_1, y_2, \cdots, y_n) dy_1 dy_2 \cdots dy_n}{\overbrace{\int \cdots \int}^{n}_{[u]^0} u(y_1, y_2, \cdots, y_n) dy_1 dy_2 \cdots dy_n}$$

$$= \frac{\overbrace{\int \cdots \int}^{n}_{[u]^0+\{\hat{\mathbf{a}}\}} y_i \cdot u(y_1, y_2, \cdots, y_n) dy_1 dy_2 \cdots dy_n}{\overbrace{\int \cdots \int}^{n}_{[u]^0} u(y_1, y_2, \cdots, y_n) dy_1 dy_2 \cdots dy_n} + a_i$$

$$= C_i(u) + a_i$$

so, Conclusion 1) holds.

Let $\alpha \neq 0$. By the definition of scalar multiplication on E^n and $K(R^n)$, we have that $[\alpha u]^0 = \overline{\{\mathbf{x} \in R^n |\ (\alpha u)(\mathbf{x}) > 0\}} = \overline{\{\mathbf{x} \in R^n |\ u(\frac{1}{\alpha}\mathbf{x}) > 0\}} = \overline{\{\alpha\mathbf{y} \in R^n |\ u(\mathbf{y}) > 0\}} = \alpha \cdot \overline{\{\mathbf{y} \in R^n |\ u(\mathbf{y}) > 0\}} = \alpha[u]^0$. Then, for any $i = 1, 2, \cdots, n$, by Equation (1) we have that

$$C_i(\alpha u) = \frac{\overbrace{\int \cdots \int}^{n}_{[\alpha u]^0} x_i \cdot (\alpha u)(x_1, x_2, \cdots, x_n) dx_1 dx_2 \cdots dx_n}{\overbrace{\int \cdots \int}^{n}_{[\alpha u]^0} (\alpha u)(x_1, x_2, \cdots, x_n) dx_1 dx_2 \cdots dx_n}$$

$$= \frac{\overbrace{\int \cdots \int}^{n}_{\alpha[u]^0} x_i \cdot u(\frac{1}{\alpha}x_1, \frac{1}{\alpha}x_2, \cdots, \frac{1}{\alpha}x_n) dx_1 dx_2 \cdots dx_n}{\overbrace{\int \cdots \int}^{n}_{\alpha[u]^0} u(\frac{1}{\alpha}x_1, \frac{1}{\alpha}x_2, \cdots, \frac{1}{\alpha}x_n) dx_1 dx_2 \cdots dx_n}$$

$$= \frac{\alpha^n \overbrace{\int \cdots \int}^{n}_{\alpha[u]^0} x_i \cdot u(\frac{1}{\alpha}x_1, \frac{1}{\alpha}x_2, \cdots, \frac{1}{\alpha}x_n) d(\frac{1}{\alpha}x_1) d(\frac{1}{\alpha}x_2) \cdots d(\frac{1}{\alpha}x_n)}{\alpha^n \overbrace{\int \cdots \int}^{n}_{\alpha[u]^0} u(\frac{1}{\alpha}x_1, \frac{1}{\alpha}x_2, \cdots, \frac{1}{\alpha}x_n) d(\frac{1}{\alpha}x_1) d(\frac{1}{\alpha}x_2) \cdots d(\frac{1}{\alpha}x_n)}$$

$$= \frac{\alpha^n \overbrace{\int \cdots \int}^{n}_{[u]^0} (\alpha y_i) \cdot u(y_1, y_2, \cdots, y_n) dy_1 dy_2 \cdots dy_n}{\alpha^n \overbrace{\int \cdots \int}^{n}_{[u]^0} u(y_1, y_2, \cdots, y_n) dy_1 dy_2 \cdots dy_n}$$

$$= \alpha \cdot \frac{\overbrace{\int \cdots \int}^{n}_{[u]^0} y_i \cdot u(y_1, y_2, \cdots, y_n) dy_1 dy_2 \cdots dy_n}{\overbrace{\int \cdots \int}^{n}_{[u]^0} u(y_1, y_2, \cdots, y_n) dy_1 dy_2 \cdots dy_n}$$
$$= \alpha C_i(u)$$

On the other hand, as $\alpha = 0$, it is obvious that $\mathbf{C}(\alpha u) = \alpha \mathbf{C}(u)$ holds, so Conclusion 2) holds. □

4 Calculation for centroid of fuzzy cell number

Although the definition of centroid of multi dimensional fuzzy number is not very complicated in form, it is not convenient to use this definition expression directly in some theoretical studies and application calculations. In the following, for fuzzy cell numbers, we are going to establish a new formula for calculating the centroid so that it can be used conveniently in theoretical studies and applications. For this reason, we first give the following lemma:

Lemma 4.1. *Let $u \in E^n$, $\Omega = \{(x_1, x_2, \cdots, x_n, r) \in R^{n+1} \mid (x_1, x_2, \cdots, x_n) \in [u]^0,\ 0 \le r \le u(x_1, x_2, \cdots, x_n)\}$. then the ith component $C_i(u)$ of centroid $\mathbf{C}(u) = (C_1(u), C_2(u), \cdots, C_n(u))$ is*

$$C_i(u) = \frac{\overbrace{\int \cdots \int}^{n+1}_{\Omega} x_i dx_1 dx_2 \cdots dx_n dr}{\overbrace{\int \cdots \int}^{n+1}_{\Omega} dx_1 dx_2 \cdots dx_n dr}, \tag{2}$$

for any $i = 1, 2, \cdots, n$.

Proof. For any $i = 1, 2, \cdots, n$, we have that

$$\frac{\overbrace{\int \cdots \int}^{n+1}_{\Omega} x_i dx_1 dx_2 \cdots dx_n dr}{\overbrace{\int \cdots \int}^{n+1}_{\Omega} dx_1 dx_2 \cdots dx_n dr} = \frac{\overbrace{\int \cdots \int}^{n}_{[u]^0} (\int_0^{u(x_1, x_2, \cdots, x_n)} x_i dr) dx_1 dx_2 \cdots dx_n}{\overbrace{\int \cdots \int}^{n}_{[u]^0} (\int_0^{u(x_1, x_2, \cdots, x_n)} dr) dx_1 dx_2 \cdots dx_n}$$
$$= \frac{\overbrace{\int \cdots \int}^{n}_{[u]^0} x_i u(x_1, x_2, \cdots, x_n) dx_1 dx_2 \cdots dx_n}{\overbrace{\int \cdots \int}^{n}_{[u]^0} u(x_1, x_2, \cdots, x_n) dx_1 dx_2 \cdots dx_n} = C_i(u).$$

The proof of the lemma is completed. □

For fuzzy cell numbers, by Lemma 4.1, we can obtain the following calculation formula of centroid:

Theorem 4.2. *Let $u = \mathbf{F}_c(u_1, u_2, \cdots, u_n) \in \mathbf{C}(E^n)$. Then for any $i = 1, 2, \cdots, n$, the ith component $C_i(u)$ of centroid $\mathbf{C}(u) = (C_1(u), C_2(u), \cdots, C_n(u))$ is*

$$C_i(u) = \frac{\frac{1}{2} \int_0^1 (\underline{u_i}(r) + \overline{u_i}(r)) \Pi_{j=1}^n (\overline{u_j}(r) - \underline{u_j}(r))\, dr}{\int_0^1 \Pi_{j=1}^n (\overline{u_j}(r) - \underline{u_j}(r))\, dr} \tag{3}$$

where $\Pi_{j=1}^{n}(\overline{u_j}(r)-\underline{u_j}(r))$ *is the multiplication of the real numbers* $(\overline{u_1}(r)-\underline{u_1}(r))$, $(\overline{u_2}(r)-\underline{u_2}(r))$, $\cdots$, $(\overline{u_n}(r)-\underline{u_n}(r))$.

Proof. Denoting $\Omega=\{(x,r)=(x_1,x_2,\cdots,x_n,r)\in R^{n+1}\mid x=(x_1,x_2,\cdots,x_n)\in[u]^0,\ 0\le r\le u(x)=u(x_1,x_2,\cdots,x_n)\}$, we see that

$$\begin{aligned}\Omega&=\{(x_1,x_2,\cdots,x_n,r)\in R^{n+1}\mid \underline{u_i}(0)\le x_i\le\overline{u_i}(0), i=1,2,\cdots,n \text{ and } 0\le r\le u(x_1,x_2,\cdots,x_n)\}\\&=\{(x_1,x_2,\cdots,x_n,r)\in R^{n+1}\mid\ 0\le r\le 1 \text{ and } \underline{u_i}(r)\le x_i\le\overline{u_i}(r),\ \ i=1,2,\cdots,n\}.\end{aligned}$$

Denoting $\Xi=\{(x_1,x_2,\cdots,x_n)\in R^n\mid\ \underline{u_i}(r)\le x_i\le\overline{u_i}(r), i=1,2,\cdots,n\}$, then, by the Lemma 4.1, we have that

$$\begin{aligned}C_i(u)&=\frac{\overbrace{\int\cdots\int_{\Omega}}^{n+1} x_i dx_1dx_2\cdots dx_n dr}{\overbrace{\int\cdots\int_{\Omega}}^{n+1} dx_1dx_2\cdots dx_n dr}\\&=\frac{\int_0^1(\overbrace{\int\cdots\int_{\Xi}}^{n} x_i dx_1dx_2\cdots dx_n)dr}{\int_0^1(\overbrace{\int\cdots\int_{\Xi}}^{n} dx_1dx_2\cdots dx_n)dr}\\&=\frac{\int_0^1(\int_{\underline{u_1}(r)}^{\overline{u_1}(r)}\int_{\underline{u_2}(r)}^{\overline{u_2}(r)}\cdots\int_{\underline{u_n}(r)}^{\overline{u_n}(r)} x_i dx_1dx_2\cdots dx_n)dr}{\int_0^1(\int_{\underline{u_1}(r)}^{\overline{u_1}(r)}\int_{\underline{u_2}(r)}^{\overline{u_2}(r)}\cdots\int_{\underline{u_n}(r)}^{\overline{u_n}(r)} dx_1dx_2\cdots dx_n)dr}\\&=\frac{\int_0^1(\int_{\underline{u_1}(r)}^{\overline{u_1}(r)} dx_1\cdots\int_{\underline{u_i}(r)}^{\overline{u_i}(r)} x_i dx_i\cdots\int_{\underline{u_n}(r)}^{\overline{u_n}(r)} dx_n)dr}{\int_0^1(\int_{\underline{u_1}(r)}^{\overline{u_1}(r)} dx_1\cdots\int_{\underline{u_n}(r)}^{\overline{u_n}(r)} dx_n)dr}\\&=\frac{\frac{1}{2}\int_0^1(\overline{u_i}(r)+\underline{u_i}(r))\Pi_{j=1}^{n}(\overline{u_j}(r)-\underline{u_j}(r))dr}{\int_0^1\Pi_{j=1}^{n}(\overline{u_j}(r)-\underline{u_j}(r))\ dr}\end{aligned}$$

where, $\Pi_{j=1}^{n}(\overline{u_j}(r)-\underline{u_j}(r))$ is the multiplication of the real numbers $(\overline{u_1}(r)-\underline{u_1}(r))$, $(\overline{u_2}(r)-\underline{u_2}(r))$, $\cdots$, $(\overline{u_n}(r)-\underline{u_n}(r))$. The proof of the theorem is completed. □

Property 4.3. *Let* $u,v\in\mathbf{C}(E^n)$ *with* $u=\mathbf{F}_c(u_1,u_2,\cdots,u_n)$ *and* $v=\mathbf{F}_c(v_1,v_2,\cdots,v_n)$. *Then* $u\le v\Longrightarrow\mathbf{C}(u)\le\mathbf{C}(v)$, *i.e.,* $C_i(u)\le C(v_i)$ *for any* $i=1,2,\cdots,n$.

Proof. From $\underline{u_i}(r)\le\underline{v_i}(r)$, $\overline{u_i}(r)\le\overline{v_i}(r)$ (since $u\le v$), $\underline{u_i}(r)\le\overline{u_i}(r)$ and $\underline{u_i}(r)\le\overline{u_i}(r)$ for any $r\in[0,1]$ and $i=1,2,\cdots,n$, we can see that $\underline{u_i}(r)+\overline{u_i}(r)\le\underline{v_i}(r)+\overline{v_i}(r)$ and

$$\Pi_{j=1}^{n}(\overline{u_j}(s)-\underline{u_j}(s))\cdot\Pi_{j=1}^{n}(\overline{v_j}(t)-\underline{v_j}(t))\ge 0$$

for any $r,s,t\in[0,1]$. Denoting $D=[0,1]\times[0,1]$ (the cartesian product of $[0,1]$ and $[0,1]$), then we can obtain that

$$\int_0^1(\underline{u_i}(r)+\overline{u_i}(r))dr\le\int_0^1(\underline{v_i}(r)+\overline{v_i}(r))dr$$

and

$$\begin{aligned}&\iint_D\Pi_{j=1}^{n}(\overline{v_j}(s)-\underline{v_j}(s))\cdot\Pi_{j=1}^{n}(\overline{u_j}(t)-\underline{u_j}(t))dsdt\\&=\int_0^1\Pi_{j=1}^{n}(\overline{v_j}(s)-\underline{v_j}(s))ds\int_0^1\Pi_{j=1}^{n}(\overline{u_j}(t)-\underline{u_j}(t))dt\\&=\int_0^1\Pi_{j=1}^{n}(\overline{v_j}(t)-\underline{v_j}(t))dt\int_0^1\Pi_{j=1}^{n}(\overline{u_j}(s)-\underline{u_j}(s))ds\\&=\iint_D\Pi_{j=1}^{n}(\overline{v_j}(t)-\underline{v_j}(t))\cdot\Pi_{j=1}^{n}(\overline{u_j}(s)-\underline{u_j}(s))dsdt\\&=\iint_D\Pi_{j=1}^{n}(\overline{u_j}(s)-\underline{u_j}(s))\cdot\Pi_{j=1}^{n}(\overline{v_j}(t)-\underline{v_j}(t))dsdt\\&\ge 0\end{aligned}$$

It implies that

$$\begin{aligned}&\int_0^1(\underline{u_i}(r)+\overline{u_i}(r))dr\iint_D\Pi_{j=1}^{n}(\overline{u_j}(s)-\underline{u_j}(s))\cdot\Pi_{j=1}^{n}(\overline{v_j}(t)-\underline{v_j}(t))dsdt\\\le&\int_0^1(\underline{v_i}(r)+\overline{v_i}(r))dr\iint_D\Pi_{j=1}^{n}(\overline{v_j}(s)-\underline{v_j}(s))\cdot\Pi_{j=1}^{n}(\overline{u_j}(t)-\underline{u_j}(t))dsdt\end{aligned}$$

i.e.,

$$\iiint_{[0,1]^3}(\underline{u_i}(r)+\overline{u_i}(r))\Pi_{j=1}^n(\overline{u_j}(s)-\underline{u_j}(s))\cdot\Pi_{j=1}^n((\overline{v_j}(t)-\underline{v_j}(t))drdsdt$$
$$\le\iiint_{[0,1]^3}(\underline{v_i}(r)+\overline{v_i}(r))\Pi_{j=1}^n(\overline{v_j}(s)-\underline{v_j}(s))\cdot\Pi_{j=1}^n(\overline{u_j}(t)-\underline{u_j}(t))drdsdt$$

where $[0,1]^3=[0,1]\times D$ (the cartesian product of $[0,1]$ and $[0,1]$ and $[0,1]$), so

$$(\underline{u_i}(r)+\overline{u_i}(r))\Pi_{j=1}^n(\overline{u_j}(s)-\underline{u_j}(s))\cdot\Pi_{j=1}^n(\overline{v_j}(t)-\underline{v_j}(t))$$
$$\le(\underline{v_i}(r)+\overline{v_i}(r))\Pi_{j=1}^n(\overline{v_j}(s)-\underline{v_j}(s))\cdot\Pi_{j=1}^n(\overline{u_j}(t)-\underline{u_j}(t))$$

almost hold for (r,s,t) everywhere on $[0,1]^3$. Therefore there exists $\Delta\subset[0,1]^3$ with $\mu([0,1]^3\setminus\Delta)$ (Lebesgue measure) $=0$ such that

$$\underline{u_i}(r)+\overline{u_i}(r)\Pi_{j=1}^n(\overline{u_j}(s)-\underline{u_j}(s))\cdot\Pi_{j=1}^n((\overline{v_j}(t)-\underline{v_j}(t))$$
$$\le\underline{v_i}(r)+\overline{v_i}(r)\Pi_{j=1}^n(\overline{v_j}(s)-\underline{v_j}(s))\cdot\Pi_{j=1}^n(\overline{u_j}(t)-\underline{u_j}(t))$$

holds for all $(r,s,t)\in\Delta$. From $\mu([0,1]^3\backslash\Delta)=0$, we see that Δ is dense in $[0,1]^3$. Therefore, for any $(r_0,s_0,t_0)\in[0,1]\backslash\Delta$, there exists $\{r_m\}_{m=1}^\infty\subset[0,1]$ which is non decreasing as $r_0\neq0$ and non increasing as $r_0=0$, $\{s_m\}_{m=1}^\infty\subset[0,1]$ which is non decreasing as $s_0\neq0$ and non increasing as $s_0=0$ and $\{t_m\}_{m=1}^\infty\subset[0,1]$ which is non decreasing as $t_0\neq0$ and non increasing as $t_0=0$ such that $\{(r_m,s_m,t_m)\}_{m=1}^\infty\subset\Delta$ and $\lim_{m\to\infty}(r_m,s_m,t_m)=(r_0,s_0,t_0)$. By the left continuity of $\underline{u}(r),\overline{u}(r),\underline{v}(r)$ and $\overline{v}(r)$ at $r\neq0$ and the right continuity of $\underline{u}(r),\overline{u}(r),\underline{v}(r)$ and $\overline{v}(r)$ at $r=0$ (see Theorem 3.1 in [29]), from

$$\underline{u_i}(r_m)+\overline{u_i}(r_m)\Pi_{j=1}^n(\overline{u_j}(s_m)-\underline{u_j}(s_m))\cdot\Pi_{j=1}^n((\overline{v_j}(t_m)-\underline{v_j}(t_m))$$
$$\le\underline{v_i}(r_m)+\overline{v_i}(r_m)\Pi_{j=1}^n(\overline{v_j}(s_m)-\underline{v_j}(s_m))\cdot\Pi_{j=1}^n(\overline{u_j}(t_m)-\underline{u_j}(t_m))$$

we have that

$$\underline{u_i}(r_0)+\overline{u_i}(r_0)\Pi_{j=1}^n(\overline{u_j}(s_0)-\underline{u_j}(s_0))\cdot\Pi_{j=1}^n((\overline{v_j}(t_0)-\underline{v_j}(t_0))$$
$$\le\underline{v_i}(r_0)+\overline{v_i}(r_0)\Pi_{j=1}^n(\overline{v_j}(s_0)-\underline{v_j}(s_0))\cdot\Pi_{j=1}^n(\overline{u_j}(t_0)-\underline{u_j}(t_0))$$

Thus, we know that

$$\underline{u_i}(r)+\overline{u_i}(r)\Pi_{j=1}^n(\overline{u_j}(s)-\underline{u_j}(s))\cdot\Pi_{j=1}^n((\overline{v_j}(t)-\underline{v_j}(t))$$
$$\le\underline{v_i}(r)+\overline{v_i}(r)\Pi_{j=1}^n(\overline{v_j}(s)-\underline{v_j}(s))\cdot\Pi_{j=1}^n(\overline{u_j}(t)-\underline{u_j}(t))$$

hold for any $(r,s,t)\in[0,1]^3$. Taking $s=r$, we obtain that

$$\underline{u_i}(r)+\overline{u_i}(r)\Pi_{j=1}^n(\overline{u_j}(r)-\underline{u_j}(r))\cdot\Pi_{j=1}^n((\overline{v_j}(t)-\underline{v_j}(t))$$
$$\le\underline{v_i}(r)+\overline{v_i}(r)\Pi_{j=1}^n(\overline{v_j}(r)-\underline{v_j}(r))\cdot\Pi_{j=1}^n(\overline{u_j}(t)-\underline{u_j}(t))$$

so

$$\iint_D(\underline{u_i}(r)+\overline{u_i}(r))\Pi_{j=1}^n(\overline{u_j}(r)-\underline{u_j}(r))\cdot\Pi_{j=1}^n((\overline{v_j}(t)-\underline{v_j}(t))drdt$$
$$\le\iint_D(\underline{v_i}(r)+\overline{v_i}(r))\Pi_{j=1}^n(\overline{v_j}(r)-\underline{v_j}(r))\cdot\Pi_{j=1}^n(\overline{u_j}(t)-\underline{u_j}(t))drdt$$

It implies that

$$\int_0^1(\underline{u_i}(r)+\overline{u_i}(r))\Pi_{j=1}^n(\overline{u_j}(r)-\underline{u_j}(r))dr\cdot\int_0^1\Pi_{j=1}^n((\overline{v_j}(t)-\underline{v_j}(t))dt$$
$$\le\int_0^1(\underline{v_i}(r)+\overline{v_i}(r))\Pi_{j=1}^n(\overline{v_j}(r)-\underline{v_j}(r))dr\cdot\int_0^1\Pi_{j=1}^n(\overline{u_j}(t)-\underline{u_j}(t))dt$$

i.e.,

$$\frac{\int_0^1(\underline{u_i}(r)+\overline{u_i}(r))\Pi_{j=1}^n(\overline{u_j}(r)-\underline{u_j}(r))dr}{\int_0^1\Pi_{j=1}^n(\overline{u_j}(t)-\underline{u_j}(t))dt}\le\frac{\int_0^1(\underline{v_i}(r)+\overline{v_i}(r))\Pi_{j=1}^n(\overline{v_j}(r)-\underline{v_j}(r))dr}{\int_0^1\Pi_{j=1}^n(\overline{v_j}(t)-\underline{v_j}(t))dt}$$

so,

$$\frac{\frac{1}{2}\int_0^1(\underline{u_i}(r)+\overline{u_i}(r))\Pi_{j=1}^n(\overline{u_j}(r)-\underline{u_j}(r))dr}{\int_0^1\Pi_{j=1}^n(\overline{u_j}(r)-\underline{u_j}(r))dr}\le\frac{\frac{1}{2}\int_0^1(\underline{v_i}(r)+\overline{v_i}(r))\Pi_{j=1}^n(\overline{v_j}(r)-\underline{v_j}(r))dr}{\int_0^1\Pi_{j=1}^n(\overline{v_j}(r)-\underline{v_j}(r))dr}$$

Thus, by Theorem 4.1, we obtain that $C_i(u)\le C(v_i)$ for any $i=1,2,\cdots,n$, i.e., we complete the proof of the property. □

5 Calculation for centroid of fuzzy ellipsoid number

In this section, for fuzzy ellipsoid numbers, we are also going to establish a new formula for calculating the centroid so that it can be used conveniently in theoretical studies and applications.

Theorem 5.1. *let $u \in \mathbf{E}(E^n)$ with $u = \mathbf{F}_e(u_1, u_2, \cdots, u_n)$. Then for any $i = 1, 2, \cdots, n$, the ith component $C_i(u)$ of centroid $\mathbf{C}(u) = (C_1(u), C_2(u), \cdots, C_n(u))$ is*

$$C_i(u) = \frac{\frac{1}{2}\int_0^1 (\underline{u_i}(r) + \overline{u_i}(r))\Pi_{j=1}^n(\overline{u_j}(r) - \underline{u_j}(r))\,dr}{\int_0^1 \Pi_{j=1}^n(\overline{u_j}(r) - \underline{u_j}(r))\,dr}, \tag{4}$$

where $\Pi_{j=1}^n(\overline{u_j}(r) - \underline{u_j}(r))$ is the multiplication of the real numbers $(\overline{u_1}(r) - \underline{u_1}(r))$, $(\overline{u_2}(r) - \underline{u_2}(r))$, $\cdots$, $(\overline{u_n}(r) - \underline{u_n}(r))$.

Proof. Let $\Omega = \{(x, r) = (x_1, x_2, \cdots, x_n, r) \in R^{n+1} \mid x = (x_1, x_2, \cdots, x_n) \in [u]^0,\ 0 \le r \le u(x) = u(x_1, x_2, \cdots, x_n)\}$. We have that

$$\begin{aligned}\Omega &= \left\{(x_1, x_2, \cdots, x_n, r) \in R^{n+1} \mid \sum_{i=1}^n \frac{(x_i - \frac{\underline{u_i}(0)+\overline{u_i}(0)}{2})^2}{(\frac{\overline{u_i}(0)-\underline{u_i}(0)}{2})^2} \le 1,\ 0 \le r \le u(x_1, x_2, \cdots, x_n)\right\}\\ &= \left\{(x_1, x_2, \cdots, x_n, r) \in R^{n+1} \mid\ 0 \le r \le 1, \sum_{i=1}^n \frac{(x_i - \frac{\underline{u_i}(r)+\overline{u_i}(r)}{2})^2}{(\frac{\overline{u_i}(r)-\underline{u_i}(r)}{2})^2} \le 1\right\},\end{aligned}$$

Denoting $\Xi_r = \left\{(x_1, x_2, \cdots, x_n) \in R^n \mid\ \sum_{i=1}^n \frac{(x_i - \frac{\underline{u_i}(r)+\overline{u_i}(r)}{2})^2}{(\frac{\overline{u_i}(r)-\underline{u_i}(r)}{2})^2} \le r\right\}$, then, for any $i = 1, 2, \cdots, n$, by the Lemma 4.1, we have that

$$C(u_i) = \frac{\overbrace{\int\cdots\int_{\Omega}}^{n+1} x_i dx_1 \cdots dx_n dr}{\overbrace{\int\cdots\int_{\Omega}}^{n+1} dx_1 \cdots dx_n dr} = \frac{\int_0^1 (\overbrace{\int\cdots\int_{\Xi_r}}^{n} x_i dx_1 \cdots dx_n) dr}{\int_0^1 (\overbrace{\int\cdots\int_{\Xi_r}}^{n} dx_1 \cdots dx_n) dr}.$$

Denoting $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$, for any $r \in (0, 1]$, by the Formula of volume of $n-$dimentional ellipsoid (see Theorem 2.1 in [30]), we see that

$$\overbrace{\int\cdots\int_{\Xi_r}}^{n} dx_1 \cdots dx_n = \frac{2\pi^{\frac{n}{s}}}{n\Gamma(\frac{n}{2})}\Pi_{i=1}^n \frac{\overline{u}_i(r) - \underline{u}_i(r)}{2}. \tag{5}$$

For any $r \in (0, 1]$, in order to compute $\overbrace{\int\cdots\int_{\Xi_r}}^{n} x_i dx_1 \cdots dx_n$, we denote $a_i(r) = \frac{\overline{u}_i(r) - \underline{u}_i(r)}{2}$ and $b_i(r) = \frac{\overline{u}_i(r) + \underline{u}_i(r)}{2}$ $(i = 1, 2, \ldots, n)$, and make the generalized polar transformation:

$$\begin{cases} x_1 = b_1(r) + a_1(r)\rho \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{n-2} \sin\theta_{n-1}\\ x_2 = b_2(r) + a_2(r)\rho \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{n-2} \cos\theta_{n-1}\\ x_3 = b_3(r) + a_3(r)\rho \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{n-3} \cos\theta_{n-2}\\ \quad\vdots\\ x_{n-1} = b_{n-1}(r) + a_{n-1}(r)\rho \sin\theta_1 \cos\theta_2\\ x_n = b_n(r) + a_n(r)\rho \cos\theta_1 \end{cases}$$

where $0 \le \rho \le 1;\ \theta_i \in [0, \pi],\ i = 1, \cdots, n-2;\ \theta_{n-1} \in [0, 2\pi]$. Then the Jacobian determinant is

$$J_n(r) = \frac{\partial(x_1, x_2, \cdots, x_n)}{\partial(\rho, \theta_1, \cdots, \theta_n)} = (\Pi_{i=1}^n a_i(r))\rho^{n-1} \sin^{n-2}\theta_1\ \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2}.$$

As $i = 1$, for any $r \in (0, 1]$, we have that

$$\begin{aligned}&\overbrace{\int\cdots\int_{\Xi_r}}^{n} x_1 dx_1 \cdots dx_n\\ &= \overbrace{\int\cdots\int_{\Xi_r}}^{n} [b_1(r) + a_1(r)\rho \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{n-2} \sin\theta_{n-1}]\, J_n(r) d\rho d\theta_1 \cdots d\theta_{n-1}\end{aligned}$$

$$= b_1(r)\overbrace{\int\cdots\int_{\Xi_r}}^{n} J_n(r)d\rho d\theta_1\cdots d\theta_{n-1} + \overbrace{\int\cdots\int_{\Xi_r}}^{n}\left[a_1(r)\rho\sin\theta_1\sin\theta_2\ldots\sin\theta_{n-2}\sin\theta_{n-1}\right]J_n(r)d\rho d\theta_1\cdots d\theta_{n-1}$$
$$= b_1(r)\overbrace{\int\cdots\int_{\Xi_r}}^{n} dx_1\cdots dx_n + a_1(r)\left(\Pi_{i=1}^n a_i(r)\right)\overbrace{\int\cdots\int_{\Xi_r}}^{n}\left(\rho^n\sin^{n-1}\theta_1\ \sin^{n-2}\theta_2\cdots\sin^2\theta_{n-2}\sin\theta_{n-1}\right)d\rho d\theta_1\cdots d\theta_{n-1}$$
$$= b_1(r)\frac{2\pi^{\frac{n}{s}}}{n\Gamma(\frac{n}{2})}\Pi_{i=1}^n\frac{\overline{u}_i(r)-\underline{u}_i(r)}{2} + a_1(r)(\Pi_{i=1}^n a_i(r))\Big(\int_0^1\rho^n d\rho\Big)\left(\int_0^\pi\sin^{n-1}\theta_1 d\theta_1\right)\cdots\left(\int_0^\pi\sin^2\theta_{n-2}d\sin\theta_{n-2}\right)\Big(\int_0^{2\pi}\sin\theta_{n-1}d\theta_{n-1}\Big)$$
$$= b_1(r)\frac{2\pi^{\frac{n}{s}}}{n\Gamma(\frac{n}{2})}\Pi_{i=1}^n\frac{\overline{u}_i(r)-\underline{u}_i(r)}{2} + 0$$
$$= b_1(r)\frac{2\pi^{\frac{n}{s}}}{n\Gamma(\frac{n}{2})}\Pi_{i=1}^n\frac{\overline{u}_i(r)-\underline{u}_i(r)}{2}$$

In addition, as $i = 2, 3, \cdots, n$, for any $r \in (0, 1]$, we have that

$$\overbrace{\int\cdots\int_{\Xi_r}}^{n} x_i dx_1\cdots dx_n$$
$$= \overbrace{\int\cdots\int_{\Xi_r}}^{n}\left[b_i(r) + a_i(r)\rho\sin\theta_1\cdots\sin\theta_{n-i}\cos\theta_{n-i+1}\right]J_n(r)d\rho d\theta_1\cdots d\theta_{n-1}$$
$$= b_i(r)\overbrace{\int\cdots\int_{\Xi_r}}^{n} J_n(r)d\rho d\theta_1\cdots d\theta_{n-1} + \overbrace{\int\cdots\int_{\Xi_r}}^{n}\left[a_i(r)\rho\sin\theta_1\cdots\sin\theta_{n-i}\cos\theta_{n-i+1}\right]J_n(r)d\rho d\theta_1\cdots d\theta_{n-1}$$
$$= b_i(r)\overbrace{\int\cdots\int_{\Xi_r}}^{n} dx_1\cdots dx_n + a_i(r)\big(\Pi_{j=1}^n a_j(r)\big)\overbrace{\int\cdots\int_{\Xi_r}}^{n}\big(\rho^n\sin^{n-1}\theta_1\cdots\sin^i\theta_{n-i}\cos\theta_{n-i+1}\sin^{i-2}\theta_{n-i+1}\cdots\sin\theta_{n-2}\big)\,d\rho d\theta_1\cdots d\theta_{n-1}$$

By $\int_0^\pi\cos\theta_{n-i+1}\sin^{i-2}\theta_{n-i+1}d\theta_{n-i+1} = 0$ $(i = 2, 3, \cdots, n)$, we know that

$$\overbrace{\int\cdots\int_{\Xi_r}}^{n}\big(\rho^n\sin^{n-1}\theta_1\cdots\sin^i\theta_{n-i}\cos\theta_{n-i+1}\sin^{i-2}\theta_{n-i+1}\cdots\sin\theta_{n-2}\big)\,d\rho d\theta_1\cdots d\theta_{n-1} = 0$$

Therefore, By Equation (5), we have that

$$\overbrace{\int\cdots\int_{\Xi_r}}^{n} x_i dx_1\cdots dx_n = b_i(r)\frac{2\pi^{\frac{n}{s}}}{n\Gamma(\frac{n}{2})}\Pi_{i=1}^n\frac{\overline{u}_i(r)-\underline{u}_i(r)}{2},\ i = 1, 2, \cdots, n$$

Thus, for any $i = 1, 2, \cdots, n$, $\overbrace{\int\cdots\int_{\Xi_r}}^{n} x_i dx_1\cdots dx_n = b_i(r)\frac{2\pi^{\frac{n}{s}}}{n\Gamma(\frac{n}{2})}\Pi_{i=1}^n\frac{\overline{u}_i(r)-\underline{u}_i(r)}{2}$ holds. So, for any $i = 1, 2, \cdots, n$, we have that

$$C(u_i) = \frac{\int_0^1(\overbrace{\int\cdots\int_{\Xi_r}}^{n} x_i dx_1\cdots dx_n)dr}{\int_0^1(\overbrace{\int\cdots\int_{\Xi_r}}^{n} dx_1\cdots dx_n)dr} = \frac{\int_0^1 b_i(r)\frac{2\pi^{\frac{n}{s}}}{n\Gamma(\frac{n}{2})}\Pi_{i=1}^n\frac{\overline{u}_i(r)-\underline{u}_i(r)}{2}dr}{\int_0^1\frac{2\pi^{\frac{n}{s}}}{n\Gamma(\frac{n}{2})}\Pi_{i=1}^n\frac{\overline{u}_i(r)-\underline{u}_i(r)}{2}dr}$$
$$= \frac{\int_0^1 b_i(r)\Pi_{i=1}^n\frac{\overline{u}_i(r)-\underline{u}_i(r)}{2}dr}{\int_0^1\Pi_{i=1}^n\frac{\overline{u}_i(r)-\underline{u}_i(r)}{2}dr} = \frac{\frac{1}{2}\int_0^1(\underline{u_i}(r)+\overline{u_i}(r))\Pi_{j=1}^n(\overline{u_j}(r)-\underline{u_j}(r))\ dr}{\int_0^1\Pi_{j=1}^n(\overline{u_j}(r)-\underline{u_j}(r))\ dr}$$

The proof of theorem is completed. □

Remark 5.2. *For any given high-dimensional fuzzy cell number (resp. high-dimensional fuzzy ellipsoid number) u, by the Theorem 3.2 in [21] (resp. Theorem 8 in [23]), we can work out its edge membership functions u_i $(i = 1, 2, \cdots, n)$ such that $u = \mathbf{F}_c(u_1, u_2, \cdots, u_n)$ (resp. $u = \mathbf{F}_e(u_1, u_2, \cdots, u_n)$). Then obtain the functions of level value r on $[0, 1]$ $\underline{u_i}(r)$ and $\overline{u_i}(r)$ which are respectively the left endpoint and the right endpoint of the $r-$cut set $[u]^r$ of u. In this way, we can use Formula (3) (resp. Formula (4)) to calculate the centroid $\mathbf{C}(u)$ of fuzzy number u only by calculating definite integrals, which avoids the trouble of calculating $n-$multiple integrals by directly using definition formula (1). Moreover, in theoretical research and application, the high-dimensional fuzzy numbers discussed or used are usually given directly in the form of cut-set functions (i.e. $\underline{u_i}(r)$ and $\overline{u_i}(r)$) Therefore, obtained Formulas (3) and (4) are more important.*

From Theorems 1 and 2, we can see that for $u_c = \mathbf{F}_c(u_1, u_2, \cdots, u_n) \in \mathbf{C}(E^n)$ and $u_e = \mathbf{F}_e(u_1, u_2, \cdots, u_n) \in \mathbf{E}(E^n)$, the calculation formulas of $\mathbf{C}(u_c)$ and $\mathbf{C}(u_e)$ are the same. So, the centroid of fuzzy ellipsoid number also has the same property with the centroid of fuzzy cell number:

Property 5.3. *Let $u, v \in \mathbf{E}(E^n)$ with $u = \mathbf{F}_e(u_1, u_2, \cdots, u_n)$ and $v = \mathbf{F}_e(v_1, v_2, \cdots, v_n)$. Then $u \le v \Longrightarrow \mathbf{C}(u) \le \mathbf{C}(v)$, i.e., $C_i(u) \le C(v_i)$ for any $i = 1, 2, \cdots, n$.*

6 Applications

In [20, 24, 25, 27, 28], we used the mean of fuzzy numbers to construct processing methods such as recognizing, classifying and ranking for multi channel uncertain or imprecise digital information. In order to show that results obtained have also a strong application background, as an example, in this section, we are going to give a fuzzy order in the fuzzy ellipsoid space based on the concept of centroid of fuzzy number, which can be used to set up a more objective and reasonable dynamic method of ranking multi-channel uncertain or imprecise information than the dynamic method proposed in [28].

For $u, v \in E^n$ and $p = (p_1, p_2, \cdots, p_n) \in R^n$ with $p_i > 0$ and $\sum_{i=1}^n p_i = 1$, it is obvious that $\mathbf{C}(u) = \mathbf{C}(v) \Longleftrightarrow \sum_{i=1}^n p_i |C_i(v) - C_i(u)| = 0$.

For any $u, v \in E^n$, we denote $C_i^M(u, v) = \max\{C_i(u), C_i(v)\}$, $i = 1, 2, \cdots, n$, where $C_i(u)$ and $C_i(v)$ are respectively the ith components of the centroid $\mathbf{C}(u)$ of u and the centroid $\mathbf{C}(v)$ of v.

Definition 6.1. *Let $p = (p_1, p_2, \cdots, p_n) \in R^n$ with $p_i > 0$ and $\sum_{i=1}^n p_i = 1$. We define a fuzzy binary relation $\prec_p^F$ on E^n, i.e., a mapping $\prec_p^F\colon E^n \times E^n \to [0, 1]$ as following:*

$$\prec_p^F (u, v) = \begin{cases} \frac{\sum_{i=1}^n p_i \left(C_i^M(u,v) - C_i(u)\right)}{\sum_{i=1}^n p_i |C_i(v) - C_i(u)|}, & \mathbf{C}(u) \ne \mathbf{C}(v) \\ \frac{1}{2}, & \mathbf{C}(u) = \mathbf{C}(v) \end{cases} \tag{6}$$

for any $(u, v) \in E^n \times E^n$. For $u, v \in E^n$, if $\prec_p^F (u, v) = a$, then we denote $u \prec_p^a v$.

Property 6.2. *Let $p = (p_1, p_2, \cdots, p_n) \in R^n$ with $p_i > 0$ and $\sum_{i=1}^n p_i = 1$. Then*

1) $\prec_p^F (u, v) + \prec_p^F (v, u) = 1$ for any $u, v \in E^n$;

2) $u \le v$ and $\mathbf{C}(u) \ne \mathbf{C}(v) \Longrightarrow \prec_p^F (u, v) = 1$ (it is equivalent to that $u \le v$ and $\mathbf{C}(u) \ne \mathbf{C}(v) \Longrightarrow \prec_p^F (v, u) = 0$) for any $u, v \in \mathbf{C}(E^n)$ or $\mathbf{E}(E^n)$;

3) $\prec_p^F (u, v) \ge \min\{\prec_p^F (u, w), \prec_p^F (w, v)\}$ (it is equivalent to that $\prec_p^F (u, w) \ge r$ and $\prec_p^F (w, v) \ge r \Longrightarrow \prec_p^F (u, v) \ge r$ for any $r \in [0, 1]$) for any $u, v, w \in E^n$;

4) $\prec_p^F (u, v) = \frac{1}{2} \Leftrightarrow \sum_{i=1}^n p_i C_i(u) = \sum_{i=1}^n p_i C_i(v)$ for any $u, v \in L(E^n) or E(E^n)$;

5) $\prec_p^F (u + \hat{a}, v + \hat{a}) = \prec_p^F (u, v)$ for any $u, v \in E^n$ and $a \in R$;

6) $\prec_p^F (au, av) = \prec_p^F (u, v)$ for any $u, v \in E^n$ and $a \in R$.

Proof. The proof of 1): As $\mathbf{C}(u) = \mathbf{C}(v)$, by the definition of fuzzy binary relation $\prec_p^F$, we see that $\prec_p^F (u, v) + \prec_p^F (v, u) = \frac{1}{2} + \frac{1}{2} = 1$; As $\mathbf{C}(u) \ne \mathbf{C}(v)$, by the definition of fuzzy binary relation $\prec_p^F$, we have that

$$\begin{aligned} & \prec_p^F (u, v) + \prec_p^F (v, u) \\ = & \frac{\sum_{i=1}^n p_i \left(C_i^M(u,v) - C_i(u)\right)}{\sum_{i=1}^n p_i |C_i(v) - C_i(u)|}) + \frac{\sum_{i=1}^n p_i \left(C_i^M(v,u) - C_i(v)\right)}{\sum_{i=1}^n p_i |C_i(u) - C_i(v)|} \\ = & \frac{\sum_{i=1}^n p_i (\max\{C_i(u), C_i(v)\} - C_i(u)) + \sum_{i=1}^n p_i (\max\{C_i(u), C_i(v)\} - C_i(v))}{\sum_{i=1}^n p_i |C_i(v) - C_i(u)|} \\ = & \frac{\sum_{i=1}^n p_i (2\max\{C_i(u), C_i(v)\} - C_i(u) - C_i(v))}{\sum_{i=1}^n p_i |C_i(v) - C_i(u)|} \\ = & \frac{\sum_{i=1}^n p_i |C_i(v) - C_i(u)|}{\sum_{i=1}^n p_i |C_i(v) - C_i(u)|} \\ = & 1 \end{aligned}$$

The proof of 2): By the definition of fuzzy binary relation $\prec_p^F$ and Properties 2 and 3, we have $\prec_p^F(u,v) = \frac{\sum_{i=1}^n p_i\left(C_i^M(u,v)-C_i(u)\right)}{\sum_{i=1}^n p_i|C(v_i)-C(u_i)|} = \frac{\sum_{i=1}^n p_i(\max\{C_i(u),C_i(v)\}-C_i(u))}{\sum_{i=1}^n p_i(C(v_i)-C(u_i))} = \frac{\sum_{i=1}^n p_i(C(v_i)-C(u_i))}{\sum_{i=1}^n p_i(C(v_i)-C(u_i))} = 1$, so Conclusion 2) holds.

The proof of 3): It is easy to prove that $\prec_p^F(u,v) \geq \min\{\prec_p^F(u,w), \prec_p^F(w,v)\}$ is equivalent to that of $\prec_p^F(u,w) \geq r$ and $\prec_p^F(w,v) \geq r \Longrightarrow \prec_p^F(u,v) \geq r$ for any $r \in [0,1]$, so we only need to prove $\prec_p^F(u,v) \geq \min\{\prec_p^F(u,w), \prec_p^F(w,v)\}$.

If $\mathbf{C}(u) = \mathbf{C}(v)$, then by Conclusion 1), we have that $\prec_p^F(u,v) = \frac{1}{2} = \frac{\prec_p^F(u,w)+\prec_p^F(w,u)}{2} = \frac{\prec_p^F(u,w)+\prec_p^F(w,v)}{2} \geq \min\{\prec_p^F(u,w), \prec_p^F(w,v)\}$;

If $\mathbf{C}(u) \neq \mathbf{C}(v)$ and $\mathbf{C}(u) = \mathbf{C}(w)$, then $\prec_p^F(u,v) = \prec_p^F(w,v) \geq \min\{\prec_p^F(u,w), \prec_p^F(w,v)\}$;

If $\mathbf{C}(u) \neq \mathbf{C}(v)$ and $\mathbf{C}(v) = \mathbf{C}(w)$, then $\prec_p^F(u,v) = \prec_p^F(u,w) \geq \min\{\prec_p^F(u,w), \prec_p^F(w,v)\}$;

If $\mathbf{C}(u) \neq \mathbf{C}(v)$, $\mathbf{C}(u) \neq \mathbf{C}(v)$ and $\mathbf{C}(v) \neq \mathbf{C}(w)$, then, by the definition of fuzzy binary relation $\prec_p^F$ and Lemma 4.1 in [28], we have that

$$\begin{aligned}
&\prec_p^F(u,v) \\
&= \frac{\sum_{i=1}^n p_i\left(C_i^M(u,v)-C_i(u)\right)}{\sum_{i=1}^n p_i|C_i(v)-C_i(u)|}) \\
&= \frac{\sum_{i=1}^n p_i(\max\{C_i(u),C_i(v)\}-C_i(u))}{\sum_{i=1}^n p_i|C_i(v)-C_i(u)|} \\
&= \frac{\sum_{i=1}^n p_i(|C_i(v)-C_i(u)|+C_i(v)-C_i(u))}{2\sum_{i=1}^n p_i|C_i(v)-C_i(u)|} \\
&= \frac{1}{2} + \frac{\sum_{i=1}^n p_i(C_i(v)-C_i(u))}{2\sum_{i=1}^n p_i|C_i(v)-C_i(u)|} \\
&= \frac{1}{2} + \frac{\sum_{i=1}^n p_i(C_i(v)-C_i(w)+C_i(w)-C_i(u))}{2\sum_{i=1}^n p_i|C_i(v)-C_i(w)+C_i(w)-C_i(u)|} \\
&\geq \frac{1}{2} + \frac{\sum_{i=1}^n p_i(C_i(w)-C_i(u))+\sum_{i=1}^n p_i(C_i(v)-C_i(w))}{2\sum_{i=1}^n p_i|C_i(w)-C_i(u)|+2\sum_{i=1}^n p_i|C_i(v)-C_i(w)|} \\
&\geq \frac{1}{2} + \min\left\{\frac{\sum_{i=1}^n p_i(C_i(w)-C_i(u))}{2\sum_{i=1}^n p_i|C_i(w)-C_i(u)|}, \frac{\sum_{i=1}^n p_i(C_i(v)-C_i(w))}{2\sum_{i=1}^n p_i|C_i(v)-C_i(w)|}\right\} \\
&= \min\left\{\frac{1}{2} + \frac{\sum_{i=1}^n p_i(C_i(w)-C_i(u))}{2\sum_{i=1}^n p_i|C_i(w)-C_i(u)|}, \frac{1}{2} + \frac{\sum_{i=1}^n p_i(C_i(v)-C_i(w))}{2\sum_{i=1}^n p_i|C_i(v)-C_i(w)|}\right\} \\
&= \min\left\{\frac{\sum_{i=1}^n p_i[|C_i(w)-C_i(u)|+C_i(w)-C_i(u)]}{2\sum_{i=1}^n p_i|C_i(w)-C_i(u)|}, \frac{\sum_{i=1}^n p_i[|C_i(v)-C_i(w)|C_i(v)-C_i(w)]}{2\sum_{i=1}^n p_i|C_i(v)-C_i(w)|}\right\} \\
&= \min\{\frac{\sum_{i=1}^n p_i[\max\{C_i(u),C_i(w)\}-C_i(u)]}{\sum_{i=1}^n p_i|C_i(w)-C_i(u)|}, \frac{\sum_{i=1}^n p_i[\max\{C_i(w),C_i(v)\}-C_i(w)]}{\sum_{i=1}^n p_i|C_i(v)-C_i(w)|}\} \\
&= \min\{\frac{\sum_{i=1}^n p_i\left(C_i^M(u,w)-C_i(u)\right)}{\sum_{i=1}^n p_i|C_i(w)-C_i(u)|}, \frac{\sum_{i=1}^n p_i\left(C_i^M(w,v)-C_i(w)\right)}{\sum_{i=1}^n p_i|C_i(v)-C_i(w)|}\} \\
&= \min\{\prec_p^F(u,w), \prec_p^F(w,v)\}
\end{aligned}$$

so Conclusion 3) holds.

The proof of 4): If $\mathbf{C}(u) \neq \mathbf{C}(v)$, then by $\prec_p^F(u,v) = \frac{1}{2} + \frac{\sum_{i=1}^n p_i(C_i(v)-C_i(u))}{2\sum_{i=1}^n p_i|C_i(v)-C_i(u)|} = \frac{1}{2} + \frac{\sum_{i=1}^n p_iC_i(v)-\sum_{i=1}^n p_iC_i(u)}{2\sum_{i=1}^n p_i|C_i(v)-C_i(u)|}$ (see the proof of Conclusion 3)), we have that $\prec_p^F(u,v) = \frac{1}{2} \Leftrightarrow \frac{\sum_{i=1}^n p_iC_i(v)-\sum_{i=1}^n p_iC_i(u)}{2\sum_{i=1}^n p_i|C_i(v)-C_i(u)|} = 0 \Leftrightarrow \sum_{i=1}^n p_iC_i(u) = \sum_{i=1}^n p_iC_i(v)$; However, if $\mathbf{C}(u) = \mathbf{C}(v)$, then $\prec_p^F(u,v) = \frac{1}{2}$ and $\sum_{i=1}^n p_iC_i(u) = \sum_{i=1}^n p_iC_i(v)$ are all hold, so $\prec_p^F(u,v) = \frac{1}{2} \Leftrightarrow \sum_{i=1}^n p_iC_i(u) = \sum_{i=1}^n p_iC_i(v)$ is also correct. Therefore Conclusion 4) holds.

The proofs of Conclusions 5) and 6) can be directly completed by the definition of fuzzy binary relation $\prec_p^F$ and the Property 3.2. So the proof of the property is completed. □

From the meaning of the definition of fuzzy binary relation $\prec_p^F$ (Equation (6)), Conclusions (2) and (3), we can easily see that $\prec_p^F(u,v)$ can reasonably reflect the credibility in which fuzzy number v is "larger" than fuzzy number u. So it can be used to set up a more objective and reasonable dynamic method of ranking multi-channel uncertain or imprecise information than the dynamic method proposed in [28] (since the centroid of multi dimensional fuzzy number can characterize the multi dimensional fuzzy number more reasonably than its mean).

Example 6.3. *In this example, what we are going to do is ranking the four Manufacturing enterprises* $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ *according to the following four evaluating evidences:*

Evidence 1*(denoted by* E_1*) : the score of Asset Operating Status of the enterprise;*

Evidence 2*(denoted by* E_2*) : the score of Product Quality of the enterprise;*

Evidence 3*(denoted by* E_3*) : the score of Production Capacity of the enterprise;*

Evidence 4*(denoted by* E_4*) : the score of Environmental Protection Evaluation of the enterprise.*

For simplicity, suppose 10 *experts are invited to give the corresponding scores (the percentage system is adopted) to each Manufacturing enterprise of the four Manufacturing enterprises for the four evaluating evidences. For example,* *the following data set is the scores of Manufacturing enterprise $\mathcal{A}$:*

	E_1	E_2	E_3	E_4
Expert 1 :	87	86	65	87
Expert 2 :	74	63	69	74
Expert 3 :	79	64	48	70
Expert 4 :	92	59	86	51
Expert 5 :	88	78	52	67
Expert 6 :	60	46	61	66
Expert 7 :	79	87	75	69
Expert 8 :	68	62	57	63
Expert 9 :	68	73	59	69
Expert 10 :	74	67	60	65

Then we can use method introduced in [20] to construct a fuzzy 4-ellipsoid number to represent enterprise $\mathcal{A}$. The specific steps are as follows:

Step 1: By $\mu_{A_j} = \frac{1}{10}\sum_{i=1}^{10} a_{ij}$ $(j = 1,2,3,4)$, we work out $\mu_{A1} = 76.9$, $\mu_{A2} = 68.5$, $\mu_{A3} = 63.2$, $\mu_{A1} = 68.1$. And by $L\sigma_{A_j} = \frac{1}{N_{L_{A_j}}}\sum_{a_{ij}<u_{A_j}}(\mu_{A_j} - a_{ij})$ and $R\sigma_{A_j} = \frac{1}{N_{R_{A_j}}}\sum_{a_{ij}>u_{A_j}}(\mu_{A_j} - a_{ij})$ $(j = 1,2,3,4)$, we can work out $L\sigma_{A_1} = 8.1$, $R\sigma_{A_1} = 8.1$, $L\sigma_{A_2} = 8.3$, $R\sigma_{A_2} = 12.5$, $L\sigma_{A_3} = 7.0$, $R\sigma_{A_3} = 10.5$, $L\sigma_{A_4} = 5.7$, $R\sigma_{A_4} = 5.7$, where $N_{L_{A_j}}$ and $N_{R_{A_j}}$ are the number of the values a_{ij} which satisfy $a_{ij} < \mu_{A_j}$ and the number of the values which satisfy $a_{ij} > \mu_{A_j}$, respectively.

Step 2: Taking $[\alpha_{A_i}, \beta_{A_i}] = [0, 100]$ $(i = 1,2,3,4)$ and $\lambda = 2$, we can work out:

$$\begin{array}{ll} uA_1 - \lambda L_{\sigma_{A_1}} = 60.7, & uA_1 + \lambda R_{\sigma_{A_1}} = 93.1 \\ uA_2 - \lambda L_{\sigma_{A_2}} = 51.8, & uA_2 + \lambda R_{\sigma_{A_2}} = 93.5 \\ uA_3 - \lambda L_{\sigma_{A_3}} = 49.1, & uA_3 + \lambda R_{\sigma_{A_3}} = 84.3 \\ uA_4 - \lambda L_{\sigma_{A_4}} = 56.7, & uA_4 + \lambda R_{\sigma_{A_4}} = 79.5 \end{array}$$

so, we have

$$\begin{array}{ll} (\underline{\max})_{A_1} = \max\{\mu_{A_1} - \lambda L_{\sigma_{A_1}}, \alpha_{A_1}\} = 60.7, & (\overline{\min})_{A_1} = \min\{\mu_{A_1} + \lambda R_{\sigma_{A_1}}, \beta_{A_1}\} = 93.1 \\ (\underline{\max})_{A_2} = \max\{\mu_{A_2} - \lambda L_{\sigma_{A_2}}, \alpha_{A_2}\} = 51.8, & (\overline{\min})_{A_2} = \min\{\mu_{A_2} + \lambda R_{\sigma_{A_2}}, \beta_{A_2}\} = 93.5 \\ (\underline{\max})_{A_3} = \max\{\mu_{A_3} - \lambda L_{\sigma_{A_3}}, \alpha_{A_3}\} = 49.1, & (\overline{\min})_{A_3} = \min\{\mu_{A_3} + \lambda R_{\sigma_{A_3}}, \beta_{A_3}\} = 84.3 \\ (\underline{\max})_{A_4} = \max\{\mu_{A_4} - \lambda L_{\sigma_{A_4}}, \alpha_{A_4}\} = 56.7, & (\overline{\min})_{A_4} = \min\{\mu_{A_4} + \lambda R_{\sigma_{A_4}}, \beta_{A_4}\} = 79.5 \end{array}$$

It is obvious that the four components (E_j, $j = 1,2,3,4$) should be all of Two-sided type. Therefore, by $u_A = CT\left((u_{A_i} - \lambda L_{\sigma_{A_i}})|_{(\underline{\max})_{A_i}}, u_{A_i}, (u_{A_i} + \lambda R_{\sigma_{A_i}})|_{(\overline{\min})_{A_i}}\right)_{i=1}^{4}$, we construct CT-fuzzy 4-ellipsoid number (see Definition 5 in [22]) u_A as

$$u_A = CT\begin{pmatrix} 60.7 & 76.9 & 93.1 \\ 51.8 & 68.5 & 93.5 \\ 49.1 & 63.2 & 84.3 \\ 56.7 & 68.1 & 79.5 \end{pmatrix}$$

to express the Manufacturing enterprise $\mathcal{A}$.

By using the same method, we can work out u_B, u_C, u_D to respectively express Manufacturing enterprises $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$. Suppose that

$$u_B = CT\begin{pmatrix} 54.0 & 62.5 & 76.4 \\ 47.1 & 61.7 & 76.3 \\ 45.3 & 63.9 & 82.5 \\ 52.6 & 69.4 & 80.6 \end{pmatrix}$$

$$u_C = CT\begin{pmatrix} 45.0 & 73.0 & 85.0 \\ 59.7 & 72.7 & 85.7 \\ 59.5 & 71.2 & 88.8 \\ 55.4 & 73.0 & 90.6 \end{pmatrix}$$

$$u_D = CT\begin{pmatrix} 47.7 & 66.3 & 84.9 \\ 58.1 & 65.3 & 82.0 \\ 56.8 & 72.2 & 82.5 \\ 51.9 & 65.7 & 79.5 \end{pmatrix}$$

Then, we have that

$$
\begin{aligned}
\underline{u_{A_1}}(r) &= 60.7 + (76.9 - 60.7)r\\
\overline{u_{A_1}}(r) &= 93.1 - (93.1 - 76.9)r\\
\underline{u_{A_2}}(r) &= 51.8 + (68.5 - 51.8)r\\
\overline{u_{A_2}}(r) &= 93.5 - (93.5 - 68.5)r\\
\underline{u_{A_3}}(r) &= 49.1 + (63.2 - 49.1)r\\
\overline{u_{A_3}}(r) &= 84.3 - (84.3 - 63.2)r\\
\underline{u_{A_4}}(r) &= 56.7 + (68.1 - 56.7)r\\
\overline{u_{A_4}}(r) &= 79.5 - (79.5 - 68.1)r
\end{aligned}
$$

$$
\begin{aligned}
\underline{u_{B_1}}(r) &= 54.0 + (62.5 - 54.0)r\\
\overline{u_{B_1}}(r) &= 76.4 - (76.4 - 62.5)r\\
\underline{u_{B_2}}(r) &= 47.1 + (61.7 - 47.1)r\\
\overline{u_{B_2}}(r) &= 76.3 - (76.3 - 61.7)r\\
\underline{u_{B_3}}(r) &= 45.3 + (63.9 - 45.3)r\\
\overline{u_{B_3}}(r) &= 82.5 - (82.5 - 63.9)r\\
\underline{u_{B_4}}(r) &= 52.6 + (69.4 - 52.6)r\\
\overline{u_{B_4}}(r) &= 80.6 - (80.6 - 69.4)r
\end{aligned}
$$

$$
\begin{aligned}
\underline{u_{C_1}}(r) &= 45.0 + (73.0 - 45.0)r\\
\overline{u_{C_1}}(r) &= 85.0 - (85.0 - 73.0)r\\
\underline{u_{C_2}}(r) &= 59.7 + (72.7 - 59.7)r\\
\overline{u_{C_2}}(r) &= 85.7 - (85.7 - 72.7)r\\
\underline{u_{C_3}}(r) &= 59.5 + (71.2 - 59.5)r\\
\overline{u_{C_3}}(r) &= 88.8 - (88.8 - 71.2)r\\
\underline{u_{C_4}}(r) &= 55.4 + (73.0 - 55.4)r\\
\overline{u_{C_4}}(r) &= 90.6 - (90.6 - 73.0)r
\end{aligned}
$$

$$
\begin{aligned}
\underline{u_{D_1}}(r) &= 47.7 + (66.3 - 47.7)r\\
\overline{u_{D_1}}(r) &= 84.9 - (84.9 - 66.3)r\\
\underline{u_{D_2}}(r) &= 58.1 + (65.3 - 58.1)r\\
\overline{u_{D_2}}(r) &= 82.0 - (82.0 - 65.3)r\\
\underline{u_{D_3}}(r) &= 56.8 + (72.2 - 56.8)r\\
\overline{u_{D_3}}(r) &= 82.5 - (82.5 - 72.2)r\\
\underline{u_{D_4}}(r) &= 51.9 + (65.7 - 51.9)r\\
\overline{u_{D_4}}(r) &= 79.5 - (79.5 - 65.7)r
\end{aligned}
$$

By Formula (4)*, we can work out* $\mathbf{C}(u_A) = (C(u_{A1}), C(u_{A2}), C(u_{A3}), C(u_{A4}))$*,* $\mathbf{C}(u_B)$*,* $\mathbf{C}(u_C)$*, and* $\mathbf{C}(u_D)$ *as follows:*

$$
\begin{aligned}
C(u_A) &= (76.9, 71.9, 66.1, 68.1)\\
C(u_B) &= (64.7, 61.7, 63.9, 67.1)\\
C(u_C) &= (66.3, 72.7, 73.7, 73.0)\\
C(u_D) &= (66.3, 69.3, 70.1, 65.7)
\end{aligned}
$$

Taking weight $\mathbf{p} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$*, by the Definition 6.1 (i.e., Equation (6)), we can work out that*

$$
\begin{aligned}
\prec_p^F (u_A, u_B) &= 0.000 \quad (i.e., \prec_p^F (u_B, u_A) = 1.000)\\
\prec_p^F (u_A, u_C) &= 0.556 \quad (i.e., \prec_p^F (u_C, u_A) = 0.444)\\
\prec_p^F (u_A, u_D) &= 0.204 \quad (i.e., \prec_p^F (u_D, u_A) = 0.796)\\
\prec_p^F (u_B, u_C) &= 1.000 \quad (i.e., \prec_p^F (u_C, u_B) = 0.000)\\
\prec_p^F (u_B, u_D) &= 0.917 \quad (i.e., \prec_p^F (u_D, u_B) = 0.083)\\
\prec_p^F (u_C, u_D) &= 0.000 \quad (i.e., \prec_p^F (u_D, u_C) = 1.000)
\end{aligned}
$$

So, we have that

$$u_B \prec_p^{0.917} u_D \prec_p^{0.796} u_A \prec_p^{0.556} u_C$$

Thus, we can get a ranking result (from excellent to bad) of the four Manufacturing enterprises: $\mathcal{C}$*,* $\mathcal{A}$*,* $\mathcal{D}$*,* $\mathcal{B}$*. And we also see that the degrees of which* $\mathcal{C}$ *is superior to* $\mathcal{A}$*,* $\mathcal{A}$ *is superior to* $\mathcal{D}$ *and* $\mathcal{D}$ *is superior to* $\mathcal{B}$ *are* $0.556, 0.796$ *and* 0.917*, respectively.*

In addition, if we take the level $r = 0.555$, $r = 0.798$ and $r = 0.918$, we can obtain the following three dynamic rankings (from excellent to bad) which changes with level r for the four Manufacturing enterprises: (1) First: $\mathcal{C}$, Second: $\mathcal{A}$, Third: $\mathcal{D}$, Fourth: $\mathcal{B}$ as $r = 0.555$; (2) First: $\mathcal{A}$ and $\mathcal{C}$, Second: $\mathcal{D}$, Third: $\mathcal{B}$ as $r = 0.798$; (3) First: $\mathcal{A}$ and $\mathcal{C}$ and $\mathcal{D}$, Second: $\mathcal{B}$ as $r = 0.918$.

7 Conclusion

In this paper, we proposed the conception of centroid (Definition 3.1) of multi dimensional fuzzy number via regarding the membership function of the multi dimensional fuzzy number as the density function on its support set, and obtained two conclusions (Property 3.2) about its general properties. Then, for fuzzy cell numbers and fuzzy ellipsoid number, we obtained a calculation formula (Theorems 1 and 2) of centroid which is easy to use in theory and application by using a lemma (Lemma 4.1) given by us and a special property (Properties 4.3 and 5.2) of the centroid by using the obtained calculation formula, respectively. Finally, as an example of application, by using the centroid of multi dimensional fuzzy number, we defined a fuzzy order (Definition 6.1) on n-dimensional fuzzy number space, which can be used to rank uncertain or imprecise multichannel digital information, and obtained some properties (Property 6.2) of the defined fuzzy order.

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