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Structural topology in a category

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Abstract

Several fuzzy topologies are defined and studied by different authors. In this article, we unify five of the most common fuzzy topologies existing in the literature, as well as the standard topology. This is done by introducing the notion of structural topology on objects in a category and proving that topologies on a set as well as fuzzy topologies on fuzzy sets and fuzzy topologies on fuzzy subsets are all structural topologies. We also introduce the notion of structural continuity and we show that the fuzzy continuity defined in the literature in all the above mentioned cases, as well as the standard continuity are structural.

Keywords: Topology, fuzzy topology, structural topology, continuity, fuzzy continuity, structural continuity.

1 Introduction

Topology has been around for a while and topological ideas are present in almost all areas of today's mathematics, see [12]. The notion of fuzzy set was first introduced by Professor Zadeh in 1965. Many others have studied fuzzy sets since then. Several definitions of fuzzy topology have been given and studied by many authors as well, see [2], [3], [4], [5], [6], [7], [8], [9], [10] and [13], to mention a few. What we try to achieve in this article is to unify all these topology and fuzzy topology notions in one notion, the so called structural topology, in a categorical setting.

2 Preliminaries

In the current section, we give a few definitions and introduce necessary notations which will be used throughout the paper.

In Section 2, we give the definition of a topological structure on an object X of a category and we use that to define the concept of a structural topology on X. We also introduce the notion of structural continuity of a morphism and we show that the identity and the composition of structurally continuous morphisms are structurally continuous.

In Section 3, we show the concept of topology is indeed structural with respect to a particular topological structure. We also establish that continuity of functions is structural.

In the subsequent Sections 4, 5, 6, 7 and 8, we investigate five different notions of fuzzy topology and we prove that they are all structural. Whenever continuity is defined we show it is structural and if no definition of continuity is available in the literature, we introduce one.

To this end we recall:

- For sets X, Y and $y \in Y$ by $\underline{y} : X \longrightarrow Y$ we mean the constant function with value y.
- For a set X, a fuzzy set in X is a function with domain X and values in $I = [0, 1]$. The set of all the fuzzy sets in X is denoted by I^X.

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- For fuzzy sets A, B in X, B is said to be a fuzzy subset of A, written $B \subseteq A$, if for all $x \in X$, $B(x) \leqslant A(x)$ in I. The family of all fuzzy subsets of A is denoted by $\mathcal{F}_A$.

3 Structural topology in a category

The notions of topological structure and structural topology as well as structural continuity, on a categorical basis, are introduced. We also show that the identity morphism and the composition of structurally continuous morphisms are structurally continuous. For categorical concepts we refer you to [1].

Definition 3.1. *Let $\mathcal{E}$ and $\mathcal{C}$ be categories.*

a) *A pair of functors $\mathcal{E}^{op} \xrightarrow{\mathbb{P}} \mathcal{C} \xrightarrow{P} \mathcal{C}$, with $\mathcal{C}$ finitely complete, is called a base on $(\mathcal{E}, \mathcal{C})$.*

b) *A topological structure relative to a base $(\mathbb{P}, P)$ or a $(\mathbb{P}, P)$-topological structure, on an object X of $\mathcal{E}$, is a quadruple $\mathsf{S}_X = (b_X, t_X, \bigwedge_X, \bigvee_X)$, where*

$$1 \xrightarrow{b_X} \mathbb{P}(X)\ , \quad 1 \xrightarrow{t_X} \mathbb{P}(X)\ ,$$
$$\mathbb{P}(X) \times \mathbb{P}(X) \xrightarrow{\bigwedge_X} \mathbb{P}(X) \quad \text{and} \quad P(\mathbb{P}(X)) \xrightarrow{\bigvee_X} \mathbb{P}(X)$$

are $\mathcal{C}$-morphisms. Here 1 denotes the terminal object of $\mathcal{C}$.

c) *A structural topology or an S-topology on X with respect to a structure S_X, is a pair (τ_X, i_X), where $T_X \rightarrowtail^{\tau_X} \mathbb{P}(X)$ and $P(T_X) \xrightarrow{i_X} P(\mathbb{P}(X))$ are $\mathcal{C}$-morphisms, with T_X a $\mathcal{C}$-object and τ_X mono, such that the morphisms b_X, t_X, $\bigwedge_X$ and $\bigvee_X$ factor through T_X in the following sense.*

There are morphisms $1 \xrightarrow{b_X} T_X$, $1 \xrightarrow{t_X} T_X$, $T_X \times T_X \xrightarrow{\bigwedge_X} T_X$ and $P(T_X) \xrightarrow{\bigvee_X} T_X$ rendering commutative the following diagrams.

$$\begin{array}{ccc} & & T_X \\ & \overset{b_X}{\nearrow} & \downarrow{\scriptstyle \tau_X} \\ 1 & \xrightarrow[t_X]{} & \mathbb{P}(X) \end{array} \qquad\qquad \begin{array}{ccc} & & T_X \\ & \overset{t_X}{\nearrow} & \downarrow{\scriptstyle \tau_X} \\ 1 & \xrightarrow[b_X]{} & \mathbb{P}(X) \end{array}$$

$$\begin{array}{ccc} T_X \times T_X & \xrightarrow{\bigwedge_X} & T_X \\ {\scriptstyle \tau_X \times \tau_X}\downarrow & /// & \downarrow{\scriptstyle \tau_X} \\ \mathbb{P}(X) \times \mathbb{P}(X) & \xrightarrow[\bigwedge_X]{} & \mathbb{P}(X) \end{array} \qquad\qquad \begin{array}{ccc} P(T_X) & \xrightarrow{\bigvee_X} & T_X \\ {\scriptstyle i_X}\downarrow & /// & \downarrow{\scriptstyle \tau_X} \\ P(\mathbb{P}(X)) & \xrightarrow[\bigvee_X]{} & \mathbb{P}(X) \end{array}$$

In this case, (X, τ_X, i_X) is called a structural topological space or an S-topological space with respect to S_X.

Definition 3.2. *Let $(\mathbb{P}, P)$ be a base on $(\mathcal{E}, \mathcal{C})$. Given S-topological spaces (X, τ_X, i_X) and (Y, τ_Y, i_Y) with respect to $(\mathbb{P}, P)$-structures $\mathsf{S}_X = (b_X, t_X, \bigwedge_X, \bigvee_X)$ and $\mathsf{S}_Y = (b_Y, t_Y, \bigwedge_Y, \bigvee_Y)$, respectively, an $\mathcal{E}$-morphism $f : X \longrightarrow Y$ is said to be structurally continuous or S-continuous, if there exists a $\mathcal{C}$-morphism $\bar{f} : T_Y \longrightarrow T_X$ such that following diagram commutes.*

$$\begin{array}{ccc} T_X & \overset{\tau_X}{\rightarrowtail} & \mathbb{P}(X) \\ {\scriptstyle \bar{f}}\uparrow & /// & \uparrow{\scriptstyle \mathbb{P}(f)} \\ T_Y & \underset{\tau_Y}{\rightarrowtail} & \mathbb{P}(Y). \end{array}$$

In this case we write $f : (X, \tau_X, i_X) \longrightarrow (Y, \tau_Y, i_Y)$.

Theorem 3.3. *Let $(\mathbb{P}, P)$ be a base on $(\mathcal{E}, \mathcal{C})$. Given* S*-topological spaces (X, τ_X, i_X), (Y, τ_Y, i_Y) and (Z, τ_Z, i_Z) with respect to $(\mathbb{P}, P)$-structures $\mathsf{S}_X = (b_X, t_X, \bigwedge_X, \bigvee_X)$, $\mathsf{S}_Y = (b_Y, t_Y, \bigwedge_Y, \bigvee_Y)$ and $\mathsf{S}_Z = (b_Z, t_Z, \bigwedge_Z, \bigvee_Z)$, respectively,*

1. *the identity morphism* $(X, \tau_X, i_X) \xrightarrow{1_X} (X, \tau_X, i_X)$ *is* S*-continuous.*

2. *the composition of the* S*-continuous morphisms* $(X, \tau_X, i_X) \xrightarrow{f} (Y, \tau_Y, i_Y)$ *and* $(Y, \tau_Y, i_Y) \xrightarrow{g} (Z, \tau_Z, i_Z)$ *is* S*-continuous.*

Proof. 1. Obvious.

2. Since f and g are continuous, there exist morphisms $\bar{f}$ and $\bar{g}$ rendering commutative the following diagrams.

$$\begin{array}{ccc} T_X & \xrightarrow{\tau_X} & \mathbb{P}(X) \\ {\scriptstyle \bar{f}}\uparrow & /// & \uparrow{\scriptstyle \mathbb{P}(f)} \\ T_Y & \xrightarrow[\tau_Y]{} & \mathbb{P}(Y) \end{array} \qquad\qquad \begin{array}{ccc} T_Y & \xrightarrow{\tau_Y} & \mathbb{P}(Y) \\ {\scriptstyle \bar{g}}\uparrow & /// & \uparrow{\scriptstyle \mathbb{P}(g)} \\ T_Z & \xrightarrow[\tau_Z]{} & \mathbb{P}(Z) \end{array}$$

Set $\overline{g \circ f} = \bar{f} \circ \bar{g}$. We have

$$\begin{aligned} \tau_X \circ (\overline{g \circ f}) &= \tau_X \circ (\bar{f} \circ \bar{g}) = (\tau_X \circ \bar{f}) \circ \bar{g} = (\mathbb{P}(f) \circ \tau_Y) \circ \bar{g} = \mathbb{P}(f) \circ (\tau_Y \circ \bar{g}) \\ &= \mathbb{P}(f) \circ (\mathbb{P}(g) \circ \tau_Z) = (\mathbb{P}(f) \circ \mathbb{P}(g)) \circ \tau_Z = \mathbb{P}(g \circ f) \circ \tau_Z \end{aligned}$$

that is, the following diagram commutes.

$$\begin{array}{ccc} T_X & \xrightarrow{\tau_X} & \mathbb{P}(X) \\ {\scriptstyle \overline{g\circ f}}\uparrow & /// & \uparrow{\scriptstyle \mathbb{P}(g\circ f)} \\ T_Z & \xrightarrow[\tau_Z]{} & \mathbb{P}(Z) \end{array}$$

implying the continuity of $g \circ f$.

□

4 Topology as a structural topology

For the notions of topology on a set and continuity of functions, we refer you to [12]. Let $Set^{op} \xrightarrow{\mathbb{P}} Set \xrightarrow{P} Set$ be the contravariant and covariant powerset functors on the category *Set* of sets and functions, so that both take a set X to its power set $\mathcal{P}(X)$, while $\mathbb{P}$, respectively, P takes a function to its inverse image, respectively direct image function. Now for a set X, define the structure $\mathsf{S}_X = (b_X, t_X, \bigwedge_X, \bigvee_X)$ relative to $(\mathbb{P}, P)$, called the topological structure, as follows:

$$1 \xrightarrow{b_X} \mathcal{P}(X) \ \text{ taking } 1 \longmapsto \emptyset$$

$$1 \xrightarrow{t_X} \mathcal{P}(X) \ \text{ taking } 1 \longmapsto X$$

$$\mathcal{P}(X) \times \mathcal{P}(X) \xrightarrow{\bigwedge_X} \mathcal{P}(X) \ \text{ taking } (A, B) \longmapsto A \cap B$$

$$\mathcal{P}(\mathcal{P}(X)) \xrightarrow{\bigvee_X} \mathcal{P}(X) \ \text{ taking } Z \longmapsto \bigcup_{A \in Z} A$$

We have:

Theorem 4.1. *A subset T_X of $\mathcal{P}(X)$ is a topology on X if and only if (τ_X, i_X) is a structural topology on X with respect to the topological structure, where $T_X \overset{\tau_X}{\hookrightarrow} \mathcal{P}(X)$ and $\mathcal{P}(T_X) \overset{i_X}{\hookrightarrow} \mathcal{P}(\mathcal{P}(X))$ are set inclusions.*

Proof. (⇒) Suppose T_X is a topology on X. Then $\emptyset, X \in T_X$. So there exist arrows $1 \xrightarrow{b_X} T(X)$ and $1 \xrightarrow{t_X} T(X)$ in $\mathcal{C}$ making the following diagrams commutative:

$$\begin{array}{ccc} & & T_X \\ & {}^{b_X}\nearrow \;\; /// & \downarrow{\scriptstyle \tau_X} \\ 1 & \xrightarrow[b_X]{} & \mathcal{P}(X) \end{array} \qquad\qquad \begin{array}{ccc} & & T_X \\ & {}^{t_X}\nearrow \;\; /// & \downarrow{\scriptstyle \tau_X} \\ 1 & \xrightarrow[t_X]{} & \mathcal{P}(X) \end{array}$$

Since T_X is a topology, for U and V in T_X, $U \cap V \in T_X$ and for $\{U_i\}_{i\in J} \subseteq T_X$ where J is an index set, $\bigcup_{i\in J} U_i \in T_X$, implying the existence of arrows $T_X \times T_X \xrightarrow{\wedge_X} T_X$ and $\mathcal{P}(T_X) \xrightarrow{\vee_X} T_X$ in $\mathcal{C}$ making the following diagrams commutative:

$$\begin{array}{ccc} T_X \times T_X & \xrightarrow{\wedge_X} & T_X \\ {\scriptstyle \tau_X \times \tau_X}\big\downarrow & & \big\downarrow{\scriptstyle \tau_X} \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow[\wedge_X]{} & \mathcal{P}(X) \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{P}(T_X) & \xrightarrow{\vee_X} & T_X \\ {\scriptstyle i_X}\big\downarrow & & \big\downarrow{\scriptstyle \tau_X} \\ \mathcal{P}(\mathcal{P}(X)) & \xrightarrow[\vee_X]{} & \mathcal{P}(X) \end{array}$$

Thus (τ_X, i_X) with $\tau_X : T_X \hookrightarrow I^X$ and $i_X : \mathcal{P}(T_X) \hookrightarrow \mathcal{P}(I^X)$ is a structural topology on X with respect to the topological structure.

(⇐) Now suppose $\tau_X : T_X \hookrightarrow \mathcal{P}(X)$ and $i_X : \mathcal{P}(T_X) \hookrightarrow \mathcal{P}(\mathcal{P}(X))$ is a structural topology on X with respect to the topological structure. So there exist arrows $1 \xrightarrow{b_X} T_X$, $1 \xrightarrow{t_X} T_X$, $T_X \times T_X \xrightarrow{\wedge_X} T_X$ and $\mathcal{P}(T_X) \xrightarrow{\vee_X} T_X$ in $\mathcal{C}$ making the above diagrams commutative. The commutativity of the two triangles implies that $\emptyset, X \in T_X$ and the commutativity of the squares yields that T_X is closed under finite intersection and arbitrary unions. Thus T_X is a topology on X. □

Theorem 4.2. *Suppose T_X and T_Y are topologies on X and Y with corresponding* S*-topologies (τ_X, i_X) and (τ_Y, i_Y), respectively. A function $f : (X, T_X) \longrightarrow (Y, T_Y)$ is continuous if and only if $f : (X, \tau_X, i_X) \longrightarrow (Y, \tau_Y, i_Y)$ is structurally continuous.*

Proof. (⇒) Let $(X, T_X) \xrightarrow{f} (Y, T_Y)$ be a continuous function. So $B \in T_Y$ yields $f^{-1}(B) \in T_X$. Therefore $\bar{f} = f^{-1}(-) : T_Y \longrightarrow T_X$ makes the following diagram commutative:

$$\begin{array}{ccc} T_X & \overset{\tau_X}{\hookrightarrow} & \mathcal{P}(Y) \\ {\scriptstyle \bar{f}}\big\uparrow & /// & \big\uparrow{\scriptstyle \mathbb{P}(f) = f^{-1}(-)} \\ T_Y & \underset{\tau_Y}{\hookrightarrow} & \mathcal{P}(Y) \end{array}$$

Thus f is structurally continuous.

(⇐) Since $(X, \tau_X, i_X) \xrightarrow{f} (Y, \tau_Y, i_Y)$ is structurally continuous, there exists an arrow $\bar{f}$ making the following diagram commutative:

$$\begin{array}{ccc} T_X & \overset{\tau_X}{\hookrightarrow} & I^X \\ {\scriptstyle \bar{f}}\big\uparrow & /// & \big\uparrow{\scriptstyle \mathbb{P}(f) = f^{-1}(-)} \\ T_Y & \underset{\tau_Y}{\hookrightarrow} & I^Y \end{array}$$

So for any $B \in T_Y$, we have $f^{-1}(B) = \bar{f}(B) \in T_X$. Thus f is continuous. □

5 Type one fuzzy topology as a structural topology

The definition of fuzzy topology on a set given in [2], [3], [6], [7], [10] and [13], is what we call type one fuzzy topology as given below.

Definition 5.1. [2] *For a set X, a family T_X of fuzzy sets on X is called type one fuzzy topology if $\underline{0}, \underline{1} \in T_X$, for U and V in T_X, $U \bigwedge V \in T_X$ and for $\{U_i\}_{i\in J} \subseteq T_X$ where J is an index set, $\bigvee_{i\in J} U_i \in T_X$.*

Now let $Set^{op} \xrightarrow{\mathbb{P}} Set \xrightarrow{P} Set$ be functors, where $\mathbb{P}$ takes an object Y to the set of all the fuzzy sets I^Y, a morphism f to $-\circ f$, the composition with f function, and $P = \mathcal{P}$ is the covariant powerset functor. For a set X, define the structure $\mathsf{S}_X = (b_X, t_X, \bigwedge_X, \bigvee_X)$ relative to $(\mathbb{P}, P)$, called type one fuzzy topological structure, as follows:

$$1 \xrightarrow{b_X} I^X \text{ taking } 1 \longmapsto \underline{0}$$

$$1 \xrightarrow{t_X} I^X \text{ taking } 1 \longmapsto \underline{1}$$

$$I^X \times I^X \xrightarrow{\bigwedge_X} I^X \text{ taking } (A, B) \longmapsto A \bigwedge B$$

$$\mathcal{P}(I^X) \xrightarrow{\bigvee_X} I^X \text{ taking } Z \longmapsto \bigvee_{A\in Z} A$$

Theorem 5.2. *A family T_X of fuzzy sets on X is a type one fuzzy topology on X if and only if (τ_X, i_X) is a structural topology on X with respect to the type one fuzzy topological structure, where $T_X \overset{\tau_X}{\hookrightarrow} I^X$ and $\mathcal{P}(T_X) \overset{i_X}{\hookrightarrow} \mathcal{P}(I^X)$ are set inclusions.*

Proof. $(\Rightarrow)$ Suppose T_X is a type one fuzzy topology on X. Then $\underline{0}, \underline{1} \in T_X$. So there exist arrows $1 \xrightarrow{b_X} T(X)$ and $1 \xrightarrow{t_X} T(X)$ in $\mathcal{C}$ making the following diagrams commutative:

Since T_X is a type one fuzzy topology, for U and V in T_X, $U \bigwedge V \in T_X$ and for $\{U_i\}_{i\in J} \subseteq T_X$ where J is an index set, $\bigvee_{i\in J} U_i \in T_X$, implying the existence of arrows $T_X \times T_X \xrightarrow{\bigwedge_X} T_X$ and $\mathcal{P}(T_X) \xrightarrow{\bigvee_X} T_X$ in $\mathcal{C}$ making the following diagrams commutative:

Thus (τ_X, i_X) with $\tau_X : T_X \hookrightarrow I^X$ and $i_X : \mathcal{P}(T_X) \hookrightarrow \mathcal{P}(I^X)$ is a structural topology on X with respect to type one fuzzy topological structure.

$(\Leftarrow)$ Now suppose $\tau_X : T_X \hookrightarrow I^X$ and $i_X : \mathcal{P}(T_X) \hookrightarrow \mathcal{P}(I^X)$ is a structural topology on X with respect to the type one fuzzy topological structure. So there exist arrows $1 \xrightarrow{b_X} T_X$, $1 \xrightarrow{t_X} T_X$, $T_X \times T_X \xrightarrow{\bigwedge_X} T_X$ and $\mathcal{P}(T_X) \xrightarrow{\bigvee_X} T_X$ in $\mathcal{C}$ making the above diagrams commutative. The commutativity of the two triangles implies that $\underline{0}, \underline{1} \in T_X$ and the commutativity of the squares yields that T_X is closed under finite intersection and arbitrary unions. Thus T_X is a type one fuzzy topology. □

The following definition of fuzzy continuity, that we call type one fuzzy continuity is given in [2], [3] and [13].

Definition 5.3. [2] *Let T_X and T_Y be type one fuzzy topologies on X and Y, respectively. The function $X \xrightarrow{f} Y$ is said to be type one fuzzy continuous if for any U in T_Y, $f^{-1}(U) = U \circ f$ is in T_X.*

Theorem 5.4. *Suppose T_X and T_Y are type one fuzzy topologies on X and Y with corresponding* S*-topologies (τ_X, i_X) and (τ_Y, i_Y) with respect to type one fuzzy topological structure, respectively. A function $f : (X, T_X) \longrightarrow (Y, T_Y)$ is type one fuzzy continuous if and only if $f : (X, \tau_X, i_X) \longrightarrow (Y, \tau_Y, i_Y)$ is structurally continuous.*

Proof. (⇒) Let $(X, T_X) \xrightarrow{f} (Y, T_Y)$ be type one fuzzy continuous. So $B \in T_Y$ yields $f^{-1}(B) = B \circ f \in T_X$. Therefore $\bar{f} = - \circ f$ makes the following diagram commutative:

$$\begin{array}{ccc} T_X & \xhookrightarrow{\tau_X} & I^X \\ \bar{f}\uparrow & /// & \uparrow \mathbb{P}(f) \\ T_Y & \xhookrightarrow[\tau_Y]{} & I^Y \end{array}$$

Thus f is structurally continuous.

(⇐) Since $(X, \tau_X, i_X) \xrightarrow{f} (Y, \tau_Y, i_Y)$ is structurally continuous, there exists an arrow $\bar{f}$ making the following diagram commutative:

$$\begin{array}{ccc} T_X & \xhookrightarrow{\tau_X} & I^X \\ \bar{f}\uparrow & /// & \uparrow \mathbb{P}(f) \\ T_Y & \xhookrightarrow[\tau_Y]{} & I^Y \end{array}$$

So for any $B \in T_Y$, we have $f^{-1}(B) = B \circ f \in T_X$. Thus f is type one fuzzy continuous. □

6 Type two fuzzy topology as a structural topology

The definition of fuzzy topology on a set given in [11], that we call type two fuzzy topology, is given below.

Definition 6.1. [11] *For a set X, a family T_X of fuzzy sets on X is called type two fuzzy topology if for all $\alpha \in I$, $\underline{\alpha} \in T_X$, for U and V in T_X, $U \bigwedge V \in T_X$ and for $\{U_i\}_{i \in J} \subseteq T_X$ where J is an index set, $\bigvee_{i \in J} U_i \in T_X$.*

Considering the discrete type one fuzzy topological space $(1, I)$ with the corresponding structural topological space $(1, 1_I, 1_{\mathcal{P}(I)})$, where $1_I : I \longrightarrow I$ is the identity function, we have:

Lemma 6.2. *Let X be a set and T_X be a type one fuzzy topology on X. The following are equivalent.*

- **a)** *for all $\alpha \in I$, $\underline{\alpha} \in T_X$.*
- **b)** *the unique map $! : (X, T_X) \longrightarrow (1, I)$ is type one fuzzy continuous.*
- **c)** *the map $! : (X, \tau_X, i_X) \longrightarrow (1, 1_I, 1_{\mathcal{P}(I)})$ is structurally continuous.*

Proof. The equivalency of **(a)** and **(b)** follows from Definition 5.3 and the fact that for each $\alpha \in I$, we have $\underline{\alpha} : 1 \longrightarrow I$ and $!^{-1}(\underline{\alpha}) = \underline{\alpha} \circ ! = \underline{\alpha} : X \longrightarrow I$.

The equivalency of **(b)** and **(c)** follows from Theorem 5.4 □

Theorem 6.3. *A family T_X of fuzzy sets on X is a type two fuzzy topology on X if and only if (τ_X, i_X) is a structural topology on X with respect to the type one fuzzy topological structure, where $T_X \xhookrightarrow{\tau_X} I^X$ and $\mathcal{P}(T_X) \xhookrightarrow{i_X} \mathcal{P}(I^X)$ are set inclusions and the unique map $! : (X, T_X) \longrightarrow (1, I)$ is type one fuzzy continuous.*

Proof. ($\Rightarrow$) Suppose T_X is a type two fuzzy topology on X. Since $\underline{0}, \underline{1} \in T_X$, T_X is a type one fuzzy topology on X. Therefore by Theorem 5.2, (X, τ_X, i_X) is a structural topology with respect to the type one fuzzy topological structure. The second assertion follows from Lemma 6.2.

($\Leftarrow$) Suppose (τ_X, i_X) is a structural topology on X with respect to the type one fuzzy topological structure and that $! : (X, T_X) \longrightarrow (1, I)$ is type one continuous. The type one continuity of ! yields that for all $\alpha \in I$, $\underline{\alpha} \in T_X$. Being a type one fuzzy topology, proves that T_X is a type two fuzzy topology on X. □

The type two fuzzy continuity is the same as type one fuzzy continuity, as defined in Definition 5.3 and thus Theorem 5.4 stays valid when replacing "type one" by "type two" in its statement.

7 Type three fuzzy topology as a structural topology

The definition of intuitionistic fuzzy topology on a set given in [5] and [8], that we call type three fuzzy topology, is given below, with a slight change of notation.

Definition 7.1. [5] *For a set X, a family T_X of intuitionistic fuzzy sets on X is called type three fuzzy topology if $\underline{0}, \underline{1} \in T_X$, for U and V in T_X, $U \bigwedge V \in T_X$ and for $\{U_i\}_{i \in J} \subseteq T_X$ where J is an index set, $\bigvee_{i \in J} U_i \in T_X$.*

Now let $Set^{op} \xrightarrow{\mathbb{P}} Set \xrightarrow{P} Set$ be functors, where $\mathbb{P}$ takes an object Y to the set of all the intuitionistic fuzzy sets $II^Y := \{(U, V) \in I^Y \times I^Y : \forall y \in Y, 0 \leq U(y) + V(y) \leq 1\}$, a morphism f to $(- \circ f, - \circ f)$ and $P = \mathcal{P}$ is the covariant powerset functor. For a set X, define the structure $\mathsf{S}_X = (b_X, t_X, \bigwedge_X, \bigvee_X)$ relative to $(\mathbb{P}, P)$, called type three fuzzy topological structure, as follows:

$$1 \xrightarrow{b_X} II^X \text{ taking } 1 \longmapsto \underline{0}$$

$$1 \xrightarrow{t_X} II^X \text{ taking } 1 \longmapsto \underline{1}$$

$$II^X \times II^X \xrightarrow{\bigwedge_X} II^X \text{ taking } (A, B) \longmapsto A \bigwedge B$$

$$\mathcal{P}(II^X) \xrightarrow{\bigvee_X} II^X \text{ taking } Z \longmapsto \bigvee_{A \in Z} A$$

Theorem 7.2. *A family T_X of intuitionistic fuzzy sets on a set X is a type three fuzzy topology on X if and only if (τ_X, i_X) is a structural topology on X with respect to the type three fuzzy topological structure, where $T_X \xhookrightarrow{\tau_X} II^X$ and $\mathcal{P}(T_X) \xhookrightarrow{i_X} \mathcal{P}(II^X)$ are set inclusions.*

Proof. Similar to the proof of Theorem 5.2. □

The following definition of fuzzy continuity, that we call type three fuzzy continuity is given in [5].

Definition 7.3. [5] *Let T_X and T_Y be type three fuzzy topologies on X and Y, respectively. The function $X \xrightarrow{f} Y$ is said to be type three fuzzy continuous if for any U in T_Y, $f^{-1}(U) = (U \circ f, U \circ f)$ is in T_X.*

Theorem 7.4. *Suppose T_X and T_Y are type three fuzzy topologies on X and Y with corresponding S-topologies (τ_X, i_X) and (τ_Y, i_Y) with respect to type three fuzzy topological structure, respectively. A function $f : (X, T_X) \longrightarrow (Y, T_Y)$ is type three fuzzy continuous if and only if $f : (X, \tau_X, i_X) \longrightarrow (Y, \tau_Y, i_Y)$ is structurally continuous.*

Proof. Similar to Theorem 5.4, with $f^{-1}(U)$ as given in Definition 5.3. □

8 Type four fuzzy topology as a structural topology

The definition of fuzzy topology on a fuzzy set given in [9] is what we call type four fuzzy topology as given below.

Definition 8.1. [9] *For a fuzzy set X on a set $\mathbf{U}$, a set T_X of fuzzy subsets of X, (that is, $T_X \subseteq \mathcal{F}_X$), is called type four fuzzy topology on X if $\underline{0}, X \in T_X$, for $U, V \in T_X$, $U \bigwedge V \in T_X$ and for $\{U_i\}_{i \in J} \subseteq T_X$ where J is an indexed set, $\bigvee_{i \in J} U_i \in T_X$.*

Let $\mathbf{U}$ be a set. Considering the partially ordered set $I^{\mathbf{U}}$ as a category, let $I^{\mathbf{U}} \xrightarrow{\mathbb{P}} Set \xrightarrow{P} Set$ be functors, where $\mathbb{P}$ takes an object Y to the set of all the fuzzy subsets $\mathcal{F}_Y$ of $\mathbf{U}$, a morphism f to i_f, the set inclusion, and $P = \mathcal{P}$ is the covariant powerset functor. For a fuzzy set X, define the structure $\mathsf{S}_X = (b_X, t_X, \bigwedge_X, \bigvee_X)$ relative to $(\mathbb{P}, P)$, called type four fuzzy topological structure, as follows:

$$1 \xrightarrow{b_X} \mathcal{F}_X \text{ taking } 1 \longmapsto \underline{0}$$

$$1 \xrightarrow{t_X} \mathcal{F}_X \text{ taking } 1 \longmapsto X$$

$$\mathcal{F}_X \times \mathcal{F}_X \xrightarrow{\bigwedge_X} \mathcal{F}_X \text{ taking } (A, B) \longmapsto A \bigwedge B$$

$$\mathcal{P}(\mathcal{F}_X) \xrightarrow{\bigvee_X} \mathcal{F}_X \text{ taking } Z \longmapsto \bigvee_{A \in Z} A$$

Theorem 8.2. *A family T_X of fuzzy subsets of a fuzzy set X is a type four fuzzy topology on X if and only if (τ_X, i_X) is a structural topology on X with respect to the type four fuzzy topological structure, where $T_X \overset{\tau_X}{\hookrightarrow} \mathcal{F}_X$ and $\mathcal{P}(T_X) \overset{i_X}{\hookrightarrow} \mathcal{P}(\mathcal{F}_X)$ are set inclusions.*

Proof. Similar to the proof of Theorem 5.2. □

Definition 8.3. *Let (X, T_X) and (Y, T_Y) be type four fuzzy topological spaces. A morphism $f : X \longrightarrow Y$ in $I^{\mathbf{U}}$ is said to be type four fuzzy continuous, if $T_X \subseteq T_Y$.*

Theorem 8.4. *Suppose T_X and T_Y are type four fuzzy topologies on X and Y with corresponding S-topologies (τ_X, i_X) and (τ_Y, i_Y) with respect to type four fuzzy topological structure, respectively. A function $f : (X, T_X) \longrightarrow (Y, T_Y)$ is type four fuzzy continuous if and only if $f : (X, \tau_X, i_X) \longrightarrow (Y, \tau_Y, i_Y)$ is structurally continuous.*

Proof. (⇒) Let $(X, T_X) \xrightarrow{f} (Y, T_Y)$ be type four fuzzy continuous. So $T_X \subseteq T_Y$. Now the inclusion $\bar{i}_f : T_X \hookrightarrow T_Y$ makes the following diagram commutative:

$$\begin{array}{ccc} T_Y & \overset{\tau_Y}{\hookrightarrow} & \mathcal{F}_Y \\ {\scriptstyle \bar{i}_f} \uparrow & /// & \uparrow {\scriptstyle i_f} \\ T_X & \underset{\tau_X}{\hookrightarrow} & \mathcal{F}_X \end{array}$$

Thus f is structurally continuous.

(⇐) Since $(X, \tau_X, i_X) \xrightarrow{f} (Y, \tau_Y, i_Y)$ is structurally continuous, there exists an arrow $\bar{i}_f$ making the following diagram commutative:

$$\begin{array}{ccc} T_Y & \overset{\tau_Y}{\hookrightarrow} & \mathcal{F}_Y \\ {\scriptstyle \bar{i}_f} \uparrow & /// & \uparrow {\scriptstyle i_f} \\ T_X & \underset{\tau_X}{\hookrightarrow} & \mathcal{F}_X \end{array}$$

The arrow $\bar{i}_f$ is necessarily the inclusion, so that $T_X \subseteq T_Y$. Thus f is type four fuzzy continuous. □

9 Type five fuzzy topology as a structural topology

The definition of fuzzy topology on a fuzzy set given in [4] is what we call type five fuzzy topology as given below.

Definition 9.1. [4] *For a fuzzy set X on a set $\mathbf{U}$, a set T_X of fuzzy subsets of X, (that is, $T_X \subseteq \mathcal{F}_X$), is called type five fuzzy topology on X if for all $\alpha \in I$, $\underline{\alpha} \bigwedge X \in T_X$, for $U, V \in T_X$, $U \bigwedge V \in T_X$ and for $\{U_i\}_{i \in J} \subseteq T_X$ where J is an indexed set, $\bigvee_{i \in J} U_i \in T_X$.*

Considering the type four fuzzy topological space (X, T'_X), where $T'_X = \{\underline{\alpha} \bigwedge X | \alpha \in I\}$, with the corresponding structural topological space (X, τ'_X, i'_X), we have:

Lemma 9.2. *Let X be a set and T_X be a type four fuzzy topology on X. The following are equivalent.*

a) *for all $\alpha \in I$, $\underline{\alpha} \bigwedge X \in T_X$.*

b) *the identity map $1_X : (X, T'_X) \longrightarrow (X, T_X)$ is type four fuzzy continuous.*

c) *the identity map $1_X : (X, \tau'_X, i'_X) \longrightarrow (X, \tau_X, i_X)$ is structurally continuous.*

Proof. The equivalency of (a) and (b) follows from Definition 8.3 and the fact that for all $\alpha \in I$, $\underline{\alpha} \bigwedge X \in T_X$ if and only if $T'_X \subseteq T_X$.

The equivalency of (b) and (c) follows from Theorem 8.4. □

Theorem 9.3. *A family T_X of fuzzy subsets of X is a type five fuzzy topology on X if and only if (τ_X, i_X) is a structural topology on X with respect to the type four fuzzy topological structure, where $T_X \overset{\tau_X}{\hookrightarrow} \mathcal{F}_X$ and $\mathcal{P}(T_X) \overset{i_X}{\hookrightarrow} \mathcal{P}(\mathcal{F}_X)$ are set inclusions and the map $1_X : (X, T'_X) \longrightarrow (X, T_X)$ is type four fuzzy continuous.*

Proof. $(\Rightarrow)$ Suppose T_X is a type five fuzzy topology on X. Since $\underline{0} = \underline{0} \bigwedge X \in T_X$ and $X = \underline{1} \bigwedge X \in T_X$, T_X is a type four fuzzy topology on X. Therefore by Theorem 8.2, (X, τ_X, i_X) is a structural topology with respect to the type four fuzzy topological structure. The second assertion follows from Lemma 9.2.

$(\Leftarrow)$ Suppose (τ_X, i_X) is a structural topology on X with respect to the type four fuzzy topological structure and that $1_X : (X, T'_X) \longrightarrow (X, T_X)$ is type four fuzzy continuous. The type four continuity of 1_X yields that for all $\alpha \in I$, $\underline{\alpha} \in T_X$. Being a type four fuzzy topology, proves that T_X is a type five fuzzy topology on X. □

The type five fuzzy continuity is the same as type four fuzzy continuity, as defined in Definition 8.3 and thus Theorem 8.4 stays valid when replacing "type four" by "type five" in its statement.

10 Conclusions

In this article, we have introduced the notion of structural topology on objects in a category. We have then used this to unify five of the most common fuzzy topologies existing in the literature, as well as the standard topology. We have also introduced the notion of structural continuity and we have shown that the fuzzy continuity defined in the literature in all the above mentioned cases, as well as the standard continuity are structural.

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