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# A normalized distribution mechanism under multi-criteria situations and fuzzy behavior

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**Abstract**

In general, agents always face an increasing need to focus on multiple aims efficiently under their operational processes. However, agents might take different activity levels to participate and might represent administrative areas of different scales. Therefore, this paper proposes a normalized index considering multi-criteria situations and supreme-utilities among fuzzy activity level (decision, strategy) vectors. Three existing notions of traditional game theory are reinterpreted in the framework of multi-criteria fuzzy transferable utility games. First, the normalized index could be represented as an alternative formulation in terms of excess functions. Second, an axiomatic result is proposed to present the rationality of this normalized index based on the reduced game and related consistency. Finally, two dynamic processes are introduced to illustrate that this normalized index could be reached by agents who start from an arbitrary efficient payoff vector and make successive adjustments.

*Keywords:* Multi-criteria situation, reduced game, excess function, dynamic process.

## 1 Introduction

In the framework of traditional transferable-utility (TU) games, the power (or utility) indexes have been defined to measure the political power of each member of a system. A member in a voting system is, e.g., a party in a parliament or a country in a confederation. Each member will have a certain number of votes, and so its power will be different. Results of the power indexes may be found in, e.g., Dubey and Shapley [10], Haller [14], Lehrer [21], van den Brink and van der Laan [5] and so on. Banzhaf [3] defined a power index on voting games that was essentially identical to that given by Coleman [9]. This value was later extended to arbitrary games by Owen [30, 31]. Briefly speaking, the Banzhaf-Coleman index is a mechanism that collects each agent's average marginal contribution from all coalitions in which he/she/it has participated.

In the axiomatic formulation for solutions in game theory, *consistency* is an important property. Consistency states the independence of a value with respect to fixing some agents with their assigned payoffs. It asserts that the recommendation made for any problem should always agree with the recommendation made in the subproblem that appears when the payoffs of some agents are settled on. It has been introduced in different ways depending upon how the payoffs of the agents that leave the bargaining are defined. This property has been investigated in various problems by always applying *reduced games*. In addition to the axiomatizations for solutions, *dynamic processes* can lead the agents to that solution, starting from an arbitrary efficient payoff vector. The foundation of a dynamic theory was laid by Stearns [34]. Related dynamic results may be found in, e.g., Billera [4], Maschler and Owen [25], and Hwang and Liao [17].

In a traditional TU game, each agent is either fully involved or totally out of participation with some other agents. In a *fuzzy* TU *game*, each agent is permitted to participate with infinite various activity decisions (or levels). The theory of fuzzy TU games commenced with the investigation of Aubin [1, 2] where the opinions of a fuzzy TU game and

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the fuzzy core are introduced. Many fuzzy solutions have been applied wildly, e.g., Butnariu and Kroupa [6], Cheng et al. [8], Hwang [15], Li and Zhang [22], Meng and Zhang [26], Tijs et al. [36], Vasilev [37], Wei et al. [36], and so on. Here this paper focuses on the Banzhaf-Coleman index. Several fuzzy Banzhaf values and related results have been proposed, such as Fernández et al. [11], Gallego et al. [12, 13], Meng et al. [27], Tan et al. [35], and so on.

In different fields, from sciences to industry, engineering, and the social sciences, managers face an increasing need to focus on multiple aims efficiently in their operational processes. Related situations include analyzing distribution trade-offs, selecting optimal decision or process designs, or any other condition where you need an efficient solution with trade-offs between two or more aims. In many cases, these real-world efficient situations could be formulated as multi-criteria mathematical optimization models. The solutions of such situations require appropriate techniques to offer optimal results that, unlike traditional viewpoints or methods, take several properties of the aims into account. This article would like to provide different necessary mathematical foundations of multi-criteria optimal solutions to analyze problems with multiple goals and fuzzy behavior simultaneously in a grand coalition. Several pre-existing results considered multi-criteria games with fuzzy goals or fuzzy matrix games multi-criteria model. For example, Campos [7] proposed a method for its solution based on the establishment of a fuzzy linear programming problem for each player under multi-criteria conditions. The method is shown as a generalization of that conventionally used in the solution of a classical game. Khorram et al. [19] proposed an approach to solve a nonlinear multi-objective problem subject to fuzzy relation inequalities with max-Archimedean-t-norm composition by a genetic algorithm. The additive generator of Archimedean t-norms is utilized to reform the existent genetic algorithm to solve the constrained nonlinear multi-objective optimization problems. The research due to Kroenke and Wilhelm [20] aimed to evaluate the placement of accounting companies in the iron and steel by means of multi-criteria fuzzy game. The positioning accounting reflects the results obtained are perceived in detailed analysis of the data under accounting perspective. By considering decision making problems of the organizations or competitive systems, Nishizaki and Sakawa [29] adopted multi-criteria fuzzy games to analyze competition and partial cooperation among the decision makers when players must cope with decision making problems in organizations which consist of decision makers with conflicting interests. To handle engineering problems associated with optimal alternative selection for multi-criteria decision-making, Peldschus [32] proposed a solution concept by taking the relationship between fuzzy sets and matrix game theories.

Based on the notions of fuzzy TU behavior and multi-criteria situations simultaneously, Liao [23] considered the framework of *multi-criteria fuzzy TU games* and related results as follows.

- Liao [23] defined the *fuzzy Banzhaf-Coleman index* and its efficient extension by extending the *Banzhaf-Coleman index* to multi-criteria fuzzy TU games.
- Inspired by average-reduction due to Hwang and Liao [17], Liao [23] defined two types of reduction and related consistency to present the rationality of the fuzzy Banzhaf-Coleman index and its efficient extension respectively.
- Liao [23] adopted the *excess functions* to present alternative viewpoint of the fuzzy efficient Banzhaf-Coleman index.
- Liao [23] adopted the excess functions and the reductions to construct different dynamic processes of the fuzzy efficient Banzhaf-Coleman index.

Inspired by Liao [23], Liao and Chung [24] considered different results as follows.

- Liao and Chung [24] proposed the *fuzzy Banzhaf-Owen index*, its efficient extension and normalization by extending the *Banzhaf-Owen index* to multi-criteria fuzzy TU games. The major difference is that the fuzzy Banzhaf-Coleman index is based on average-behavior", and the fuzzy Banzhaf-Owen index is based on sum-behavior". Further, the normalized notion does not appear in Liao [23].
- Inspired by sum-reduction due to Hwang and Liao [17], Liao and Chung [24] defined two types of reduction and related consistency to present the rationality of the fuzzy Banzhaf-owen index and its efficient extension respectively.
- Liao and Chung [24] adopted the excess functions to present alternative viewpoints of the fuzzy efficient Banzhaf-owen index and the fuzzy normalized Banzhaf-owen index respectively.
- Liao and Chung [24] adopted the excess functions and the reductions to construct different dynamic processes of the fuzzy efficient Banzhaf-owen index and the fuzzy normalized Banzhaf-owen index respectively.

The above pre-existing results raise one motivation:

- Inspired by Liao and Chung [24], whether there exists a normalization and related results of the fuzzy Banzhaf-Coleman index due to Liao [23] under multi-criteria situations and fuzzy behavior.

The paper is devoted to investigate this motivation. The main results are as follows.

- By building on the results due to Liao [23] and Liao and Chung [24] on multi-criteria fuzzy TU games, the *fuzzy normalized Banzhaf-Coleman index* is proposed in Section 2.
- In order to present the rationality, this paper adopt an extended reduction defined by Liao [23] to show that the fuzzy normalized Banzhaf-Coleman index is the only solution satisfying related properties of consistency and standard for games.
- Based on different viewpoint, this paper also presents alternative formulation for the fuzzy normalized Banzhaf-Coleman index in terms of excess functions in Section 3.
- In Section 4, this paper adopts reduction and excess function to show that the fuzzy normalized Banzhaf-Coleman index can be reached by agents who start from an arbitrary efficient payoff vector. Some more applications, comparisons and discussions are also provided in Sections 2, 5 and 6.

## 2 The fuzzy normalized Banzhaf-Coleman index

Let $U$ be the universe of agents. For $i \in U$ and $b_i \in (0,1]$, $B_i = [0, b_i]$ could be treated as the level (decision) space of agent $i$ and $B_i^+ = (0, b_i]$, where 0 denotes no participation. For $N \subseteq U$, $N \neq \emptyset$, let $B^N = \prod_{i \in N} B_i$ be the product set of the level (decision) spaces of all agents of $N$. A **fuzzy coalition** is a vector $\alpha \in B^N$. The $i$-th coordinate $\alpha_i$ of $\alpha$ is called the participation level of agent $i$ in the fuzzy coalition $\alpha$. A fuzzy coalition could be described as a collection of economic agents, i.e., agents, who transfer fractions of their representation to a collective decision maker, the fuzzy coalition. The term fuzzy coalition also arises when the possibility of graduating the membership of an agent in a coalition is considered. For all $T \subseteq N$, one would define that $\theta^T \in \mathbb{R}^N$ is the vector with $\theta_i^T = 1$ if $i \in T$, and $\theta_i^T = 0$ if $i \in N \setminus T$. Denote $0_N$ the zero vector in $\mathbb{R}^N$. For $m \in \mathbb{N}$, let $0_m$ be the zero vector in $\mathbb{R}^m$ and $\mathbb{N}_m = \{1, 2, \cdots, m\}$.

A **fuzzy TU game**[1] is a triple $(N, b, v)$, where $N \neq \emptyset$ is finite set of agents, $b \in \prod_{i \in N} B_i^+$ is the vector that presents the highest levels for each agent, and $v : B^N \to \mathbb{R}$ is a characteristic map with $v(0_N) = 0$ which assigns to each $\alpha = (\alpha_i)_{i \in N} \in B^N$ the worth that the agents can gain when each agent $i$ takes level $\alpha_i$. Given a fuzzy TU game $(N, b, v)$ and $\alpha \in B^N$, one could write $A(\alpha) = \{i \in N | \; \alpha_i \neq 0\}$ and $\alpha_T$ to be the restriction of $\alpha$ at $T$ for each $T \subseteq N$. Further, one could define that $v_*(T) = \sup_{\alpha \in B^N}\{v(\alpha) | A(\alpha) = T\}$ is the **supreme-utility**[2] among all vector $\alpha$ with $A(\alpha) = T$. A **multi-criteria fuzzy TU game** is a triple $(N, b, V^m)$, where $m \in \mathbb{N}$, $V^m = (v^t)_{t \in \mathbb{N}_m}$ and $(N, b, v^t)$ is a fuzzy TU game for all $t \in \mathbb{N}_m$. Denote the set of all multi-criteria fuzzy TU games by $\Gamma$.

Let $(N, b, V^m) \in \Gamma$. A **payoff vector** of $(N, b, V^m)$ is a vector $x = (x^t)_{t \in \mathbb{N}_m}$ and $x^t = (x_i^t)_{i \in N} \in \mathbb{R}^N$, where $x_i^t$ denotes the payoff to agent $i$ in $(N, b, v^t)$ for all $t \in \mathbb{N}_m$ and for all $i \in N$. A payoff vector $x$ of $(N, b, V^m)$ is **multi-criteria efficient** if $\sum_{i \in N} x_i^t = v_*^t(N)$ for all $t \in \mathbb{N}_m$. The set of all multi-criteria efficient vector of $(N, b, V^m)$ is denoted by $E(N, b, V^m)$. A **solution** is a map $\sigma$ assigning to each $(N, b, V^m) \in \Gamma$ an element

$$\sigma\big(N, b, V^m\big) = \big(\sigma^t\big(N, b, V^m\big)\big)_{t \in \mathbb{N}_m},$$

where $\sigma^t\big(N, b, V^m\big) = \big(\sigma_i^t\big(N, b, V^m\big)\big)_{i \in N} \in \mathbb{R}^N$ and $\sigma_i^t\big(N, b, V^m\big)$ is the payoff of the agent $i$ assigned by $\sigma$ in $\big(N, b, v^t\big)$.

Liao [23] defined the fuzzy Banzhaf-Coleman index as follows.

**Definition 2.1.** *The* **fuzzy Banzhaf-Coleman index (FBCI, Liao [23])**, $\kappa$, *is defined by*

$$\kappa_i^t(N, b, V^m) = \frac{1}{2^{|N|-1}} \sum_{\substack{S \subseteq N \\ i \in S}} \big[v_*^t(S) - v_*^t(S \setminus \{i\})\big],$$

*for all* $(N, b, V^m) \in \Gamma$, *for all* $t \in \mathbb{N}_m$ *and for all* $i \in N$. *Under the solution* $\kappa$, *all agents receive its* **average marginal contributions** *related to supreme-utilities in each* $S \subseteq N$, *respectively.*

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[1] A fuzzy TU game, which is defined by Aubin [1, 2], is a pair $(N, v^a)$, where $N$ is a coalition and $v^a$ is a map such that $v^a : [0,1]^N \longrightarrow \mathbb{R}$ and $v^a(0_N) = 0$. In fact, $(N, v^a) = (N, \theta^N, v)$ .

[2] From now on one could consider bounded fuzzy TU games, defined as those games $(N, b, v)$ such that, there exists $K_v \in \mathbb{R}$ such that $v(\alpha) \leq K_v$ for all $\alpha \in B^N$. One could adopt it to ensure that $v_*(T)$ is well-defined.

A solution $\sigma$ satisfies **multi-criteria efficiency (MEFF)** if $\sum_{i\in N}\sigma_i^t(N,b,V^m)=v_*^t(N)$ for all $(N,b,V^m)\in\Gamma$ and for all $t\in\mathbb{N}_m$. Property MEFF asserts that all agents allocate all the utility completely. It is easy to check that the FBCI violates MEFF. Therefore, Liao [23] considered an efficient extension of the FBCI as follows.

**Definition 2.2.** *The* **fuzzy efficient Banzhaf-Coleman index (FEBCI, Liao [23])**, $\overline{\kappa}$, *is defined by*

$$\overline{\kappa_i^t}(N,b,V^m)=\kappa_i^t(N,b,V^m)+\frac{1}{|N|}\big[v_*^t(N)-\sum_{k\in N}\kappa_k^t(N,b,V^m)\big],$$

*for all* $(N,b,V^m)\in\Gamma$, *for all* $t\in\mathbb{N}_m$ *and for all* $i\in N$. *Under the solution* $\overline{\kappa}$, *all agents firstly receive its average marginal contributions related to supreme-utilities in each* $S\subseteq N$, *and further allocate the remaining utility equally.*

Here one could consider a normalization of the FBCI as follows.

**Definition 2.3.** *The* **fuzzy normalized Banzhaf-Coleman index (FNBCI)**, $\overline{\rho}$, *is defined by*

$$\overline{\rho_i^t}(N,b,V^m)=\frac{v_*^t(N)}{\sum\limits_{k\in N}\kappa_k^t(N,b,V^m)}\cdot\kappa_i^t(N,b,V^m),$$

*for all* $(N,b,V^m)\in\Gamma^*$, *for all* $t\in\mathbb{N}_m$ *and for all* $i\in N$, *where* $\Gamma^*=\{(N,b,V^m)\in\Gamma\mid\sum\limits_{i\in N}\kappa_i^t(N,b,V^m)\neq 0$ *for all* $t\in\mathbb{N}_m\}$. *Under the notion of* $\overline{\rho}$, *all agents allocate the supreme-utility of the grand coalition proportionally by applying the FBCI of all agents.*

**Lemma 2.4.** *The FNBCI satisfies MEFF on* $\Gamma^*$.

*Proof.* For all $(N,b,V^m)\in\Gamma^*$ and for all $t\in\mathbb{N}_m$,

$$\sum_{i\in N}\overline{\rho_i^t}(N,b,V^m)=\sum_{i\in N}\Big[\frac{v_*^t(N)}{\sum_{k\in N}\kappa_k^t(N,b,V^m)}\cdot\kappa_i^t(N,b,V^m)\Big]=\frac{v_*^t(N)}{\sum_{k\in N}\kappa_k^t(N,b,V^m)}\cdot\Big[\sum_{i\in N}\kappa_i^t(N,b,V^m)\Big]=v_*^t(N).$$

Thus, the FNBCI satisfies MEFF on $\Gamma^*$. □

As one mentioned in the introduction, multi-criteria analysis (also known as multi-attribute analysis, multi-objective analysis, etc.) is a notion of multiple criteria analysis that is concerned with the simultaneous optimization of conditions involving more than one aim. Multi-criteria analysis has been applied in many areas, including engineering, politics, economics, and logistics where efficient decisions need to be used in the presence of trade-offs among two or more aims. For example, simultaneously minimizing the cost and maximizing comfort while buying a central air-conditioning system, and maximizing efficiency while minimizing energy consumption and emission of pollutants are examples of multi-criteria efficient problems involving two and three aims, respectively. In many situations, there can be more than three aims. On the other hand, each agent could be allowed to participate with infinite various activity decisions (levels, strategies) in real situations. Therefore, this paper considers the framework of multi-criteria fuzzy TU games.

# 3 Reduction and axiomatic results

In this section, one would show that there exists corresponding reduced game that could be adopted to analyze the FNBCI.

Liao [23] defined reductions and related consistency properties as follows. Let $\psi$ be a solution, $(N,b,V^m)\in\Gamma$ and $S\subseteq N$. The **1-average-reduced game** $(S,b_S,V^{1,m}_{S,\psi})$ is defined by $V^{1,m}_{S,\psi}=(v^{1,t}_{S,\psi})_{t\in\mathbb{N}_m}$ and

$$v^{1,t}_{S,\psi}(\alpha)=\begin{cases}0 & ,\ \alpha=0_S,\\ \frac{1}{2^{|N\setminus S|}}\cdot\sum\limits_{Q\subseteq N\setminus S}\big[v_*^t\big(A(\alpha)\cup Q\big)-\sum\limits_{i\in Q}\psi_i^t(N,b,V^m)\big] & ,\ \text{otherwise.}\end{cases}$$

$\psi$ satisfies **1-average-consistency (1ACON)** if for all $(N,b,V^m)\in\Gamma$, for all $S\subseteq N$, for all $t\in\mathbb{N}_m$ and for all $i\in S$, $\psi_i^t(S,b_S,V^{1,m}_{S,\psi})=\psi_i^t(N,b,V^m)$. $\psi$ satisfies **1-standard for fuzzy games (1SFG)** if $\psi(N,b,V^m)=\kappa(N,b,V^m)$ for all $(N,b,V^m)\in\Gamma$ with $|N|\leq 2$.

Let $\psi$ be a solution, $(N, b, V^m) \in \Gamma$ and $S \subseteq N$. The **2-average-reduced game** $(S, b_S, V^{2,m}_{S,\psi})$ is defined by $V^{2,m}_{S,\psi} = (v^{2,t}_{S,\psi})_{t\in\mathbb{N}_m}$ and

$$v^{2,t}_{S,\psi}(\alpha) = \begin{cases} 0 & , \ \alpha = 0_S, \\ v^t_*(N) - \sum\limits_{i\in N\setminus S} \psi^t_i(N, b, V^m) & , \ A(\alpha) = S, \\ \frac{1}{2^{|N\setminus S|}} \cdot \sum\limits_{Q\subseteq N\setminus S} \big[v^t_*\big(A(\alpha)\cup Q\big) - \sum\limits_{i\in Q} \psi^t_i(N, b, V^m)\big] & , \text{otherwise.} \end{cases}$$

$\psi$ satisfies **2-average-consistency (2ACON)** if for all $(N, b, V^m) \in \Gamma$, for all $S \subseteq N$, for all $t \in \mathbb{N}_m$ and for all $i \in S$, $\psi^t_i(S, b_S, V^{2,m}_{S,\psi}) = \psi^t_i(N, b, V^m)$. $\psi$ satisfies **2-standard for fuzzy games (2SFG)** if $\psi(N, b, V^m) = \overline{\kappa}(N, b, V^m)$ for all $(N, b, V^m) \in \Gamma$ with $|N| \leq 2$.

Liao [23] provided several axiomatic results as follows.

1. The FBCI and FEBCI satisfy 1ACON and 2ACON on $\Gamma$ respectively.
2. On $\Gamma$, the FBCI is the only solution satisfying 1SFG and 1ACON.
3. On $\Gamma$, the FEBCI is the only solution satisfying 2SFG and 2ACON.

The major difference is that the results of this paper are based on the "normalization" of the fuzzy Banzhaf-Coleman index, and Liao's [23] results are based on the "efficient extension" of the fuzzy Banzhaf-Coleman index.

Unfortunately, it is also easy to see that $(S, b_S, V^m_{S,\psi})$ does not exist if $\sum_{i\in S} \kappa^t_i(N, b, V^m) = 0$. Thus, one could consider the **3-average-consistency** as follows. $\psi$ satisfies **3-consistency (3ACON)** if $(S, b_S, V^m_{S,\psi}) \in \Gamma^*$ for some $(N, b, V^m) \in \Gamma$ and for some $S \subseteq N$, it holds that $\psi^t_i(S, b_S, V^m_{S,\psi}) = \psi^t_i(N, b, V^m)$ for all $t \in \mathbb{N}_m$ and for all $i \in S$. Further, $\psi$ satisfies **3-standard for fuzzy games (3SFG)** if $\psi(N, b, V^m) = \overline{\rho}(N, b, V^m)$ for all $(N, b, V^m) \in \Gamma$ with $|N| \leq 2$.

Next, one would characterize the FNBCI by means of 3ACON and 3SFG.

**Theorem 3.1.**

1. *The FNBCI satisfies 3ACON on $\Gamma^*$.*
2. *If $\psi$ satisfies 3SFG and 3ACON, then it also satisfies MEFF on $\Gamma^*$.*
3. *On $\Gamma^*$, the FNBCI is the only solution satisfying 3SFG and 3ACON.*

*Proof.* To verify result 1, let $(N, b, V^m) \in \Gamma^*$ and $S \subseteq N$. If $|N| = 1$, then the proof is completed. Assume that $|N| \geq 2$, $S = \{i, j\}$ for some $i, j \in N$ and $(S, b_S, V^{2,m}_{S,\overline{\rho}}) \in \Gamma^*$. Let $a^t = \frac{v^t(N)}{\sum\limits_{k\in N} \kappa^t_k(N,b,V^m)}$ and $a^t_S = \frac{(v^{2,t}_{S,x})_*(S)}{\sum\limits_{k\in S} \kappa^t_k(S,b_S,V^m_{S,\overline{\rho}})}$ for all $t \in \mathbb{N}_m$. By applying the proof of Liao's [23],

$$\kappa^t_l(S, b_S, V^{2,m}_{S,\overline{\rho}}) = \kappa^t_l(N, b, V^m), \tag{1}$$

for all $t \in \mathbb{N}_m$ and for all $l \in S$. So,

$$a^t_S = \frac{(v^{2,t}_{S,\overline{\rho}})_*(S)}{\sum\limits_{k\in S} \kappa^t_k(S,b_S,V^{2,m}_{S,\overline{\rho}})} = \frac{v^t_*(N) - \sum\limits_{i\in N\setminus S} \overline{\rho^t_i}(N,b,V^m)}{\sum\limits_{k\in S} \kappa^t_k(S,b_S,V^{2,m}_{S,\overline{\rho}})} = \frac{\sum\limits_{i\in S} \overline{\rho^t_i}(N,b,V^m)}{\sum\limits_{k\in S} \kappa^t_k(S,b_S,V^{2,m}_{S,\overline{\rho}})} = \frac{v^t(N)}{\sum\limits_{k\in N} \kappa^t_k(N,b,V^m)} = a^t, \tag{2}$$

for all $t \in \mathbb{N}_m$. By equations (1), (2) and definitions of $\overline{\rho}$ and $(S, b_S, V^{2,m}_{S,\overline{\kappa}})$,

$$\overline{\kappa^t_l}(S, b_S, V^{2,m}_{S,\overline{\kappa}}) = a^t_S \cdot \kappa^t_l(S, b_S, V^{2,m}_{S,\overline{\rho}}) = a^t \cdot \kappa^t_l(N, b, V^m) = \overline{\kappa^t_l}(N, b, V^m).$$

for all $t \in \mathbb{N}_m$ and for all $l \in S$. That is, $\overline{\kappa}$ satisfies 3ACON.

To prove result 2, suppose that $\psi$ satisfies 3SFG and 3ACON. Let $(N, b, V^m) \in \Gamma^*$ and $t \in \mathbb{N}_m$. If $|N| \leq 2$, it is trivial that $\psi$ satisfies MEFF by 3SFG. The case $|N| > 2$: Assume, on the contrary, that there exists $(N, b, V^m) \in \Gamma^*$ such that $\sum_{i\in N} \psi^t_i(N, b, V^m) \neq v^t_*(N)$. This means that there exist $i \in N$ and $j \in N$ such that $[v^t_*(N) - \sum_{k\in N\setminus\{i,j\}} \psi^t_k(N, b, V^m)] \neq [\psi^t_i(N, b, V^m) + \psi^t_j(N, b, V^m)]$. By 3ACON and $\psi$ satisfies MEFF for two-person games, this contradicts with

$$\psi^t_i(N, b, V^m) + \psi^t_j(N, b, V^m) \quad = \psi^t_i(\{i, j\}, b_{\{i,j\}}, v^{2,t}_{\{i,j\},\psi}) + \psi^t_j(\{i, j\}, b_{\{i,j\}}, v^{2,t}_{\{i,j\},\psi}) = v^t_*(N) - \sum_{k\in N\setminus\{i,j\}} \psi^t_k(N, b, V^m).$$

Hence $\psi$ satisfies MEFF.

Next, one would prove result 3. By result 1, the FNBCI satisfies 3ACON. Clearly, the FNBCI satisfies 3SFG. To prove uniqueness, suppose that $\psi$ satisfies 3ACON and 3SFG on $\Gamma^*$. By result 2, $\psi$ satisfies MEFF on $\Gamma^*$. Let $(N,b,V^m)\in\Gamma^*$. One would complete the proof by induction on $|N|$. If $|N|\leq 2$, it is trivial that $\psi(N,b,V^m)=\overline{\rho}(N,b,V^m)$ by 3SFG. Assume that it holds if $|N|\leq r-1$, $r\geq 3$. The case $|N|=r$: Let $i,j\in N$ with $i\neq j$ and $t\in\mathbb{N}_m$. By Definition 2.3, $\overline{\rho^t_k}(N,b,V^m)=\frac{v^t_*(N)}{\sum_{h\in N}\kappa^t_h(N,b,V^m)}\cdot\kappa^t_k(N,b,V^m)$ for all $k\in N$. Assume that $\alpha^t_k=\frac{\kappa^t_k(N,b,V^m)}{\sum_{h\in N}\kappa^t_h(N,b,V^m)}$ for all $k\in N$ and for all $t\in\mathbb{N}_m$. Therefore,

$$\begin{aligned}
\psi^t_i(N,b,V^m) &= \psi^t_i\big(N\setminus\{j\},b_{N\setminus\{j\}},v^{2,t}_{N\setminus\{j\},\psi}\big) && \textbf{(by 3ACON of } \psi\textbf{)}\\
&= \overline{\rho^t_i}\big(N\setminus\{j\},b_{N\setminus\{j\}},v^{2,t}_{N\setminus\{j\},\psi}\big) && \textbf{(by 3SFG of } \psi\textbf{)}\\
&= \frac{\big(v^{2,t}_{N\setminus\{j\},\psi}\big)_*\big(N\setminus\{j\}\big)}{\sum_{k\in N\setminus\{j\}}\kappa^t_k\big(N\setminus\{j\},b_{N\setminus\{j\}},v^{2,t}_{N\setminus\{j\},\psi}\big)}\cdot\kappa^t_i\Big(N\setminus\{j\},b_{N\setminus\{j\}},v^{2,t}_{N\setminus\{j\},\psi}\Big)\\
&= \frac{v^t_*(N)-\psi^t_i(N,b,V^m)}{\sum_{k\in N\setminus\{j\}}\kappa^t_k(N,b,V^m)}\cdot\rho^t_i(N,b,V^m) && \textbf{(by equation (1))}\\
&= \frac{v^t_*(N)-\psi^t_i(N,b,V^m)}{-\kappa^t_j(N,b,V^m)+\sum_{k\in N}\kappa^t_k(N,b,V^m)}\cdot\kappa^t_i(N,b,V^m).
\end{aligned}\tag{3}$$

By equation (3),

$$\begin{aligned}
& \psi^t_i(N,b,V^m)\cdot[1-\alpha^t_j]=[v^t_*(N)-\psi^t_j(N,b,V^m)]\cdot\alpha^t_j\\
\Longrightarrow\ & \sum_{i\in N}\psi^t_i(N,b,V^m)\cdot[1-\alpha^t_j]=[v^t_*(N)-\psi^t_j(N,b,V^m)]\cdot\sum_{i\in N}\alpha^t_j\\
\Longrightarrow\ & v^t_*(N)\cdot[1-\alpha^t_j]=[v^t_*(N)-\psi^t_j(N,b,V^m)]\cdot 1 && \textbf{(by MEFF of } \psi\textbf{)}\\
\Longrightarrow\ & v^t_*(N)-v^t_*(N)\cdot\alpha^t_j=v^t_*(N)-\psi^t_j(N,b,V^m)\\
\Longrightarrow\ & \overline{\rho^t_j}(N,b,V^m)=\psi^t_j(N,b,V^m).
\end{aligned}$$

The proof is completed. □

The following examples are to show that each of the axioms adopted in Theorem 3.1 is logically independent of the remaining axioms.

**Example 3.2.** *Define a solution $\psi$ by for all $(N,b,V^m)\in\Gamma^*$, for all $t\in\mathbb{N}_m$ and for all $i\in N$, $\psi^t_i(N,b,V^m)=0$. On $\Gamma^*$, $\psi$ satisfies 3ACON, but it violates 3SFG.*

**Example 3.3.** *Define a solution $\psi$ by for all $(N,b,V^m)\in\Gamma^*$, for all $t\in\mathbb{N}_m$ and for all $i\in N$,*

$$\psi^t_i(N,b,V^m)=\begin{cases}\overline{\rho^t_i}(N,b,V^m) & \text{if } |N|\leq 2,\\ 0 & \text{otherwise.}\end{cases}$$

*On $\Gamma^*$, $\psi$ satisfies 3SFG, but it violates 3ACON.*

# 4 Excess formulation and dynamic processes

In this section, one would adopt excess functions and reductions to propose dynamic processes for the FNBCI.

In order to establish consistency of the FNBCI, it would be useful to present alternative formulation for the FNBCI in terms of *excess*. Let $(N,b,V^m)\in\Gamma$, $S\subseteq N$ and $x$ be a payoff vector in $(N,b,V^m)$. Define that $x^t(S)=\sum_{i\in S}x^t_i$ for all $t\in\mathbb{N}_m$. The **excess** of a coalition $S\subseteq N$ at $x$ is the vector

$$e(S,V^m,x)=(e(S,v^t,x^t))_{t\in\mathbb{N}_m}\text{ and }e(S,v^t,x^t)=v^t_*(S)-x^t(S).\tag{4}$$

The value $e(S,v^t,x^t)$ could be treated as the **complaint** of coalition $S$ when all agents receive its payoffs from $x^t$ in $(N,b,v^t)$.

**Lemma 4.1.** *Let $(N,b,V^m)\in\Gamma^*$ and $x\in E(N,b,V^m)$. Then for all $i,j\in N$ and $t\in\mathbb{N}_m$,* $\sum_{S\subseteq N\setminus\{i,j\}}e(S\cup\{i\},v^t,\frac{x^t}{a^t})=\sum_{S\subseteq N\setminus\{i,j\}}e(S\cup\{j\},v^t,\frac{x^t}{a^t})\Longleftrightarrow x=\overline{\rho}(N,b,V^m)$*, where* $a^t=\frac{v^t(N)}{\sum_{k\in N}\kappa^t_k(N,b,V^m)}$.

*Proof.* Let $(N,b,V^m)\in\Gamma^*$ and $x\in E(N,b,V^m)$. For all pairs $i,j\in N$ and $t\in\mathbb{N}_m$,

$$\begin{aligned}&\sum_{S\subseteq N\setminus\{i,j\}} e(S\cup\{i\},v^t,\tfrac{x^t}{a^t}) = \sum_{S\subseteq N\setminus\{i,j\}} e(S\cup\{j\},v^t,\tfrac{x^t}{a^t})\\
\Longleftrightarrow &\sum_{S\subseteq N\setminus\{i,j\}}\Big[v^t(S\cup\{i\})-\tfrac{x^t(S\cup\{i\})}{a^t}\Big] = \sum_{S\subseteq N\setminus\{i,j\}}\Big[v^t(S\cup\{j\})-\tfrac{x^t(S\cup\{j\})}{a^t}\Big]\\
\Longleftrightarrow &\Bigg[\sum_{S\subseteq N\setminus\{i,j\}} v^t(S\cup\{i\})\Bigg]-\tfrac{2^{|N|-2}\cdot x_i^t}{a^t} = \Bigg[\sum_{S\subseteq N\setminus\{i,j\}} v^t(S\cup\{j\})\Bigg]-\tfrac{2^{|N|-2}\cdot x_j^t}{a^t}\\
\Longleftrightarrow &\, x_i^t-x_j^t=\tfrac{a^t}{2^{|N|-2}}\cdot\sum_{S\subseteq N\setminus\{i,j\}}\Big[v^t(S\cup\{i\})-v^t(S\cup\{j\})\Big].\end{aligned}\tag{5}$$

By definition of $\overline{\rho}$,

$$\overline{\rho_i^t}(N,b,V^m)-\overline{\rho_j^t}(N,b,V^m) = \tfrac{a^t}{2^{|N|-2}}\cdot\sum_{S\subseteq N\setminus\{i,j\}}\Big[v^t(S\cup\{i\})-v^t(S\cup\{j\})\Big].\tag{6}$$

By equations (5) and (6),

$$x_i^t-x_j^t=\overline{\rho_i^t}(N,b,V^m)-\overline{\rho_j^t}(N,b,V^m).$$

Hence,

$$\sum_{j\neq i}\Big[x_i^t-x_j^t\Big]=\sum_{j\neq i}\Big[\overline{\rho_i^t}(N,b,V^m)-\overline{\rho_j^t}(N,b,V^m)\Big].$$

That is, $\Big(|N|-1\Big)\cdot x_i^t-\sum_{j\neq i}x_j^t=\Big(|N|-1\Big)\cdot\overline{\rho_i^t}(N,b,V^m)-\sum_{j\neq i}\overline{\rho_j^t}(N,b,V^m)$. Since $x\in E(N,b,V^m)$ and $\overline{\rho}$ satisfies MEFF, $|N|\cdot x_i^t-v_*^t(N)=|N|\cdot\overline{\rho_i^t}(N,b,V^m)-v_*^t(N)$. Therefore, $x_i^t=\overline{\rho_i^t}(N,b,V^m)$ for all $(N,b,V^m)\in\Gamma^*$, for all $i\in N$ and for all $t\in\mathbb{N}_m$. □

In order to establish the dynamic processes of the FNBCI, one would firstly define correction functions by means of excess functions. The correction functions are based on the idea that, each agent shortens the complaint relating to its own and others' non-participation, and adopts these regulations to correct the original payoff.

**Definition 4.2.** *Let $(N,b,V^m)\in\Gamma$ and $i\in N$. The* **correction function** *is defined to be $f=(f^t)_{t\in\mathbb{N}_m}$, where $f^t=(f_i^t)_{i\in N}$ and $f_i^t:E(N,b,V^m)\to\mathbb{R}$ is define by*

$$f_i^t(x)=x_i^t+w\sum_{j\in N\setminus\{i\}}\Bigg[\frac{a^t}{2^{|N|-2}}\sum_{S\subseteq N\setminus\{i,j\}}\Big[e(S\cup\{i\},v^t,\frac{x^t}{a^t})-e(S\cup\{j\},v^t,\frac{x^t}{a^t})\Big]\Bigg],$$

*where $a^t=\frac{v_*^t(N)}{\sum_{k\in N}\kappa_k^t(N,b,V^m)}$ and $w\in\mathbb{R}$ is a fixed positive number, which reflects the assumption that agent $i$ does not ask for full correction (when $w=1$) but only (usually) a fraction of it. Define $[x]^0=x,\ [x]^1=f([x]^0),\cdots,\ [x]^q=f([x]^{q-1})$ for all $q\in\mathbb{N}$.*

**Lemma 4.3.** *$f(x)\in E(N,b,V^m)$ for all $(N,b,V^m)\in\Gamma$ and for all $x\in E(N,b,V^m)$.*

*Proof.* Let $(N,b,V^m)\in\Gamma$, $t\in\mathbb{N}_m$, $i,j\in N$, $x\in E(N,b,V^m)$ and $a^t=\frac{v_*^t(N)}{\sum_{k\in N}\kappa_k^t(N,b,V^m)}$. Similar to equation (5),

$$\begin{aligned}&\sum_{j\in N\setminus\{i\}}\tfrac{a^t}{2^{|N|-2}}\sum_{Q\subseteq N\setminus\{i,j\}}\Big[e(Q\setminus\{j\},v^t,\tfrac{x^t}{a^t})-e(Q\setminus\{i\},v^t,\tfrac{x^t}{a^t})\Big]\\
=\;&\sum_{j\in N\setminus\{i\}}\Big[\Big(\tfrac{a^t}{2^{|N|-2}}\sum_{Q\subseteq N\setminus\{i,j\}}\big[v^t(Q\setminus\{j\})-v^t(Q\setminus\{i\})\big]\Big)-x_i^t+x_j^t\Big].\end{aligned}\tag{7}$$

By definition of $\overline{\rho}$,

$$\overline{\rho_i^t}(N,b,V^m)-\overline{\rho_j^t}(N,b,V^m)=\frac{a^t}{2^{|N|-2}}\sum_{Q\subseteq N\setminus\{i,j\}}\big[v^t(Q\setminus\{j\})-v^t(Q\setminus\{i\})\big].\tag{8}$$

By equations (7) and (8),

$$\begin{aligned}
& \sum_{j\in N\setminus\{i\}} \tfrac{a^t}{2^{|N|-2}} \sum_{Q\subseteq N\setminus\{i,j\}} \Big[e(Q\setminus\{j\},v^t,\tfrac{x^t}{a^t}) - e(Q\setminus\{i\},v^t,\tfrac{x^t}{a^t})\Big] \\
= & \sum_{j\in N\setminus\{i\}} \Big(\overline{\rho_i^t}(N,b,V^m) - \overline{\rho_j^t}(N,b,V^m) - x_i^t + x_j^t\Big) \\
= & \Big((|N|-1)\big(\overline{\rho_i^t}(N,b,V^m) - x_i^t\big) - \sum_{j\in N\setminus\{i\}} \overline{\rho_j^t}(N,b,V^m) + \sum_{j\in N\setminus\{i\}} x_j^t\Big) \\
= & \Big(|N|\big(\overline{\rho_i^t}(N,b,V^m) - x_i^t\big) - \sum_{j\in N} \overline{\rho_j^t}(N,b,V^m) + \sum_{j\in N} x_j^t\Big) \\
= & \Big(|N|\big(\overline{\rho_i^t}(N,b,V^m) - x_i^t\big) - v_*^t(N) + v_*^t(N)\Big) \qquad \Big(\textbf{by MEFF of } \overline{\rho}\textbf{,}\ x\in E(N,b,V^m)\Big) \\
= & |N|\cdot\Big(\overline{\rho_i^t}(N,b,V^m) - x_i^t\Big).
\end{aligned} \tag{9}$$

Moreover,

$$\begin{aligned}
& \sum_{i\in N}\sum_{j\in N\setminus\{i\}} \tfrac{a^t}{2^{|N|-2}} \sum_{Q\subseteq N\setminus\{i,j\}} \Big[e(Q\setminus\{j\},v^t,\tfrac{x^t}{a^t}) - e(Q\setminus\{i\},v^t,\tfrac{x^t}{a^t})\Big] \\
= & \sum_{i\in N} |N|\cdot\Big(\overline{\rho_i^t}(N,b,V^m) - x_i^t\Big) \\
= & |N|\cdot\Big(\sum_{i\in N}\overline{\rho_i^t}(N,b,V^m) - \sum_{i\in N} x_i^t\Big) \\
= & |N|\cdot\big(v_*^t(N) - v_*^t(N)\big) \qquad \Big(\textbf{by MEFF of } \overline{\rho}\textbf{,}\ x\in E(N,b,V^m)\Big) \\
= & 0.
\end{aligned} \tag{10}$$

So,

$$\begin{aligned}
\sum_{i\in N} f_i^t(x) & = \sum_{i\in N}\Big[x_i^t + w\sum_{j\in N\setminus\{i\}} \tfrac{a^t}{2^{|N|-2}} \sum_{Q\subseteq N\setminus\{i,j\}} \Big[e(Q\setminus\{j\},v^t,\tfrac{x^t}{a^t}) - e(Q\setminus\{i\},v^t,\tfrac{x^t}{a^t})\Big]\Big] \\
& = \sum_{i\in N} x_i^t + w\sum_{i\in N}\sum_{j\in N\setminus\{i\}} \tfrac{a^t}{2^{|N|-2}} \sum_{Q\subseteq N\setminus\{i,j\}} \Big[e(Q\setminus\{j\},v^t,\tfrac{x^t}{a^t}) - e(Q\setminus\{i\},v^t,\tfrac{x^t}{a^t})\Big] \\
& = v_*^t(N) + 0 \\
& \Big(\textbf{by equation (10) and } x\in E(N,b,V^m)\Big) \\
& = v_*^t(N).
\end{aligned}$$

Hence, $f(x)\in E(N,b,V^m)$ if $x\in E(N,b,V^m)$. □

**Theorem 4.4.** *Let* $(N,b,V^m)\in\Gamma$. *If* $0 < t < \frac{2}{|N|}$, *then* $\{[x]^q\}_{q=1}^{\infty}$ *converges geometrically to* $\overline{\rho}(N,b,V^m)$ *for each* $x\in E(N,b,V^m)$.

*Proof.* Let $(N,b,V^m)\in\Gamma$, $t\in\mathbb{N}_m$, $i\in N$ and $x\in E(N,b,V^m)$. By equation (9) and definition of $f$,

$$f_i^t(x) - x_i^t = w\sum_{j\in N\setminus\{i\}} \tfrac{a^t}{2^{|N|-2}} \sum_{Q\subseteq N\setminus\{i,j\}} \Big[e(Q\setminus\{j\},v^t,\tfrac{x^t}{a^t}) - e(Q\setminus\{i\},v^t,\tfrac{x^t}{a^t})\Big] = w\cdot|N|\cdot\Big(\overline{\kappa_i^t}(N,b,V^m) - x_i^t\Big).$$

Hence,

$$\begin{aligned}
\overline{\kappa_i^t}(N,b,V^m) - f_i^t(x) & = \overline{\kappa_i^t}(N,b,V^m) - x_i^t + x_i^t - f_i^t(x) \\
& = \overline{\kappa_i^t}(N,b,V^m) - x_i^t - w\cdot|N|\cdot(\overline{\kappa_i^t}(N,b,V^m) - x_i^t) \\
& = \Big(1 - w\cdot|N|\Big)\Big[\overline{\kappa_i^t}(N,b,V^m) - x_i^t\Big].
\end{aligned}$$

So, for all $q\in\mathbb{N}$,

$$\overline{\kappa}(N,b,V^m) - [x]^q = \Big(1 - w\cdot|N|\Big)^q\Big[\overline{\kappa}(N,b,V^m) - x\Big].$$

If $0 < w < \frac{2}{|N|}$, then $-1 < \Big(1 - w\cdot|N|\Big) < 1$ and $\{[x]^q\}_{q=1}^{\infty}$ converges geometrically to $\overline{\kappa}(N,b,V^m)$. □

Inspired by Maschler and Owen [25], one would find a dynamic process under reductions.

**Definition 4.5.** *Let $\psi$ be a solution, $(N,b,V^m)\in\Gamma$, $S\subseteq N$ and $x\in E(N,b,V^m)$. The $(x,\psi)$*-**reduced game** *$(S,b_S,V^{r,m}_{\psi,S,x})$ is given by $V^{r,m}_{\psi,S,x}=(v^{r,t}_{\psi,S,x})_{t\in\mathbb{N}_m}$ and for all $T\subseteq S$,*

$$v^{r,t}_{\psi,S,x}(\alpha)=\begin{cases} v^t_*(N)-\sum_{i\in N\setminus S}x^t_i & ,A(\alpha)=S,\\ v^{2,t}_{S,\psi}(\alpha) & ,\text{otherwise.}\end{cases}$$

Inspired by Maschler and Owen [25], one would also define different correction function as follow. The **R-correction function** to be $g=(g^t)_{t\in\mathbb{N}_m}$, where $g^t=(g^t_i)_{i\in N}$ and $g^t_i:E(N,b,V^m)\to\mathbb{R}$ is define by

$$g^t_i(x)=x^t_i+w\sum_{k\in N\setminus\{i\}}\Big(\overline{\rho^t_i}\big(\{i,k\},v^t_{\overline{\rho},\{i,k\},x}\big)-x^t_i\Big).$$

Define $[\theta]^0=x,\ [\theta]^1=g([\theta]^0),\cdots,\ [\theta]^q=g([\theta]^{q-1})$ for all $q\in\mathbb{N}$.

**Lemma 4.6.** *$g\big(x\big)\in E(N,b,V^m)$ for all $(N,b,V^m)\in\Gamma$ and for all $x\in E(N,b,V^m)$.*

*Proof.* Let $(N,b,V^m)\in\Gamma$, $t\in\mathbb{N}_m$, $i,k\in N$ and $x\in E(N,b,V^m)$. Let $S=\{i,k\}$, by MEFF of $\overline{\rho}$ and Definition 4.5,

$$\overline{\rho^t_i}(S,b_S,V^{r,m}_{\overline{\rho},S,x})+\overline{\rho^t_k}(S,b_S,V^{r,m}_{\overline{\rho},S,x})=x^t_i+x^t_k.$$

By 3ACON and 3SFG of $\overline{\rho}$,

$$\overline{\rho^t_i}(S,b_S,V^{r,m}_{\overline{\rho},S,x})-\overline{\rho^t_k}(S,b_S,V^{r,m}_{\overline{\rho},S,x})=\overline{\rho^t_i}(S,b_S,V^{2,m}_{S,\overline{\rho}})-\overline{\rho^t_k}(S,b_S,V^{2,m}_{S,\overline{\rho}})=\overline{\rho^t_i}(N,b,V^m)-\overline{\rho^t_k}(N,b,V^m).$$

Therefore,

$$2\cdot\Big[\overline{\rho^t_i}(S,b_S,V^{r,m}_{\overline{\rho},S,x})-x^t_i\Big]=\overline{\rho^t_i}(N,b,V^m)-\overline{\rho^t_k}(N,b,V^m)-x^t_i+x^t_k. \tag{11}$$

By definition of $g$ and equation (11),

$$\begin{aligned} g^t_i(x)=&\ \ x^t_i+\tfrac{w}{2}\cdot\Big[\sum_{k\in N\setminus\{i\}}\overline{\rho^t_i}(N,b,V^m)-\sum_{k\in N\setminus\{i\}}x^t_i-\sum_{k\in N\setminus\{i\}}\overline{\rho^t_k}(N,b,V^m)+\sum_{k\in N\setminus\{i\}}x^t_k\Big]\\ =&\ \ x^t_i+\tfrac{w}{2}\cdot\Big[\big(|N|-1\big)\overline{\rho^t_i}(N,b,V^m)-\big(|N|-1\big)x^t_i-\big(v^t_*(N)-\overline{\rho^t_i}(N,b,V^m)\big)+\big(v^t_*(N)-x^t_i\big)\Big]\\ &\ \ \Big(\textbf{by MEFF of }\overline{\rho},\ x\in E(N,b,V^m)\Big)\\ =&\ \ x^t_i+\tfrac{|N|\cdot w}{2}\cdot\Big[\overline{\rho^t_i}(N,b,V^m)-x^t_i\Big]. \end{aligned} \tag{12}$$

So,

$$\sum_{i\in N}g^t_i(x)=\sum_{i\in N}x^t_i+\tfrac{|N|\cdot w}{2}\cdot\Big[\sum_{i\in N}\overline{\rho^t_i}(N,b,V^m)-\sum_{i\in N}x^t_i\Big]=v^t_*(N)+\tfrac{|N|\cdot w}{2}\cdot\big[v^t_*(N)-v^t_*(N)\big]=v^t_*(N).$$

Thus, $g\big(x\big)\in E(N,b,V^m)$ for all $x\in E(N,b,V^m)$. □

**Theorem 4.7.** *Let $(N,b,V^m)\in\Gamma$. If $0<\alpha<\frac{4}{|N|}$, then $\{[\theta]^q\}^\infty_{q=1}$ converges to $\overline{\rho}(N,b,V^m)$ for each $x\in E(N,b,V^m)$.*

*Proof.* Let $(N,b,V^m)\in\Gamma$, $t\in\mathbb{N}_m$ and $x\in E(N,b,V^m)$. By equation (12), $g^t_i(x)=x^t_i+\frac{|N|\cdot w}{2}\cdot\Big[\overline{\rho^t_i}(N,b,V^m)-x^t_i\Big]$ for all $i\in N$. Therefore,

$$\big(1-\frac{|N|\cdot w}{2}\big)\cdot\big[\overline{\rho^t_i}(N,b,V^m)-x^t_i\big]=\big[\overline{\rho^t_i}(N,b,V^m)-g^t_i(x)\big].$$

So, for all $q\in\mathbb{N}$,

$$\overline{\rho}(N,b,V^m)-[\theta]^q=\Big(1-\frac{|N|\cdot w}{2}\Big)^q\Big[\overline{\rho}(N,b,V^m)-x\Big].$$

If $0<w<\frac{4}{|N|}$, then $-1<\Big(1-\frac{|N|\cdot w}{2}\Big)<1$ and $\{[\theta]^q\}^\infty_{q=1}$ converges to $\overline{\rho}(N,b,v)$ for all $(N,b,V^m)\in\Gamma$, for all $t\in\mathbb{N}_m$ and for all $i\in N$. □

## 5 Managerial insights and discussion

First, a brief application of multi-criteria fuzzy TU games is provided in the setting of "management. This kind of problem can be formulated as follows. Let $N = \{1, 2, \cdots, n\}$ be a set of all agents of a grand management system $(N, b, V^m)$. The function $v^t$ could be treated as a utility function that assigns to each level vector $\alpha = (\alpha_i)_{i \in N} \in B^N$ the worth that the agents can obtain when each agent $i$ participates at an operation strategy $\alpha_i \in B_i$ in the sub-management system $(N, b, v^t)$. Modeled in this way, the grand management system $(N, b, V^m)$ could be considered as a multi-criteria fuzzy TU game, with $v^t$ being each characteristic function and $B_i$ being the set of all operation strategies of the agent $i$. In previous sections, this paper would like to show that the FNBCI could provide "*optimal distribution mechanism* among all agents, in the sense that this organization can get a payoff from each combination of operation strategies of all agents under multi-criteria situations and fuzzy behavior.

In the following, a practical application of power distribution in a national parliament is also provided. Let $N = \{1, 2, \cdots, n\}$ be a set of all members of a national parliament of a certain country. In the national parliament of a certain country, all members of the parliament are elected by voting or recommendation by parties. All members have the power to propose, discuss, establish, and veto all bills. They dedicate different levels of attention and participation to different bills depending on their academic expertise and the public opinion they represent. The level of involvement is also closely associated with the alliance strategy formed for the interests of different political parties. For the aforementioned reasons, the strategies adopted by each member of the parliament show distinct levels of participation and certain amounts of ambiguity. The function $v$ could be treated as a power function that assigns to each level vector $\alpha = (\alpha_i)_{i \in N} \in B^N$ the power that the members can dedicate when each member $i$ participates at an operation strategy $\alpha_i \in B_i$. Modeled in this way, the national parliament operational system $(N, b, v)$ could be considered as a fuzzy TU game, with $v$ being each characteristic function and $B_i$ being the set of all operation strategies of the member $i$. To evaluate the influence of each member in the national parliament, using the power indicators proposed in this work, this paper first assessed the supreme influence that each parliament member has averaged over previous bill meetings based on various and ambiguous behaviors, which is the fuzzy Banzhaf-Coleman index mentioned in Definition 2.1. The remaining shared power distribution should also be allocated equally by all members, which is the fuzzy efficient Banzhaf-Coleman index mentioned in Definition 2.2. As each parliament member has different academic expertise and represents different public opinions, they naturally carry different levels of importance; hence, it makes complete sense for them to derive different levels of importance from the fuzzy efficient Banzhaf-Coleman index. It is reasonable that the supreme-utility of the grand coalition should also be allocated in proportion to the importance derived for each member, which is the fuzzy normalized Banzhaf-Coleman index mentioned in Definition 2.3.

Next, an example with real data is provided as follows. Let $(N, b, V^m) \in \Gamma$ with $N = \{i, j, k\}$, $B_i = [0, 0.91]$, $B_j = [0, 0.83]$, $B_k = [0, 1]$ and $m = 2$. Thus, $(N, b, V^m) = \Big(\{i, j, k\}, (0.91, 0.83, 1), V^2\Big)$. Further, let $v^1_*(\{i, j, k\}) = 9$, $v^1_*(\{i\}) = 7$, $v^1_*(\{j\}) = -3$, $v^1_*(\{k\}) = 6$, $v^1_*(\{i, j\}) = 8$, $v^1_*(\{i, k\}) = -2$, $v^1_*(\{j, k\}) = 3$, $v^2_*(\{i, j, k\}) = 4$, $v^2_*(\{i\}) = 2$, $v^2_*(\{j\}) = 8$, $v^2_*(\{k\}) = 6$, $v^2_*(\{i, j\}) = 5$, $v^2_*(\{i, k\}) = 3$, $v^2_*(\{j, k\}) = -1$ and $v^1_*(\emptyset) = v^2_*(\emptyset) = 0$. By Definitions 2.1, 2.2 and 2.3,

$$
\begin{array}{rclrclrcl}
\kappa^1_i(N, b, V^m) & = & 4, & \kappa^1_j(N, b, V^m) & = & \frac{3}{2}, & \kappa^1_k(N, b, V^m) & = & 1, \\
\overline{\kappa^1_i}(N, b, V^m) & = & \frac{29}{6}, & \overline{\kappa^1_j}(N, b, V^m) & = & \frac{7}{3}, & \overline{\kappa^1_k}(N, b, V^m) & = & \frac{11}{6}, \\
\overline{\rho^1_i}(N, b, V^m) & = & \frac{72}{13}, & \overline{\rho^1_j}(N, b, V^m) & = & \frac{27}{13}, & \overline{\rho^1_k}(N, b, V^m) & = & \frac{18}{13}, \\
\kappa^2_i(N, b, V^m) & = & \frac{1}{4}, & \kappa^2_j(N, b, V^m) & = & \frac{5}{4}, & \kappa^2_k(N, b, V^m) & = & \frac{-3}{4}, \\
\overline{\kappa^2_i}(N, b, V^m) & = & \frac{4}{3}, & \overline{\kappa^2_j}(N, b, V^m) & = & \frac{7}{3}, & \overline{\kappa^2_k}(N, b, V^m) & = & \frac{1}{3}, \\
\overline{\rho^2_i}(N, b, V^m) & = & \frac{4}{3}, & \overline{\rho^2_j}(N, b, V^m) & = & \frac{20}{3}, & \overline{\rho^2_k}(N, b, V^m) & = & -4.
\end{array}
$$

The advantages and the disadvantages of this proposed results over the existing ones could be provided as follows.

- The advantages of the method presented in this paper are as follows.
  - Solution concepts on traditional TU games have only discussed the participation or non-participation of all members. Thus, this paper assumes that all members have various and ambiguous levels of participation.
  - The solution concepts of real-world situations require appropriate techniques to offer optimal results that, unlike traditional viewpoints or methods, take several properties of multiple aims into account. Therefore, this paper considers multi-criteria situations.
  - In a multitude of fuzzy game literature on solution concepts, although it might be also assumed that all members have various and ambiguous levels of participation, some literature evaluated the power or value of a given member presented with a given level of participation, such as Hwang and Liao [16, 18]. Thus,

this paper considers the overall value or power that each member exerts with various and ambiguous levels of participation.

  - Under the fuzzy efficient Banzhaf-Coleman index due to Liao [23], any additional fixed utility should be distributed equally among the agents who are concerned. By considering many real-world situations, this paper proposes the fuzzy normalized Banzhaf-Owen index to allocate the supreme-utility of the grand coalition among the agents in proportion to its fuzzy Banzhaf-Coleman indexes.

- The disadvantage of the method presented in this paper is as follows. As stated in the advantages above, each member has various and ambiguous participation levels. Although this paper can evaluate the overall value or power that each member exerts, it is impossible to evaluate the power or value of a given member with a given level of participation. Therefore, in future researches, this paper will propose different solution concepts based on the simultaneous consideration of the overall value and the specific level of participation.

# 6 Conclusions

This paper proposed different solution in the framework of multi-criteria fuzzy TU games. Several main results of this paper are as follows.

- By considering supreme-utilities under fuzzy behavior, this paper proposed the fuzzy normalized Banzhaf-Coleman index.
- Further, this paper adopted excess functions to present alternative viewpoint for the fuzzy normalized Banzhaf-Coleman index.
- In order to present the rationality of the fuzzy normalized Banzhaf-Coleman index, this paper adopted the 2-average-reduction and 3-average-consistency property to characterize the fuzzy normalized Banzhaf-Coleman index.
- By applying excess functions and reductions respectively, this paper proposed two correction functions and related dynamic processes to illustrate that the fuzzy normalized Banzhaf-Coleman index can be approached by agents who start from an arbitrary efficient payoff vector.

One should compare our results with related pre-existing results.

- The fuzzy normalized Banzhaf-Coleman index and related results are introduced initially in the frameworks of fuzzy TU games and multi-criteria fuzzy TU games. Furthermore,
  - Different from the Banzhaf-Coleman index and other solutions on traditional TU games, this paper proposed the fuzzy normalized Banzhaf-Coleman index to analyze power distribution under multi-criteria situations and fuzzy behavior simultaneously.
  - Different from the fuzzy Banzhaf-Coleman index and the fuzzy efficient Banzhaf-Coleman index due to Liao [23] on multi-criteria fuzzy TU games, this paper defined the fuzzy normalized Banzhaf-Coleman index to analyze different efficient extension of the fuzzy Banzhaf-Coleman index.
  - By considering supreme-utilities under multi-criteria situations and fuzzy behavior simultaneously, Liao and Chung [24] extended the Banzhaf-Owen index to multi-criteria fuzzy TU games. The definitions of solutions, reductions, correction functions and properties are different between Liao and Chung [24] and this paper. Although the notions of axiomatic results and dynamic processes are similar, the techniques of the proofs are still different. The major difference is that the results of this paper are based on the “Banzhaf-Coleman index”, but Liao and Chung’s [24] results are based on the “Banzhaf-Owen index”.
- Inspired by Maschler and Owen [25], this paper proposed dynamic processes for the fuzzy normalized Banzhaf-Coleman index. The major difference is that the correction function of this paper (Definition 4.2) is based on “excess functions”, and Maschler and Owen’s [25] correction function is based on “reductions”.

The results proposed in this paper raise two questions.

- Whether there exist different modifications and related results for some more solutions on multi-criteria fuzzy TU games.

- Whether there exist alternative formulations and related results for some more solutions on multi-criteria fuzzy TU games.

These issues are left to the readers.

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## References

[1] J. P. Aubin, *Coeur et valeur des jeux flous á paiements latéraux*, Comptes Rendus de l'Académie des Sciences, **279** (1974), 891-894.

[2] J. P. Aubin, *Cooperative fuzzy games*, Mathematics of Operations Research, **6** (1981), 1-13.

[3] J. F. Banzhaf, *Weighted voting doesn't work: A mathematical analysis*, Rutgers Law Review, **19** (1965), 317-343.

[4] L. J. Billera, *Global stability in n-person games*, Transactions of the American Mathematical Society, **172** (1972), 45-56.

[5] R. van den. Brink, G. van der. Lann, *Axiomatizations of the normalized banzhaf value and the Shapley value*, Social Choice and Welfare, **15** (1998), 567-582.

[6] D. Butnariu, T. Kroupa, *Shapley mappings and the cumulative value for n-person games with fuzzy coalitions*, European Journal of Operational Research, **186** (2008), 288-299.

[7] L. Campos, *Fuzzy linear programming models to solve fuzzy matrix games*, Fuzzy Sets and Systems, **32** (1989), 275-289.

[8] C. Y. Cheng, E. C. Chi, K. Chen, H. L. Lin, Y. H. Liao, *A power mensuration and its normalization under multicriteria situations*, IAENG International Journal of Applied Mathematics, **50** (2020), 262-267.

[9] J. S. Coleman, *Control of collectivities and the power of a collectivity to act*, B. Lieberman, Ed., Social Choice, Gordon and Breach, 269-300, London, U.K., 1971.

[10] P. Dubey, L. S. Shapley, *Mathematical properties of the Banzhaf power index*, Mathematics of Operations Research, **4** (1979), 99-131.

[11] J. R. Fernández, I. Gallego, A. Jiménez-Losada, M. Ordóñez, *A Banzhaf value for games with a proximity relation among the agents*, International Journal of Approximate Reasoning, **88** (2017), 192-208.

[12] I. Gallego, J. R. Fernández, A. Jiménez-Losada, M. Ordóñez, *A Banzhaf value for games with fuzzy communication structure: Computing the power of the political groups in the European Parliament*, Fuzzy Sets and Systems, **255** (2014), 128-145.

[13] I. Gallego, J. R. Fernández, A. Jiménez-Losada, M. Ordóñez, *A symmetric Banzhaf cooperation value for games with a proximity relation among the agents*, Symmetry, **12** (2020), 1196.

[14] H. Haller, *Collusion properties of values*, International Journal of Game Theory, **23** (1994), 261-281.

[15] Y. A. Hwang, *Fuzzy games: A characterization of the core*, Fuzzy Sets and Systems, **158** (2007), 2480-2493.

[16] Y. A. Hwang, Y. H. Liao, *The consistent value of fuzzy games*, Fuzzy Sets and Systems, **160** (2009), 644-656.

[17] Y. A. Hwang, Y. H. Liao, *Consistency and dynamic approach of indexes*, Social Choice and Welfare, **34** (2010), 679-694.

[18] Y. A. Hwang, Y. H. Liao, *The core configuration for fuzzy games*, Journal of Intelligent and Fuzzy Systems, **27** (2014), 3007-3014.

[19] E. Khorram, R. Ezzati, Z. Valizadeh, *Solving nonlinear multi-objective optimization problems with fuzzy relation inequality constraints regarding Archimedean triangular norm compositions*, Fuzzy Optimization and Decision Making, **11** (2012), 299-335.

[20] A. Kroenke, V. E. Wilhelm, *Multicriteria fuzzy games as a tool to the evaluation of Brazilian companies*, Applied Mathematical Sciences, **8** (2014), 7453-7467.

[21] E. Lehrer, *An axiomatization of the Banzhaf value*, International Journal of Game Theory, **17** (1988), 89-99.

[22] S. Li, Q. Zhang, *A simplified expression of the Shapley function for fuzzy game*, European Journal of Operational Research, **196** (2009), 234-245.

[23] Y. H. Liao, *Power mensuration mechanism under fuzzy behavior and multicriteria situation*, Journal of Intelligent and Fuzzy Systems, **36** (2019), 6631-6640.

[24] Y. H. Liao, L. Y. Chung, *Power allocation rules under fuzzy behavior and multicriteria situations*, Iranian Journal of Fuzzy Systems, **17** (2020), 187-198.

[25] M. Maschler, G. Owen, *The consistent Shapley value for hyperplane games*, International Journal of Game Theory, **18** (1989), 389-407.

[26] F. Meng, Q. Zhang, *The Shapley value on a kind of cooperative fuzzy games*, Journal of Computational Information Systems, **7** (2011), 1846-1854.

[27] F. Meng, J. Zhao, Q. Zhang, *The Banzhaf value for fuzzy games with fuzzy payoffs*, Information Technology Journal, **11** (2012), 262-268.

[28] S. Muto, S. Ishihara, S. Fukuda, S. Tijs, R. Branzei, *Generalized cores and stable sets for fuzzy games*, International Game Theory Review, **8** (2006), 95-109.

[29] I. Nishizaki, M. Sakawa, *Fuzzy and multiobjective games for conflict resolution*, Physica-Verlag, Heidelberg, 2001

[30] G. Owen, *Multilinear extensions and the Banzhaf value*, Naval Research Logistics Quart, **22** (1975), 741-750.

[31] G. Owen, *Characterization of the Banzhaf-Coleman index*, SIAM Journal on Applied Mathematics, **35** (1978), 315-327.

[32] F. Peldschus, *Fuzzy matrix games multi-criteria model for decision-making in engineering*, Informatica, **16** (2005), 107-120.

[33] L. S. Shapley, *A value for n-person game*, In: Kuhn, H.W., Tucker, A.W. (Eds.), Contributions to the Theory of Games II (Annals of Mathematics Studies, Princeton, **28** (1953), 307-317.

[34] R. E. Stearns, *Convergent transfer schemes for n-person games*, Transactions of the American Mathematical Society, **134** (1968), 449-459.

[35] C. Tan, Z. Z. Jiang, X. Chen, W. H. Ip, *A Banzhaf function for a fuzzy game*, IEEE Transactions on Fuzzy Systems, **22** (2014), 1489-1502.

[36] S. Tijs, R. Branzei, S. Ishihara, S. Muto, *On cores and stable sets for fuzzy games*, Fuzzy Sets and Systems, **146** (2004), 285-296.

[37] V. A. Vasilev, *Unblocked imputations of fuzzy games. II: Nonemptiness of cores of two market games*, Journal of Mathematical Sciences, **253** (2021), 455-469.

[38] H. C. Wei, A. T. Li, W. N. Wang, Y. H. Liao, *Sustainability for utility allocation: A game-theoretical mechanism*, Journal of Intelligent and Fuzzy Systems, (2021), preprint.