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Interval discrete fractional calculus and its application to interval fractional difference equations

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Abstract

In this work, we present some useful results of discrete fractional calculus for interval-valued functions. The composition rules for interval fractional operators are introduced, which are used to construct the general form of the solutions to nonlinear interval fractional difference equations. An illustrative example is provided in which the method of recursive iterations is applied to obtain explicit formulas for the solutions of linear interval fractional difference equations.

Keywords: Discrete fractional calculus, interval discrete fractional calculus, interval fractional difference equations, composition rules.

1 Introduction

Fractional calculus has attracted the attention of many researchers since the late 19^{th} century. Mathematicians such as Liouville, Grünwald, Letnikov and Riemann have done several distinguished works on fractional calculus in the continuous setting. Since then, real-world phenomena and a variety of applied problems have been studied and modelled using fractional calculus. The non-locality of fractional difference operator makes fractional models more practical than the usual ones, especially for systems which involve memory; see for example [6, 7, 12, 27].

Discrete fractional calculus is a new area for scientists. Miller and Ross were pioneers in this field [27]. Atici and Eloe published some works that contained a variety of results on discrete fractional calculus [4, 5, 6]. Atici and Şengül [7] used discrete fractional difference equations to model tumor growth as an initial work in this field. Holm [16] developed the theory of composition of fractional sums and differences that has many applications, most notably in fractional difference equations. Some possible definitions of fractional difference have been proposed for various applications. Riemann-Liouville forward fractional difference is one of the more common ones.

On the other hand, in several mathematical models, the knowledge about the parameters of real world systems is imprecise or uncertain; generally, we cannot observe or exactly measure these parameters. In these situations, the parameters cannot be represented by real numbers. This shortcoming can be overcome using interval models. Moreover, interval (fuzzy) fractional differential equations provide a mathematical model for real-world phenomena in which the uncertainty and the memory effect are inevitable [1, 2, 3, 10, 13, 14, 15, 22, 23, 24, 25, 28, 32].

Most of the real-world phenomena, which appear in many branches of sciences such as physics, signal and image processing, mechanics and dynamical systems, biology, material, economics and so on, are frequently modeled as difference equations [5, 6, 7, 16, 24, 31]. In these cases the nonlocality (or memory effect) and uncertainty of the parameters and initial values naturally exist, therefore we can utilize interval fractional difference equations to model such phenomena. For example in [7], fractional difference equations have been used to model the tumor growth. Due to the lack of information or experimental errors the initial size of tumor or the available parameters may have uncertainties, which are taken in terms of interval numbers. Therefore, the tumor growth can be remodeled by interval fractional difference equations to get the solutions that describe better the behavior of the model.

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Discrete fractional calculus for interval-valued functions are defined in [18] and the solutions of fractional discrete time Malthusian equations with and without delay are presented. The purpose of this study is to investigate some useful properties of interval discrete fractional calculus and interval fractional operators. We are aiming to present the fundamental composition rules for interval fractional operators, which will be used to solve nonlinear interval fractional difference equations and construct the general form of the solutions.

The outline of this paper is as follows. In Section 2, we provide some definitions and preliminary results that will be used throughout this paper. In Section 3, we introduce discrete fractional calculus for interval-valued functions and present some composition rules. We also introduce higher order fractional difference for interval-valued functions. The mono- tonicity of interval fractional sum operator are investigated. In Section 4, we study interval discrete fractional initial value problems and present their solutions. We provide a general example in which the solutions of linear interval fractional difference are presented. Finally, conclusions are given.

2 Preliminaries

In this section, we provide some definitions and useful results and introduce the necessary notation which will be used throughout the paper; see for example [16, 30].

Let $\mathbb{I}(\mathbb{R})$ denote the set of (closed bounded) intervals of the real line. If $A=[\underline{a},\overline{a}]\in\mathbb{I}(\mathbb{R})$, we denote by $len(A)=\overline{a}-\underline{a}$, the length (or diameter) of the interval A. Given two intervals $A,B\in\mathbb{I}(\mathbb{R})$ and $k\in\mathbb{R}$, the usual interval arithmetic operations, i.e., the Minkowski addition and scalar multiplication, are defined by $A+B=\{a+b|a\in A,b\in B\}$ and $kA=\{ka|a\in A\}$. If $k=-1$, scalar multiplication gives the interval $-A=(-1)A=\{-a|a\in A\}$ that is the opposite of the interval A. In general, $A+(-A)\neq\{0\}$, i.e., an arbitrary interval $A=[\underline{a},\overline{a}]$, with $\underline{a}\neq\overline{a}$, has no inverse element with respect to the Minkowski addition (if $\underline{a}=\overline{a}$, then A is a singleton). Some simplifications are not valid in the space of the interval numbers such as $(A+B)-B\neq A$, where $A-B=A+(-1)B$ is the Minkowski difference and $A,B\in\mathbb{I}(\mathbb{R})$.

To partially overcome this situation, using the results of Hukuhara introduced in [19], we define the Hukuhara difference (H-difference, for short) as the interval C for which $A\ominus B=C\Longleftrightarrow A=B+C$. An important property of $\ominus$ is that $A\ominus A=\{0\}$, $\forall A\in\mathbb{I}(\mathbb{R})$ and $(A+B)\ominus B=A$, $\forall A,B\in\mathbb{I}(\mathbb{R})$.
The H-difference is unique, but it does not always exist (a necessary condition for $A\ominus B$ to exist is that A contains a translation $\{c\}+B$ of B) [30].

Recently, a generalization of the Hukuhara difference was proposed in [8] that always exists for any two intervals.

Definition 2.1. [8] *Given two intervals $A,B\in\mathbb{I}(\mathbb{R})$, the generalized Hukuhara difference (gH-difference, for short) is the interval $C\in\mathbb{I}(\mathbb{R})$, such that*

$$A\ominus_{gH}B=C\Longleftrightarrow\left\{\begin{array}{ll}\quad(1) & A=B+C,\\ or\ (2) & B=A+(-1)C.\end{array}\right.$$

In the following propositions, some properties of the space $\mathbb{I}(\mathbb{R})$ are given.

Proposition 2.2. [26] *Let A and B be arbitrary intervals and α and β be real numbers. Then*

(a) $\alpha(A+B)=\alpha A+\alpha B$,

(b) $(\alpha+\beta)A=\alpha A+\beta A$, *if* $\alpha\beta\geq 0$,

(c) $\alpha(\beta A)=(\alpha\beta)A$.

Proposition 2.3. [30] *Let $A,B\in\mathbb{I}(\mathbb{R})$. Then the following properties hold.*

(a) $A\ominus_{gH}A=\{0\}$,

(b) $(A+B)\ominus_{gH}B=A$,

(c) $\alpha\left(A\ominus_{gH}B\right)=\alpha A\ominus_{gH}\alpha B$, $\alpha\in\mathbb{R}$.

The following proposition can be directly obtained by the definition of gH-difference.

Proposition 2.4. *Let $A\in\mathbb{I}(\mathbb{R})$ and $\alpha,\beta\in\mathbb{R}$. If $\alpha>\beta>0$, or $\alpha<\beta<0$, then*

$$\alpha A\ominus_{gH}\beta A=(\alpha-\beta)A.$$

Proposition 2.5. *Let $A, B, C, D \in \mathbb{I}(\mathbb{R})$. Then the following properties hold,*

(a) If $(A \ominus B) \ominus C$ exists, then
$$(A \ominus B) \ominus C = (A \ominus C) \ominus B.$$

(b) If $A \ominus (B \ominus C)$ and $(B \ominus A)$ exist, then
$$A \ominus (B \ominus C) = C \ominus (B \ominus A).$$

(c) If $(A \ominus B)$ and $(C \ominus D)$ exist, then
$$(A \ominus B) + (C \ominus D) = (A + C) \ominus (B + D).$$

(d) If $A \ominus (B + C)$ exists, then
$$A \ominus (B + C) = (A \ominus B) \ominus C.$$

Proof. Let us prove Case (a), the proofs of the other cases are analogous.
Suppose that $(A \ominus B) \ominus C$ exists. We denote $(A \ominus B) \ominus C = W$. Then, the definition of the H-difference implies $A = B + C + W$. It follows that $A \ominus C = B + W$, which yields $(A \ominus C) \ominus B = W$. $\square$

Definition 2.6. [26] *Let $f : T \longrightarrow \mathbb{I}(\mathbb{R})$ be an interval-valued function such that $f(t) = \left[\underline{f}(t), \overline{f}(t)\right]$ and $T \subseteq \mathbb{R}$. Then f is said to be w-increasing (w-decreasing) if the real-valued function $len(f(t)) = \overline{f}(t) - \underline{f}(t)$ is increasing (decreasing).*

3 Interval discrete fractional calculus

In this section, we provide some definitions related to discrete fractional calculus for interval-valued functions. Then, taking into account some restrictions of interval arithmetic, the concepts of fractional sum and difference are extended to the interval case. Analogous to [12], we consider interval-valued functions defined on a shift of the natural numbers as follows
$$f : \mathbb{N}_a \longrightarrow \mathbb{I}(\mathbb{R}),$$
where
$$\mathbb{N}_a := \{a\} + \mathbb{N}_0 = \{a, a+1, a+2, ...\}, \ a \in \mathbb{R}.$$

Definition 3.1. [18] *Let $f : \mathbb{N}_a \longrightarrow \mathbb{I}(\mathbb{R})$ be an interval-valued function. The first order difference of f is defined by*
$$(\Delta f)(t) := f(t+1) \ominus_{gH} f(t), \qquad \forall t \in \mathbb{N}_a.$$

So, we have two possibilities
$$(\Delta_1 f)(t) = f(t+1) \ominus f(t) \ \text{or} \ (\Delta_2 f)(t) = (-1)\left(f(t) \ominus f(t+1)\right).$$

Moreover, for $N \in \mathbb{N}$,
$$\left(\Delta^N f\right)(t) := \left(\Delta \Delta^{N-1} f\right)(t), \qquad \forall t \in \mathbb{N}_a$$
is called the whole order difference (N^{th}-order) of the interval-valued function f. We know that the gH-difference of two arbitrary intervals always exists [30]*. Hence, the N^{th}-order difference of an interval-valued function always exists.*

Definition 3.2. [12] *The function $\Gamma : (0, \infty) \longrightarrow \mathbb{R}$, defined by*
$$\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu - 1} dt, \quad \forall \nu \in \mathbb{C} \backslash (-\mathbb{N}_0),$$
is called Euler's Gamma function.

Definition 3.3. [12] *Let $t, \nu \in \mathbb{R}$ be given. The factorial function or generalized falling function is given by*
$$t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}, \tag{1}$$
such the right-hand side of (1) *makes sense.*

Based on the definitions of fractional sum and difference for real-valued functions [5], the concepts of fractional sum and difference for interval-valued functions are given.

Definition 3.4. [18] *Let $f : \mathbb{N}_a \longrightarrow \mathbb{I}(\mathbb{R})$ and $\nu > 0$ be given. The ν^{th}-order fractional sum of f is given by*

$$\left(\Delta_a^{-\nu} f\right)(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-\sigma(s))^{(\nu-1)} f(s), \quad \forall t \in \mathbb{N}_{a+\nu},$$

where $\sigma(s) = s+1$ is the forward jump operator. Also, the trivial sum is defined by $\left(\Delta_a^{-0} f\right)(t) := f(t)$, for $t \in \mathbb{N}_a$. In particular, for case $\nu = 1$, we get the summation operator

$$\left(\Delta_a^{-1} f\right)(t) = \sum_{s=a}^{t-1} f(s).$$

Definition 3.5. [18] *Let $f : \mathbb{N}_a \longrightarrow \mathbb{I}(\mathbb{R})$ and $\nu > 0$ be given, and let $N \in \mathbb{N}$ be chosen such that $N-1 < \nu \le N$. The Riemann-Liouville fractional difference of f is defined by*

$$\left(\Delta_a^{\nu} f\right)(t) := \left(\Delta^N \Delta_a^{-(N-\nu)} f\right)(t), \quad \forall t \in \mathbb{N}_{a+N-\nu}. \tag{2}$$

Remark 3.6. [16] *According to Definition 3.4, $\left(\Delta_a^{-\nu} f\right)(t)$ at each $t \in \mathbb{N}_{a+\nu}$ is a linear combination of the values $\{f(a), f(a+1), ..., f(t-\nu)\}$, hence, the fractional sum of f at a specific time depends on the values of f at all previous times. According to the definition of fractional difference, the fractional difference is defined by using the fractional sum, therefore, the memory property comes from the fact that the fractional difference at time t also depends on all previous values.*

In the following remark, the domains of the fractional sum and difference operators are discussed.

Remark 3.7. [16] *The definite sum $\Delta_a^{-1} f(t)$ represents the area under the curve f from a to t, where the height on the interval $[t, t+1]$ is given by the value $f(t)$. Now, we find that*

$$\Delta_a^{-1} f(a) = \frac{1}{\Gamma(1)} \sum_{s=a}^{t-1} (t-\sigma(s))^{(1-1)} f(s)\Big|_{t=a} = \sum_{s=a}^{a-1} f(s).$$

Therefore, by the convention $\sum_{s=a}^{a-1} f(s) = 0$, we consider $\Delta_a^{-1} f(a) = 0$ as the legitimate value. Then, by this convention the values of $\Delta_a^N f$ at the N points $t = a, a+1, \cdots, a+N-1$ must be zero. Therefore, the first nontrivial value of $\Delta_a^N f$ occurs at the point $t = a+N$.

For the fractional-order case, let $\nu > 0$ such that $N-1 < \nu \le N$, then $\Delta_a^{-\nu} f$ satisfies

$$\Delta_a^{-\nu} f(a+\nu-N) = \Delta_a^{-\nu} f(a+\nu-N+1) = \cdots = \Delta_a^{-\nu} f(a+\nu-1) = 0.$$

Moreover, the first nontrivial value of $\Delta_a^{-\nu} f$ occurs at the point $t = a+\nu$.
The above discussion leads us to define the domain of $\Delta_a^{-\nu} f$ to be $\mathcal{D}\{\Delta_a^{-\nu} f\} := \mathbb{N}_{a+\nu}$.

The whole-order differences are domain preserving operators, i.e. $\mathcal{D}\left\{\Delta^N f\right\} = \mathcal{D}\{f\}$, for all $N \in \mathbb{N}_0$. Therefore, we can utilize the fractional sum domains to easily determine the domain of the ν^{th}-order fractional difference as

$$\mathcal{D}\{\Delta_a^{\nu} f\} = \mathcal{D}\left\{\Delta^N \Delta_a^{-(N-\nu)} f\right\} = \mathcal{D}\left\{\Delta_a^{-(N-\nu)} f\right\} = \mathbb{N}_{a+N-\nu}.$$

In the following lemma, we show that the fractional difference of a constant functions is not equal to zero, in general.

Lemma 3.8. *Let $f : \mathbb{N}_a \longrightarrow \mathbb{I}(\mathbb{R})$ be such that $f(t) = c \in \mathbb{I}(\mathbb{R})$, $\forall t \in \mathbb{N}_a$ and $0 < \nu < 1$. Then,*

$$\left(\Delta_a^{\nu} c\right)(t) = \frac{c}{\Gamma(1-\nu)} (t-a)^{-\nu}, \qquad \forall t \in \mathbb{N}_{a+1-\nu}.$$

Proof. The proof is immediate by Definition 3.5 and Proposition 2.4. □

Lemma 3.9. *The fractional sum is a linear operator.*

Proof. The proof is immediate due to the fact that $(t-\sigma(s))^{(\nu-1)} > 0$ for $t \in \mathbb{N}_{a+\nu}$, and $s = a, a+1, \ldots, t-\nu$. □

In general, the fractional difference is not a linear operator. Analogous to the results obtained in [21], it possess the linear property in some particular cases.

Lemma 3.10. *Let f and g be interval-valued functions on $\mathbb{N}_a$ and $\alpha, \beta \in \mathbb{R}$. If either f and g are w-increasing or $\Delta_a^{-(1-\nu)} f$ and $\Delta_a^{-(1-\nu)} g$ are w-decreasing, then we have*

$$\left(\Delta_a^\nu(\alpha f + \beta g)\right)(t) = \alpha \left(\Delta_a^\nu f\right)(t) + \beta \left(\Delta_a^\nu g\right)(t).$$

Proof. The proof is immediate by Proposition 2.5 (c). □

Lemma 3.11. [4] *(Power rule) Let $a \in \mathbb{R}$, $\nu > 0$ and $\mu \in \mathbb{R} \backslash (-\mathbb{N})$ be given. Then,*

$$\Delta_{a+\mu}^{-\nu}(t-a)^{\mu} = \mu^{-\nu}(t-a)^{\mu+\nu}, \qquad \forall t \in \mathbb{N}_{a+\mu+\nu}, \tag{3}$$

and

$$\Delta_{a+\mu}^{\nu}(t-a)^{\mu} = \mu^{\nu}(t-a)^{\mu-\nu}, \qquad \forall t \in \mathbb{N}_{a+\mu+N-\nu}. \tag{4}$$

The following two lemmas can be obtained by Definition 3.4.

Lemma 3.12. *Let $f : \mathbb{N}_a \longrightarrow \mathbb{I}(\mathbb{R})$ be a w-increasing function and $\nu \geq 0$. Then, the function*

$$\Delta_a^{-\nu} f : \mathbb{N}_{a+\nu} \longrightarrow \mathbb{I}(\mathbb{R}),$$

is w-increasing.

Lemma 3.13. *Let $f : \mathbb{N}_a \longrightarrow \mathbb{I}(\mathbb{R})$ be a w-decreasing function. Then,*

(a) *If $\nu \geq 1$, then the function $\Delta_a^{-\nu} f : \mathbb{N}_{a+\nu} \longrightarrow \mathbb{I}(\mathbb{R})$ is w-increasing.*

(b) *If $0 < \nu < 1$ and f satisfies the condition*

$$(w_1): \ (t-a)^{(\nu-1)} len(f(a)) \geq \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} \left(len(f(s)) - len(f(s+1))\right),$$

for all $t \in \mathbb{N}_{a+\nu}$, then the function $\Delta_a^{-\nu} f : \mathbb{N}_{a+\nu} \longrightarrow \mathbb{I}(\mathbb{R})$ is w-increasing.

(c) *If $0 < \nu < 1$ and f satisfies the condition*

$$(w_2): (t-a)^{(\nu-1)} len(f(a)) < \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} \left(len(f(s)) - len(f(s+1))\right),$$

for all $t \in \mathbb{N}_{a+\nu}$, then the function $\Delta_a^{-\nu} f : \mathbb{N}_{a+\nu} \longrightarrow \mathbb{I}(\mathbb{R})$ is w-decreasing.

Remark 3.14. *Note that the function $\Delta_a^{-\nu}$ is always w-increasing for $\nu \geq 1$ without consideration of w-monotonic property of f. For $0 < \nu < 1$, we have different cases that are indicated in Lemma 3.13.*

In the following, some specific cases of fractional sum and difference composition rules are presented. These composition rules will be used in the main results.

Lemma 3.15. *Let $f : \mathbb{N}_a \longrightarrow \mathbb{I}(\mathbb{R})$ be given and suppose $0 < \nu, \mu \leq 1$. Then*

$$\left(\Delta_{a+\mu}^{-\nu} \Delta_a^{-\mu} f\right)(t) = \left(\Delta_a^{-\nu-\mu} f\right)(t) = \left(\Delta_{a+\nu}^{-\mu} \Delta_a^{-\nu} f\right)(t), \quad \forall t \in \mathbb{N}_{a+\mu+\nu}. \tag{5}$$

Proof. The proof is analogous to Theorem 2.46 in [12]. □

Lemma 3.16. [18] *Let $f : \mathbb{N}_a \longrightarrow \mathbb{I}(\mathbb{R})$ be given. Then*

$$\left(\Delta_1 \Delta_a^{-1} f\right)(t) = f(t), \qquad \forall t \in \mathbb{N}_{a+1}. \tag{6}$$

Lemma 3.17. [18] *(Leibniz sum law) If $f : \mathbb{N}_a \longrightarrow \mathbb{I}(\mathbb{R})$ is w-monotone, then*

$$\left(\Delta_a^{-1} \Delta f\right)(t) = f(t) \ominus_{gH} f(a), \qquad \forall t \in \mathbb{N}_{a+1}. \tag{7}$$

Remark 3.18. *In Definition 3.1, since we do not have an explicit formula for the N^{th}-order difference ($N \geq 2$) of interval-valued functions, therefore, in the following, we only study the explicit formulas for the composition rules involved difference operators of order $0 < \nu \leq 1$.*

Theorem 3.19. *Let $f : \mathbb{N}_a \longrightarrow \mathbb{I}(\mathbb{R})$ and $0 < \nu \leq 1$. If f is w-increasing, then*

$$\left(\Delta_a^{-\nu}\Delta_1 f\right)(t) = \left(\Delta_1\Delta_a^{-\nu} f\right)(t) \ominus \frac{1}{\Gamma(\nu)}(t-a)^{(\nu-1)} f(a), \qquad \forall t \in \mathbb{N}_{a+\nu}. \tag{8}$$

If f is w-decreasing and satisfies (w_1), then

$$\left(\Delta_a^{-\nu}\Delta_2 f\right)(t) = \frac{(-1)}{\Gamma(\nu)}(t-a)^{(\nu-1)} f(a) \ominus (-1)\left(\Delta_1\Delta_a^{-\nu} f\right)(t), \ \forall t \in \mathbb{N}_{a+\nu}. \tag{9}$$

Proof. By Definition 3.4, we have

$$\left(\Delta_a^{-\nu}\Delta f\right)(t) = \frac{1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-\sigma(s))^{(\nu-1)}\Delta f(s) = \frac{1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-\sigma(s))^{(\nu-1)}\left(f(t+1)\ominus_{gH} f(t)\right).$$

According to the definition of the generalized Hukuhara difference, we have the following two cases.
Case I. Let f be w-increasing. Then, we have $f(t+1)\ominus_{gH} f(t) = f(t+1)\ominus f(t)$ and by Proposition 2.5 (a) and (c), we obtain

$$\begin{aligned}
&\left(\Delta_a^{-\nu}\Delta_1 f\right)(t) = \frac{1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-\sigma(s))^{(\nu-1)}\left(f(s+1)\ominus f(s)\right)\\
&= \left(\frac{1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-\sigma(s))^{(\nu-1)} f(s+1)\right)\ominus\left(\frac{1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-\sigma(s))^{(\nu-1)} f(s)\right)\\
&= \left(\frac{1}{\Gamma(\nu)}\sum_{s=a}^{t+1-\nu}(t-s)^{(\nu-1)} f(s)\ominus\frac{1}{\Gamma(\nu)}(t-a)^{(\nu-1)} f(a)\right)\ominus\\
&\quad\left(\frac{1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-\sigma(s))^{(\nu-1)} f(s)\right)\\
&= \left(\frac{1}{\Gamma(\nu)}\sum_{s=a}^{t+1-\nu}(t-s)^{(\nu-1)} f(s)\ominus\frac{1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-\sigma(s))^{(\nu-1)} f(s)\right)\ominus\\
&\quad\frac{1}{\Gamma(\nu)}(t-a)^{(\nu-1)} f(a)\\
&= \left(\left(\Delta_a^{-\nu} f\right)(t+1)\ominus\left(\Delta_a^{-\nu} f\right)(t)\right)\ominus\frac{1}{\Gamma(\nu)}(t-a)^{(\nu-1)} f(a)\\
&= \left(\Delta_1\Delta_a^{-\nu} f\right)(t)\ominus\frac{1}{\Gamma(\nu)}(t-a)^{(\nu-1)} f(a).
\end{aligned}$$

Therefore,

$$\left(\Delta_1\Delta_a^{-\nu} f\right)(t)\ominus\left(\Delta_a^{-\nu}\Delta_1 f\right)(t) = \frac{1}{\Gamma(\nu)}(t-a)^{(\nu-1)} f(a), \quad \forall t \in \mathbb{N}_{a+\nu}. \tag{10}$$

Case II. Let f be w-decreasing. Then, we have $f(t+1)\ominus_{gH} f(t) = (-1)\left(f(t)\ominus f(t+1)\right)$ and using Proposition 2.5 (c), we obtain

$$\begin{aligned}
&\left(\Delta_a^{-\nu}\Delta_2 f\right)(t) = \frac{1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-\sigma(s))^{(\nu-1)}(-1)\left(f(s)\ominus f(s+1)\right)\\
&= (-1)\left[\left(\frac{1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-\sigma(s))^{(\nu-1)} f(s)\right)\ominus\left(\frac{1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-\sigma(s))^{(\nu-1)} f(s+1)\right)\right]\\
&= (-1)\left(\frac{1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-\sigma(s))^{(\nu-1)} f(s)\right)\ominus\\
&\quad(-1)\left(\frac{1}{\Gamma(\nu)}\sum_{s=a}^{t+1-\nu}(t-s)^{(\nu-1)} f(s)\ominus\frac{1}{\Gamma(\nu)}(t-a)^{(\nu-1)} f(a)\right).
\end{aligned}$$

According to the assumption that f is w-decreasing and satisfies (w_1), by Lemma 3.13, we conclude that $\Delta_a^{-\nu} f$ is w-increasing. So, the H-difference $(\Delta_a^{-\nu} f)(t+1) \ominus (\Delta_a^{-\nu} f)(t)$ exists. Therefore, using Proposition 2.5 (b), we obtain

$$\begin{aligned}
& \left(\Delta_a^{-\nu} \Delta_2 f\right)(t) \\
= \; & (-1) \frac{1}{\Gamma(\nu)} (t-a)^{(\nu-1)} f(a) \ominus \\
& (-1) \left(\frac{1}{\Gamma(\nu)} \sum_{s=a}^{t+1-\nu} (t-s)^{(\nu-1)} f(s) \ominus \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-\sigma(s))^{(\nu-1)} f(s) \right) \\
= \; & (-1) \frac{1}{\Gamma(\nu)} (t-a)^{(\nu-1)} f(a) \ominus (-1) \Big(\left(\Delta_a^{-\nu} f\right)(t+1) \ominus \left(\Delta_a^{-\nu} f\right)(t) \Big) \\
= \; & (-1) \frac{1}{\Gamma(\nu)} (t-a)^{(\nu-1)} f(a) \ominus (-1) \left(\Delta_1 \Delta_a^{-\nu} f\right)(t).
\end{aligned}$$

This completes the proof. □

Remark 3.20. *We demonstrate that* (8) *and* (9) *can be unified via the definition of gH-difference as*

$$\left(\Delta_a^{-\nu} \Delta f\right)(t) = \left(\Delta_1 \Delta_a^{-\nu} f\right)(t) \ominus_{gH} \frac{1}{\Gamma(\nu)} (t-a)^{(\nu-1)} f(a), \qquad \forall t \in \mathbb{N}_{a+\nu}. \tag{11}$$

provided that f is either w-increasing or w-decreasing and satisfies (w_1).

In Theorem 3.19, the composition of fractional sums and first order differences is considered. In the following, we obtain the composition rule for fractional sums and fractional differences.

Corollary 3.21. *Let $f : \mathbb{N}_a \longrightarrow \mathbb{I}(\mathbb{R})$ and $0 < \mu \leq 1, \nu > 0$.*

(a) *If f is w-increasing, then*

$$\left(\Delta_{a+1-\mu}^{-\nu} \Delta_a^{\mu} f\right)(t) = \left(\Delta_1 \Delta_a^{-(\nu+1-\mu)} f\right)(t) \ominus \frac{1}{\Gamma(\nu)} (t-a)^{(\nu-1)} f(a), \tag{12}$$

for all $t \in \mathbb{N}_{a+1-\mu+\nu}$.

(b) *Suppose that $\nu > 0$ and $0 < \mu \leq 1$ be such that $\nu \geq \mu$. If f is w-decreasing and satisfies (w_2) for order $1-\mu$, then*

$$\left(\Delta_{a+1-\mu}^{-\nu} \Delta_a^{\mu} f\right)(t) = \frac{(-1)}{\Gamma(\nu)} (t-a)^{(\nu-1)} f(a) \ominus (-1) \left(\Delta_1 \Delta_a^{-(\nu+1-\mu)} f\right)(t), \tag{13}$$

for all $t \in \mathbb{N}_{a+1-\mu+\nu}$.

Analogous to Remark 3.20 the unified formula for (12) *and* (13), *is given by*

$$\left(\Delta_{a+1-\mu}^{-\nu} \Delta_a^{\mu} f\right)(t) = \left(\Delta_1 \Delta_a^{-(1-\mu+\nu)} f\right)(t) \ominus_{gH} \frac{1}{\Gamma(\nu)} (t-a)^{(\nu-1)} f(a), \tag{14}$$

Proof. The proof is quite straightforward by Theorem 3.19 and the composition rule of two fractional sum operators. □

The following lemma will be used in the proof of the main result of this paper.

Lemma 3.22. *Let m be an arbitrary positive integer and $0 < \nu < 1$, then*

$$\sum_{i=1}^{m} \frac{\Gamma(m-i-\nu+1)\Gamma(\nu+i-1)}{\Gamma(m-i+1)\Gamma(i)} = \Gamma(1-\nu)\Gamma(\nu). \tag{15}$$

Proof. Since $m-i-\nu > -1$ and $\nu+i-2 > -1$, we can use Laplace transform [12] to obtain

$$\Gamma(m-i-\nu+1) = s^{m-i-\nu+1} \mathcal{L}\left\{t^{m-i-\nu}\right\}(s), \quad \Gamma(\nu+i-1) = s^{\nu+i-1} \mathcal{L}\left\{t^{\nu+i-2}\right\}(s).$$

In the following, we use the definition of convolution product of $t^{m-i-\nu}$ and $t^{\nu+i-2}$ as

$$\mathcal{L}\left\{t^{m-i-\nu}\right\}(s) \mathcal{L}\left\{t^{\nu+i-2}\right\}(s) = \mathcal{L}\left\{\int_0^t \tau^{\nu+i-2}(t-\tau)^{m-i-\nu} d\tau\right\}(s).$$

The above convolution product and the linearity of Laplace transform imply that

$$\begin{aligned}
&\sum_{i=1}^{m}\frac{\Gamma(m-i-\nu+1)\Gamma(\nu+i-1)}{\Gamma(m-i+1)\Gamma(i)}\\
&=s^m\sum_{i=1}^{m}\frac{\mathcal{L}\left\{t^{m-i-\nu}\right\}(s)\mathcal{L}\left\{t^{\nu+i-2}\right\}(s)}{\Gamma(m-i+1)\Gamma(i)}\\
&=s^m\sum_{i=1}^{m}\frac{\mathcal{L}\left\{\int_0^t\tau^{\nu+i-2}(t-\tau)^{m-i-\nu}d\tau\right\}(s)}{\Gamma(m-i+1)\Gamma(i)}\\
&=s^m\sum_{i=1}^{m}\frac{\mathcal{L}\left\{\int_0^t\tau^{\nu+i-2}(t-\tau)^{-\nu}\sum_{j=0}^{m-i}\binom{m-i}{j}t^j(-\tau)^{m-i-j}d\tau\right\}(s)}{\Gamma(m-i+1)\Gamma(i)}\\
&=s^m\sum_{i=1}^{m}\frac{1}{(m-i)!(i-1)!}\sum_{j=0}^{m-i}\binom{m-i}{j}(-1)^{m-i-j}\mathcal{L}\left\{\int_0^t\frac{t^j\tau^{m+\nu-2-j}}{(t-\tau)^{\nu}}d\tau\right\}(s)\\
&=s^m\sum_{i=1}^{m}\frac{1}{(m-i)!(i-1)!}\sum_{j=0}^{m-i}\binom{m-i}{j}(-1)^{m-i-j}\frac{s^{-m}\Gamma(m)\Gamma(1-\nu)\Gamma(m+\nu-j-1)}{\Gamma(m-j)}\\
&=\Gamma(m)\Gamma(1-\nu)\sum_{i=1}^{m}\frac{1}{(i-1)!}\sum_{j=0}^{m-i}\frac{(-1)^{m-i-j}\Gamma(m+\nu-j-1)}{j!(m-i-j)!(m-j-1)!}\\
&=\Gamma(m)\Gamma(1-\nu)\sum_{i=1}^{m}\frac{\Gamma(m-i+\nu)}{(i-1)!(m-i)!}\sum_{j=0}^{m-i}\frac{(-1)^{m-i-j}}{j!(m-i-j)!}. \qquad (16)
\end{aligned}$$

It is easy to see that for $i=m$, we have

$$\sum_{j=0}^{m-i}\frac{(-1)^{m-i-j}}{j!(m-i-j)!}=1. \qquad (17)$$

We show that

$$\sum_{j=0}^{m-i}\frac{(-1)^{m-i-j}}{j!(m-i-j)!}=0,\qquad i=1,...,m-1. \qquad (18)$$

First, we show that (18) holds for $i=1$, namely

$$\sum_{j=0}^{m-1}\frac{(-1)^{m-1-j}}{j!(m-1-j)!}=0,\qquad m\geq 2.$$

We see that $m=1$ yields (17). The other cases, $i=2,...,m-1$, can be proven analogously.
In order to prove (18), firstly, we assume that m is an even number. So, (18) is the sum of the pairwise opposite terms due to that fact that $\binom{m-1}{j}=\binom{m-1}{m-1-j}$, therefore

$$\sum_{j=0}^{m-1}\frac{(-1)^{m-1-j}}{j!(m-1-j)!}=\frac{(-1)^{m-1}}{(m-1)!}+\frac{(-1)^{m-2}}{(m-2)!}+...+\frac{(-1)^{1}}{(m-2)!}+\frac{(-1)^{0}}{(m-1)!}=0.$$

Secondly, we assume that m is an odd number. Let $m=2n+1$, where $n\geq 1$, then

$$\sum_{j=0}^{m-1}\frac{(-1)^{m-1-j}}{j!(m-1-j)!}=\sum_{j=0}^{2n}\frac{(-1)^{j}}{j!(2n-j)!}=(2n)!\sum_{j=0}^{2n}(-1)^j\binom{2n}{j}=(2n)!(-1+1)^{2n}=0.$$

Therefore, (16), (17) and (18) imply that

$$\sum_{i=1}^{m}\frac{\Gamma(m-i-\nu+1)\Gamma(\nu+i-1)}{\Gamma(m-i+1)\Gamma(i)}=\Gamma(m)\Gamma(1-\nu)\sum_{i=1}^{m}\frac{\Gamma(m-i+\nu)}{(i-1)!(m-i)!}\sum_{j=0}^{m-i}\frac{(-1)^{m-i-j}}{j!(m-i-j)!}=\Gamma(1-\nu)\Gamma(\nu).$$

This completes the proof. □

4 Interval fractional difference equations

Consider the following interval fractional difference equation with an initial condition

$$\begin{cases} \Delta^{\nu}_{a+\nu-1}y(t) = f(t+\nu-1, y(t+\nu-1)), \quad t \in \mathbb{N}_a, \\ \Delta^{-(1-\nu)}_{a+\nu-1}y(t)\Big|_{t=a} = a_0, \end{cases} \tag{19}$$

where $\nu \in (0,1]$, $f : \mathbb{N}_{a+\nu-1} \times \mathbb{I}(\mathbb{R}) \longrightarrow \mathbb{I}(\mathbb{R})$, and $a_0 \in \mathbb{I}(\mathbb{R})$.
Note that, using the definition of fractional sum, we get

$$\begin{aligned} \Delta^{-(1-\nu)}_{a+\nu-1}y(t)\Big|_{t=a} &= \frac{1}{\Gamma(1-\nu)} \sum_{s=a+\nu-1}^{t-(1-\nu)} (t-s-1)^{(-\nu)}y(s)\Big|_{t=a} \\ &= \frac{1}{\Gamma(1-\nu)} \sum_{s=a+\nu-1}^{a-1+\nu} (a-s-1)^{(-\nu)}y(s) \\ &= y(a+\nu-1) \text{ (By Remark 3.11).} \end{aligned}$$

Definition 4.1. *We say that an interval-valued function $y : \mathbb{N}_{a+\nu-1} \times \mathbb{I}(\mathbb{R}) \longrightarrow \mathbb{I}(\mathbb{R})$ is a solution to the initial value problem* (19) *if it satisfies the equation $\left(\Delta^{\nu}_{a+\nu-1}y\right)(t) = f(t+\nu-1, y(t+\nu-1))$ for $t \in \mathbb{N}_a$ and the initial condition $y(a+\nu-1) = a_0$.*

Theorem 4.2. *Let $f : \mathbb{N}_{a+\nu-1} \times \mathbb{I}(\mathbb{R}) \longrightarrow \mathbb{I}(\mathbb{R})$ be given and $0 < \nu \leq 1$. Then a w-increasing solution to* (19) *is given by*

$$y(t) = \left(\Delta^{-\nu}_{a+\nu-1}f(s,y(s))\right)(t+\nu-1) + \frac{1}{\Gamma(\nu)}(t-a)^{(\nu-1)}a_0, \ t \in \mathbb{N}_{a+\nu-1},$$

and a w-decreasing solution to (19) *is given by*

$$y(t) = \frac{1}{\Gamma(\nu)}(t-a)^{(\nu-1)}a_0 \ominus (-1)\left(\Delta^{-\nu}_{a+\nu-1}f(s,y(s))\right)(t+\nu-1), \ t \in \mathbb{N}_{a+\nu-1},$$

provided that it satisfies the condition (w_2) *and the corresponding H-difference exists.*

Proof. We apply the operator $\Delta^{-\nu}_a$ to both sides of (19) to obtain

$$\left(\Delta^{-\nu}_a\Delta^{\nu}_{a+\nu-1}y\right)(t) = \left(\Delta^{-\nu}_a f(s+\nu-1, y(s+\nu-1))\right)(t), \ t \in \mathbb{N}_{a+\nu}.$$

The change of variable $r = s+\nu-1$ leads to

$$\left(\Delta^{-\nu}_a\Delta^{\nu}_{a+\nu-1}y\right)(t) = \left(\Delta^{-\nu}_{a+\nu-1}f(r,y(r))\right)(t+\nu-1), \ t \in \mathbb{N}_{a+\nu}. \tag{20}$$

If we apply Theorem 3.19 to the left-hand side of (20), we have two cases.

Case I. Let y be a w-increasing solution of (19). Then, by Lemma 3.12, $\Delta^{-(1-\nu)}_{a+\nu-1}y$ is w-increasing, and we have

$$\begin{aligned} \left(\Delta^{-\nu}_a\Delta^{\nu}_{a+\nu-1}y\right)(t) &= \left(\Delta^{-\nu}_a\left(\Delta_1\Delta^{-(1-\nu)}_{a+\nu-1}y\right)\right)(t) \\ &= \left(\Delta^{-\nu}_a\Delta_1\left(\Delta^{-(1-\nu)}_{a+\nu-1}y\right)\right)(t) \\ &= \left(\Delta_1\Delta^{-\nu}_a\left(\Delta^{-(1-\nu)}_{a+\nu-1}y\right)\right)(t) \ominus \frac{1}{\Gamma(\nu)}(t-a)^{(\nu-1)}\left(\Delta^{-(1-\nu)}_{a+\nu-1}y\right)(a) \\ &= \left(\Delta_1\Delta^{-1}_a y\right)(t) \ominus \frac{1}{\Gamma(\nu)}(t-a)^{(\nu-1)}\left(\Delta^{-(1-\nu)}_{a+\nu-1}y\right)(a) \quad \text{(By Lemma 3.15)} \\ &= y(t) \ominus \frac{1}{\Gamma(\nu)}(t-a)^{(\nu-1)}a_0. \end{aligned}$$

So, we have

$$\left(\Delta^{-\nu}_a\Delta^{\nu}_{a+\nu-1}y\right)(t) = y(t) \ominus \frac{1}{\Gamma(\nu)}(t-a)^{(\nu-1)}a_0, \ t \in \mathbb{N}_{a+\nu}. \tag{21}$$

Equations (20) and (21) imply that the solution of (19) is the solution of the summation equation

$$y(t) = \left(\Delta_{a+\nu-1}^{-\nu} f(s, y(s))\right)(t+\nu-1) + \frac{1}{\Gamma(\nu)}(t-a)^{(\nu-1)}a_0,\ t \in \mathbb{N}_{a+\nu-1}. \tag{22}$$

Reciprocally, let y be the solution of (22). First we compute the values $y(a+\nu-1), y(a+\nu), ..., y(t+\nu-1), y(t+\nu))$ directly from the summation equation, and then we multiply these values by $(t-a-\nu+1)^{(-\nu)}, (t-a-\nu)^{(-\nu)}, ..., (1-\nu)^{(-\nu)}, (-\nu)^{(-\nu)}$, respectively. So, we obtain the following equations

$$(t-a-\nu+1)^{(-\nu)}y(a+\nu-1) = (t-a-\nu+1)^{(-\nu)}y(a+\nu-1),$$

$$(t-a-\nu)^{(-\nu)}y(a+\nu) = (t-a-\nu)^{(-\nu)}\frac{(\nu-1)^{(\nu-1)}}{\Gamma(\nu)}f(a+\nu-1, y(a+\nu-1)) + (t-a-\nu)^{(-\nu)}\frac{(\nu)^{(\nu-1)}}{\Gamma(\nu)}y(a+\nu-1),$$

$$\vdots$$

$$\begin{aligned}(1-\nu)^{(-\nu)}y(t+\nu-1) = (1-\nu)^{(-\nu)}\frac{(t+\nu-1-a)^{(\nu-1)}}{\Gamma(\nu)}&y(a+\nu-1)\\ &+ \frac{(1-\nu)^{(-\nu)}}{\Gamma(\nu)}\sum_{s=a+\nu-1}^{t+\nu-2}(t+2\nu-2-\sigma(s))^{\nu-1}f(s, y(s)),\end{aligned}$$

$$(-\nu)^{(-\nu)}y(t+\nu) = (-\nu)^{(-\nu)}\frac{(t+\nu-a)^{(\nu-1)}}{\Gamma(\nu)}y(a+\nu-1) + \frac{(-\nu)^{(-\nu)}}{\Gamma(\nu)}\sum_{s=a+\nu-1}^{t+\nu-1}(t+2\nu-1-\sigma(s))^{\nu-1}f(s, y(s)).$$

Then, by adding the above equations and simplifying, we obtain

$$\begin{aligned}\Delta_{a+\nu-1}^{-(1-\nu)}y(t+1) = \frac{1}{\Gamma(1-\nu)\Gamma(\nu)}\Bigg[&\sum_{s=a+\nu-2}^{t+\nu-1}(t-\sigma(s))^{-(\nu)}(s-a+1)^{(\nu-1)}y(a+\nu-1)+\\ &\sum_{i=0}^{t-a-1}\sum_{s=a+\nu-1+i}^{t+\nu-1}(t-\sigma(s))^{-(\nu)}(s-a-i)^{(\nu-1)}f(a+\nu-1+i, y(a+\nu-1+i))\Bigg]\\ &+ f(t+\nu-1, y(t+\nu-1)).\end{aligned} \tag{23}$$

Now, apply Lemma 3.22 to conclude that

$$\sum_{i=0}^{t-a-1}\sum_{s=a+\nu-1+i}^{t+\nu-1}(t-\sigma(s))^{-(\nu)}(s-a-i)^{(\nu-1)} = \sum_{i=0}^{t-a-1}\sum_{s=a+\nu+i}^{t+\nu-1}(t-\sigma(s))^{-(\nu)}(s-a-1-i)^{(\nu-1)}. \tag{24}$$

The definition of $\Delta_{a+\nu-1}^{-(1-\nu)}y(t)$ and (24) imply that

$$\begin{aligned}&\Delta_{a+\nu-1}^{-(1-\nu)}y(t+1) \ominus \Delta_{a+\nu-1}^{-(1-\nu)}y(t) = f(t+\nu-1, y(t+\nu-1)),\\ &\Delta\Delta_{a+\nu-1}^{-(1-\nu)}y(t) = f(t+\nu-1, y(t+\nu-1)),\\ &\Delta_{a+\nu-1}^{\nu}y(t) = f(t+\nu-1, y(t+\nu-1)).\end{aligned}$$

This indicates that y is the (i)-solution of the problem (19).

Case II. Let y be a w-decreasing solution of (19), such that it satisfies the condition (w_2) for order $(1-\nu)$. So, according to Lemma 3.13, $\Delta_{a+\nu-1}^{-(1-\nu)}y$ is w-decreasing, and we have

$$\begin{aligned}\left(\Delta_a^{-\nu}\Delta_{a+\nu-1}^{\nu}y\right)(t) &= \left(\Delta_a^{-\nu}\left(\Delta_2\Delta_{a+\nu-1}^{-(1-\nu)}y\right)\right)(t)\\ &= \left(\Delta_a^{-\nu}\Delta_2\left(\Delta_{a+\nu-1}^{-(1-\nu)}y\right)\right)(t)\\ &= (-1)\frac{1}{\Gamma(\nu)}(t-a)^{(\nu-1)}\left(\Delta_{a+\nu-1}^{-(1-\nu)}y\right)(a) \ominus (-1)\left(\Delta_1\Delta_a^{-1}y\right)(t)\\ &= (-1)\frac{1}{\Gamma(\nu)}(t-a)^{(\nu-1)}a_0 \ominus (-1)y(t),\end{aligned}$$

So, we get

$$\left(\Delta_a^{-\nu}\Delta_{a+\nu-1}^{\nu}y\right)(t) = (-1)\frac{1}{\Gamma(\nu)}(t-a)^{(\nu-1)}a_0 \ominus (-1)y(t),\ t \in \mathbb{N}_{a+\nu-1}. \tag{25}$$

Equations (20) and (25) imply that the solution of (19) is the solution of the summation equation

$$y(t) = \frac{1}{\Gamma(\nu)}(t-a)^{(\nu-1)}a_0 \ominus (-1)\left(\Delta_{a+\nu-1}^{-\nu}f(s,y(s))\right)(t+\nu-1),\ t \in \mathbb{N}_{a+\nu-1}. \tag{26}$$

Reciprocally, let y be the solution of (26). The rest of the proof can be handled in a manner analogous to what has been done in the previous case. Therefore, we have

$$\begin{aligned}(-1)\Delta_{a+\nu-1}^{-(1-\nu)}y(t+1) = {} & \frac{(-1)}{\Gamma(1-\nu)\Gamma(\nu)}\sum_{s=a+\nu-2}^{t+\nu-1}(t-\sigma(s))^{-(\nu)}(s-a+1)^{(\nu-1)}y(a+\nu-1)\\ & \ominus\left(\left[\frac{1}{\Gamma(1-\nu)\Gamma(\nu)}\sum_{i=0}^{t-a-1}\sum_{s=a+\nu-1+i}^{t+\nu-1}(t-\sigma(s))^{-(\nu)}(s-a-i)^{(\nu-1)}\right.\right.\\ & \left.\left. f(a+\nu-1+i,y(a+\nu-1+i))\right]+f(t+\nu-1,y(t+\nu-1))\right),\end{aligned}$$

then, the definition of Hukuhara difference and (24) imply

$$\begin{aligned}&(-1)\Delta_{a+\nu-1}^{-(1-\nu)}y(t+1) = (-1)\Delta_{a+\nu-1}^{-(1-\nu)}y(t)\ominus f(t+\nu-1,y(t+\nu-1)),\\ &\Delta_2\Delta_{a+\nu-1}^{-(1-\nu)}y(t) = f(t+\nu-1,y(t+\nu-1)),\\ &\Delta_{a+\nu-1}^{\nu}y(t) = f(t+\nu-1,y(t+\nu-1)).\end{aligned}$$

This indicates that y is the (ii)-solution of the problem (27). □

Remark 4.3. *One of the methods to obtain the solutions of* (22) *and* (26)(*or equally the* (i),(ii)*-solution of the problem* (19)), *is the method of recursive iterations as follows*
(i)*-solution:*

$$\begin{aligned}&y_0(t) = \frac{1}{\Gamma(\nu)}(t-a)^{(\nu-1)}a_0,\\ &y_m(t) = \left(\Delta_{a+\nu-1}^{-\nu}f(s,y_{m-1}(s))\right)(t+\nu-1)+y_0(t).\end{aligned}$$

(ii)*-solution:*

$$\begin{aligned}&y_0(t) = \frac{1}{\Gamma(\nu)}(t-a)^{(\nu-1)}a_0,\\ &y_m(t) = y_0(t)\ominus(-1)\left(\Delta_{a+\nu-1}^{-\nu}f(s,y_{m-1}(s))\right)(t+\nu-1).\end{aligned}$$

We represent the solutions by $y(t) = \sum_{i=0}^{\infty}y_m(t)$. *Moreover, we denote by* (i)*-solution, the* w*-increasing and* (ii)*-solution, the* w*-decreasing solutions of* (27).

Definition 4.4. [17] *Let* $\lambda \in \mathbb{R}$ *and* $\nu, t \in \mathbb{C}$ *with* $Re(\nu) > 0$. *The discrete Mittag-Leffler function is defined by*

$$E_{\nu,\nu}(t,\lambda) = \sum_{i=0}^{m}\lambda^i\frac{(t+i(\nu-1))^{(i\nu+\nu+1)}}{\Gamma((i+1)\nu)}.$$

Definition 4.5. [20] *Let* $\lambda \in \mathbb{R}$, $N \in \mathbb{N}$ *and* $\nu, t \in \mathbb{C}$ *with* $Re(\nu) > 0$. *We define*

$$E^*_{\nu,\nu-N+j+1}(t,\lambda,j) = \sum_{i=0}^{\infty}\lambda^i\frac{(t+i(\nu-k)^{(i\nu+\nu-N+j)})}{\Gamma(i\nu+\nu-N+j+1)},$$

where $j = \{0,1,\cdots,N-1\}$.

In [7], the authors have used the linear fractional difference equations to model tumor growth. The initial size of tumor or the available parameters may have uncertainty due to the lack of information or experimental errors, therefore the following example proposes an appropriate technique which can be used to remodel the tumor growth.

Example 4.6. *Let us consider the linear ν^{th}-order interval fractional difference equation*

$$\begin{cases} \left(\Delta^{\nu}_{a+\nu-1} y\right)(t) = \lambda y(t+\nu-1) + A, \quad t \in \mathbb{N}_a, \\ y(a+\nu-1) = a_0, \end{cases} \tag{27}$$

where $0 < \nu \leq 1$, $A \in \mathbb{I}(\mathbb{R})$, and $\lambda \in \mathbb{R}$ such that $|\lambda| < \nu$.

We consider (27) in two cases $\lambda < 0$ and $\lambda \geq 0$.

Case I. $\lambda < 0$: by Theorem 4.2, the (i)-solution is the solution of the summation equation

$$y(t) = \left(\Delta^{-\nu}_{a+\nu-1}\left(\lambda y(s) + A\right)\right)(t+\nu-1) + \frac{1}{\Gamma(\nu)}(t-a)^{(\nu-1)} a_0, \qquad t \in \mathbb{N}_{a+\nu-1}. \tag{28}$$

We apply the recursive iteration to (28) to obtain

$$\begin{aligned} y_0(t) &= \frac{1}{\Gamma(\nu)}(t-a)^{(\nu-1)} a_0 + \frac{A}{\Gamma(1+\nu)}(t-a)^{(\nu)}, \\ y_m(t) &= \left(\Delta^{-\nu}_{a+\nu-1} \lambda y_{m-1}(s)\right)(t+\nu-1) + y_0(t), \ m = 1, 2, ... \end{aligned}$$

The lower and upper endpoints of $y_m(t)$ is given by

$$\underline{y}_m(t) = \frac{(t-a)^{\nu-1}}{\Gamma(\nu)} \underline{a}_0 + \frac{(t-a)^{(\nu)}}{\Gamma(1+\nu)} \underline{A} + \lambda \left(\Delta^{-\nu}_{a+\nu-1} \overline{y}_{m-1}(s)\right)(t+\nu-1),$$

and

$$\overline{y}_m(t) = \frac{(t-a)^{\nu-1}}{\Gamma(\nu)} \overline{a}_0 + \frac{(t-a)^{(\nu)}}{\Gamma(1+\nu)} \overline{A} + \lambda \left(\Delta^{-\nu}_{a+\nu-1} \underline{y}_{m-1}(s)\right)(t+\nu-1).$$

By the successive applications of the power rule (Lemma 3.11), we obtain a closed form to $\underline{y}_m(t)$ and $\overline{y}_m(t)$ as

$$\begin{aligned} \underline{y}_m(t) = \underline{a}_0 &\sum_{i=0}^{\left[\frac{m}{2}\right]} \frac{\lambda^{2i}}{\Gamma((2i+1)\nu)}(t+2i(\nu-1))^{(2i\nu+\nu-1)} \\ &+ \overline{a}_0 \sum_{i=0}^{\left[\frac{m}{2}\right]+1} \frac{\lambda^{2i+1}}{\Gamma(2i\nu)}(t-a+(2i+1)(\nu-1))^{((2i+2)\nu-1)} \\ &+ \underline{A} \sum_{i=0}^{\left[\frac{m}{2}\right]} \frac{\lambda^{2i}(t+2i(\nu-1)-a)^{(2i\nu+\nu)}}{\Gamma((2i+1)\nu+1)} \\ &+ \overline{A} \sum_{i=0}^{\left[\frac{m}{2}\right]+1} \frac{\lambda^{2i+1}(t+(2i+1)(\nu-1)-a)^{(2i\nu+2\nu)}}{\Gamma((2i+2)\nu+1)}, \end{aligned}$$

and

$$\begin{aligned} \overline{y}_m(t) = \overline{a}_0 &\sum_{i=0}^{\left[\frac{m}{2}\right]} \frac{\lambda^{2i}}{\Gamma((2i+1)\nu)}(t+2i(\nu-1))^{(2i\nu+\nu-1)} \\ &+ \underline{a}_0 \sum_{i=0}^{\left[\frac{m}{2}\right]+1} \frac{\lambda^{2i+1}}{\Gamma(2i\nu)}(t-a+(2i+1)(\nu-1))^{((2i+2)\nu-1)} \\ &+ \overline{A} \sum_{i=0}^{\left[\frac{m}{2}\right]} \frac{\lambda^{2i}(t+2i(\nu-1)-a)^{(2i\nu+\nu)}}{\Gamma((2i+1)\nu+1)} \\ &+ \underline{A} \sum_{i=0}^{\left[\frac{m}{2}\right]+1} \frac{\lambda^{2i+1}(t+(2i+1)(\nu-1)-a)^{(2i\nu+2\nu)}}{\Gamma((2i+2)\nu+1)}, \end{aligned}$$

where [ ] denotes the floor function.
Therefore, passing to the limit $m \longrightarrow \infty$, the (i)-solution of FIVP (27) is given by

$$\begin{aligned}
y_m(t) &= \left(a_0 \sum_{i=0}^{\infty} \frac{\lambda^{2i}(t+2i(\nu-1))^{(2i\nu+\nu-1)}}{\Gamma((2i+1)\nu)} + A \sum_{i=0}^{\infty} \frac{\lambda^{2i}(t+2i(\nu-1)-a)^{(2i\nu+\nu)}}{\Gamma((2i+1)\nu+1)} \right) \\
&\quad \ominus (-1) \left(a_0 \sum_{i=0}^{\infty} \frac{\lambda^{2i+1}}{\Gamma(2i\nu)} (t-a+(2i+1)(\nu-1))^{((2i+2)\nu-1)} + A \sum_{i=0}^{\infty} \frac{\lambda^{2i+1}(t+(2i+1)(\nu-1)-a)^{(2i\nu+2\nu)}}{\Gamma((2i+2)\nu+1)} \right) \\
&= \left(a_0 {}^{e}E_{\nu,\nu}(t-a,\lambda) + A {}^{e}E^{*}_{\nu,\nu+1}(t-a,\lambda,1) \right) \ominus (-1) \left(a_0 {}^{o}E_{\nu,\nu}(t-a,\lambda) + A {}^{o}E^{*}_{\nu,\nu+1}(t-a,\lambda,1) \right),
\end{aligned}$$

provided that the H-differences exist. The functions ${}^{e}E_{\nu,\nu}$ and ${}^{e}E^{*}_{\nu,\nu+1}$ denotes the summation of the even terms and also ${}^{o}E_{\nu,\nu}$ and ${}^{o}E^{*}_{\nu,\nu+1}$ denote the summation of the odd terms of $E_{\nu,\nu}$ and $E^{*}_{\nu,\nu+1}$, respectively. Also, $|\lambda| < \nu$ guarantees the convergence of functions $E_{\nu,\nu}$ and $E^{*}_{\nu,\nu+1}$, and the convergence of the solution as well [17, 20].

Now, by Theorem 4.2, the (ii)-solution is the solution of the summation equation

$$\begin{aligned}
y(t) &= \frac{(t-a)^{(\nu-1)}a_0}{\Gamma(\nu)} \ominus (-1) \left(\Delta^{-\nu}_{a+\nu-1} (\lambda y(s) + A) \right)(t+\nu-1) \\
&= \frac{(t-a)^{(\nu-1)}a_0}{\Gamma(\nu)} \ominus \left((-\lambda) \left(\Delta^{-\nu}_{a+\nu-1} y(s) \right)(t+\nu-1) + (-1) \frac{A(t-a)^{(\nu)}}{\Gamma(1+\nu)} \right),
\end{aligned} \tag{29}$$

provided that the H-difference exists. Since the H-difference in (29) is considered to be existed, then Proposition 2.5 (d) yields that

$$y(t) = \left(\frac{(t-a)^{(\nu-1)}a_0}{\Gamma(\nu)} \ominus (-1) \frac{A(t-a)^{(\nu)}}{\Gamma(1+\nu)} \right) \ominus (-\lambda) \left(\Delta^{-\nu}_{a+\nu-1} y(s) \right)(t+\nu-1). \tag{30}$$

Analogous to the previous case, we apply the recursive iterations to (29) as

$$\begin{aligned}
y_0(t) &= \frac{(t-a)^{(\nu-1)}a_0}{\Gamma(\nu)} \ominus (-1) \frac{A(t-a)^{(\nu)}}{\Gamma(1+\nu)}, \\
y_m(t) &= y_0(t) \ominus (-1) \left(\Delta^{-\nu}_{a+\nu-1} \lambda y_{m-1}(s) \right)(t+\nu-1), \ m = 1, 2, ...
\end{aligned}$$

Therefore, we have

$$\underline{y}_m(t) = \frac{(t-a)^{\nu-1}}{\Gamma(\nu)} \underline{a}_0 + \frac{(t-a)^{(\nu)}}{\Gamma(1+\nu)} \overline{A} + \lambda \left(\Delta^{-\nu}_{a+\nu-1} \underline{y}_{m-1}(s) \right)(t+\nu-1),$$

and

$$\overline{y}_m(t) = \frac{(t-a)^{\nu-1}}{\Gamma(\nu)} \overline{a}_0 + \frac{(t-a)^{(\nu)}}{\Gamma(1+\nu)} \underline{A} + \lambda \left(\Delta^{-\nu}_{a+\nu-1} \overline{y}_{m-1}(s) \right)(t+\nu-1).$$

By the successive applications of the power rule, we get the following close forms

$$\underline{y}_m(t) = \underline{a}_0 \sum_{i=0}^{m} \frac{\lambda^i (t-a+i(\nu-1))^{(i\nu+\nu+1)}}{\Gamma((i+1)\nu)} + \overline{A} \sum_{i=0}^{m} \frac{\lambda^i (t+i(\nu-1)-a)^{(i\nu+\nu)}}{\Gamma((i+1)\nu+1)},$$

and

$$\overline{y}_m(t) = \overline{a}_0 \sum_{i=0}^{m} \frac{\lambda^i (t-a+i(\nu-1))^{(i\nu+\nu+1)}}{\Gamma((i+1)\nu)} + \underline{A} \sum_{i=0}^{m} \frac{\lambda^i (t+i(\nu-1)-a)^{(i\nu+\nu)}}{\Gamma((i+1)\nu+1)}.$$

Therefore, passing to the limit $m \to \infty$, we obtain

$$\begin{aligned}
\underline{y}(t) &= \underline{a}_0 \sum_{i=0}^{\infty} \frac{\lambda^i (t-a+i(\nu-1))^{(i\nu+\nu+1)}}{\Gamma((i+1)\nu)} + \overline{A} \sum_{i=0}^{\infty} \frac{\lambda^i (t+i(\nu-1)-a)^{(i\nu+\nu)}}{\Gamma((i+1)\nu+1)} \\
&= \underline{a}_0 E_{\nu,\nu}(t-a,\lambda) + \overline{A} E^{*}_{\nu,\nu+1}(t-a,\lambda,1),
\end{aligned}$$

and

$$\begin{aligned}
\overline{y}(t) &= \overline{a}_0 \sum_{i=0}^{\infty} \frac{\lambda^i (t-a+i(\nu-1))^{(i\nu+\nu+1)}}{\Gamma((i+1)\nu)} + \underline{A} \sum_{i=0}^{\infty} \frac{\lambda^i (t+i(\nu-1)-a)^{(i\nu+\nu)}}{\Gamma((i+1)\nu+1)} \\
&= \overline{a}_0 E_{\nu,\nu}(t-a,\lambda) + \underline{A} E^{*}_{\nu,\nu+1}(t-a,\lambda,1).
\end{aligned}$$

Finally, we obtain

$$y(t) = a_0 E_{\nu,\nu}(t-a,\lambda) \ominus (-1) A E^*_{\nu,\nu+1}(t-a,\lambda,1), \tag{31}$$

provided that the H-difference exists.

Now, according to Theorem 4.2, $y(t)$ in (29) must satisfy the condition (w_2) for order $1-\nu$, so we have

$$\begin{aligned}
\frac{(t-a-\nu+1)^{(-\nu)}}{\Gamma(1-\nu)} len(a_0) &< \Big(\Delta^{-(1-\nu)}_{a+\nu-1}\left(len(y(s)) - len(y(s+1))\right)\Big)(t) \\
&= \Big(\Delta^{-(1-\nu)}_{a+\nu-1} \frac{(s-a)^{(\nu-1)}}{\Gamma(\nu)} len(a_0)\Big)(t) - \Big(\Delta^{-(1-\nu)}_{a-\nu-1}\left(\Delta^{-\nu}_{a+\nu-1} len(A)\right)(s+\nu-1)\Big)(t) \\
&\quad + \lambda\Big(\Delta^{-(1-\nu)}_{a+\nu-1}\left(\Delta^{-\nu}_{a+\nu-1} len(y(r))\right)(s+\nu-1)\Big)(t) \\
&\quad - \Big(\Delta^{-(1-\nu)}_{a+\nu-1} \frac{(s+1-a)^{(\nu-1)}}{\Gamma(\nu)} len(a_0)\Big)(t) + \Big(\Delta^{-(1-\nu)}_{a+\nu-1}\left(\Delta^{-\nu}_{a+\nu-1} len(A)\right)(s+\nu)\Big)(t) \\
&\quad - \lambda\Big(\Delta^{-(1-\nu)}_{a+\nu-1}\left(\Delta^{-\nu}_{a+\nu-1} len(y(r))\right)(s+\nu)\Big)(t) \\
&= \frac{len(a_0)}{\Gamma(\nu)}\Big(\Gamma(\nu)-\Gamma(\nu)\Big) + len(A)\Big((t+1-a)^{(1)} - (t-a)^{(1)}\Big) \\
&\quad + \lambda\left[\left(\Delta^{-1}_{a+\nu-1} len(y(s))\right)(t+\nu-1) - \left(\Delta^{-1}_{a+\nu-1} len(y(s))\right)(t+\nu)\right] \\
&= 0 + 0 - \lambda len(y(t+\nu-1)).
\end{aligned}$$

So, we get

$$\lambda < -\frac{(t-a-\nu+1)^{(-\nu)} len(a_0)}{\Gamma(1-\nu) len(y(t+\nu-1))}, \qquad \forall t \in \mathbb{N}_{a+\nu-1}. \tag{32}$$

Besides, we can conclude that

$$\lambda < \frac{-1}{\nu\Gamma(\nu)\Gamma(1-\nu)}, \tag{33}$$

this follows from the fact that $(t-a-\nu+1)^{(-\nu)}$ is an increasing function with respect to t. Therefore, if λ satisfies (33), then the (ii)-solution satisfies (w_2).

Case II. $\lambda \geq 0$: Analogous to the Case I, the lower and upper endpoints of $y_m(t)$ is given by

$$\underline{y}_m(t) = \frac{(t-a)^{\nu-1}}{\Gamma(\nu)} \underline{a}_0 + \frac{(t-a)^{(\nu)}}{\Gamma(1+\nu)} \underline{A} + \lambda\left(\Delta^{-\nu}_{a+\nu-1} \overline{y}_{m-1}(s)\right)(t+\nu-1),$$

and

$$\overline{y}_m(t) = \frac{(t-a)^{\nu-1}}{\Gamma(\nu)} \overline{a}_0 + \frac{(t-a)^{(\nu)}}{\Gamma(1+\nu)} \overline{A} + \lambda\Big(\Delta^{-\nu}_{a+\nu-1} \underline{y}_{m-1}(s)\Big)(t+\nu-1).$$

Now, we can proceed in an analogous way to Case I for the (ii)-solution to conclude that

$$\begin{aligned}
y(t) &= \Bigg(a_0 \sum_{i=0}^{\infty} \frac{\lambda^{2i}}{\Gamma((2i+1)\nu)} (t+2i(\nu-1))^{(2i\nu+\nu-1)} + A \sum_{i=0}^{\infty} \frac{\lambda^{2i+1}(t+(2i+1)(\nu-1)-a)^{(2i\nu+2\nu)}}{\Gamma((2i+2)\nu+1)}\Bigg) \\
&\quad \ominus (-1)\Bigg(a_0 \sum_{i=0}^{\infty} \frac{\lambda^{2i+1}}{\Gamma((2i+2)\nu)} (t-a+(2i+1)(\nu-1))^{((2i+2)\nu-1)} + A \sum_{i=0}^{\infty} \frac{\lambda^{2i}(t+2i(\nu-1)-a)^{(2i\nu+\nu)}}{\Gamma((2i+1)\nu+1)}\Bigg) \\
&= \left(a_0\, {}^{e}E_{\nu,\nu}(t-a,\lambda) + A\, {}^{o}E^*_{\nu,\nu+1}(t-a,\lambda,1)\right) \ominus (-1)\left(a_0\, {}^{o}E_{\nu,\nu}(t-a,\lambda) + A\, {}^{e}E^*_{\nu,\nu+1}(t-a,\lambda,1)\right),
\end{aligned}$$

provided that the H-differences exist. According to the convergence of $E_{\nu,\nu}$ and $E^*_{\nu,\nu+1}$ the assumption $0<\lambda<\nu<1$ guarantees the convergence of the solution. Moreover, according to Theorem 4.2, the (ii)-solution $y(t)$ must satisfies (w_2). Similar to Case I, we get

$$\lambda > \frac{1}{\nu\Gamma(\nu)\Gamma(1-\nu)}. \tag{34}$$

Therefore, if λ satisfies (34), then the (ii)-solution satisfies (w_2).

5 Conclusions

In this paper, a study on discrete fractional calculus for interval-valued functions have been made. Indeed, we have presented some composition rules for interval fractional operators, which were used to solve nonlinear interval fractional difference equations and construct the general form of the solutions. We have also introduced higher order fractional difference operators for interval-valued functions. The monotonicity of interval fractional sum operator is investigated. Furthermore, we have considered the initial value problem for nonlinear interval difference equations and investigated the existence of w-increasing and w-decreasing solutions to these problems. Moreover, as an example, we have presented linear interval fractional difference equations with positive and negative constant coefficients and investigated the solutions.

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