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# Fuzzy arithmetic with product t-norm

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**Abstract**

Fuzzy arithmetic performed with the product t-norm is the focus of this paper. The subject is handled from both practical and theoretical perspectives. Explicit formulas for product-sum and product-multiplication of triangular fuzzy numbers are obtained. These formulas can effectively replace the computational methods proposed so far. The issue that these operations are not shape preserving is solved by the presentation of appropriate approximations. Finally, the product arithmetic is compared in detail to the arithmetic performed with the boundary t-norms, namely the minimum and drastic sum.

*Keywords:* Fuzzy arithmetic, product t-norm, fuzzy number.

## 1 Introduction

Arithmetic with fuzzy numbers has a wide range of application in many different directions especially in engineering and decision making. The demand for fuzzy arithmetic in these different disciplines has attracted many researchers. Most of these applications use the minimum t-norm $T_M$ as connective referring to Zadeh's extension principle [17]. For this arithmetic type the $\alpha$-level method [12] is easy to exploit thanks to the monotonicity of arithmetic operators. For non-monotonic operators, discrete computational methods are introduced such as in [3]. Fairly new tools for fuzzy arithmetic are the Relative Distance Measure (RDM) arithmetic that uses horizontal membership function representations and the M-IT2-F arithmetic linked to Piegat and Landowski [13, 14]. This approach seems very useful for solving equations [8], but is not suitable for use with t-norms other than the minimum connective. Arithmetic with different types of t-norms plays an important role in applications. However, fuzzy arithmetic with different type of t-norms -a generalization of Zadeh's extension by replacing the minimum operator with a t-norm- is facing serious difficulties in practice. These difficulties arise because supremums of infinitely many combinations for each resulting single point are involved in the definition of the extension principle. On the other hand, for particular t-norms and operations, there are simple and useful results. For example, in the case of addition with the product t-norm $T_P$ Fullėr [2] calculates the limit of product-sum series, Triesch [16] develops the results for the finite case with sufficient conditions and Hong [4] completes the study of Triesch with the necessary conditions. Mesiar [10] takes a step further and generalizes the results to a family of t-norms under sufficient conditions, followed by Markova's work [9] about the necessary conditions. It should be emphasized that in all these papers the input fuzzy numbers have common left and right-spreads, a rather strong condition. Multiplication with the product t-norm is a more complex task, Hong [5] proves that the only shape preserving multiplication is the drastic-multiplication. In [15] Seresht and Fayek introduce a computational method for arithmetic on triangular fuzzy numbers with $T_P$ where they relax the condition of common spreads of the input variables. Unfortunately, the method only works for a one-time operation, with more than two fuzzy inputs, repetitive operations are too complex to be performed so far. Moreover, the results are computed with the help of an algorithm only at discrete points. This leads to the requirement of an interpolation process.

Despite all the difficulties with $T_P$-arithmetic there is a demand for it in certain fields of applications. For example

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in [1] it is proven that the $T_P$ t-norm is the unique one for which fuzzy constrained optimization problems are scale-invariant. In [7] the authors show the effect of using $T_P$ in a fuzzy controller.

The motivation of this study is twofold: the first is to provide product-arithmetic tools for the potential applications and secondly to present a theoretical insight into the mechanism of the arithmetic. The main contribution of the study is the introduction of explicit formulas for the product-sum and product-multiplication of triangular fuzzy numbers. Thereby, the problems of the computational method in [15] are solved. Since the results are computed continuously rather than at discrete points an interpolation procedure is no more necessary.

There are two main directions to deal with t-norm fuzzy arithmetic: the generalized $\alpha$-cut method and the generalized extension principle which are of equivalent complexity. We follow the extension principle. The results are compared with $T_M$ and the drastic t-norm $T_\omega$ based operations in terms of fuzziness. In order to enable multiple calculations, the problem of lack of shape preservation is solved by introducing well-suited, appropriate approximations. Moreover, the distances of the approximations to the original results are computed and a close relation between $T_P$-arithmetic and $T_\omega$-arithmetic is discovered.

# 2 Preliminaries

Throughout the text it will be used only conventional notations, nevertheless a brief list of some definitions can be find in this section.

**Definition 2.1.** *A triangular fuzzy number $A$ denoted by $(a, b, c)$ has the membership function $\mu_A$ defined on the set of real numbers $\mathbb{R}$ by,*

$$\mu_A(x) = \begin{cases} \frac{x-a}{b-a}, & a \le x \le b, \\ \frac{c-x}{c-b}, & b < x \le c, \end{cases} \quad \forall x \in \mathbb{R}.$$

Let $A = (a, b, c)$ be a fuzzy number. The left and right spreads of this fuzzy number will be denoted respectively by $\alpha = b - a$, and $\beta = c - b$.

**Definition 2.2.** *The additive inverse of a fuzzy number $A$ is denoted by $-A$ and has the following membership function,*

$$\mu_{-A}(x) = \mu_A(-x), \ \forall x \in \mathbb{R}.$$

**Definition 2.3.** *The reciprocal of a fuzzy number $A$ is denoted by $A^{-1}$ and has the following membership function,*

$$\mu_{A^{-1}}(x) = \mu_A(1/x), \ \forall x \in \mathbb{R}.$$

**Definition 2.4.** *The core of a fuzzy number $A$ is denoted by $core(A)$ and is defined as,*

$$core(A) = \{x \in \mathbb{R} : \mu_A(x) = 1\}.$$

**Definition 2.5.** *The support of a fuzzy number $A$ is denoted by $supp(A)$ and is defined as,*

$$supp(A) = \{x \in \mathbb{R} : \mu_A(x) > 0\}.$$

**Remark 2.6.** *Although for a triangular fuzzy number $A = (a, b, c)$, $supp(A)$ is the open interval $(a, c)$, we will use the convention to use its closure $[a, c]$.*

Another useful concept is the L-R representation of fuzzy numbers. The L-R representation of a fuzzy number carries the information of the modal value $b$, the left and right spreads $\alpha, \beta$ and finally the "shapes" $L, R$ of both sides of the modal value of the membership function. In this setting fuzzy numbers are denoted by $\langle b, \alpha, \beta \rangle_{L,R}$.

**Definition 2.7.** *The L-R representation of a triangular fuzzy number $A = (a, b, c)$ is $\langle b, \alpha, \beta \rangle_{1-x,1-x}$.*

# 3 Addition with the product t-norm

Besides that this section presents an explicit formula for the addition using $T_P$, the results can be transformed easily to the case of subtraction by using the identity $A - B = A + (-B)$, which is also valid in fuzzy arithmetic.

Let $A$ and $B$ be two triangular fuzzy numbers with, $A = (a_1, b_1, c_1)$ and $B = (a_2, b_2, c_2)$.

The spreads of these fuzzy numbers will be denoted by, $b_1 - a_1 = \alpha_1,\ c_1 - b_1 = \beta_1,\ b_2 - a_2 = \alpha_2$ and $c_2 - b_2 = \beta_2$.

For the product t-norm it is known that $supp[A + B] = [a_1 + a_2, c_1 + c_2]$ and $core[A + B] = b_1 + b_2$.

The increasing part of the membership function of a fuzzy number will briefly be called the left-side ($L$) and the decreasing part the right-side ($R$). The study will begin with the construction of the left-side of $A+B$.

Let $x$ be a point in the domain of the left-side of the sum, so $x \in [a_1+a_2, b_1+b_2]$. According to Zadeh's Extension Principle the membership value of $x$ is

$$\mu_{A+B}(x) = \bigvee_{x=y+z} \mu_A(y) \cdot \mu_B(z).$$

There are three possibilities:

1. $x$ is the sum of a point in the domain of the left-side of $A$ and a point in the domain of the left-side of $B$ ($LL$).
2. $x$ is the sum of a point in the domain of the left-side of $A$ and a point in the domain of the right-side of $B$ ($LR$).
3. $x$ is the sum of a point in the domain of the right-side of $A$ and a point in the domain of the left-side of $B$ ($RL$).

So, for each particular choice of $x$ the question is: which combination leads to a greater membership, $LL$, $LR$ or $RL$?

To find the answer, the optimum value for each particular case has to be calculated and compared with each other. Without loss of generality, throughout it is assumed that $\alpha_1 \le \alpha_2$ and $\beta_1 \le \beta_2$.

The coordinates $c_L = a_1 + a_2 + 2\alpha_1$ and $c_R = c_1 + c_2 - 2\beta_1$ will be useful in the sequel.

i. **Case:** $LL$.

Let $y \in [a_1, b_1]$, $z \in [a_2, b_2]$ and $x = y + z$. There are two subcases:
$x \in [a_1 + a_2, c_L]$ or $x \in (c_L, b_1 + b_2]$. The subcase $x \in [a_1 + a_2, c_L]$ is discussed as follows:
Assuming that $x = a_1 + a_2 + t$, for a $k \in [0, \alpha_1]$ one may write $y = a_1 + k$ and $z = a_2 + t - k$. It can be observed that the corresponding membership degrees are as follows,

$$\mu_A(y) = \frac{k}{\alpha_1} \text{ and } \mu_B(z) = \frac{t-k}{\alpha_2}.$$

Keeping in mind that it is needed to find the value, $\mu(x) = \bigvee_{x=y+z} \{\mu_A(y) \cdot \mu_B(z)\}$, the following function is defined,

$$f_t(k) = \mu_A(y) \cdot \mu_B(z) = \frac{k}{\alpha_1} \cdot \frac{t-k}{\alpha_2} = \frac{tk - k^2}{\alpha_1 \alpha_2}.$$

In order to calculate $\mu(x)$ it remains to optimize $f_t(k)$. We derivate $f_t(k)$ w.r.t. $k$ and observe, $\frac{d}{dk}[f_t(k)] = 0$ if $k = \frac{t}{2}$. The maximum value then is, $f_t(\frac{t}{2}) = \frac{t^2}{4\alpha_1\alpha_2}$. At this step it should be noted that since $k \in [0, \alpha_1]$ it holds that $t \in [0, 2\alpha_1]$ and therefore $x \in [a_1 + a_2, c_L]$.

For the subcase $x \in (c_L, b_1 + b_2]$ we have $t > 2\alpha_1$ and this together with $k \in [0, \alpha_1]$ means that,

$$\frac{d}{dk}[f_t(k)] = \frac{t-2k}{\alpha_1\alpha_2} > 0.$$

Therefore $f_t(k)$ is increasing w.r.t $k$ and attains its maximum at $k = \alpha_1$ with the value $f_t(\alpha_1) = \frac{t-\alpha_1}{\alpha_2}$. Substituting backwards, $t = x - (a_1 + a_2)$ the result for the ($LL$) case will be obtained as following,

$$\mu_1(x) = \begin{cases} \frac{(x-(a_1+a_2))^2}{4\alpha_1\alpha_2}, & x \in [a_1+a_2, c_L] \\ \frac{x-(b_1+a_2)}{\alpha_2}, & x \in (c_L, b_1+b_2]. \end{cases} \tag{1}$$

The result above has to be compared with the cases ($RL$) and ($LR$) to obtain the supremum value of $\mu_A(y) \cdot \mu_B(z)$.

ii. **Case:** $RL$.

Let $y \in [b_1, c_1]$, $z \in [a_2, b_2]$ and $x = y + z$. Since $x \in [b_1 + a_2, b_1 + b_2]$, it may be assumed that $x = b_1 + a_2 + t$, $y = b_1 + k$ and $z = a_2 + (t - k)$, with the restrictions $k \in [0, \beta_1]$ and $(t - k) \in [0, \alpha_2]$. In this case the corresponding membership values are as follows,

$$\mu_A(y) = \frac{\beta_1 - k}{\beta_1} \text{ and } \mu_B(z) = \frac{t-k}{\alpha_2}.$$

Then $f_t(k)$ becomes

$$f_t(k) = \mu_A(y) \cdot \mu_B(z) = \frac{\beta_1 - k}{\beta_1} \cdot \frac{t-k}{\alpha_2}.$$

Derivating w.r.t. $k$, it is observed that,

$$\frac{d}{dk}[f_t(k)] = -\frac{(t-k) + (\beta_1 - k)}{\beta_1 \alpha_2} \le 0.$$

So $f_t(k)$ is a decreasing function and attains its maximum value at $k = 0$. For $k = 0$, $\mu(y) \cdot \mu(z) = \frac{t}{\alpha_2}$. Substituting backwards, $t = x - (b_1 + a_2)$ the result for the $(RL)$ case is obtained as,

$$\mu_2(x) = \frac{x - (b_1 + a_2)}{\alpha_2}, \; x \in [b_1 + a_2, b_1 + b_2]\cdot \tag{2}$$

It should be noted that in this case the resulting membership function is linear.

iii. **Case:** $LR$

Let $y \in [a_1, b_1]$, $z \in [b_2, c_2]$ and $x = y+z$. Since $x \in [a_1+b_2, b_1+b_2]$, it is assumed that $x = a_1+b_2+t$, $y = a_1+(t-k)$ and $z = a_2 + \alpha_2 + k$, with the restrictions $k \in [0, \beta_2]$ and $(t - k) \in [0, \alpha_1]$. The corresponding membership values are as follows,

$$\mu_A(y) = \frac{t-k}{\alpha_1}, \; \mu_B(z) = \frac{\beta_2 - k}{\beta_2}.$$

The procedure is similar to the case $(RL)$,

$$f_t(k) = \mu_A(y) \cdot \mu_B(z) = \frac{t-k}{\alpha_1} \cdot \frac{\beta_2 - k}{\beta_2}.$$

Derivating $f_t(k)$ it can be seen that,

$$\frac{d}{dk}[f_t(k)] = \frac{-\left((t-k) + (\beta_2 - k)\right)}{\beta_1 \alpha_2} \leq 0.$$

So $f_t(k)$ is a decreasing function and attains its maximum value at $k = 0$. For $k = 0$, $\mu_A(y) \cdot \mu_B(z) = \frac{t}{\alpha_1}$. Substituting backwards, $t = x - (a_1 + b_2)$ the result for the $(RL)$ case is obtained to be

$$\mu_3(x) = \frac{x - (a_1 + b_2)}{\alpha_1}, \; x \in [a_1 + b_2, b_1 + b_2]\cdot \tag{3}$$

The case $(LR)$ returns a linear membership function as well.

Since $\mu_{A+B}(x) = \max\{\mu_1(x), \mu_2(x), \mu_3(x)\}$, it remains to compare the functions $\mu_1, \mu_2, \mu_3$ on their common domain. Keeping in mind that $\alpha_1 \leq \alpha_2$ it is obvious that $\mu_3 \leq \mu_2$ for all $x \leq b_1 + b_2$, therefore it will be sufficient to compare $\mu_1$ with $\mu_2$.

The claim is that $\mu_2(x) \leq \mu_1(x), \forall x \in \mathbb{R}$. This can be verified by observing that the only solution to the equation $\mu_1(x) = \mu_2(x)$ is $x_0 = 2\alpha_1 + a_1 + a_2 = c_L$. The result is $\mu_3 \leq \mu_2 \leq \mu_1$ and considering the domains of these functions completes the discussion.

It is easy to check that all the results obtained above can be transformed symmetrically to the right-side of the fuzzy number $A + B$. The conclusion of the discussion is summarized in the following theorem.

**Theorem 3.1.** *Given two triangular fuzzy numbers $A = (a_1, b_1, c_1)$ and $B = (a_2, b_2, c_2)$, let $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$. The sum $A + B$ under the product t-norm is a fuzzy number with the following membership function,*

$$\mu_{A+B}(x) = \begin{cases} \frac{(x-(a_1+a_2))^2}{4\alpha_1\alpha_2}, & a_1 + a_2 \leq x \leq a_1 + a_2 + 2\alpha_1, \\ \frac{x-(b_1+a_2)}{\alpha_2}, & a_1 + a_2 + 2\alpha_1 \leq x \leq b_1 + b_2, \\ \frac{-x+(b_1+c_2)}{\beta_2}, & b_1 + b_2 \leq x \leq c_1 + c_2 - 2\beta_1, \\ \frac{(x-(c_1+c_2))^2}{4\beta_1\beta_2}, & c_1 + c_2 - 2\beta_1 \leq x \leq c_1 + c_2. \end{cases} \tag{4}$$

**Remark 3.2.** *If $\alpha_1 > \alpha_2$ and/or $\beta_1 > \beta_2$ all indexes for $\alpha, a, b$ in the left part of the result and/or $\beta, b, c$ in the right part of the result should we swapped.*

**Corollary 3.3.** *If $\alpha_1 = \alpha_2$ then $\mu_{A+B}(x) = \frac{(x-(a_1+a_2))^2}{4\alpha_1\alpha_2}, \forall x \in [a_1 + a_2, b_1 + b_2]$.*

**Corollary 3.4.** *If $\beta_1 = \beta_2$ then $\mu_{A+B}(x) = \frac{(x-(c_1+c_2))^2}{4\beta_1\beta_2}, \forall x \in [b_1 + b_2, c_1 + c_2]$.*

It means that in case of equal left (right) spreads of the input fuzzy numbers the left (right) spread of their sum consists only of a quadratic term. As it was mentioned in the introduction, the case of equal spreads of the summands was already covered in the literature. For instance Corollary 1 in [10] can be exploited as following. Since $\log(1-x)$, the logarithm of both left and right shape functions, is concave, the membership function of $A+B$ is (setting $\alpha=\alpha_1=\alpha_2$ and $\beta=\beta_1=\beta_2$),

$$\mu_{A+B}(x)=\begin{cases} L^2(\frac{b_1+b_2-x}{2\alpha}) & , b_1+b_2-2\alpha\le x\le b_1+b_2,\\ R^2(\frac{x-(b_1+b_2)}{2\beta}) & , b_1+b_2\le x\le b_1+b_2+2\beta.\end{cases}$$

This is equal to,

$$\mu_{A+B}(x)=\begin{cases} (1-(\frac{b_1+b_2-x}{2\alpha}))^2 & , b_1+b_2-2\alpha\le x\le b_1+b_2,\\ (1-(\frac{x-(b_1+b_2)}{2\beta}))^2 & , b_1+b_2\le x\le b_1+b_2+2\beta.\end{cases}$$

And since $b_1+b_2=a_1+a_2+2\alpha=c_1+c_2-2\beta$, after simple computation we observe,

$$\mu_{A+B}(x)=\begin{cases} \left(\frac{x-(a_1+a_2)}{2\alpha}\right)^2 & , a_1+a_2\le x\le b_1+b_2,\\ \left(\frac{(x-(c_1+c_2)}{2\beta}\right)^2 & , b_1+b_2\le x\le c_1+c_2,\end{cases}$$

which agrees with the results claimed in Corollary 1 and Corollary 2.

**Example 3.5.** *We consider the triangular fuzzy numbers $A=(3,6,8)$ and $B=(10,14,17)$. The sum of $A$ and $B$ is computed by the formula (4) as following:*

$$\mu_{A+B}(x)=\begin{cases} \frac{(x-13)^2}{48}, & 13\le x\le 19,\\ \frac{x-16}{4}, & 19\le x\le 20,\\ \frac{-x+23}{3}, & 20\le x\le 21,\\ \frac{(x-25)^2}{24}, & 21\le x\le 25.\end{cases}$$

*The support of $A+B$ is $[13,25]$ with core $\{20\}$. Since $2\alpha_{\min}=6$ and $2\beta_{\min}=4$ we observe in Figure 1 that in $[13,19]$*

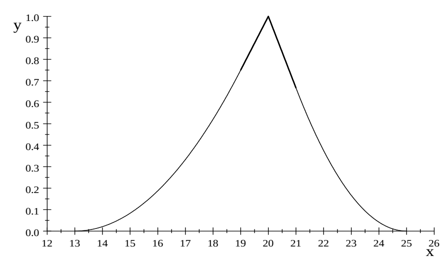


Figure 1: The membership function of the sum $A+B$ .

*and $[21,25]$ the membership degrees are quadratic (normal lines). In the interval $[19,21]$ we have linear membership degrees (thick lines).*

# 4 Multiplication with the product t-norm

The change of the arithmetic operator under investigation will not change the approach to the solution of the problem. Assuming that $A,B>0$, first the left-side of $A\cdot B$ is going to be constructed. Obviously, $supp[A\cdot B]=[a_1\cdot a_2,c_1\cdot c_2]$ and $core[A\cdot B]=b_1\cdot b_2$. Let $x$ be a point in the domain of the left-side of the sum ($x\in[a_1a_2,b_1b_2]$). According to Zadeh's Extension Principle the value,

$$\mu_{A\cdot B}(x)=\bigvee_{x=y\cdot z}\mu_A(y)\cdot\mu_B(z).$$

has to be calculated. Similarly to the case of addition there will be three cases that will be analyzed:

i. **Case** $LL$**:**

Let $y\in[a_1,b_1]$, $z\in[a_2,b_2]$ and $x=y\cdot z$.

The corresponding membership degrees are as follows,

$$\mu_A(y)=\frac{y-a_1}{\alpha_1},\ \mu_B(z)=\mu\left(\frac{x}{y}\right)=\frac{x-a_2y}{\alpha_2y}.$$

We define,

$$f_x(y) = \mu_A(y) \cdot \mu_B(z) = \left(\frac{y - a_1}{\alpha_1}\right)\left(\frac{x - a_2 y}{\alpha_2 y}\right).$$

The maxima of $f_x(y)$ is determined by

$$\frac{d}{dy}[f_x(y)] = 0 \Longleftrightarrow y = \sqrt{\frac{a_1 x}{a_2}}.$$

Then,

$$f_x(\sqrt{\frac{a_1 x}{a_2}}) = \frac{(\sqrt{x} - \sqrt{a_1 a_2})^2}{\alpha_1 \alpha_2}.$$

This way we define,

$$\mu_1(x) = \frac{(\sqrt{x} - \sqrt{a_1 a_2})^2}{\alpha_1 \alpha_2}.$$

It is still needed to find a valid domain for this function. Consider the following constraints: since $y = \sqrt{\frac{a_1 x}{a_2}}$ one has,

$$a_1 \leq \sqrt{\frac{a_1 x}{a_2}} \leq b_1 \Leftrightarrow a_1 a_2 \leq x \leq \frac{a_2 b_1^2}{a_1}, \tag{5}$$

and since $z = \sqrt{\frac{x a_2}{a_1}}$,

$$a_2 \leq \sqrt{\frac{x a_2}{a_1}} \leq b_2 \Leftrightarrow a_2 a_1 \leq x \leq \frac{b_2^2 a_1}{a_2}. \tag{6}$$

Combining (5) and (6) the domain of $\mu_1$ is obtained as,

$$a_1 a_2 \leq x \leq \min\left\{\frac{a_1 b_2^2}{a_2}, \frac{a_2 b_1^2}{a_1}\right\}.$$

So the result for the $LL$ case is,

$$\mu_1(x) = \frac{(\sqrt{x} - \sqrt{a_1 a_2})^2}{\alpha_1 \alpha_2}, \; x \in \left[a_1 a_2, \; \min\left\{\frac{a_1 b_2^2}{a_2}, \frac{a_2 b_1^2}{a_1}\right\}\right]. \tag{7}$$

ii. **Case:** $LR$**:**

Let $y \in [a_1, b_1]$, $z \in [b_2, c_2]$ and $x = y \cdot z$. Substituting $y = \frac{x}{z}$ it is observed that, $\mu_A\left(\frac{x}{z}\right) = \frac{x - a_1 z}{\alpha_1 z}$, and $\mu_A$ is decreasing with respect to $z$. On the other hand $\mu_B(z) = \frac{b_2 + \beta_2 - z}{\beta_2}$, is decreasing with respect to $z$ as well, so the product of these functions attains its maximum value at the minimum value of $z$ which is $z = b_2$. Since

$$\mu_A\left(\frac{x}{b_2}\right) \cdot \mu_B(b_2) = \frac{x - a_1 b_2}{\alpha_1 b_2},$$

it can be concluded that in the $LR$ case the membership is the following linear function,

$$\mu_2(x) = \frac{x - a_1 b_2}{\alpha_1 b_2}, \; x \in [a_1 b_2, b_1 b_2]. \tag{8}$$

iii. **Case:** $RL$**:**

Let $y \in [b_1, c_2]$, $z \in [a_2, b_2]$ and $x = y \cdot z$. Substituting $z = \frac{x}{y}$ it may be observed that $\mu_A(y) = \frac{b_1 + \beta_1 - y}{\beta_1}$, is decreasing with respect to $y$. On the other hand, $\mu_B\left(\frac{x}{y}\right) = \frac{x - a_2 y}{\alpha_2 y}$, is decreasing with respect to $y$ as well, so the product of these functions attains its maximum value at the minimum value of $y$ which is $y = b_1$. Since

$$\mu_A(b_1) \cdot \mu_B(b_1) = \frac{x - a_2 b_1}{\alpha_2 b_1},$$

it can be concluded that in the case $LR$ the membership is the following linear function,

$$\mu_3(x) = \frac{x - a_2 b_1}{\alpha_2 b_1}, \; x \in [a_2 b_1, b_1 b_2]. \tag{9}$$

Now the functions (7),(8) and (9) will be compared on their common domain.

**Lemma 4.1.** *If* $\min\left\{\frac{a_1b_2^2}{a_2}, \frac{a_2b_1^2}{a_1}\right\} = \frac{a_1b_2^2}{a_2}$ *then* $\mu_3 \le \mu_2 \le \mu_1, \forall x \in x \in [a_1a_2, b_1b_2]$.

*Proof.* Let $\min\left\{\frac{a_1b_2^2}{a_2}, \frac{a_2b_1^2}{a_1}\right\} = \frac{a_1b_2^2}{a_2}$. By the observation,

$$\frac{a_1b_2^2}{a_2} \le \frac{a_2b_1^2}{a_1} \Leftrightarrow a_1b_2 \le a_2b_1,$$

we conclude that the root of $\mu_2$ is less than the root of $\mu_3$ and since the functions intersect at $(b_1b_2, 1)$ it is seen that $\mu_3 \le \mu_2$ in the domain under discussion. On the other hand solving the equality $\mu_2 = \mu_1$ it may be observed that the equality

$$\frac{x - a_1b_2}{\alpha_1b_2} = \frac{(\sqrt{x} - \sqrt{a_1a_2})^2}{\alpha_1\alpha_2},$$

leads to the quadratic equation

$$x\left(\alpha_2 - b_2\right) + 2b_2\sqrt{a_1a_2}\sqrt{x} - \left(a_1b_2\alpha_2 + b_2a_1a_2\right) = 0,$$

for which the discriminant is

$$(2b_2\sqrt{a_1a_2})^2 - 4\left(a_2\left(a_1b_2\alpha_2 + b_2a_1a_2\right)\right) = 0.$$

Since the discriminant of the quadratic equation is zero, it is concluded that $\mu_2$ is tangent to $\mu_1$ and therefore $\mu_2 \le \mu_1$. □

Once it is observed that $\mu_2$ is tangent to $\mu_1$, one can compute the $x-$coordinate of the tangent intersection by solving the following equality, $\frac{d}{dx}\mu_1(x) = \frac{d}{dx}\mu_2(x)$, to obtain the solution $x = \frac{a_1b_2^2}{a_2}$.

All the results obtained above can easily be transformed symmetrically to the right-side of the fuzzy number $A \cdot B$. The conclusion of the discussion is summarized in the following theorem.

**Theorem 4.2.** *Given two triangular fuzzy numbers $A = (a_1, b_1, c_1)$ and $B = (a_2, b_2, c_2)$ their product $A \cdot B$ under the product t-norm is a fuzzy number with the following membership function,*

$$\mu_{A\cdot B}(x) = \begin{cases} \frac{(\sqrt{x}-\sqrt{a_1a_2})^2}{\alpha_1\alpha_2}, & a_1a_2 \le x \le \min\left\{\frac{a_1b_2^2}{a_2}, \frac{a_2b_1^2}{a_1}\right\}, \\ \frac{x-a_1b_2}{\alpha_1b_2}, & \frac{a_1b_2^2}{a_2} \le x \le b_1b_2 \text{ and } a_1b_2 \le a_2b_1, \\ \frac{x-a_2b_1}{\alpha_2b_1}, & \frac{a_2b_1^2}{a_1} \le x \le b_1b_2 \text{ and } a_2b_1 \le a_1b_2, \\ \frac{c_2b_1-x}{\beta_2b_1}, & b_1b_2 \le x \le \frac{c_2b_1^2}{c_1} \text{ and } c_1b_2 \le c_2b_1, \\ \frac{c_1b_2-x}{\beta_1b_2}, & b_1b_2 \le x \le \frac{c_1b_2^2}{c_2} \text{ and } c_2b_1 \le c_1b_2, \\ \frac{(\sqrt{x}-\sqrt{c_1c_2})^2}{\beta_1\beta_2}, & \max\left\{\frac{c_1b_2^2}{c_2}, \frac{c_2b_1^2}{c_1}\right\} \le x \le c_1c_2. \end{cases} \tag{10}$$

The following two propositions ensure that the conditional linear parts of $\mu_{A\cdot B}$ match up on the common boundary conditions.

**Proposition 4.3.** *If* $a_1b_2 = a_2b_1$ *then* $\frac{x-a_1b_2}{\alpha_1b_2} = \frac{x-a_2b_1}{\alpha_2b_1}$.

*Proof.* Let $a_1b_2 = a_2b_1$. We use the identities $b_2 = a_2 + \alpha_2$ and $b_1 = a_1 + \alpha_1$ to observe $a_1\alpha_2 = a_2\alpha_1$. By adding the term $\alpha_1\alpha_2$, to both sides of the last equality we observe $\alpha_2b_1 = \alpha_1b_2$, and therefore,

$$\frac{x - a_1b_2}{\alpha_1b_2} = \frac{x - a_2b_1}{\alpha_2b_1}.$$

So we obtain that $\mu_3 = \mu_2$ on their common domain.

Since $a_1b_2 = a_2b_1$ also implies $\frac{a_1b_2^2}{a_2} = \frac{a_2b_1^2}{a_1}$, we see that their domains are also equal and we can conclude that $\mu_3 = \mu_2$. □

**Proposition 4.4.** *If* $c_1b_2 = c_2b_1$ *then* $\frac{c_2b_1-x}{\beta_2b_1} = \frac{c_1b_2-x}{\beta_1b_2}$, $\forall x \in \left[b_1b_2, \frac{c_2b_1^2}{c_1}\right]$.

*Proof.* The proof is similar to the proof of the foregoing proposition and will be omitted. □

For both $A < 0$ and $B < 0$, multiplication can be performed by the identity $A \cdot B = (-A) \cdot (-B)$. If $A < 0$, $B > 0$, then $A \cdot B = -((-A) \cdot (B))$.

It should be pointed out that division can not be performed directly by using $A \div B = A \cdot B^{-1}$ as claimed in [15] since the term $B^{-1}$ will not be a triangular fuzzy number. For division, a solution is to use the tangent approximation of $B^{-1}$ ([3]) and then make use of $A \div B = A \cdot B^{-1}$. The tangent approximation of an either positive or negative triangular fuzzy number $B = (a, b, c)$ is

$$B^{-1} = \left\langle \frac{1}{b}, \frac{\beta}{b^2}, \frac{\alpha}{b^2} \right\rangle_{1-x}.$$

**Example 4.5.** *We consider the triangular fuzzy numbers,* $A = (2, 3, 4)$ *and* $B = (5, 7, 10)$. *Here* $a_1b_2 = 14, a_2b_1 = 15, c_1b_2 = 28c_2b_1 = 30$, *therefore the product of A and B is computed by formula (10) as following:*

$$\mu_{A\cdot B}(x) = \begin{cases} \frac{(\sqrt{x}-\sqrt{10})^2}{2}, & 10 \leq x \leq 19.6, \\ \frac{x-14}{7}, & 19.6 \leq x \leq 21, \\ \frac{30-x}{9}, & 21 \leq x \leq 22.5, \\ \frac{(\sqrt{x}-\sqrt{40})^2}{3}, & 22.5 \leq x \leq 40. \end{cases}$$

*The support of* $A \cdot B$ *is* $[10, 40]$ *with core* $\{21\}$. *Since* $\min\left\{\frac{a_1b_2^2}{a_2}, \frac{a_2b_1^2}{a_1}\right\} = 19.6$ *and* $\max\left\{\frac{c_1b_2^2}{c_2}, \frac{c_2b_1^2}{c_1}\right\} = 22.5$ *we observe*

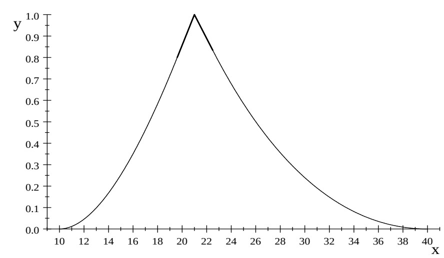


Figure 2: The membership function of the product $A \cdot B$ .

*in Figure 2 that in* $[10, 19.6]$ *and* $[22.5, 40]$ *the membership degrees are quadratic (normal lines), whereas in the interval* $[19.6, 22.5]$ *we have linear membership degrees again(thick lines).*

## 5 Some algebraic properties

In this section we review some key properties of the product arithmetic. Algebraic properties of $T$-arithmetic of fuzzy numbers are studied in detail in [6]. Here we list the modified results for the particular case of product arithmetic. We just prove some of them to show how the results in [6] can be transformed. Maybe the most important observation is that the sum (or product) of two normal and convex fuzzy numbers is also a normal and convex fuzzy number.

**Proposition 5.1.** *Product-sum and product-multiplication are commutative, i.e. let* $* \in \{+, \cdot\}$

$$A * B = B * A.$$

*Proof.* Let $Z = A * B$. The commutativities of $*$ and the product t-norm imply,

$$\mu_Z(z) = \sup_{z=x*y} (\mu_A(x) \cdot \mu_B(y)) = \sup_{z=y*x} (\mu_B(y) \cdot \mu_A(x)).$$

□

**Proposition 5.2.** *Product-sum and product-multiplication are associative, i.e. let* $* \in \{+, \cdot\}$

$$(A * B) * C = A * (B * C).$$

*Proof.* Let $V = (A * B) * C$ and $V^{'} = A * (B * C)$. The associativities of $*$ and the product t-norm provide,

$$\begin{aligned}\mu_V(v) &= \sup_{v=s*z}((\sup_{s=x*y}\mu_A(x)\cdot\mu_B(y))\cdot\mu_C(z))\\ &= \sup_{v=(x*y)*z}(\mu_A(x)\cdot\mu_B(y))\cdot\mu_C(z)\\ &= \sup_{v=x*(y*z)}\mu_A(x)\cdot(\mu_B(y)\cdot\mu_C(z))\\ &= \sup_{v=x*t}(\mu_A(x)\cdot(\sup_{t=y*z}\mu_B(y)\cdot\mu_C(z)))\\ &= \mu_{V'}(v).\end{aligned}$$

□

**Proposition 5.3.** *The crisp numbers* 0 *and* 1 *are neutral elements of product addition and multiplication respectively:*

$$A + 0 = A,\ A \cdot 1 = A.$$

**Proposition 5.4.** *Product sum and product multiplication are not invertible, i.e. there exist no objects (as long as A is non-crisp)* $A_+, A_\times$ *with,*

$$A + A_+ = 0,\ A \cdot A_\times = 1.$$

**Proposition 5.5.** *Product arithmetic is weak distributive:*

$$A \times (B + C) \subset A \times B + A \times C.$$

The paper [6] includes an example of violation of exact distributivity for the case of product arithmetic. In the light of these observations we can conclude that fuzzy numbers equipped with product arithmetic form commutative monoids. The lack of complete distributivity hinders a semiring structure.

## 6 Measure of fuzziness

In this section we compute the measures of fuzziness of the results and compare them with minimum t-norm arithmetic and weak t-norm arithmetic. As measure of fuzziness we adopt the cardinality:

$$Car(\mu) = \int_X \mu(x)dx.$$

The calculation will be performed for the left-hand side of the result that can be transformed to the right-hand side as both cases are equivalent.

$$\int_{-\infty}^{b_1+b_2} \mu_{A+B}(x)dx = \int_{a_1+a_2}^{a_1+a_2+2\alpha_1} \frac{(x-(a_1+a_2))^2}{4\alpha_1\alpha_2}dx + \int_{a_1+a_2+2\alpha_1}^{b_1+b_2} \frac{x-(b_1+a_2)}{\alpha_2}dx = \frac{\alpha_1^2+3\alpha_2^2}{6\alpha_2}. \tag{11}$$

It is easy to verify that the cardinality measure for the same fuzzy inputs with the minimum t-norm arithmetic is $\frac{\alpha_1+\alpha_2}{2}$ and with the weak t-norm arithmetic is $\frac{\alpha_2}{2}$.

The gain factor of the fuzzy addition performed with product t-norm over the addition performed with minimum t-norm can then be obtained as:

$$\left[(1-\left(\frac{(\alpha_1^2+3\alpha_2^2)}{3\alpha_2(\alpha_1+\alpha_2)}\right)\times 100\right]\%$$

**Example 6.1.** *The sum and product of the triangular fuzzy numbers* $A = (2, 3, 5)$ *and* $B = (6, 8, 11)$ *with the* $T_P$ *t-norm and their comparisons with* $T_M$ *and* $T_\omega$ *are discussed below.*
*For the given inputs (4) returns the following membership function:*

$$\mu_{A+B}(x) = \begin{cases} \frac{(x-8)^2}{8}, & 8 \le x \le 10,\\ \frac{x-9}{2}, & 10 \le x \le 11,\\ \frac{-x+14}{3}, & 11 \le x \le 12,\\ \frac{(x-16)^2}{24}, & 12 \le x \le 16. \end{cases}$$

*Figure 1 illustrates the membership functions of the sum $A+B$ performed with the three t-norms $T_P, T_M$ and $T_\omega$.*

*Their cardinalities are,* 2.80, 4.00 *and* 2.50 *respectively. The gain factor of the addition performed with $T_P$ over the addition performed with $T_M$ is* 30%. *At this point it should be noted that for any t-norm weaker or equal to the Luckasiewicz t-norm, the addition of triangular fuzzy numbers coincide with their $T_w$ addition (see, e.g. Theorem 2(a) in [11]).*

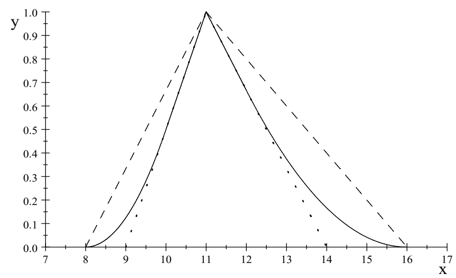

Figure 3: $A+B$, Dashed: $T_M$ Dots: $T_\omega$. Solid: $T_P$

*For the given inputs (10) returns the following membership function:*

$$\mu_{A\cdot B}(x)=\begin{cases}\frac{(\sqrt{x}-\sqrt{12})^2}{2}, & 12\le x\le \frac{64}{3},\\ \frac{x-16}{8}, & \frac{64}{3}\le x\le 24,\\ \frac{40-x}{16}, & 24\le x\le \frac{320}{11},\\ \frac{(\sqrt{x}-\sqrt{55})^2}{6}, & \frac{320}{11}\le x\le 55.\end{cases}$$

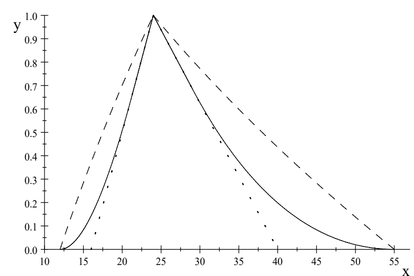

Figure 4: $A\cdot B$, Dashed:$T_M$ Dots:$T_\omega$. Solid:$T_P$

*Figure 2 illustrates the membership functions of the product $A\cdot B$ performed with the three t-norms $T_P, T_M$ and $T_\omega$.*

*Their cardinalities are rounded to,* 14.15, 20.83 *and,* 12 *respectively. The gain factor of the multiplication performed with $T_P$ over the multiplication performed with $T_M$ is about* 32%.

# 7 Triangular approximations and conclusion

Generally the addition of two triangular fuzzy numbers with product t-norm results in fuzzy numbers with piece-wise linear and quadratic membership functions. Only when both input variables have the same left and right spreads respectively, their sum has an entirely quadratic membership function. This particular case can be expressed in terms of shape functions with the following equality:

$$\langle b_1,\alpha_1,\beta_1\rangle_{1-x,1-x}+_{T_P}\langle b_2,\alpha_1,\beta_1\rangle_{1-x,1-x}=\langle b_1+b_2,2\alpha_1,2\beta_1\rangle_{1-x^2,1-x^2}\,.$$

It means that if there are more than one addition tasks to perform, the calculations will become rather complex. Especially when the left and right spreads differ, it seems quite difficult to find a general formula for $\sum_{i=1}^{n}A_i$, where $A_i$ is a triangular fuzzy number for all $i$, when the connective is the product t-norm.

The task is even more difficult in case of the product of triangular fuzzy numbers. As a result, to perform multiple arithmetic operations, the set of triangular fuzzy numbers needs to be closed under addition and multiplication with the product t-norm. This can be provided by using triangular approximations to the non-linear results. Consider the membership function of addition (4), a natural way to approximate this function is to use the linear parts involved and extend their domain. The resulting approximation denoted by $\widetilde{\mu}_{A+B}$ is then as following,

$$\widetilde{\mu}_{A+B}(x) = \begin{cases} \frac{x-(b_1+a_2)}{\alpha_2}, & a_1 + a_2 + \alpha_1 \leq x \leq b_1 + b_2, \\ \frac{-x+(b_1+c_2)}{\beta_2}, & b_1 + b_2 < x \leq c_1 + c_2 - \beta_1. \end{cases} \tag{12}$$

The L-R representation of this membership function is

$$\langle b_1 + b_2, \alpha_2, \beta_2 \rangle_{1-x,1-x} .$$

So for the general case;

$$\langle b_1, \alpha_1, \beta_1 \rangle_{1-x,1-x} +_{T_P} \langle b_2, \alpha_2, \beta_2 \rangle_{1-x,1-x} \cong \langle b_1 + b_2, \max\{\alpha_1, \alpha_2\}, \max\{\beta_1, \beta_2\} \rangle_{1-x,1-x} .$$

The distance of the approximation (12) to the exact result (4) with the integral metric is,

$$\int_{\mathbb{R}} |\mu_{A+B}(x) - \widetilde{\mu}_{A+B}(x)|\, dx = \frac{1}{6}\left(\frac{\alpha_1^2}{\alpha_2} + \frac{\beta_1^2}{\beta_2}\right),$$

and the relative error of the approximation can be calculated by the ratio of the distance to the measure of fuzziness,

$$\left( \frac{\frac{1}{6}(\frac{\alpha_1^2}{\alpha_2} + \frac{\beta_1^2}{\beta_2})}{\frac{\alpha_1^2+3\alpha_2^2}{6\alpha_2} + \frac{\beta_1^2+3\beta_2^2}{6\beta_2}} \times 100 \right) \%.$$

**Example 7.1.** *We consider the sum of the triangular fuzzy numbers* $A = (2, 3, 5)$ *and* $B = (6, 8, 11)$ *given in Example 1.*

*The sum has the following membership function:*

$$\mu_{A+B}(x) = \begin{cases} \frac{(x-8)^2}{8}, & 8 \leq x \leq 10, \\ \frac{x-9}{2}, & 10 \leq x \leq 11, \\ \frac{-x+14}{3}, & 11 \leq x \leq 12, \\ \frac{(x-16)^2}{24}, & 12 \leq x \leq 16. \end{cases}$$

*The membership function of the approximation is,*

$$\widetilde{\mu}_{A+B}(x) = \begin{cases} \frac{x-9}{2}, & 9 \leq x \leq 11, \\ \frac{-x+14}{3}, & 11 < x \leq 14. \end{cases}$$

*Then the relative error can be computed to be about* 10%. *The dotted graph in Figure 1 corresponds to the approximation.*

The most important observation is that this approximation equals the sum of $A$ and $B$ with the weak t-norm (see for instance [11]).

For the multiplication process consider the membership function (10), once again a natural approximation to this fuzzy number can be obtained by using the linear pieces in this membership function and extend their domain. For example if $a_1b_2 \leq a_2b_1$ and $c_2b_1 \leq c_1b_2$, the related approximation then is,

$$\mu_{A \cdot B}(x) \simeq \begin{cases} \frac{x-a_1b_2}{\alpha_1 b_2}, & a_1b_2 \leq x \leq b_1b_2, \\ \frac{c_1b_2-x}{\beta_1 b_2}, & b_1b_2 \leq x \leq c_1b_2. \end{cases}$$

This fuzzy number can be described in the L-R setting as

$$\langle b_1b_2, b_1b_2 - a_1b_2, c_1b_2 - b_1b_2 \rangle_{1-x,1-x} = \langle b_1b_2, \alpha_1b_2, \beta_1b_2 \rangle_{1-x,1-x} .$$

Since $a_1b_2 \leq a_2b_1 \Leftrightarrow \alpha_2b_1 \leq \alpha_1b_2$ and $c_2b_1 \leq c_1b_2 \Leftrightarrow \beta_2b_1 \leq \beta_1b_2$, the membership function of the approximation in L-R form can be written as

$$\langle b_1b_2, \max\{\alpha_2b_1, \alpha_1b_2\}, \max\{\beta_2b_1, \beta_1b_2\} \rangle_{1-x,1-x} .$$

This again agrees with the membership function of $A \cdot B$ with the weak t-norm which can be found in [5].

In the light of these facts it can be concluded that whenever multiple arithmetic operations have to be performed, the use of $T_P$ could be replaced with the arithmetic operations with $T_\omega$. It should also be stressed that recent software algorithms designed to perform fuzzy arithmetic based on t-norms with additive generators are capable to deal with product sum and multiplication quite reasonably.

## 8 Future works

A generalization of the results obtained in this paper regarding addition remains to be a challenging problem. However, a comparison with the results in the literature covering the case of fixed spreads of the input variables yields to the following postulate:

**Proposition 8.1.** *Let $A_i = \langle a_i, \alpha_i, \beta_i \rangle_{L,R}$ be fuzzy numbers for $i \in \{1, 2, ..., n\}$. Let $T$ be a t-norm with additive generator $f$. Let the composite functions $f \circ L$ and $f \circ R$ be convex. Then the $T$ sum of $A_i$ has the membership function*

$$\left[\sum_{i=1}^{n} A_i\right](x) = \begin{cases} f^{[-1]}\left(nf\left(L\left(\frac{c-x}{\sum_{i=1}^{n}\alpha_i}\right)\right)\right) & \text{if } c - \sum_{i=1}^{n}\alpha_i \leq x \leq c - n\alpha, \\ \text{linear} & \text{if } c - n\alpha \leq x \leq c, \\ \text{linear} & \text{if } c \leq x \leq c + n\beta, \\ f^{[-1]}\left(nf\left(R\left(\frac{x-c}{\sum_{i=1}^{n}\beta_i}\right)\right)\right) & \text{if } c + n\beta \leq x \leq c + \sum_{i=1}^{n}\beta_i, \end{cases}$$

*where $c = \sum_{i=1}^{n} a_i$, $\alpha = \min\{\alpha_i\}$ and $\beta = \min\{\beta_i\}$.*

Even a proof of this proposition for the particular case $T = T_P$ (so $f$ can be chosen as $-\log x$) would provide exact solutions to iterative additions and also would enable to generalize the results to all strict t-norms. On the other hand, the transformation of the potential generalizations mentioned above to the case of multiplication is also an subject of interest. For this purpose, the equality $A \times_T B = \exp(\log A +_T \log B)$ might be useful.

## References

[1] J. C. Figueroa-García, M. Ceberio, V. Kreinovich, *Algebraic product is the only t-norm for which optimization under fuzzy constraints is scale-invariant*, in: Constraint Programming and Decision Making: Theory and Applications, Springer, (2018), 51-54.

[2] R. Fullér, *On product-sum of triangular fuzzy numbers*, Fuzzy Sets and Systems, **41**(1) (1991), 83-87.

[3] M. Hanss, *Applied fuzzy arithmetic*, Springer-Verlag Berlin Heidelberg, 2005.

[4] D. H. Hong, *A note on product-sum of L-R fuzzy numbers*, Fuzzy Sets and Systems, **66**(3) (1994), 381-382.

[5] D. H. Hong, *Shape preserving multiplications of fuzzy numbers*, Fuzzy Sets and Systems, **123**(1) (2001), 81-84.

[6] M. F. Kawaguchi, T. Da-te, *Some algebraic properties of weakly non-interactive fuzzy numbers*, Fuzzy Sets and Systems, **68**(3) (1994), 281-291.

[7] A. Khurshid, A. Mesiarová-Zemánková, *Choosing t-norms and t-conorms for fuzzy controllers*, in: Fourth International Conference on Fuzzy Systems and Knowledge Discovery (FSKD 2007), **2** (2007), 641-646.

[8] M. Landowski, *Method with horizontal fuzzy numbers for solving real fuzzy linear systems*, Soft Computing, **23**(12) (2019), 3921-3933.

[9] A. Marková, *T-sum of L-R fuzzy numbers*, Fuzzy Sets and Systems, **85**(3) (1997), 379-384.

[10] R. Mesiar, *A note to the t-sum of L-R fuzzy numbers*, Fuzzy Sets and Systems, **79**(2) (1996), 259-261.

[11] R. Mesiar, *Shape preserving additions of fuzzy intervals*, Fuzzy Sets and Systems, **86**(1) (1997), 73-78.

[12] H. T. Nguyen, *A note on the extension principle for fuzzy sets*, Journal of Mathematical Analysis and Applications, **64**(2) (1978), 369-380.

[13] A. Piegat, M. Landowski, *Horizontal membership function and examples of its applications*, International Journal of Fuzzy Systems, **17**(1) (2015), 22-30.

[14] A. Piegat, M. Landowski, *Multidimensional interval type 2 epistemic fuzzy arithmetic*, Iranian Journal of Fuzzy Systems, **18**(5) (2021), 19-36.

[15] N. G. Seresht, A. R. Fayek, *Computational method for fuzzy arithmetic operations on triangular fuzzy numbers by extension principle*, International Journal of Approximate Reasoning, **106** (2019), 172-193.

[16] E. Triesch, *On the convergence of product-sum series of L-R fuzzy numbers*, Fuzzy Sets and Systems, **53**(2) (1993), 189-192.

[17] L. A. Zadeh, *The concept of a linguistic variable and its application to approximate reasoning*, in: Learning Systems and Intelligent Robots, Springer, (1974), 1-10.