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On deferred statistical $A-$convergence of fuzzy sequence and applications

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Abstract

This paper introduces the idea of deferred-statistical $A-$convergence of order β of the sequence of fuzzy numbers by using a regular matrix A and deferred Cesàro mean $D_{p,q}$. Also, we establish some relations between the proposed idea and the strong deferred $A-$summability of sequences of fuzzy numbers. As an application, we apply this newly statistical convergence for proving fuzzy Korovkin-type approximation theorem. Some illustrative examples are provided to justify the results obtained from this investigation.

Keywords: Fuzzy sequence, statistical convergence, regular matrix, fuzzy-deferred Cesàro mean, fuzzy number, deferred-statistical convergence, fuzzy type Korovkin- theorem.

1 Introduction

It is known that the convergence (in the classical sense) of a sequence is usually based on the fact that almost all terms of the sequence should be in an arbitrarily small neighborhood of the limit, which is not always possible to analyze for any arbitrary sequence. Therefore, certain general ideas on the convergence of sequences were developed which relax the above fact. The statistical convergence of sequences is one of them, which has been introduced by Fast [19] in 1951 and subsequently developed by Schoenberg [39] in 1959. The idea of statistical convergence is generally based on the convergence condition only for a majority of terms of the sequence.

A sequence (x_n) is said to be statistical converges to a limit ξ, if for given $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n}|\{k \le n : |x_k - \xi| \ge \varepsilon\}| = 0,$$

where vertical lines represent the cardinality of the set enclosed. It is noted that convergence in classical sense implies statistical convergence, but the converse is not true. Therefore, this method of convergence is more general and is used in broader perspectives of the convergence analysis. As a result, this new method of convergence has attracted many prominent researchers to explore its utilities in several fields of mathematics. It is considerably used in the summability theory [10, 15, 17, 20, 37], approximation theory and $p-$Cesaro summability theory [11, 14, 21, 29, 32], fuzzy set theory and applications [3, 4, 5, 6, 9] and many more. The reader can refer to the monographs [13] and [30] for background on the summability theory, and to the papers [40, 42, 43] and [23] for the classical sets of fuzzy valued sequences, and related topics.

In 1965, the idea of fuzzy number was developed by Zadeh [45] by introducing its arithmetic operations and was applied in sequence space theory by Matloka [27] in 1986. Further, its applications were found in various fields of summability theory such as topological structures and fuzzy metric theory and other algebraic properties of sequence spaces of fuzzy numbers by Nanda [33], Puri and Relscu [36], Talo and Başar [41], Savaş and Mursaleen [38], Tripathy

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and Nanda [44], Hazarika and Savas [22], etc. Nuray and Savas [35] introduced the statistical convergence of fuzzy sequences and then, it has been generalized and studied by several prominent authors such as Altin et al. [3], Altınok [4], Altinok and Et [6] Küçükaslan and Yılmaztürk [26], On further applications, Anastassiou [7], Anastassiou and Yılmaztürk [8], Mohiuddine et al. [29], etc. applied these ideas in approximation theory. Now, we have following definitions related to fuzzy number and deferred Cesàro mean.

1.1 Fuzzy number

A function $u : \mathbb{R} \to [0,1]$ is said to be a fuzzy number if it satisfies the following properties:

(i) u is normal, i.e., $u(t) = 1$ for some $t \in \mathbb{R}$, (the set of all real numbers),

(ii) u is fuzzy convex, i.e., $u(t) \geq \min\{u(a), u(b)\}$ where $a < t < b$,

(iii) u is upper semi continuous,

(iv) The closure of $u^0 = \{u \in \mathbb{R} : u(t) > 0\}$ is compact.

By $\mathscr{L}(\mathbb{R})$, we denote the set of all fuzzy numbers on $\mathbb{R}$. The set $\mathbb{R}$ is embedded with $\mathscr{L}(\mathbb{R})$, i.e., every $r \in \mathbb{R}$ can be considered by a fuzzy number $\overline{r}(u)$ as

$$\overline{r}(u) = \begin{cases} 1 & (u = r) \\ 0 & (u \neq r). \end{cases}$$

For $\alpha \in (0,1]$ the $\alpha-$level cut of the fuzzy number u is defined by

$$[u]_\alpha = \{u \in \mathbb{R} : u(u) \geq \alpha\},$$

and the distance between two fuzzy numbers u and v is given by

$$d(u,v) = \sup_{\alpha\in[0,1]} d_H([u]_\alpha, [v]_\alpha) = \sup_{\alpha\in[0,1]} \max\{\left|[u]_\alpha^- - [v]_\alpha^-\right|, \left|[u]_\alpha^+ - [v]_\alpha^+\right|\}.$$

Here the symbol $d_H(.,.)$ represents the Hausdorff metric and $[u]_\alpha^+$ and $[u]_\alpha^-$, respectively represent the upper and lower bounds of the $\alpha-$level cut of the fuzzy number u. The set $\mathscr{L}(\mathbb{R})$ with the metric d is a complete metric space. Let $u, v \in \mathscr{L}(\mathbb{R})$. Then the fuzzy sum $u \oplus v$ and fuzzy product $u \odot v$ are, respectively defined by

$$[u \oplus v]_\alpha = [u]_\alpha + [v]_\alpha, \text{ and } [u \odot v]_\alpha = [u]_\alpha [v]_\alpha.$$

The set $\mathscr{L}(\mathbb{R})$ constitutes a fuzzy metric space with the metric d. Indeed, the space $(\mathscr{L}(\mathbb{R}), d)$ is a complete metric space with the metric d and for every $u, v, w, x \in \mathscr{L}(\mathbb{R})$, we have

- $d(\zeta u, \zeta v) = |\zeta| d(u,v)$ for $\zeta \in \mathbb{C}$ (the set of all complex scalars),
- $d(u \oplus w, v \oplus w) = d(u,v)$,
- $d(u \oplus w, v \oplus x) \leq d(u,v) + d(w,x)$
- $|d(u,\overline{0}) - d(v,\overline{0})| \leq d(u,v) \leq d(u,\overline{0}) + d(v,\overline{0})$,

where $\overline{0}$ is the additive identity element of $\mathscr{L}(\mathbb{R})$.

1.2 Deferred Cesàro mean

Let $p = (p_n)$ and $q = (q_n)$ be two sequences of non-negative integers satisfying

(i) $p_n < q_n$ for all $n \in \mathbb{N}_0$ (the set of all non-negative integers i.e., $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$).

(ii) $\lim_{n\to\infty} q_n = \infty$, Then, the deferred Cesàro mean (see, also [1, 34]) of the sequence $u=(u_n)$ is defined by

$$(D_{p,q}u)_n = \frac{u_{p_n+1}+u_{q_n+2}+\cdots+u_{q_n}}{q_n-p_n} = \sum_{k=0}^{\infty} d_{nk}u_k,$$

where the infinite matrix $D_{p,q}=(d_{nk})$ is stated by

$$d_{nk} = \begin{cases} \frac{1}{q_n-p_n}, & (p_n < k \leq q_n) \\ 0, & (\text{ otherwise}). \end{cases}$$

It is known that (i) and (ii) are the regularity conditions for the deferred Cesàro mean $D_{p,q}$. This definition includes several known existing means such as $D_{0,n}=(C,1)$, the Cesàro mean of order one and $D_{n-\lambda_n^*+1,n}=(V,\lambda^*)$, the de la Vallée Poussin mean, etc. where $\lambda^*=(\lambda_n^*)$ being a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_n^* \leq \lambda_n^*+1$ and $\lambda_0^*=1$.

An infinite matrix $A=(a_{nk})$ is said to be conservative if the $A-$transform of x, i.e., $Ax \in c$ (the space of all convergent sequences), for $x=(x_k)\in c$. Matrix A is said to be regular if it is conservative and $\lim Ax = \lim x$. The followings are the regularity conditions for the matrix A (see [16]),

(i) $\sup_n \sum_k |a_{nk}| < \infty$,

(ii) $\lim_{n\to\infty} a_{nk}=0$, for each k,

(iii) $\lim_{n\to\infty}\sum_k a_{nk}=1$.

A sequence (x_n) is said to be statistically convergent to ξ provided that for every $\varepsilon>0$ the set $K(\varepsilon)=\{k\in\mathbb{N} : |x_k-\xi|\geq\varepsilon\}$ has the natural density zero (see Fast [19]), i.e.,

$$\delta(K(\varepsilon)) = \lim_{n\to\infty}\frac{1}{n}|\{k\leq n : |x_k-\xi|\geq\varepsilon\}| = 0.$$

A sequence (x_n) is said to be $A-$statistically convergent to ξ provided that for every $\varepsilon>0$ the set $K(\varepsilon)$ has the $A-$density zero (see Kolk [24]), i.e.,

$$\delta_A(K(\varepsilon)) = \lim_{n\to\infty}\sum_{k\in K(\varepsilon)} a_{nk} = 0.$$

It is remarked that for $A=I$ (the identity matrix) the concept of $A-$statistical convergence coincides with the convergence in classical sense, and for $A=(C,1)$ (Cesàro matrix of order one) with the usual statistical convergence (see, [18]).

Using all the definitions defined above, we define deferred $A-$ summable mean of a sequence $x=(x_k)$ as

$$s_n = (Ax)_n = \sum_{k=p_n+1}^{q_n} a_{nk}x_k. \tag{1}$$

A sequence $u=(u_n)$ of fuzzy numbers is said to be deferred statistical $A-$convergent of order β, $(0<\beta\leq 1)$ or $(S^A_{p,q,\beta})-$ convergent to a fuzzy number u_0 if for every $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{1}{(q_n-p_n)^\beta}|\{p_n<k\leq q_n : d((Au)_k,u_0)\geq\varepsilon\}| = 0, \tag{2}$$

where vertical line indicates the cardinality of the set enclosed. In this case we write $(S^A_{p,q,\beta})-\lim_n u_n = u_0$.

A sequence $u=(u_n)$ of fuzzy numbers is said to be strongly deferred $A-$summable of order β or $(W^A_{p,q,\beta})-$summable(or $(W^A_{p,q,\beta})-$ converges to u_0) if there exists a fuzzy number u_0 such that

$$\lim_{n\to\infty}\frac{1}{(q_n-p_n)^\beta}\left(\sum_{k=p_n+1}^{q_n} d((Au)_k,u_0)\right) = 0.$$

Note that the deferred statistical convergence of order β generalizes the idea of statistical convergence for the special case $q_n=n$, $p_n=0$ and $\beta=1$, and the limit is unique. It is observed that deferred statistical convergence is well-defined for the order $0<\beta\leq 1$.

A sequence of fuzzy numbers (u_n) is said to have m numbers of deferred statistical $A-$cluster points $u^{(i)}$ $i = 1, 2, \dots m$, if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{(q_n - p_n)} |\{p_n < k \leq q_n : d((Au)_k, u^{(i)}) < \varepsilon\}| \neq 0.$$

From the above definition, one can easily find some $0 < \mu_i < 1$ for $i = 1, 2, \dots m$ such that

$$|\{p_n < k \leq q_n : d((Au)_k, u^{(i)}) \geq \varepsilon\}| = \mathcal{O}((q_n - p_n)^{\mu_i}),\ i = 1, 2, 3, \dots m.$$

Remark 1.1. It is noted that the idea of deferred statistical $A-$convergence of order β for a sequence of fuzzy numbers (u_n) is well defined and associated with a unique limit if

$$0 < \mu < \beta \leq 1 \text{ or } \beta \in (\mu, 1],$$

where $\mu = \min_i\{\mu_i\}$.

To prove this we provide the following counter examples:

Example 1.2. Let us define the sequence (u_n) of fuzzy numbers by

$$u_n(t) = \begin{cases} t, & \text{if } n \text{ is odd,} \\ 1 - t, & \text{if } n \text{ is even,} \end{cases} \quad t \in [0, 1]$$

and the matrix $A = (a_{nk})$ as

$$a_{nk} = \begin{cases} \frac{1}{2}, & \text{if } n \text{ is a cube and } k = n^3, \\ \frac{1}{3}, & \text{if } n \text{ is a non cube and } k = n^3, k = n^3 + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then, it is easy to calculate that

$$(Au(t))_n = \sum_{k=1}^{\infty} a_{nk} u_k = \begin{cases} \frac{t}{2} (= u'), & \text{if } n \text{ is an odd cube,} \\ \frac{1-t}{2} (= u''), & \text{if } n \text{ is an even cube,} \\ \frac{2}{3}, & \text{otherwise.} \end{cases}$$

Therefore, we have

$$d((Au(t))_n, u') = \begin{cases} 0, & \text{if } n \text{ is an odd cube,} \\ \sup\limits_{\alpha\in[0,1]} \max\{|[u'']_\alpha^- - [u']_\alpha^-|, |[u'']_\alpha^+ - [u']_\alpha^+|\}, & \text{if } n \text{ is an even cube,} \\ \sup\limits_{\alpha\in[0,1]} \max\{|[u']_\alpha^-|, |[u']_\alpha^+|\}, & \text{otherwise.} \end{cases}$$

$$d((Au(t))_n, u'') = \begin{cases} \sup\limits_{\alpha\in[0,1]} \max\{|[u']_\alpha^- - [u'']_\alpha^-|, |[u']_\alpha^+ - [u'']_\alpha^+|\}, & \text{if } n \text{ is an odd cube,} \\ 0, & \text{if } n \text{ is an even cube,} \\ \sup\limits_{\alpha\in[0,1]} \max\{|[u'']_\alpha^-|, |[u'']_\alpha^+|\}, & \text{otherwise.} \end{cases}$$

and

$$d((Au(t))_n, 2/3) = \begin{cases} \sup\limits_{\alpha\in[0,1]} \max\{|[u']_\alpha^-|, |[u']_\alpha^+|\}, & \text{if } n \text{ is a odd cube,} \\ \sup_{\alpha\in[0,1]} \max\{|[u'']_\alpha^-|, |[u'']_\alpha^+|\}, & \text{if } n \text{ is an even cube,} \\ 0, & \text{otherwise.} \end{cases}$$

Assume that $\beta > 1, p_n = 0, q_n = n$ and for given $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{(q_n - p_n)^\beta} |\{p_n < k \leq q_n : d((Au(t))_k, u') \geq \varepsilon\}| \leq \lim_{n\to\infty} \frac{2n - \sqrt[3]{n}}{2n^\beta} = 0,$$

$$\lim_{n\to\infty}\frac{1}{(q_n-p_n)^\beta}|\{p_n<k\le q_n : d((Au(t))_k,u'')\ge\varepsilon\}|\le\lim_{n\to\infty}\frac{2n-\sqrt[3]{n}}{2n^\beta}=0,$$

and

$$\lim_{n\to\infty}\frac{1}{(q_n-p_n)^\beta}|\{p_n<k\le q_n : d((Au(t))_k,2/3)\ge\varepsilon\}|\le\lim_{n\to\infty}\frac{\sqrt[3]{n}}{n^\beta}=0.$$

From this example we have concluded that the deferred statistical $A-$summable limit of order β of a fuzzy sequence (u_n) takes different values i.e., u', u'' and $3/2$. But, as per the primary definition of deferred statistical $A-$summable limit of order β of a sequence, it must be is unique. In this example the deviation of this uniqueness is due to the poor choice of β, i.e., $\beta>1$. Otherwise, if we choose $\frac{1}{3}<\beta\le 1$, then the deferred statistical $A-$summable limit of order β of (u_n) is unique and it is found to be $2/3$, i.e.,

$$(S^A_{p,q,\beta})-\lim_n u_n=\frac{2}{3}.$$

Example 1.3. Consider the sequence (u_n) of fuzzy numbers, defined by

$$u_n(t)=\begin{cases}t, & \text{if } n \text{ is a square},\\ t/2, & \text{otherwise},\end{cases}\quad t\in[0,1],$$

and the matrix $A=(a_{nk})$, defined by

$$a_{nk}=\begin{cases}1, & \text{if } n \text{ is a square and } k=n^2,\\ \frac{1}{2}, & \text{if } n \text{ is a non square and } k=n^2, k=n^2+1,\\ 0, & \text{otherwise.}\end{cases}$$

Then, we can evaluate

$$(Au(t))_n=\begin{cases}t, & \text{if } n \text{ is a square},\\ \frac{3t}{4}, & \text{otherwise.}\end{cases}.$$

Now, we have

$$d((Au(t))_n,t)=\begin{cases}0, & \text{if } n \text{ is a square},\\ \sup\limits_{\alpha\in[0,1]}\max\{|[3t/4]^-_\alpha-[t]^-_\alpha|,|[3t/4]^+_\alpha-[t]^+_\alpha|\}, & \text{otherwise},\end{cases}$$

and

$$d((Au(t))_n,3t/4)=\begin{cases}\sup_{\alpha\in[0,1]}\max\{|[t]^-_\alpha-[3t/4]^-_\alpha|,|[t]^+_\alpha-[3t/4]^+_\alpha|\}, & \text{if } n \text{ is a square},\\ 0, & \text{otherwise.}\end{cases}$$

If $p_n=0, q_n=n$, then for given $\varepsilon>0$, we have

$$\lim_{n\to\infty}\frac{1}{(q_n-p_n)^\beta}|\{p_n<k\le q_n : d((Au(t))_k,t)\ge\varepsilon\}|\le\lim_{n\to\infty}\frac{n-\sqrt{n}}{n^\beta},$$

and

$$\lim_{n\to\infty}\frac{1}{(q_n-p_n)^\beta}|\{p_n<k\le q_n : d((Au(t))_k,3t/4)\ge\varepsilon\}|\le\lim_{n\to\infty}\frac{\sqrt{n}}{n^\beta}.$$

In order to follow the uniqueness of the deferred statistical $A-$summable limit of order β of a fuzzy sequence, we must take β from the interval $(1/2,1]$. Indeed, in this interval, the deferred statistical $A-$summable limit of order β of (u_n) is unique and

$$(S^A_{p,q,\beta})-\lim_n u_n=\frac{3t}{4},\ t\in[0,1].$$

2 Main results

In this section, we establish some results on deferred statistical $A-$summability and strongly deferred $A-$summability of order β for fuzzy sequences.

Theorem 2.1. Let $u=(u_n)$ and $v=(v_n)$ be two fuzzy sequences of real numbers, $\mathbb{C}$ be the set of all complex numbers, and $\mu<\beta\le 1$. Then the following statements hold:

(i) If $(S^A_{p,q,\beta}) - \lim_n u_n = u_0$ and $\zeta \in \mathbb{C}$, then $(S^A_{p,q,\beta}) - \lim_n \zeta u_n = \zeta u_0$.

(ii) If $(S^A_{p,q,\beta}) - \lim_n u_n = u_0$,$(S^A_{p,q,\beta}) - \lim_n v_n = v_0$ and $\zeta, \xi \in \mathbb{C}$, then $(S^A_{p,q,\beta}) - \lim_n(\zeta u_n + \xi v_n) = \zeta u_0 + \xi v_0$.

(iii) If $(S^A_{p,q,\beta}) - \lim_n u_n = u_0$ and there exist a fuzzy sequence $v = (v_n)$ such that $(Au)_k = (Av)_k$ for all most of k, $p_n < k \leq q_n$, then $(S^A_{p,q,\beta}) - \lim_n v_n = u_0$.

Proof. (i) The proof requires the following relation. For given $\varepsilon > 0$,

$$\begin{aligned}\{p_n < k \leq q_n : d((A\mu u)_k, \zeta u_0) \geq \varepsilon\} &= \{p_n < k \leq q_n : |\zeta| d((Au)_k, u_0) \geq \varepsilon\} \\ &\subseteq \left\{p_n < k \leq q_n : d((Au)_k, u_0) \geq \frac{\varepsilon}{|\zeta|}\right\}\end{aligned}$$

(ii) This is direct consequence of the following relation:

$$\begin{aligned}\{p_n < k \leq q_n : d((A(\zeta u + \xi v))_k, &\zeta u_0 + \xi v_0) \geq \varepsilon\} \\ &\subseteq \{p_n < k \leq q_n : [|\zeta| d((Au)_k, u_0) + |\xi| d((Av)_k, v_0)] \geq \varepsilon\} \\ &\subseteq \left\{p_n < k \leq q_n : |\zeta| d((Au)_k, u_0) \geq \frac{\varepsilon}{2}\right\} \\ &\quad \cup \left\{p_n < k \leq q_n : |\xi| d((Av)_k, v_0) \geq \frac{\varepsilon}{2}\right\} \\ &\subseteq \left\{p_n < k \leq q_n : d((Au)_k, u_0) \geq \frac{\varepsilon}{2|\zeta|}\right\} \\ &\quad \cup \left\{p_n < k \leq q_n : d((Av)_k, v_0) \geq \frac{\varepsilon}{2|\xi|}\right\}.\end{aligned}$$

(iii) Consider $(S^A_{p,q,\beta}) - \lim_n u_n = u_0$ and a sequence $v = (v_n)$ such that $(Au)_k = (Av)_k$ for all most of k, $p_n < k \leq q_n$. Then

$$\lim_{n\to\infty} \frac{1}{(q_n - p_n)^\beta} |\{p_n < k \leq q_n : d((Au)_k, u_0) \geq \varepsilon\}| = 0,$$

and

$$\lim_{n\to\infty} \frac{1}{(q_n - p_n)^\beta} |\{p_n < k \leq q_n : d((Au)_k, Av_k) \geq \varepsilon\}| = 0.$$

We use the following relation for the proof:

$$\begin{aligned}\{p_n < k \leq q_n : d((Av)_k, u_0) \geq \varepsilon\} \subseteq &\{p_n < k \leq q_n : d((Au)_k, u_0) \geq \varepsilon\} \\ &\cup \{p_n < k \leq q_n : d((Av)_k, u_0) \geq \varepsilon \text{ and } (Au)_k \neq (Av)_k\} \\ \subseteq &\{p_n < k \leq q_n : d((Au)_k, u_0) \geq \varepsilon\} \\ &\cup \{p_n < k \leq q_n : (Au)_k \neq Av_k\}\end{aligned}$$

□

Theorem 2.2. Let $u = (u_n)$ be a fuzzy sequence of real numbers, and $\mu < \beta \leq 1$. Then, the following statements hold:

(i) If $u = (u_k)$ is $(W^A_{p,q,\beta})-$summable to u_0, then $(S^A_{p,q,\beta}) - \lim_n u_n = u_0$, but the converse is not true in general.

(ii) If $\beta = 1, (S^A_{p,q,\beta}) - \lim_n u_n = u_0$ and $u \in A(\ell_\infty)$, then u is $(W^A_{p,q,\beta})-$summable to u_0.

(iii) For $\beta = 1$, $W^A_{p,q,\beta} \cap \ell^A_\infty = S^A_{p,q,\beta} \cap \ell^A_\infty$,

where $W^A_{p,q,\beta}, S^A_{p,q,\beta}, \ell^A_\infty$, respectively are the spaces of all strongly $(W^A_{p,q,\beta})-$summable, $A-$deferred statistical convergent and $A-$bounded sequences of fuzzy numbers (i.e.,$\{u_k : \sup_k d((Au)_k, 0) < K < \infty\}$).

Proof. (i) Suppose u is $(W^A_{p,q,\beta})-$summable to u_0. Then

$$\lim_{n\to\infty}\frac{1}{(q_n-p_n)^\beta}\left(\sum_{k=p_n+1}^{q_n} d((Au)_k,u_0)\right)=0. \tag{3}$$

Now, we can write

$$\begin{aligned}\sum_{k=p_n+1}^{q_n} d((Au)_k,u_0) &= \sum_{\substack{k=p_n+1,\\ d((Au)_k,u_0)<\epsilon}}^{q_n} d((Au)_k,u_0)+\sum_{\substack{k=p_n+1,\\ d((Au)_k,u_0)\geq\epsilon}}^{q_n} d((Au)_k,u_0)\\ &\geq \sum_{\substack{k=p_n+1,\\ d((Au)_k,u_0)\geq\epsilon}}^{q_n} d((Au)_k,u_0).\end{aligned}$$

From (3) and the definition of the deferred statistical limit of order β, we conclude that $(S^A_{p,q,\beta})-\lim_n u_n=u_0$. For the converse part, we have the following example:

Consider the sequence $((Au)_k)$, defined by

$$(Au)_k=\begin{cases}k^2, & [\sqrt{q_n-p_n}]-\rho<k\leq[\sqrt{q_n-p_n}],\\ 0, & \text{otherwise},\end{cases}$$

where ρ being a fixed natural number and $[\alpha]$ is the integral part of α. Taking $\beta=1$, we write

$$\lim_{n\to\infty}\frac{1}{(q_n-p_n)^\beta}\left|\{p_n<k\leq q_n : d((Au)_k,0)\geq\varepsilon\}\right|=\lim_{n\to\infty}\frac{\rho}{(q_n-p_n)^\beta}=0.$$

Thus, $(S^A_{p,q,\beta})-\lim_n u_n=0$. For the the second part, we have

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{(q_n-p_n)^\beta}\left(\sum_{k=p_n+1}^{q_n} d((Au)_k,0)\right) &\geq \lim_{n\to\infty}\frac{1}{(q_n-p_n)^\beta}\sum_{\substack{k=p_n+1,\\ d((Au)_k,0)\geq\epsilon}}^{q_n} d((Au)_k,0)\\ &\geq \lim_{n\to\infty}\frac{\sigma}{(q_n-p_n)^\beta}([\sqrt{q_n-p_n}]-\rho)^2\\ &=\sigma(\neq 0).\end{aligned}$$

Hence, the sequence $u=(u_k)$ is not $(W^A_{p,q,\beta})-$summable to 0.

(ii) This follows from the following relation:

$$\begin{aligned}\frac{1}{(q_n-p_n)^\beta}\left(\sum_{k=p_n+1}^{q_n} d((Au)_k,u_0)\right) &=\frac{1}{(q_n-p_n)^\beta}\sum_{\substack{k=p_n+1,\\ d((Au)_k,u_0)<\epsilon}}^{q_n} d((Au)_k,u_0)+\frac{1}{(q_n-p_n)^\beta}\sum_{\substack{k=p_n+1,\\ d((Au)_k,u_0)\geq\epsilon}}^{q_n} d((Au)_k,u_0)\\ &\leq\frac{\varepsilon}{(q_n-p_n)^\beta}\sum_{\substack{k=p_n+1,\\ d((Au)_k,u_0)<\epsilon}}^{q_n} 1+\frac{1}{(q_n-p_n)^\beta}\sum_{\substack{k=p_n+1,\\ d((Au)_k,u_0)\geq\epsilon}}^{q_n} d((Au)_k,u_0).\end{aligned}$$

Since $\beta=1$ and $(u_n)\in\ell^A_\infty$, the sequence $d((Au)_k,u_0)$ is also bounded. Taking limit $n\to\infty$ in the above relation we complete the proof. To prove (iii) we can use (i) and (ii). □

Theorem 2.3. If (u_n) is $A-$deferred statistically convergent (of order β, $\mu<\beta\leq 1$) to u_0, then it must $A-$statistical converges to u_0. But the converse part is true if the sequence $\left(\sum_{k=1}^\infty(-1)^k\binom{\beta}{k}\left(\frac{p_n}{q_n-p_n}\right)^k\right)_{n\in\mathbb{N}}$ is bounded.

Proof. The proof of 1st part is simple, hence omitted. For the second part, we use the following relation:

$$\begin{aligned}
&\frac{1}{(q_n-p_n)^\beta}|\{p_n<k\le q_n: d((Au)_k,u_0)\ge\varepsilon\}| \\
&\qquad \le \frac{1}{(q_n-p_n)^\beta}|\{0<k\le q_n: d((Au)_k,u_0)\ge\varepsilon\}| \\
&\qquad = \left(1+\frac{p_n}{q_n-p_n}\right)^\beta \frac{|\{0<k\le q_n: d((Au)_k,u_0)\ge\varepsilon\}|}{q_n^\beta} \\
&\qquad = \left(1+\sum_{k=1}^{\infty}(-1)^k\binom{\beta}{k}\left(\frac{p_n}{q_n-p_n}\right)^k\right)\frac{|\{0<k\le q_n: d(Au_k,u_0)\ge\varepsilon\}|}{q_n^\beta}.
\end{aligned}$$

□

3 Fuzzy Korovkin-type theorems

In this section, we use the $A-$deferred statistically convergence of fuzzy sequence to study the fuzzy Korovkin-type theorems. In 1960, Korovkin [25] who first developed and studied some approximation results using the text functions $f_i(t)=t^i$, $(i=0,1,2,\dots)$, and then, these results were known as the classical version of Korovkin theorem. These theorems were further studied by several authors via weighted statical convergence [31], quasi statistical convergence using lacunary sequence [28], statistical summability for periodic functions[12], statistical summability using difference sequence of fractional order [11], $\alpha\beta-$ statistical convergence [2], etc. These theorems analogous to fuzzy sequences was initially studied by Anastassiou [7]. Then these results were further extended and studied by [8, 28] in the context of statistical convergence.

Let $C[a,b]$ and $C_{\mathscr{F}}[a,b]$, respectively be the spaces of all continuous and fuzzy continuous functions on the interval $[a,b]$. A function $u:[a,b]\to\mathscr{L}(\mathbb{R})$ is said to be fuzzy continuous at t_0 in $[a,b]$ if whenever $t_k\to t_0$, then

$$d^*(u(t_k),u(t_0))\to\infty,\ \text{as } k\to\infty,$$

where $d^*(u(t_k),u(t_0))=\sup_{t\in[a,b]}d(u(t_k),u(t_0))$. An operator $\mathscr{P}:C_{\mathscr{F}}[a,b]\to C_{\mathscr{F}}[a,b]$ is said to fuzzy linear if, for every $\lambda_1,\lambda_2\in\mathbb{R}$ and $u,v\in C_{\mathscr{F}}[a,b]$, we have

$$\mathscr{P}(\lambda_1\odot u\oplus\lambda_2\odot v;t)=\lambda_1\odot\mathscr{P}(u;t)\oplus\lambda_2\odot\mathscr{P}(v;t).$$

Further, if $\mathscr{P}$ is fuzzy linear, then for every $t\in[a,b]$ and $f,g\in C_{\mathscr{F}}[a,b]$ with $u(t)\le v(t)$, we have

$$\mathscr{P}(u;t)\le\mathscr{P}(v;t).$$

Theorem 3.1. Let $(\mathscr{P}_k)$ be a sequence of fuzzy positive linear operators defined from $C_{\mathscr{F}}[a,b]$ to itself. Suppose that there exists a corresponding sequence $(\overline{\mathscr{P}}_k)$ of positive linear operators from $C[a,b]$ to itself with the relation

$$\{\mathscr{P}_k(u;t)\}_\alpha^\pm=\overline{\mathscr{P}}_k(u_\alpha^\pm;t),\ (k\in\mathbb{N}_0) \tag{4}$$

for every $t\in[a,b],\alpha\in[0,1]$ and $u\in C_{\mathscr{F}}[a,b]$. Assume for every $v_i\in C[a,b]$, we have

$$\lim_k\|\overline{\mathscr{P}}_k(v_i)-v_i\|\to0,\ (i=0,1,2\dots).$$

Then, for all $u\in C_{\mathscr{F}}[a,b]$, we have

$$\lim_k d^*(\mathscr{P}_k(u),u)=0.$$

Theorem 3.2. Let $(\mathscr{P}_k)$ be a sequence of fuzzy positive linear operators defined from $C_{\mathscr{F}}[a,b]$ to itself. Suppose that there exists a corresponding sequence $(\overline{\mathscr{P}}_k)$ of positive linear operators from $C[a,b]$ to itself with the relation (4). Assume for every $v_i\in C[a,b]$, we have

$$S_{p,q,\beta}^A-\lim_k\|(A\overline{\mathscr{P}}(v_i))_k-v_i\|\to0,\ (i=0,1,2\dots).$$

Then, for all $u\in C_{\mathscr{F}}[a,b]$, we have

$$S_{p,q,\beta}^A-\lim_k d^*((A\mathscr{P}(u))_k,u)=0.$$

Proof. Let $\alpha \in [1,0], t \in [a,b]$ and $u \in C_{\mathscr{F}}[a,b]$. Since u is fuzzy bounded, there exists a fuzzy constant $\mathcal{K}$ such that

$$|u_\alpha^\pm(t)| \leq \mathcal{K}_\alpha^\pm, \text{ for all } t \in [a,b].$$

This implies that for every $a < x, t < b$, we have

$$|u_\alpha^\pm(x) - u_\alpha^\pm(t)| \leq 2\mathcal{K}_\alpha^\pm. \tag{5}$$

Secondly, since u is fuzzy continuous on $[a,b]$, for every $x \in [a,b]$ and given $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|u_\alpha^\pm(x) - u_\alpha^\pm(t)| < \varepsilon, \tag{6}$$

whenever $|x-t| < \delta$. Combining (5) and (6), we obtain that

$$|u_\alpha^\pm(x) - u_\alpha^\pm(t)| < \varepsilon + \frac{2\mathcal{K}_\alpha^\pm}{\delta^2}|x-t|^2.$$

Applying fuzzy linear operator $(A\overline{\mathscr{P}}(1,t))_k$ on both the sides for a fixed t, it is noticed that

$$\begin{aligned}
|(A\overline{\mathscr{P}}(u_\alpha^\pm(x);t))_k - (A\overline{\mathscr{P}}(u_\alpha^\pm(t);t))_k| &= |(A\overline{\mathscr{P}}(u_\alpha^\pm(x);t))_k - u_\alpha^\pm(t)(A\overline{\mathscr{P}}(1;t))_k| \\
&< |\varepsilon(A\overline{\mathscr{P}}(1;t))_k + \frac{2K_\alpha^\pm}{\delta^2}(A\overline{\mathscr{P}}((x-t)^2;t))_k| \\
&= |\varepsilon(A\overline{\mathscr{P}}(1;t))_k \frac{2\mathcal{K}_\alpha^\pm}{\delta^2}\left((A\overline{\mathscr{P}}(x^2;t))_k - 2t(A\overline{\mathscr{P}}(x;t))_k + t^2(A\overline{\mathscr{P}}(1;t)_k)\right)|.
\end{aligned}$$

As a consequence, we have

$$\begin{aligned}
&|(A\overline{\mathscr{P}}(u_\alpha^\pm(x);t))_k - u_\alpha^\pm(t)| \\
&\quad = |(A\overline{\mathscr{P}}(u_\alpha^\pm(x);t))_k - u_\alpha^\pm(t)(A\overline{\mathscr{P}}(1;t))_k + u_\alpha^\pm(t)((A\overline{\mathscr{P}}(1;t))_k - 1)| \\
&\quad \leq |\varepsilon(A\overline{\mathscr{P}}(1;t))_k + \frac{2\mathcal{K}_\alpha^\pm}{\delta^2}\left((A\overline{\mathscr{P}}(x^2;t))_k - 2t(A\overline{\mathscr{P}}(x;t))_k + t^2(A\overline{\mathscr{P}}(1;t))_k\right)| \\
&\qquad + |u_\alpha^\pm(t)((A\overline{\mathscr{P}}(1;t))_k - 1)| \\
&\quad = |\varepsilon + \varepsilon(A\overline{\mathscr{P}}(1;t))_k - \varepsilon + \frac{2\mathcal{K}_\alpha^\pm}{\delta^2}\left((A\overline{\mathscr{P}}(x^2;t))_k - t^2 - 2t(A\overline{\mathscr{P}}(x;t))_k + 2t^2\right. \\
&\qquad \left. + t^2(A\overline{\mathscr{P}}(1;t))_k - t^2\right)| + |u_\alpha^\pm(t)((A\overline{\mathscr{P}}(1;t))_k - 1)| \\
&\quad \leq \varepsilon + \varepsilon|(A\overline{\mathscr{P}}(1;t))_k - 1| + \frac{2\mathcal{K}_\alpha^\pm}{\delta^2}|(A\overline{\mathscr{P}}(x^2;t))_k - t^2| + \frac{4\mathcal{K}_\alpha^\pm}{\delta^2}|2t(A\overline{\mathscr{P}}(x;t))_k - t| \\
&\qquad + |t^2(A\overline{\mathscr{P}}(1;t))_k - 1| + |u_\alpha^\pm(t)((A\overline{\mathscr{P}}(1;t))_k - 1)| \\
&\quad \leq \varepsilon + \left(\varepsilon + \frac{2\mathcal{K}_\alpha^\pm M^2}{\delta^2} + \mathcal{K}_\alpha^\pm\right)|(A\overline{\mathscr{P}}(1;t))_k - 1| + \frac{4\mathcal{K}_\alpha^\pm M}{\delta^2}|(A\overline{\mathscr{P}}(x;t))_k - t| \\
&\qquad + \frac{2\mathcal{K}_\alpha^\pm}{\delta^2}|(A\overline{\mathscr{P}}(x^2;t))_k - t^2| \\
&\quad \leq \varepsilon + \mathscr{M}_\alpha\left(|(A\overline{\mathscr{P}}(u_0;t))_k - u_0| + |(A\overline{\mathscr{P}}(u_1;t))_k - u_1| + |(A\overline{\mathscr{P}}(u_2;t))_k - u_2|\right),
\end{aligned}$$

Note that $M = \max\{|a|,|b|\}$ and

$$\mathscr{M}_\alpha^\pm = \max\left\{\varepsilon + \frac{2\mathcal{K}_\alpha^\pm M^2}{\delta^2} + \mathcal{K}_\alpha^\pm, \frac{4\mathcal{K}_\alpha^\pm M}{\delta^2}, \frac{2\mathcal{K}_\alpha^\pm}{\delta^2}\right\}.$$

This leads to the relation

$$\|(A\overline{\mathscr{P}}(u_\alpha^\pm))_k - u_\alpha^\pm\| \leq \varepsilon + \mathscr{M}_\alpha^\pm\left(\|(A\overline{\mathscr{P}}(u_0))_k - u_0\| + \|(A\overline{\mathscr{P}}(u_1))_k - u_1\| + \|(A\overline{\mathscr{P}}(u_2))_k - u_2\|\right).$$

From the definition of $d^*(.,.)$ and the assumption

$$\begin{aligned}
d^*((A\mathscr{P}(u))_k, u) &= \sup_{\alpha\in[1,0]} \max\{\|\{(A\mathscr{P}(u))_k\}_\alpha^- - u_\alpha^-\|, \|\{(A\mathscr{P}(u))_k\}_\alpha^+ - u_\alpha^+\|\} \\
&= \sup_{\alpha\in[1,0]} \max\{\|(A\overline{\mathscr{P}}(u_\alpha^-))_k - u_\alpha^-\|, \|\{(A\overline{\mathscr{P}}(u_\alpha^+))_k - u_\alpha^+\|\}.
\end{aligned}$$

If we take $\mathscr{M}_\alpha = \sup_{\alpha\in[1,0]} \max\{\mathscr{M}_\alpha^-, \mathscr{M}_\alpha^+\}$, then above equation becomes

$$d^*(A\mathscr{P}_k(u), u) \leq \varepsilon + \mathscr{M}_\alpha \left(\|(A\overline{\mathscr{P}}(u_0)_k) - u_0\| + \|(A\overline{\mathscr{P}}(u_1))_k - u_1\| + \|(A\overline{\mathscr{P}}(u_2))_k - u_2\| \right).$$

For given $\varepsilon^* > 0$, such that $0 < \varepsilon < \varepsilon^*$, we set

$$\mathcal{R}^{p,q} = \{p_n < k < q_n : d^*((A\mathscr{P}(u), u))_k \geq \varepsilon^*\},$$

and

$$\mathcal{R}^{p,q} \subseteq \bigcup_{j=0}^{2} \mathcal{R}_j^{p,q},$$

where

$$\mathcal{R}_0^{p,q} = \left\{ p_n < k < q_n : \|(A\overline{\mathscr{P}}(u_0))_k - u_0\| \geq \frac{\varepsilon^* - \varepsilon}{3\mathscr{M}_\alpha} \right\}$$
$$\mathcal{R}_1^{p,q} = \left\{ p_n < k < q_n : d^*\|(A\overline{\mathscr{P}}(u_1))_k - u_1\| \geq \frac{\varepsilon^* - \varepsilon}{3\mathscr{M}_\alpha} \right\}$$
$$\mathcal{R}_2^{p,q} = \left\{ p_n < k < q_n : \|(A\overline{\mathscr{P}}(u_2))_k - u_2\| \geq \frac{\varepsilon^* - \varepsilon}{3\mathscr{M}_\alpha} \right\}.$$

With the consequences of above relations and hypothesis of the theorem, we conclude that

$$S_{p,q,\beta}^A - \lim_k d^*((A\mathscr{P}(u))_k, u) = 0.$$

□

Now, in supporting to the above theorem we provide the following numerical example based on fuzzy Bernstein operators (see [8]).

Example 3.3. Consider the sequence (u_n) of fuzzy real numbers defined by

$$u_n = \begin{cases} 1, & (n = m^2) \\ \frac{1}{2}, & (\text{otherwise}). \end{cases} \quad (m = 1, 2, 3, \ldots).$$

Also, consider the matrix $A = (a_{nk})$, defined by

$$a_{nk} = \begin{cases} 1, & (n = k = m^2) \\ \frac{2}{5}, & (n \neq k) \text{ and } (k = m^2 \text{ or } k = m^2 + 1) \\ 0, & (\text{otherwise}). \end{cases}, (m = 1, 2, 3, \ldots).$$

Taking $A-$transform of the sequence (u_n), we have

$$(Au)_n = \begin{cases} 1, & (n = k = m^2) \\ \frac{3}{5}, & (\text{otherwise}). \end{cases} \quad (m = 1, 2, 3, \ldots).$$

Note that $\{(Au)_n\}$ is not convergent, but the sequence (u_n) is deferred statistically $A-$convergent (of order $\frac{1}{2} < \beta < 1$) to $3/5$.
Let $u \in C_{\mathscr{F}}[a, b]$ and $t \in [0, 1]$. Then the sequence of fuzzy Bernstein operators is defined by

$$\mathcal{B}_n^{\mathcal{L}}(u; t) = \bigoplus_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} \odot f\left(\frac{n}{i}\right),$$

and

$$\{\mathcal{B}_n^{\mathcal{L}}(u; t)\}_\alpha^\pm = \overline{\mathcal{B}}_n(u_\alpha^\pm; t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} u_\alpha^\pm \left(\frac{n}{i}\right),$$

where $\alpha \in [0,1]$ and $u_\alpha^\pm \in C[0,1]$. As per the requirement of Theorem 3.2, we define the sequence of fuzzy positive linear operators via fuzzy Bernstein operator $\mathcal{B}^{\mathcal{L}}$ as

$$A\mathscr{P}_n(u(x);t) = \left((Au)_n + \frac{2}{5}\right) \odot \mathcal{B}_n^{\mathcal{L}}(u;t),$$

where we calculate that

$$A\overline{\mathscr{P}}_n(u_\alpha^\pm;t) = \left((Au)_n + \frac{2}{5}\right) \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} u_\alpha^\pm \left(\frac{n}{i}\right).$$

$$(A\overline{\mathscr{P}}(u_0;t))_n = (Au)_n + \frac{2}{5},$$
$$A\overline{\mathscr{P}}_n(u_1;t) = \left((Au)_n + \frac{2}{5}\right) t,$$

and

$$(A\overline{\mathscr{P}}(u_2;t)_n = \left((Au)_n + \frac{2}{5}\right)\left(t^2 + \frac{t-t^2}{n}\right).$$

Since $S_{p,q,\beta}^A - \lim_n u_n = \frac{3}{5}$, thus, we deduce that

$$S_{p,q,\beta}^A - \lim_n (A\overline{\mathscr{P}}(u_0;t))_n = 1,$$
$$S_{p,q,\beta}^A - \lim_n (A\overline{\mathscr{P}}(u_1;t))_n = t,$$

and

$$S_{p,q,\beta}^A - \lim_n (A\overline{\mathscr{P}}(u_2;t))_n = t^2.$$

As a result we conclude that for $j = 0,1,2$

$$S_{p,q,\beta}^A - \lim_n \|(A\overline{\mathscr{P}}(u_j)_n - u_j\| = 0.$$

Using Theorem 3.2 we have

$$S_{p,q,\beta}^A - \lim_n d^*((A\mathscr{P}(u)_n, u) = 0.$$

Conclusion

The idea of deferred-statistical $A-$convergence of variable order β for the sequence of fuzzy numbers has been introduced, which includes several definitions of the convergence of sequences in both classical and statistical sense. One of the most valuable results of this paper is that $A-$deferred statistically convergence (of order β) implies $A-$statistical convergence which implies the statistical and usual convergence of a fuzzy sequence. An important consequence of this fact was illustrated by an example Further, an application of this idea, we have studied the fuzzy Korovkin-type theorems and justified the results by using certain example.

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