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Original paper

A note on divisible discrete triangular norms

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Abstract

Triangular norms and conorms on $[0,1]$ as well as on finite chains are characterized by 4 independent properties, namely by the associativity, commutativity, monotonicity and neutral element being one of extremal points of the considered domain (top element for t-norms, bottom element for t-conorms). In the case of $[0,1]$ domain, earlier results of Mostert and Shields on I-semigroups can be used to relax the latest three properties significantly, once the continuity of the underlying t-norm or t-conorm is considered. The aim of this short note is to show a similar result for finite chains, we significantly relax 3 basic properties of t-norms and t-conorms (up to the associativity) when the divisibility of a t-norm or of a t-conorm is considered.

Keywords: Discrete I-semigroup, divisible t-norm, divisible t-conorm, finite chain.

1 Introduction

Triangular norms (t-norms, in short) were first introduced by Menger [15] in 1942 as binary operations on $[0,1]$ (related to the domain and the range of distribution functions) in the framework of statistical metric spaces to model triangular inequality of metrics under some milder conditions (e.g., no associativity, no neutral element, but $T(a,1) > 0$ for each $a > 0$ were required, etc.; for more details see [8]). Later, Schweizer and Sklar [19, 20] have stated the actual definition of triangular norms (and of their dual operations, triangular conorms). Characterization of continuous t-norms was completed by Ling [10], where she has also mentioned the link between continuous t-norms, continuous t-conorms, and special topological semigroups, called I-semigroups, introduced and characterized by Mostert and Shields [16].

An intensive growth of interest in t-norms and t-conorms was connected with the development of fuzzy set theory [22], and particularly of fuzzy logic theory [6, 7, 18], where t-norms play the role of conjunction and t-conorms model disjunction. Fuzzy logics are special instances of many-valued logics whose study was initiated by Łukasiewicz [11, 12, 13] around 1920. The original Łukasiewicz approach has dealt with 3-valued logics, and later it was extended to n-valued logics, with truth values forming a finite chain.

Observe that t-norms and t-conorms on finite chains were deeply studied by several authors, see, e.g., [5, 14, 21], and, following [8], they are commonly called discrete t-norms and discrete t-conorms. Surprisingly the link between finite I-semigroups and t-norms (t-conorms) on finite chains is still missing. The aim of this paper is to fill this gap. Observe that one of our results can be deduced from the results related to BL-logics [7]. In particular, the redundancy of commutativity axiom was shown first by S. Lehmke [9] (in fact, this result was obtained by means of an automatic prover), and, independently, also in [4], where purely logical arguments based on some tautologies were applied. Some related results can be found also in [3]. In our approach, purely algebraic and analytical arguments are used, and the paper is self-contained. We show not only the redundancy of the commutativity axiom for discrete divisible t-norms (t-conorms), but also possible relaxation of some other axioms.

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Note that though our results are primarily interesting for the theory, bringing a much weaker axiomatic framework for divisible t-norms and t-conorms, it has also an important practical impact. To check whether a binary operation $\star$ on a finite chain L is a t-norm or a t-conorm is a problem with rather high computational complexity, what is, due to our results, not more the case of divisible t-norms, where the computational complexity is significantly reduced.

2 Preliminaries

Throughout this paper, let L be any non-trivial finite chain, i.e., L is a linearly ordered set with a bottom element $\mathsf{e} \in L$ and a top element $\mathsf{u} \in L$. For any two elements $a, b \in L$, $a \leq b$, let $[a, b]$ be defined as $[a, b] = \{x \in L : a \leq x \leq b\}$. Hence $L = [\mathsf{e}, \mathsf{u}]$.

Definition 2.1. [14] *A* t-norm *on L is a function $\mathsf{T}: L^2 \to L$ (discrete t-norm) such that conditions*

(T1) $\mathsf{T}(\mathsf{T}(x, y), z) = \mathsf{T}(x, \mathsf{T}(y, z))$*, i.e.,* T *is associative;*

(T2) $\mathsf{T}(x, y) = \mathsf{T}(y, x)$*, i.e.,* T *is commutative;*

(T3) if $y \leq z$*, then* $\mathsf{T}(x, y) \leq \mathsf{T}(x, z)$*, i.e.,* T *is monotone; and*

(T4) $\mathsf{T}(x, \mathsf{u}) = x = \mathsf{T}(\mathsf{u}, x)$*, i.e.,* u *is neutral element of* T

hold for all $x, y, z \in L$.

Example 2.2. *If we consider $L = \{0, 1, 2, 3, 4\}$, and a standard ordering $0 < 1 < 2 < 3 < 4$ (this convention will be considered also in the rest of the paper), our chain has 5 elements,* $\mathrm{card}(L) = 5$*. Then there are 22 different functions $\mathsf{T}: L^2 \to L$ satisfying axioms (T1)-(T4), i.e., 22 different t-norms on L. The functions $\mathsf{T}_i: L^2 \to L$, $i = 1, 2, 3$, are examples of them:*

T_1	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	1	2
3	0	0	1	1	3
4	0	1	2	3	4

T_2	0	1	2	3	4
0	0	0	0	0	0
1	0	1	1	1	1
2	0	1	1	1	2
3	0	1	1	1	3
4	0	1	2	3	4

T_3	*0*	*1*	*2*	*3*	*4*
0	0	0	0	0	0
1	0	1	1	1	1
2	0	1	1	2	2
3	0	1	2	3	3
4	0	1	2	3	4

Definition 2.3. [14] *A* t-conorm *on L is a function $\mathsf{S}: L^2 \to L$ (discrete t-conorm) such that conditions (T1), (T2), (T3) and a condition*

(S4) $\mathsf{S}(x, \mathsf{e}) = x$*, i.e.,* e *is neutral element of* S*,*

holds for all $x, y, z \in L$.

Observe that the axioms are independent if $\mathrm{card}(L) \geq 4$, see the next example.

Example 2.4. *Let $L = \{0, 1, 2, 3\}$. The next functions $\mathsf{K}_i: L^2 \to L$, $i = 1, 2, 3, 4$, satisfy all axioms (T1)-(T4) up to the axiom (Ti):*

K_1	0	1	2	3
0	0	0	0	0
1	0	0	2	1
2	0	2	2	2
3	0	1	2	3

K_2	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	1	2	2
3	0	1	2	3

K_3	0	1	2	3
0	0	0	0	0
1	0	2	2	1
2	0	2	2	2
3	0	1	2	3

K_4	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0

Well-known basic t-norms and t-conorms on any bounded chain $L = \{0, 1, \dots, n\}$ are given as follows:

$$\mathsf{T}_{\mathbf{M}}(x, y) = \min\{x, y\}, \quad \mathsf{T}_{\mathbf{D}}(x, y) = \begin{cases} 0, & \text{if } \max\{x, y\} \neq n, \\ \min\{x, y\}, & \text{if } \max\{x, y\} = n, \end{cases} \quad \mathsf{T}_{\mathbf{Luk}}(x, y) = \max\{0, x + y - 1\},$$

$$\mathsf{S}_{\mathbf{M}}(x, y) = \max\{x, y\}, \quad \mathsf{S}_{\mathbf{D}}(x, y) = \begin{cases} n, & \text{if } \min\{x, y\} \neq 0, \\ \max\{x, y\}, & \text{if } \min\{x, y\} = 0, \end{cases} \quad \mathsf{S}_{\mathbf{Luk}}(x, y) = \min\{x + y, 1\},$$

(we avoid the standard notation for the Łukasiewicz t-norm $\mathsf{T}_{\mathbf{L}}$ not to confuse L with the considered lattice).

Remark 2.5. *From Definitions 2.1 and 2.3, we obtain that* e *is an annihilator of any discrete t-norm and* u *is annihilator of any discrete t-conorm, i.e., we have* $\mathsf{T}(\mathsf{e}, x) = \mathsf{e}$ *and* $\mathsf{S}(\mathsf{u}, x) = \mathsf{u}$ *for all* $x \in L$*. Also,* $\mathsf{T_M}$ *is the only discrete t-norm for which* $\mathsf{T_M}(x, x) = x$ *for all* $x \in L$*, i.e., whose all elements are idempotent. Analogously,* $\mathsf{S_M}$ *is the only discrete t-conorm which all elements of L are idempotent. If L and L′ are two finite non-trivial chains with the same cardinality then for any discrete t-norm on L there is an isomorphic t-norm on L′ and vice versa.*

It is a well known fact that two finite chains L and L' are isomorphic if and only if their cardinalities coincide. As a prototypical chain L with cardinality $n+1$, $n \in \mathbb{N}$, the chain $L = \{0, 1, \ldots, n\}$ with standard ordering $0 < 1 < \cdots < n$ can be considered (or, in fuzzy framework, often $L = \{0, 1/n, 2/n, \ldots, 1\}$). Due to this fact, the study of t-norms and t-conorms on finite chains can be reduced to their study on such prototypical chains. As an advantage of this approach one should mention the possible use of the standard arithmetical operations of addition $+$ and subtraction $-$. Thus we can avoid considerations of predecessors and successors and then several proofs become more transparent and much shorter. So, since now, we will work on chains of the type $L = \{0, 1, \ldots, n\}$.

Definition 2.6. [5, 8, 14] *We say that a discrete t-norm* T *is*

- *divisible if and only if for all* $x, y \in L$ *such that* $x \le y$ *there is* $z \in L$ *such that* $x = \mathsf{T}(y, z)$*;*
- *idempotent-free if and only if the only idempotent elements of* T *are* e *and* u*;*
- *smooth if and only if*

$$\mathsf{T}(x+1, y) - \mathsf{T}(x, y) \in \{0, 1\}, \quad \textit{and} \quad \mathsf{T}(y, x+1) - \mathsf{T}(y, x) \in \{0, 1\}.$$

  for all $x, y \in L$*,* $x \neq \mathsf{u}$*; and*
- *1-Lipschitz if and only if* $|\mathsf{T}(x, y) - \mathsf{T}(x', y')| \le |x - x'| + |y - y'|$ *for all* $x, y, x', y' \in L$*.*

Analogous definitions can be introduced for discrete t-conorms. Now we give a definition of an ordinal sum of t-norms:

Definition 2.7. [5, 14] *Let L be a finite chain and* $\{a_i\}_{i=0}^k \subseteq L$ *be an increasing sequence of elements of L. More, let* T_i *be a discrete t-norm on* $[a_{i-1}, a_i]$ *for all* $i \in \{1, 2, \ldots, k\}$*. Then* $\mathsf{T}\colon L^2 \to L$ *given by*

$$\mathsf{T}(x, y) = \begin{cases} \mathsf{T}_i(x, y), & \textit{if } (x, y) \in [a_{i-1}, a_i]^2, \\ \min\{x, y\}, & \textit{otherwise,} \end{cases}$$

is called an ordinal sum of summands $\langle a_{i-1}, a_i, \mathsf{T}_i \rangle$*,* $i \in \{1, 2, \ldots, k\}$*, and is denoted by* $\mathsf{T} = (\langle a_{i-1}, a_i, \mathsf{T}_i \rangle)_{i=1}^k$*.*

Lemma 2.8. *Let L be a finite chain and* $\{a_i\}_{i=0}^k \subseteq L$ *be an increasing sequence of elements of L. More, let* T_i *be a discrete t-norm on* $[a_{i-1}, a_i]$ *for all* $i \in \{1, 2, \ldots, k\}$*. Then* $\mathsf{T} = (\langle a_{i-1}, a_i, \mathsf{T}_i \rangle)_{i=1}^k$ *is a discrete t-norm in L.*

The proof of this lemma is analogous to its continuous case, see, e.g., [8], and thus will be omitted.

The following theorem relates discrete divisible t-norms and ordinal sums [14, Thm. 7.2.16]:

Theorem 2.9. *Let* T *be a discrete t-norm on L and let* $\{a_i\}_{i=0}^k$ *be an increasing sequence of its idempotent elements. Then* T *is divisible if and only if* $\mathsf{T} = \langle [a_{i-1}, a_i], \mathsf{T}_i \rangle_{i=1}^k$ *where each* T_i *is divisible, discrete and idempotent-free t-norm on* $[a_{i-1}, a_i]$*.*

As before, let L be a finite chain and, without loss of any generality, we may assume that $L = \{0, 1, \ldots, n\}$, where $n \in \mathbb{N}$. An ordered pair $(L, \star)$ is called a *semigroup* if and only if $\star\colon L^2 \to L$ is associative.

Definition 2.10. *We say that* $(L, \star)$ *is a* topological *semigroup if and only if*

$$|x \star y - x' \star y'| \le |x - x'| + |y - y'|,$$

i.e., if and only if $\star$ *is an associative 1-Lipschitz function.*

The next definition was inspired by [16], where the topological semigroups on closed real (extended real) intervals were considered.

Definition 2.11. *We say that* $(L, \star)$ *is a discrete* I-semigroup *if and only if* $(L, \star)$ *is a topological semigroup and one of the elements* $\{0, n\}$ *acts as an idempotent element of* $\star$ *and the second one acts as an annihilator of* $\star$*.*

Note that the greatest discrete I-semigroup on $L = \{0, 1, \dots, n\}$ is related to neutral element 0 and annihilator n, and it is given by $\mathsf{S}(i, j) = \min\{n, i+j\}$ (this is a well-known discrete Łukasiewicz t-conorm). On the other hand, the smallest discrete I-semigroup on L is related to annihilator 0 and neutral element n, and it is given by $\mathsf{T}(i, j) = \max\{0, i+j-n\}$ (and it is known as discrete Łukasiewicz t-norm). Note that these two discrete I-semigroups on L are the only ones not having any non-trivial idempotent elements.

As in the case of discrete t-norms and discrete t-conorms, we say that x is an idempotent element of $\star$ if and only if $x \star x = x$.

Definition 2.12. *We say that a discrete I-semigroup $(L, \star)$ is idempotent-free if and only if the only idempotent elements of $\star$ are $\{0, n\}$. Any idempotent-free discrete I-semigroup is called an* Archimedean *discrete I-semigroup.*

Observe that in each discrete I-semigroup $(L, \star)$, if 0 is its annihilator then n is its neutral element. Similarly, if n is an annihilator then 0 is a neutral element.

Remark 2.13. *For an interested reader, the smoothness of a discrete t-norm (or, a discrete t-conorm) on a general finite chain $L = \{a_0, a_1, \dots, a_n\}$, where $a_0 < a_1 < \dots < a_n$, should be seen as 1-Lipschitz property in index set, i.e., for any $T(a_i, a_j) = a_k$ and $T(a_r, a_p) = a_q$, the inequality*

$$|k - q| \leq |i - r| + |j - p|.$$

3 Main results

In this section we will prove that any discrete I-semigroup operation $\star$ is either a discrete divisible t-norm or a discrete divisible t-conorm and vice versa, any discrete divisible t-norm or discrete divisible t-conorm (together with the career L) is a discrete I-semigroup operation. We start by recalling the next lemma, see [14, Proposition 7.3.3].

Lemma 3.1. *Let $\mathsf{T}\colon L^2 \to L$ be a discrete t-norm or a discrete t-conorm on L. Then T is divisible if and only if T is 1-Lipchitz.*

Note that the greatest t-norm $\mathsf{T}_{\mathbf{M}}$ is divisible, i.e., 1-Lipschitz, as is the Łukasiewicz t-norm $\mathsf{T}_{\mathbf{Luk}}$. More, $\mathsf{T}_{\mathbf{Luk}}$ is the smallest divisible t-norm on L. On the other hand, $\mathsf{T}_{\mathbf{D}}$, the smallest t-norm, is not divisible nor 1-Lipschitz for $n > 2$ (if $n = 1$, all discrete t-norms coincide and hence they are divisible; if $n = 2$, $\mathsf{T}_{\mathbf{Luk}} = \mathsf{T}_{\mathbf{D}}$ and we have only two discrete t-norms, both of them divisible). If $n > 2$, then

$$\left|\mathsf{T}_{\mathbf{D}}(n, n) - \mathsf{T}_{\mathbf{D}}(n-1, n-1)\right| = \left|n - 0\right| = n,$$

while $|n - (n-1)| + |n - (n-1)| = 2$, and hence $\mathsf{T}_{\mathbf{D}}$ is not 1-Lipschitz in such a case.

Using the previous lemma we can prove one side of the main theorem of the paper easily.

Proposition 3.2. *Let $\star$ be a discrete divisible t-norm or a discrete divisible t-conorm on L. Then $(L, \star)$ is a discrete I-semigroup.*

Proof. Because every discrete t-norm $\star$ on L (or, a discrete t-conorm) is an associative binary operation we obtain that $(L, \star)$ forms a semigroup. By prerequisites, the considered discrete t-(co)norm $\star$ is divisible and thus, by previous lemma, $\star$ is 1-Lipschitz which ensures that $(L, \star)$ is a topological semigroup. Lastly, because $\{0, n\}$ is the set of the annihilator and the neutral element of any discrete t-(co)norm, we obtain that $(L, \star)$ is discrete I-semigroup. □

In proving the contrary, we need to show that every discrete I-semigroup operation $\star$ satisfies axioms of discrete t-norms or discrete t-conorms. As we already know that the underlying binary operation of every discrete I-semigroup is associative we trivially obtain that the condition (T1) holds for discrete I-semigroups. More, the set $\{0, n\}$ is the set of the annihilator and the neutral element of every discrete I-semigroup and thus either n acts as a neutral element (ensuring condition (T4)) or 0 acts as a neutral element (ensuring condition (S4)). Two only remaining conditions that need to be proved for discrete I-semigroups are the commutativity (T2) and the monotonicity (T3).

We start by proving the latter, the monotonicity of the binary operation $\star$. The following lemma will be helpful in the process.

Lemma 3.3. *Let $(L, \star)$ be a discrete I-semigroup with annihilator 0. Then $x \star y \leq \min\{x, y\}$ for all $x, y \in L$.*

Proof. The assumption of 0 being the annihilator of $\star$ implies that n is a neutral element of $(L, \star)$. Note that if $x \in \{0, n\}$ then the lemma holds as a consequence of 0 being the annihilator of $\star$:

$$0 \star y = 0 \leq \min\{0, y\},$$

and as a consequence of n being a neutral element of $\star$:

$$n \star y = y \leq \min\{n, y\},$$

for all $y \in L$. Now, for any $x \in L \setminus \{0, n\}$, let

$$M_x = \{z \in L \colon u \star v \leq x \text{ for all } u, v \in L \text{ such that } u \leq z\}.$$

This set is non-empty because $0 \in M_x$ is the annihilator of $\star$. Then set $z_0 = \max M_x$ and note that $z_0 \in M_x$. Now, we prove that there exists $v_0 \in L$ such that $z_0 \star v_0 = x$. Let us start proving that there exists $\bar{v} \in L$ such that $z_0 \star \bar{v} \geq x$. If $z_0 = n$ then take $\bar{v} = n$. Now, if $z_0 \neq n$, we will prove by contrart that there exists such $\bar{v}$: Let us assume that $z_0 \star v < x$ for all $v \in L$. Let us consider a point $\bar{z} = z + 1$ and note that $\bar{z} \in L$ because $z \neq n$. Then we obtain that

$$\bar{z} \star v = (z_0 + 1) \star v \leq z_0 \star v + 1 \leq x.$$

The first inequality is due to the fact that $\star$ is 1-Lipschitz and the second one is due the fact that $z_0 \star v < x$ for all $v \in L$. But this would imply that $\bar{z} \in M_x$ contradicting the fact that z_0 is the maximum of M_x and thus there exists a point $\bar{v} \in L$ such that $z_0 \star \bar{v} \geq x$. Because $z_0 \star 0 = 0 \leq x \leq z_0 \star \bar{v}$ then, by divisibility of $\star$ (implied by Lemma 3.1), we obtain that there exists a point $v_0 \in L$ such that $z_0 \star v_0 = x$. Now, based on the facts that z_0 is an element of M_x and properties of M_x, we have that

$$x \star y = (z_0 \star v_0) \star y = z_0 \star (v_0 \star y) \leq x.$$

Analogously, one can prove that $x \star y \leq y$ implying that $x \star y \leq \min\{x, y\}$. □

The proof of the case when n is the annihilator of $\star$ can be done similarly, and then $x \star y \geq \max\{x, y\}$ for all $x, y \in L$. Now we can prove the monotonicity of any discrete I-semigroup.

Proposition 3.4. *Let $(L, \star)$ be a discrete I-semigroup. Then $\star$ is non-decreasing in both arguments, i.e., $\star$ satisfies (T3).*

Proof. Let $x, x' \in L$ be such that $x \leq x'$, and let $y \in L$ be an arbitrary element. First, let us assume that 0 is the annihilator of $\star$. Based on the facts that $0 \star x' = 0$, $n \star x' = x'$, and the divisibility of $\star$ (again, implied by Lemma 3.1), we know that there exists $x_0 \in L$ such that $x_0 \star x' = x$. Then, since $\star$ is bounded above by minimum, see Lemma 3.3, we have that

$$x \star y = (x_0 \star x') \star y = x_0 \star (x' \star y) \leq \min\{x_0, x' \star y\} \leq x' \star y,$$

and thus $\star$ is non-decreasing in the first argument. The non-decreasingness of the second argument can be shown analogously. The case when n is the annihilator of $\star$ can be proved similarly. □

Now we can prove the commutativity (T2) of any discrete I-semigroup. Note that we define powers of elements of discrete I-semigroup $(L, \star)$ recursively as follows: $x^1 = x$ and $x^{k+1} = x^k \star x$ for all $k \in \mathbb{N}$.

Proposition 3.5. *Let $(L, \star)$ be a discrete I-semigroup. Then $\star$ is commutative, i.e., $\star$ satisfies (T2).*

Proof. Let us start with the case that 0 is the annihilator and n the neutral element of $\star$. Case I: Let us assume that $\star$ is idempotent-free, i.e., $x \star x < x$ for all $x \in L \setminus \{0, n\}$. If this is the case, we prove that $(n-1)^k = n - k$ for $k \in \{1, 2, \ldots, n\}$. Note that the case $k = 1$ is trivially obtained. Now, let us assume that the theorem holds for k, i.e., $(n-1)^k = n - k$. We prove that also $(n-1)^{k+1} = n - k - 1$ holds, if $k < n - 1$. Note that from 1-Lipschitzness (and thus smoothness) and monotonicity of $\star$ we obtain that

$$n \star (n-k) - (n-1) \star (n-k) \in \{0, 1\},$$

and thus $(n-1) \star (n-k) \in \{n-k, n-k-1\}$. We show that necessarily $(n-1) \star (n-k) = n-k-1$. For the sake of a contradiction, let us assume that $(n-1) \star (n-k) = n-k$. Note that if this would be the case, then

$n-k=(n-1)^m \star (n-k)$, where $m \in \mathbb{N}$. This can be proved by induction. If $m=1$ then the equality holds. Now, let us assume that the equality holds for m, and we will prove that it also holds for $m+1$:

$$\begin{aligned} n-k &= (n-1)^m \star (n-k) = (n-1)^m \star \Big((n-1) \star (n-k)\Big) \\ &= \Big((n-1)^m \star (n-1)\Big) \star (n-k) = (n-1)^{m+1} \star (n-k), \end{aligned}$$

i.e., the equality holds also for $m+1$, in other words, $n-k=(n-1)^m \star (n-k)$ for all $m \in \mathbb{N}$. Let us choose specifically $m=k$. Then

$$n-k=(n-1)^k \star (n-k) = (n-k) \star (n-k),$$

contradicts that $\star$ is idempotent-free and thus $(n-1) \star (n-k) = n-k-1$, which completes the proof of equality $(n-1)^k = n-k$ for $k \in \{1,2,\ldots,n\}$. Now, let $x,y \in L$ be arbitrary elements. If $x \in \{0,n\}$ or $y \in \{0,n\}$ then the proof of the commutativity of $\star$ is trivial. Now let us assume that $x,y \notin \{0,n\}$. Then from the fact that $(n-1)^k = n-k$ we obtain that $x=(n-1)^{n-x}$ and $y=(n-1)^{n-y}$. Thus

$$x \star y = (n-1)^{n-x} \star (n-1)^{n-y} = (n-1)^{2n-x-y} = (n-1)^{n-y} \star (n-1)^{n-x} = y \star x,$$

i.e., $\star$ is commutative. Case II: Let us assume that $\star$ is not idempotent-free. Again, if $x \in \{0,n\}$ or $y \in \{0,n\}$, then the commutativity of $\star$ is obtained easily. Let us assume that $x,y \in L \setminus \{0,n\}$ and, without loss of any generality, that $x \leq y$. We will divide the proof into four parts:

- **Case II.1:** x is idempotent element of $\star$, i.e., $x \star x = x$. Then, we obtain that

  $$\min\{x,y\} = x = x \star x = x \star y \leq \min\{x,y\},$$

  and thus $x \star y = \min\{x,y\}$. Similarly, one can prove that $y \star x = \min\{x,y\}$ which implies that $x \star y = y \star x$ as needed.

- **Case II.2:** y is idempotent element of $\star$. Then, from the 1-Lipschitzness (and thus divisibility, see Lemma 3.1) of $\star$ we know that there are elements $u,v \in L$ such that $x = u \star y = y \star v$. But then

  $$\begin{aligned} x \star y &= (u \star y) \star y = u \star (y \star y) = u \star y = x \\ &= y \star v = (y \star y) \star v = y \star (y \star v) = y \star x, \end{aligned}$$

  and again, $x \star y = y \star x$.

- **Case II.3:** There exists $z \in L$ such that $x < z < y$ and z is idempotent. Then we have that

  $$\min\{x,z\} = x \star z \leq x \star y \leq \min\{x,y\},$$

  and thus $x \star y = \min\{x,y\}$. Analogously, one can find that $y \star x = \min\{x,y\}$ and thus $x \star y = y \star x$.

- **Case II.4:** There is no idempotent element in $[x,y]$. Now, let us choose two elements a,b such that

  $$\begin{aligned} a &= \min\{z \in P \colon y \leq z, z \star z = z\}, \\ b &= \max\{z \in P \colon z \leq x, z \star z = z\}. \end{aligned}$$

  Note that these elements exist, because n belongs to the first set and 0 belongs to the second set. Then we have that $0 \leq a < x < y < b \leq n$ and from the fact that $a = a \star a \leq u \star v \leq b \star b = b$ we obtain that for all $u,v \in L$ such that $u,v \in [x,y]$ we have that $u \star v \in [x,y]$. Then $(L', \star\lceil_{L' \times L'})$, where $L' = [x,y]$, is an idempotent-free I-semigroup that is isomorphic to some idempotent-free I-semigroup $(\bar{L}, \bar{\star})$ with $\bar{L} = \{0,1,\ldots,m\}$, where $m \in \mathbb{N}$ is some natural number. Thus the equality $x \star y = y \star x$ follows from the proof of Case I.

The proof for the case where 0 is the neutral element of $\star$ and n its annihilator is analogous. □

Note that the converse of Proposition 3.5 evidently does not hold, i.e., the commutativity of $\star$ does not force $(L,\star)$ to be a discrete I-semigroup. Consider, e.g., the t-norm $\mathsf{T}_{\mathbf{D}}$ (and $n > 2$) to see the failure of such a claim.

Now as a corollary of propositions 3.4, 3.5 and previous discussions we have the following:

Corollary 3.6. *Let $(L,\star)$ be a discrete I-semigroup. Then $\star$ is a discrete divisible t-norm or a discrete divisible t-conorm on L.*

Finally, as a consequence of Proposition 3.2 and Corollary 3.6 we can present the main result of this paper.

Theorem 3.7. *Let L be a finite chain. Then $(L, \star)$ is a discrete I-semigroup if and only if $\star$ is a discrete divisible t-norm or a discrete divisible t-conorm on L.*

Note that we can even relax the constraints of Theorem 3.7 in the following sense.

Theorem 3.8. *Let L be a finite chain. Then $(L, \star)$ is a topological semigroup on the chain L with an annihilator $\mathsf{e} \in \{0, n\}$ and an idempotent element $\mathsf{u} \in \{0, n\} \setminus \{\mathsf{e}\}$ if and only if $\star$ is a discrete divisible t-norm or a discrete divisible t-conorm on L.*

Proof. As already mentioned, we can consider $L = \{0, 1, \dots, n\}$, $n \in \mathbb{N}$. Consider $(L, \star)$ to be a topological semigroup such that $\mathsf{e} = 0$ is an annihilator of $\star$, i.e., for any $x \in L$ we have $x \star 0 = 0$. Due to 1-Lipschitzianity of $\star$, it holds

$$|n \star n - x \star n| = n - x \star n \leq n - x \quad \text{and} \quad |n \star 0 - x \star n| = x \star n \leq x,$$

ensuring $x \star n = x$. Similarly, one can show that $n \star x = x$, i.e., n is a neutral element of $\star$. Hence $(L, \star)$ is a divisible I-semigroup and thus a discrete divisible t-norm. The opposite implication is trivial. □

To stress the importance of Theorem 3.8, we reformulate it avoiding the consideration of topological (I-)semigroups.

Theorem 3.9. *Let L be a finite chain. Then a divisible mapping $\mathsf{T}\colon L^2 \to L$ is a t-norm if and only if it is*

i. associative;

ii. 0_L is annihilator of T; and

iii. 1_L is an idempotent element of T;

Recall that in [5] a complete representation of divisible t-norms as ordinal sums of summands which are idempotent-free is given. Summarizing our results and representation from [5], we have the next concluding claim of our contribution.

Corollary 3.10. *Let $L = \{0, 1, \dots, n\}$, $n \in \mathbb{N}$, and $(L, \star)$ be a topological semigroup. Let $\{a_i\}_{i=0}^k$ be a set of all idempotent elements of $\star$, $0 = a_0 < a_1 < \dots < a_k = n$. Then the following are equivalent:*

1. *0 is an annihilator of $\star$;*
2. *$(L, \star)$ is a divisible discrete I-semigroup with a neutral element n;*
3. *$\star$ is a divisible t-norm on L; and*
4. *$T = (\langle a_{i-1}, a_i, \mathsf{T}_i \rangle)_{i=1}^k$, where $\mathsf{T}_i\colon [a_{i-1}, a_i]^2 \to [a_{i-1}, a_i]$ is given by $\mathsf{T}_i(x, y) = \max\{a_{i-1}, x + y - a_i\}$.*

For the case of discrete t-conorms, similar result holds.

Remark 3.11. *Note that due to Corollary 3.10, the number of discrete divisible t-norms on $L = \{0, 1, \dots, n\}$, $n \in \mathbb{N}$, is just the same as the number of all subsets of $\{1, 2, \dots, n-1\}$, i.e., it is 2^{n-1}. Consequently, the number of all discrete I-semigroups on L (which is the same as the number of topological semigroups such that both 0 and n are idempotent and one of them is an annihilator) is 2^n.*

Remark 3.12. *Note that we cannot replace the existence of an annihilator $\mathsf{e} \in \{0, n\}$ in Theorem 3.8 by the existence of a neutral element $\mathsf{u} \in \{0, n\}$. Consider, for example, $\star\colon \{0, 1, 2\}^2 \to \{0, 1, 2\}$ given by*

$\star$	0	1	2
0	0	1	0
1	1	1	1
2	0	1	2

Then $(\{0, 1, 2\}, \star)$ is a topological semigroup, 0 and 2 are idempotent elements of $\star$ and 2 is its neutral element. Obviously, $\star$ is not monotone and thus is not a discrete t-norm.

4 Concluding remarks

In this paper we have shown the equivalence of special topological semigroups, i.e., of discrete I-semigroups, with discrete divisible t-norms and discrete divisible t-conorms on discrete chains. This equivalence was already known for the case of classical continuous t-norms and continuous t-conorms defined on the unit interval $[0,1]$ (cf., e.g., [8, Cor. 2.44]) but, to our knowledge, was not discussed in the setting of discrete chains (though the redundancy of the commutativity axiom can be logically deduced from BL-logics theory). Some of our proofs were inspired by the related proofs dealing with I-semigroups and continuous t-norms and t-conorms on $[0,1]$. This is not the case of Proposition 3, where original approaches were necessary.

Note that, for an associative divisible operation $\star\colon L^2 \to L$, to check whether $\star$ is a t-norm (t-conorm) on L due to our results it is enough to verify $n \star n = n$ and $i \star 0 = 0 = 0 \star i$ for all $i \in L$, i.e., we have to verify $2n+2$ equalities only. This is not more the case when we consider the standard axiomatic approach to t-norms, see Definition 2.1. Only to check the commutativity of $\star$ one should verify $n(n+1)/2$ equalities. And we still have to verify the monotonicity of $\star$ and all n equalities $n \star i = i$ proving n is a neutral element of $\star$. Hence, though the value of our results is important mostly for the theory, it can significantly help in huge computational complexity reduction when verifying divisible t-norms or t-conorms on a finite chain.

Our results have led to a weakening of axioms for t-norms (t-conorms) on finite chains under the constraint of divisibility. Observe that finite chains are particular bounded lattices, and that, recently, several works concerning t-norms (t-conorms) on lattices appeared, see, e.g., [1, 2, 17]. Motivated by these works, and by our results, we aim to focus our further research on t-norms and t-conorms on particular bounded lattices, in particular on possible modification of the classical axioms dealing with monotonicity, associativity, commutativity and neutral element.

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