Constructing t-norms and t-conorms by using interior and closure operators on bounded lattices

E. Aşıcı

1Department of Software Engineering, Faculty of Technology, Karadeniz Technical University, 61830 Trabzon, Turkey
emelkalin@hotmail.com

Abstract

In this paper, we propose construction methods for triangular norms (t-norms) and triangular conorms (t-conorms) on bounded lattices by using interior and closure operators, respectively. Thus, we obtain some proposed methods by Ertuğrul, Karaçal, Mesiar [15] and Çaylı [8] as results. Also, we give some illustrative examples. Finally, we show that the introduced construction methods cannot be generalized by induction to a modified ordinal sum for t-norms and t-conorms on bounded lattices. This paper has further constructed the t-norms and t-conorms on bounded lattices from a mathematical viewpoint.

Keywords: Bounded lattice, t-norms, t-conorms.

1 Introduction and motivation

Aggregation operators [18] play an important role in theories of fuzzy sets and fuzzy logic. Two basic types of aggregation functions, namely t-norms and t-conorms, were introduced by Schweizer and Sklar [26], in 1963. Although the t-norms and t-conorms were strictly defined on the unit interval [0,1], they were mostly studied on bounded lattices. The notion of ordinal sum of semigroups in Clifford’s sense [9] was further developed by Mostert and Shields [22] and later used for introducing new t-norms and conorms on the unit interval [0,1], see [20]. Note that there is a minor difference in ordinal sum construction for triangular norms (based on min operator) with those for triangular conorms (based on max operator). Since Goguen’s [17] generalization of the classical fuzzy sets (with membership values from [0,1]) to $L$-fuzzy sets (with membership values from a bounded lattice $L$), there is a growing interest in t-norms and t-conorms on bounded lattices, in particular in ordinal sum constructions.

In general topology [13], closure and interior operators on the powerset $P(X)$ of a nonempty set $X$ are common tools to construct topologies on $X$. Actually, there is a one-to-one correspondence between the set of all closure and interior operators on $P(X)$ and that of all topologies on $X$. Note that closure and interior operators on $P(X)$ are essentially defined on the inherent lattice structure on $P(X)$ with set inclusion, set intersection and set union as the partial order, the meet and the join on $P(X)$, respectively.

In 1996, Drossos and Navara [11] studied a class of t-norms and t-conorms on any bounded lattice was generated by the use of interior operators and closure operators, respectively. In 2006, Saminger [25] focused on ordinal sums of t-norms acting on some particular bounded lattice which is not necessarily a chain or an ordinal sum of lattices. Also, it was provided necessary and sufficient conditions for an ordinal sum operation yielding again a t-norm on some bounded lattice whereas the operation is determined by an arbitrary selection of subintervals as carriers for arbitrary summand t-norms. In 2012, Medina [21] presented several necessary and sufficient conditions for ensuring whether an ordinal sum on a bounded lattice of arbitrary t-norms is a t-norm.

In 2015, a modification of ordinal sums of t-norms and t-conorms resulting to a t-norms and t-conorms on an arbitrary bounded lattice was shown by Ertuğrul, Karaçal, Mesiar [15]. Further modifications were proposed by Aşıcı, Mesiar [2, 3], Aşıcı [2], Çaylı [8, 9], Ouyang, Zhang, Baets [23] and Dan, Hu, Qiao [11]. In 2020, a new ordinal sum...
construction of t-norms and t-conorms on bounded lattices based on interior and closure operators was proposed by Dvofák, Holčapek [13]. Also, the proposed method generalized several known constructions and provided a simple tool to introduce new classes of t-norms and t-conorms.

In this paper, we introduce some new constructions of t-norms and t-conorms by using interior and closure operators on bounded lattices, respectively. The rest of this paper is organized as follows. In Section 2, some basic concepts and results about t-norms, t-conorms, lattices and t-conorms on bounded lattices, respectively. The rest of this paper is organized as follows. In Section 2, we present some basic facts about lattices, t-norms and t-conorms. Extremal t-norms

\[ T(x, y) = x \land y, \quad T_W(x, y) = \begin{cases} x \land y & \text{if } 1 \in \{x, y\}, \\ 0 & \text{otherwise.} \end{cases} \]

Similarly, the t-conorms \( S_V \) and \( S_W \) on \( L \) are defined as follows, respectively:

\[ S_V(x, y) = x \lor y, \quad S_W(x, y) = \begin{cases} x \lor y & \text{if } 0 \in \{x, y\}, \\ 1 & \text{otherwise.} \end{cases} \]

The following definition of an ordinal sum of t-norms defined on subintervals of a bounded lattice \((L, \leq, 0, 1)\) has been extracted from [25], which generalizes the methods given in [20] on subintervals of \([0, 1]\).

**Definition 2.3.** [25] Let \((L, \leq, 0, 1)\) be a bounded lattice and fix some subinterval \([a, b]\) of \(L\). Let \(V\) be a t-norm on \([a, b]\). Then \(T : L^2 \to L\) defined by

\[ T(x, y) = \begin{cases} V(x, y) & \text{if } (x, y) \in [a, b]^2, \\ x \land y & \text{otherwise.} \end{cases} \]  

is an ordinal sum \((< a, b, V >)\) of \(V\) on \(L\).

**Definition 2.4.** [25] Let \((L, \leq, 0, 1)\) be a bounded lattice and fix some subinterval \([a, b]\) of \(L\). Let \(W\) be a t-conorm on \([a, b]\). Then \(S : L^2 \to L\) defined by

\[ S(x, y) = \begin{cases} W(x, y) & \text{if } (x, y) \in [a, b]^2, \\ x \lor y & \text{otherwise.} \end{cases} \]  

is an ordinal sum \((< a, b, W >)\) of \(W\) on \(L\).
However, the operation $T$ (resp. $S$) given by Formula (11) (resp. Formula (12)) need not be a t-norm (resp. t-conorm), in general. Observe that condition ensuring that $T$ (resp. $S$) given by (11) ((3)) is a t-norm (t-conorm) on $L$ are given in (22).

**Definition 2.5.** Let $(L, \leq, 0, 1)$ be a bounded lattice. A mapping $\text{cl} : L \rightarrow L$ is said to be a closure operator if for any $x, y \in L$, it satisfies the following three conditions:
(i) $x \leq \text{cl}(x)$.
(ii) $\text{cl}(x \lor y) = \text{cl}(x) \lor \text{cl}(y)$.
(iii) $\text{cl}(\text{cl}(x)) = \text{cl}(x)$.

**Definition 2.6.** Let $(L, \leq, 0, 1)$ be a bounded lattice and $b \in L$ be given. Then the mapping $\text{cl}_b : L \rightarrow L$ defined as $\text{cl}_b(x) = x \lor b$ (\forall x \in L) is a closure operator.

**Definition 2.7.** Let $(L, \leq, 0, 1)$ be a bounded lattice. The set of all universally comparable elements in $L$, denoted by $\text{UC}(L)$, be defined as
$\text{UC}(L) = \{b \in L \mid \forall c \in L$, either $b \leq c$ or $c \leq b \}$.

**Definition 2.8.** Let $(L, \leq, 0, 1)$ be a complete lattice. The mapping $\uparrow : L \rightarrow L$ defined as, for any $x \in L$,
$\uparrow (x) = \bigwedge \{b \in \text{UC}(L) \mid b \geq x\}$,
is a closure operator.

**Definition 2.9.** Let $(L, \leq, 0, 1)$ be a bounded lattice. A mapping $\text{int} : L \rightarrow L$ is said to be an interior operator if for any $x, y \in L$, it satisfies the following three conditions:
(i) $\text{int}(x) \leq x$,
(ii) $\text{int}(x \land y) = \text{int}(x) \land \text{int}(y)$,
(iii) $\text{int}(\text{int}(x)) = \text{int}(x)$.

**Definition 2.10.** Let $(L, \leq, 0, 1)$ be a bounded lattice and $b \in L$ be given. Then the mapping $\text{int}_b : L \rightarrow L$ defined as
$\text{int}_b(x) = x \land b$ (\forall x \in L),
is an interior operator.

**Definition 2.11.** Let $(L, \leq, 0, 1)$ be a complete lattice. The mapping $\downarrow : L \rightarrow L$ defined as, for any $x \in L$,
$\downarrow (x) = \bigvee \{b \in \text{UC}(L) \mid b \leq x\}$,
is an interior operator.

In the following, it is proposed a method for generating t-norms and t-conorms on bounded lattices based on interior and closure operators, respectively.

**Theorem 2.12.** Let $(L, \leq, 0, 1)$ be a bounded lattice, $\text{int} : L \rightarrow L$ and $\text{cl} : L \rightarrow L$ be an interior and a closure operators on $L$, respectively. Then, the functions $T : L^2 \rightarrow L$ and $S : L^2 \rightarrow L$ are, respectively, a t-norm and a t-conorm on $L$, where
\begin{align*}
T(x, y) &= \begin{cases} 
x \land y & \text{if } 1 \in \{x, y\}, 
\text{int}(x) \land \text{int}(y) & \text{otherwise.}
\end{cases} \\
S(x, y) &= \begin{cases} 
x \lor y & \text{if } 0 \in \{x, y\}, 
\text{cl}(x) \lor \text{cl}(y) & \text{otherwise.}
\end{cases}
\end{align*}

**3 New construction method for t-norms on bounded lattices by using interior operators**

In this section, we propose new construction method for t-norms on bounded lattices with the given t-norms by using interior operators. The main aim of this section is to present a rather effective method to construct t-norms by using interior operators on a bounded lattice. Using this method, in Corollary 3.8 and Corollary 3.10 we obtain the methods proposed by Çayh [8] and Ertuğrul, Karaçal, Mesiar [13], respectively.
**Theorem 3.1.** Let \((L, \leq, 0, 1)\) be a bounded lattice with \(a \in L\) and \(\text{int} : L \to L\) be an interior operator such that for all \(x \in I_a\) it holds \(x \wedge a = \text{int}(x \wedge a)\). Given a t-norm \(V\) on \([a, 1]\), then the function \(T : L^2 \to L\) defined as follows is a t-norm on \(L\) where

\[
T(x, y) = \begin{cases} 
V(x, y) & \text{if } (x, y) \in [a, 1]^2, \\
\text{int}(x) \wedge \text{int}(y) & \text{otherwise}. 
\end{cases}
\]

**Proof.** It is easy to see that \(T\) is commutative and has 1 as the neutral element.

i) Monotonicity: We prove that if \(x \leq y\), then \(T(x, z) \leq T(y, z)\) for all \(z \in L\). If \(z = 1\), then we have that \(T(x, z) = T(x, 1) = x \leq y = T(y, 1) = T(y, z)\) for all \(x, y \in L\). The proof can be split into all possible cases.

1. \(x \in [0, a)\),
   1.1 \(y \in [0, a)\),
      1.1.1. \(z \in [0, a)\) or \(z \in [a, 1)\) or \(z \in I_a\),
         \[T(x, z) = \text{int}(x) \wedge \text{int}(z) \leq \text{int}(y) \wedge \text{int}(z) = T(y, z),\]
   1.2. \(y \in [a, 1)\),
      1.2.1. \(z \in [0, a)\),
         \[T(x, z) = \text{int}(x) \wedge \text{int}(z) \leq \text{int}(y) \wedge \text{int}(z) = T(y, z),\]
      1.2.2. \(z \in [a, 1)\),
         \[T(x, z) = \text{int}(x) \wedge \text{int}(z) \leq x \leq a \leq V(y, z) = T(y, z),\]
      1.2.3. \(z \in I_a\),
         \[T(x, z) = \text{int}(x) \wedge \text{int}(z) \leq x \wedge z \leq a \wedge z = T(y, z),\]
   1.3. \(y \in I_a\),
      1.3.1. \(z \in [0, a)\),
         \[T(x, z) = \text{int}(x) \wedge \text{int}(z) \leq \text{int}(y) \wedge \text{int}(z) = T(y, z),\]
      1.3.2. \(z \in [a, 1)\),
         \[T(x, z) = \text{int}(x) \wedge \text{int}(z) \leq x \leq a \wedge y = T(y, z),\]
      1.3.3. \(z \in I_a\),
         \[T(x, z) = \text{int}(x) \wedge \text{int}(z) \leq x \wedge z \leq y \wedge z \wedge a = T(y, z),\]
   1.4. \(y = 1\),
      1.4.1. \(z \in [0, a)\) or \(z \in [a, 1)\) or \(z \in I_a\),
         \[T(x, z) = \text{int}(x) \wedge \text{int}(z) \leq z = T(1, z),\]

2. \(x \in [a, 1)\),
   2.1 \(y \in [a, 1)\),
      2.1.1. \(z \in [0, a)\),
         \[T(x, z) = \text{int}(x) \wedge \text{int}(z) \leq \text{int}(y) \wedge \text{int}(z) = T(y, z),\]
      2.1.2. \(z \in [a, 1)\),
         \[T(x, z) = V(x, z) \leq V(y, z) = T(y, z),\]
      2.1.3. \(z \in I_a\),
         \[T(x, z) = z \wedge a = T(y, z),\]
   2.2 \(y = 1\),
2.2.1. \( z \in [0, a) \),
\[
T(x, z) = \text{int}(x) \land \text{int}(z) \leq z = T(1, z),
\]

2.1.2. \( z \in [a, 1) \),
\[
T(x, z) = V(x, z) \leq z = T(1, z),
\]

2.1.3. \( z \in I_a \),
\[
T(x, z) = z \land a \leq z = T(1, z),
\]

3. \( x \in I_a \),

3.1. \( y \in [a, 1) \),

3.1.1. \( z \in [0, a) \),
\[
T(x, z) = \text{int}(x) \land \text{int}(z) \leq \text{int}(y) \land \text{int}(z) = T(y, z),
\]

3.1.2. \( z \in [a, 1) \),
\[
T(x, z) = x \land a \leq a \leq V(y, z) = T(y, z),
\]

3.1.3. \( z \in I_a \),
\[
T(x, z) = x \land z \land a \leq z \land a = T(y, z),
\]

3.2. \( y = 1 \),

3.2.1. \( z \in [0, a) \),
\[
T(x, z) = \text{int}(x) \land \text{int}(z) \leq z = T(1, z),
\]

3.2.2. \( z \in [a, 1) \),
\[
T(x, z) = x \land a \leq a \leq z = T(1, z),
\]

3.2.3. \( z \in I_a \),
\[
T(x, z) = x \land z \land a \leq z = T(1, z),
\]

4. \( x = 1 \),
   Then, it must be \( y = 1 \). Clearly, monotonicity holds.

ii) Associativity: We need to prove that \( T(x, T(y, z)) = T(T(x, y), z) \) for all \( x, y, z \in L \). If at least one of \( x, y, z \) in \( L \) is 1, then it is obvious. So, the proof is split into all possible cases.

1. \( x \in [0, a) \),

1.1. \( y \in [0, a) \),

1.1.1. \( z \in [0, a) \) or \( z \in [a, 1) \) or \( z \in I_a \),
\[
T(x, T(y, z)) = T(x, \text{int}(y) \land \text{int}(z)) = \text{int}(x) \land \text{int}(y) \land \text{int}(z) = T(\text{int}(x) \land \text{int}(z), z) = T(T(x, y), z),
\]

1.2. \( y \in [a, 1) \),

1.2.1. \( z \in [0, a) \),
\[
T(x, T(y, z)) = T(x, \text{int}(y) \land \text{int}(z)) = \text{int}(x) \land \text{int}(y) \land \text{int}(z) = T(\text{int}(x) \land \text{int}(z), z) = T(T(x, y), z),
\]

1.2.2. \( z \in [a, 1) \),
\[
T(x, T(y, z)) = T(x, V(y, z)) = \text{int}(x) \land \text{int}(V(y, z))
\]
\[
= \text{int}(x) = \text{int}(x) \land \text{int}(y) \land \text{int}(z)
\]
\[
= T(\text{int}(x) \land \text{int}(y), z) = T(x, y, z),
\]

1.2.3. \( z \in I_a \),
\[
T(x, T(y, z)) = T(x, z \land a) = \text{int}(x) \land \text{int}(z \land a)
\]
\[
= \text{int}(x \land z) = \text{int}(x) \land \text{int}(y) \land \text{int}(z)
\]
\[
= T(\text{int}(x) \land \text{int}(y), z) = T(x, y, z),
\]
130. \( y \in I_a \),
13.1. \( z \in [0,a) \),
\[
T(x, T(y, z)) = T(x, \text{int}(y) \land \text{int}(z)) = \text{int}(x) \land \text{int}(y) \land \text{int}(z) = T(\text{int}(x) \land \text{int}(y), z) = T(T(x, y), z),
\]
13.2. \( z \in [a,1) \),
\[
T(x, T(y, z)) = T(x, y) = \text{int}(x) \land \text{int}(y)
  = \text{int}(x) \land \text{int}(y) \land \text{int}(z)
  = T(\text{int}(x) \land \text{int}(y), z) = T(T(x, y), z),
\]
13.3. \( z \in I_a \),
\[
T(x, T(y, z)) = T(x, y) \land z = \text{int}(x) \land \text{int}(y) \land \text{int}(z) = T(\text{int}(x) \land \text{int}(y), z) = T(T(x, y), z),
\]
2. \( x \in [a,1) \),
2.1 \( y \in [0,a) \),
2.1.1. \( z \in [0,a) \) or \( z \in [a,1) \) or \( z \in I_a \),
\[
T(x, T(y, z)) = T(x, \text{int}(y) \land \text{int}(z)) = \text{int}(x) \land \text{int}(y) \land \text{int}(z) = T(\text{int}(x) \land \text{int}(y), z) = T(T(x, y), z),
\]
2.2. \( y \in [a,1) \),
2.2.1. \( z \in [0,a) \),
\[
T(x, T(y, z)) = T(x, \text{int}(y) \land \text{int}(z)) = \text{int}(x) \land \text{int}(y) \land \text{int}(z)
  = \text{int}(z) = \text{int}(V(x, y)) \land \text{int}(z)
  = T(V(x, y), z) = T(T(x, y), z),
\]
2.2.2. \( z \in [a,1) \),
\[
T(x, T(y, z)) = T(x, V(y, z)) = V(x, V(y, z)) = V(V(x, y), z) = T(V(x, y), z) = T(T(x, y), z),
\]
2.2.3. \( z \in I_a \),
\[
T(x, T(y, z)) = T(x, z \land a) = \text{int}(z \land a) = z \land a = T(V(x, y), z) = T(T(x, y), z),
\]
2.3. \( y \in I_a \),
2.3.1. \( z \in [0,a) \),
\[
T(x, T(y, z)) = T(x, \text{int}(y) \land \text{int}(z)) = \text{int}(x) \land \text{int}(y) \land \text{int}(z)
  = \text{int}(y \land z) = \text{int}(y \land a) \land \text{int}(z)
  = T(y \land a, z) = T(T(x, y), z),
\]
2.3.2. \( z \in [a,1) \),
\[
T(x, T(y, z)) = T(x, y \land a) = \text{int}(x) \land \text{int}(y \land a)
  = \text{int}(y \land a) = \text{int}(y \land a) \land \text{int}(z)
  = T(y \land a, z) = T(T(x, y), z),
\]
2.3.3. \( z \in I_a \),
\[
T(x, T(y, z)) = T(x, y \land z \land a) = \text{int}(x) \land \text{int}(y \land z \land a)
  = \text{int}(y \land z \land a) = \text{int}(y \land a) \land \text{int}(z)
  = T(y \land a, z) = T(T(x, y), z),
\]
3. \( x \in I_a, \)
   3.1 \( y \in [0, a), \)
       3.1.1. \( z \in [0, a) \) or \( z \in [a, 1) \) or \( z \in I_a, \)
       \[ T(x, T(y, z)) = T(x, \text{int}(y) \land \text{int}(z)) = \text{int}(x) \land \text{int}(y) \land \text{int}(z) = T(\text{int}(x) \land \text{int}(y), z) = T(T(x, y), z), \]
   3.2. \( y \in [a, 1), \)
       3.2.1. \( z \in [0, a), \)
       \[ T(x, T(y, z)) = T(x, \text{int}(y) \land \text{int}(z)) = \text{int}(x) \land \text{int}(y) \land \text{int}(z) \]
       \[ = \text{int}(x \land z) = \text{int}(x \land a) \land \text{int}(z) \]
       \[ = T(x \land a, z) = T(T(x, y), z), \]
   3.2.2. \( z \in [a, 1), \)
       \[ T(x, T(y, z)) = T(x, V(y, z)) = x \land a = \text{int}(x \land a) = \text{int}(x \land a) \land \text{int}(z) = T(x \land a, z) = T(T(x, y), z), \]
   3.2.3. \( z \in I_a, \)
       \[ T(x, T(y, z)) = T(x, z \land a) = \text{int}(x) \land \text{int}(z \land a) = \text{int}(x \land a) \land \text{int}(z) = T(x \land a, z) = T(T(x, y), z), \]
   3.3. \( y \in I_a, \)
       3.3.1. \( z \in [0, a), \)
       \[ T(x, T(y, z)) = T(x, \text{int}(y) \land \text{int}(z)) = \text{int}(x) \land \text{int}(y) \land \text{int}(z) \]
       \[ = \text{int}(x \land y \land a) \land \text{int}(z) = T(x \land y \land a, z) \]
       \[ = T(T(x, y), z), \]
   3.3.2. \( z \in [a, 1), \)
       \[ T(x, T(y, z)) = T(x, y \land a) = \text{int}(x) \land \text{int}(y \land a) \]
       \[ = \text{int}(x \land y \land a) = \text{int}(x \land y \land a) \land \text{int}(z) \]
       \[ = T(x \land y \land a, z) = T(T(x, y), z), \]
   3.3.3. \( z \in I_a, \)
       \[ T(x, T(y, z)) = T(x, y \land z \land a) = \text{int}(x) \land \text{int}(y \land z \land a) \]
       \[ = \text{int}(x \land y \land z \land a) = \text{int}(x \land y \land a) \land \text{int}(z) \]
       \[ = T(x \land y \land a, z) = T(T(x, y), z), \]

So, we have the fact that \( T \) is a t-norm on \( L. \) □

**Remark 3.2.** Let \( (L, \leq, 0, 1) \) be a bounded lattice with \( a \in L. \) In Theorem \( 5.7, \) observe that the condition for all \( x \in I_a \) it holds \( x \land a = \text{int}(x \land a) \) cannot be omitted, in general. The following example illustrates this fact that the function \( T : L^2 \rightarrow L \) defined by Theorem \( 5.7, \) is not a t-norm.

**Example 3.3.** Consider the lattice \( (L_1 = \{0_{L_1}, b, c, d, a, k, m, 1_{L_1}\}, \leq, 0_{L_1}, 1_{L_1}) \) in Figure 1. And we take the t-norm \( V(x, y) = x \land y \) on \([a, 1_{L_1}].\) The interior operator \( \text{int} : L_1 \rightarrow L_1 \) defined by \( \text{int}(0_{L_1}) = 0_{L_1}, \text{int}(b) = \text{int}(c) = \text{int}(d) = \text{int}(a) = \text{int}(k) = b, \text{int}(m) = m \) and \( \text{int}(1_{L_1}) = 1_{L_1}. \) For all \( x \in I_a \) it does not hold \( x \land a = \text{int}(x \land a). \) Because, \( k \land a = c \neq b = \text{int}(c) = \text{int}(k \land a). \) Then, the function \( T \) on \( L_1 \) defined by Table 1 is not a t-norm. Indeed, it does not satisfy the associativity. Because \( T(k, T(m, m)) = T(k, m) = c \neq b = T(c, m) = T(T(k, m), m). \)

**Corollary 3.4.** Let \( (L, \leq, 0, 1) \) be a bounded lattice with \( a, b \in L \) such that for all \( x \in I_a \) it holds \( x \land a = x \land a \land b \) and \( V \) be a t-norm on \([a, 1].\) Then, the function \( T : L^2 \rightarrow L \) defined by

\[
T(x, y) = \begin{cases} 
V(x, y) & \text{if } (x, y) \in [a, 1]^2, \\
\land a & \text{if } (x, y) \in [a, 1] \times I_a, \\
x \land a & \text{if } (x, y) \in I_a \times [a, 1), \\
x \land y \land a & \text{if } (x, y) \in I_a \times I_a, \\
x \land y & \text{if } x = 1 \text{ or } y = 1, \\
x \land y \land b & \text{otherwise}. 
\end{cases}
\]

is a t-norm on \( L. \)
We give next construction methods for t-norms on complete lattices from Definition 2.3 and Definition 2.4.

**Corollary 3.5.** Let \((L, \leq, 0, 1)\) be a complete lattice with \(a \in L\), \(\sqcup : L \rightarrow L\) be defined in Definition 2.4 such that for all \(x \in I_a\) it holds \(x \land a = \sqcup (x \land a)\) and \(V\) be a t-norm on \([a, 1]\). Then, the binary operation \(T : L^2 \rightarrow L\) defined by

\[
T(x, y) = \begin{cases} 
V(x, y) & \text{if } (x, y) \in [a, 1]^2, \\
y \land a & \text{if } (x, y) \in [a, 1] \times I_a, \\
x \land a & \text{if } (x, y) \in I_a \times [a, 1], \\
x \land y & \text{if } (x, y) \in I_a \times I_a, \\
x \land y & \text{if } x = 1 \text{ or } y = 1, \\
\sqcup (x) \land \sqcup (y) & \text{otherwise}.
\end{cases}
\]

is a t-norm on \(L\).

We can give an example to illustrate Corollary 3.5.

**Example 3.6.** Consider the complete lattice \((L_2 = \{0_{L_2}, t, p, q, a, s, n, 1_{L_2}\}, \leq, 0_{L_2}, 1_{L_2})\) in Figure 2. And we take the t-norm \(V(x, y) = x \land y\) on \([a, 1_{L_2}]\). It is clear that \(UC(L_2) = \{0_{L_2}, t, n, 1_{L_2}\}\). So, we obtain \(\sqcup (0_{L_2}) = 0_{L_2}\), \(\sqcup (t) = \sqcup (p) = \sqcup (q) = \sqcup (a) = \sqcup (s) = t\), \(\sqcup (n) = n\) and \(\sqcup (1_{L_2}) = 1_{L_2}\). Since for all \(x \in I_a\) it holds \(x \land a = \sqcup (x \land a)\), \(L_2\) satisfies the constraint of Corollary 3.5. That is, \(q \land a = t = \sqcup (t) = \sqcup (q \land a)\) and \(s \land a = t = \sqcup (t) = \sqcup (s \land a)\). Then the t-norm \(T : L_2^2 \rightarrow L_2\) constructed via Corollary 3.5 is given by Table 2.

**Remark 3.7.** If we take \(b = 0\) in Corollary 3.5, then it must be \(x \land a = 0\) for all \(x \in I_a\). So, we obtain corresponding t-norm as follows constructed by Çaylı [8].

**Corollary 3.8.** Let \((L, \leq, 0, 1)\) be a bounded lattice with \(a \in L \setminus \{0, 1\}\) and \(V\) be a t-norm on \([a, 1]\). Then the function \(T_1 : L^2 \rightarrow L\) is a t-norm on \(L\), where

\[
T_1(x, y) = \begin{cases} 
V(x, y) & \text{if } (x, y) \in [a, 1]^2, \\
x \land y & \text{if } x = 1 \text{ or } y = 1, \\
0 & \text{otherwise}.
\end{cases}
\]
Remark 3.9. If we take $b = 1$ in Corollary 3.8, then we obtain corresponding t-norm as follows constructed by Ertuğrul, Karačal and Mesiar [15].

Corollary 3.10. Let $(L, \leq, 0, 1)$ be a bounded lattice and $V$ be a t-norm on $[a, 1]$. Then the function $T_2 : L^2 \to L$ is a t-norm on $L$, where

$$T_2(x, y) = \begin{cases} V(x, y) & \text{if } (x, y) \in [a, 1)^2, \\ x \land y & \text{if } x = 1 \text{ or } y = 1, \\ x \land y \land a & \text{otherwise}. \end{cases}$$

Remark 3.11. It should be noted that the t-norms $T_1$ and $T_2$ in Corollary 3.8 and Corollary 3.10, respectively are different from the t-norm $T$ in Theorem 3.1. To show that this claim, we shall consider the bounded lattice $(L_2 = \{0_{L_2}, t, p, q, a, s, n, 1_{L_2}\}, \leq, 0_{L_2}, 1_{L_2})$ described in Figure 2., we take the t-norm $V(x, y) = x \land y$ on $[a, 1_{L_2}]$ and the interior operator $\text{int} : L_2 \to L_2$ defined by $\text{int}(0_{L_2}) = 0_{L_2}$, $\text{int}(t) = \text{int}(p) = \text{int}(q) = \text{int}(a) = \text{int}(s) = t$, $\text{int}(n) = n$ and $\text{int}(1_{L_2}) = 1_{L_2}$. According to the Table 2, Table 3 and Table 4, it is clear that the t-norms $T$, $T_1$ and $T_2$ different from each other.
Table 4: The t-norm $T_2$ on $L_2$

<table>
<thead>
<tr>
<th>$T_2$</th>
<th>$0_{L_2}$</th>
<th>$t$</th>
<th>$p$</th>
<th>$q$</th>
<th>$s$</th>
<th>$n$</th>
<th>$1_{L_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0_{L_2}$</td>
<td>$0_{L_2}$</td>
<td>$0_{L_2}$</td>
<td>$0_{L_2}$</td>
<td>$0_{L_2}$</td>
<td>$0_{L_2}$</td>
<td>$0_{L_2}$</td>
<td>$0_{L_2}$</td>
</tr>
<tr>
<td>$t$</td>
<td>$t$</td>
<td>$t$</td>
<td>$t$</td>
<td>$t$</td>
<td>$t$</td>
<td>$t$</td>
<td>$t$</td>
</tr>
<tr>
<td>$p$</td>
<td>$p$</td>
<td>$p$</td>
<td>$p$</td>
<td>$p$</td>
<td>$p$</td>
<td>$p$</td>
<td>$p$</td>
</tr>
<tr>
<td>$q$</td>
<td>$q$</td>
<td>$q$</td>
<td>$q$</td>
<td>$q$</td>
<td>$q$</td>
<td>$q$</td>
<td>$q$</td>
</tr>
<tr>
<td>$s$</td>
<td>$s$</td>
<td>$s$</td>
<td>$s$</td>
<td>$s$</td>
<td>$s$</td>
<td>$s$</td>
<td>$s$</td>
</tr>
<tr>
<td>$n$</td>
<td>$n$</td>
<td>$n$</td>
<td>$n$</td>
<td>$n$</td>
<td>$n$</td>
<td>$n$</td>
<td>$n$</td>
</tr>
<tr>
<td>$1_{L_2}$</td>
<td>$1_{L_2}$</td>
<td>$1_{L_2}$</td>
<td>$1_{L_2}$</td>
<td>$1_{L_2}$</td>
<td>$1_{L_2}$</td>
<td>$1_{L_2}$</td>
<td>$1_{L_2}$</td>
</tr>
</tbody>
</table>

4 New construction method for t-conorms on bounded lattices by using closure operators

In this section, we propose new construction method for t-conorms on bounded lattices with the given t-conorms by using closure operators. The main aim of this section is to present a rather effective method to construct t-conorms by using closure operators on a bounded lattice. Using this method, in Corollary [4.2] and Corollary [4.3], we obtain the methods proposed by Ertuğrul, Karaçal, Mesiar [4.15] and Çayh [4.8], respectively.

Theorem 4.1. Let $(L, \leq, 0, 1)$ be a bounded lattice with $a \in L$ such that for all $x \in I_a$ it holds $x \lor a = \text{cl}(x \lor a)$ and $\text{cl} : L \rightarrow L$ be a closure operator. Given a t-conorm $W$ on $[0, a]$, then the function $S : L^2 \rightarrow L$ defined as follows is a t-conorm on $L$

\[
S(x, y) = \begin{cases} 
W(x, y) & \text{if } (x, y) \in (0, a]^2, \\
y \lor a & \text{if } (x, y) \in (0, a] \times I_a, \\
x \lor a & \text{if } (x, y) \in I_a \times (0, a], \\
x \lor y \lor a & \text{if } (x, y) \in I_a \times I_a, \\
x \lor y & \text{if } x = 0 \text{ or } y = 0, \\
\text{cl}(x) \lor \text{cl}(y) & \text{otherwise}.
\end{cases}
\]

Remark 4.2. Let $(L, \leq, 0, 1)$ be a bounded lattice with $a \in L$. In Theorem 4.1, observe that the condition for all $x \in I_a$ it holds $x \lor a = \text{cl}(x \lor a)$ can not be omitted, in general. The following example illustrates this fact that the function $S : L^2 \rightarrow L$ defined by Theorem 4.1 is not a t-conorm.

Example 4.3. Consider the lattice $(L_3 = \{0_{L_3}, t, a, n, p, s, q, 1_{L_3}\}, \leq, 0_{L_3}, 1_{L_3})$ in Figure 3. And we take the t-conorm $W(x, y) = x \lor y$ on $[0_{L_3}, a]$. The closure operator $\text{cl} : L_3 \rightarrow L_3$ defined by $\text{cl}(0_{L_3}) = 0_{L_3}$, $\text{cl}(t) = t$, $\text{cl}(n) = \text{cl}(a) = \text{cl}(s) = \text{cl}(p) = \text{cl}(q) = q$, and $\text{cl}(1_{L_3}) = 1_{L_3}$. For all $x \in I_a$ it does not hold $x \lor a = \text{cl}(x \lor a)$. Because, $n \lor a = p \neq q = \text{cl}(p) = \text{cl}(n \lor a)$. Then, the function $S$ on $L_3$ defined by Table 5 is not a t-conorm. Indeed, it does not satisfy the associativity. Because $S(n, S(t, t)) = S(n, t) = p \neq q = S(p, t) = S(S(n, t), t)$.

![Figure 3: The lattice $L_3$](image)
Table 5: The t-function $S$ on $L_3$

<table>
<thead>
<tr>
<th>$S$</th>
<th>$0_{L_3}$</th>
<th>$t$</th>
<th>$a$</th>
<th>$n$</th>
<th>$p$</th>
<th>$s$</th>
<th>$q$</th>
<th>$1_{L_3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0_{L_3}$</td>
<td>$t$</td>
<td>$a$</td>
<td>$n$</td>
<td>$p$</td>
<td>$s$</td>
<td>$q$</td>
<td>$1_{L_3}$</td>
<td></td>
</tr>
<tr>
<td>$t$</td>
<td>$t$</td>
<td>$a$</td>
<td>$n$</td>
<td>$p$</td>
<td>$q$</td>
<td>$q$</td>
<td>$q$</td>
<td>$1_{L_3}$</td>
</tr>
<tr>
<td>$a$</td>
<td>$a$</td>
<td>$a$</td>
<td>$n$</td>
<td>$p$</td>
<td>$q$</td>
<td>$q$</td>
<td>$q$</td>
<td>$1_{L_3}$</td>
</tr>
<tr>
<td>$n$</td>
<td>$n$</td>
<td>$p$</td>
<td>$p$</td>
<td>$p$</td>
<td>$q$</td>
<td>$q$</td>
<td>$q$</td>
<td>$1_{L_3}$</td>
</tr>
<tr>
<td>$p$</td>
<td>$p$</td>
<td>$q$</td>
<td>$q$</td>
<td>$q$</td>
<td>$q$</td>
<td>$q$</td>
<td>$q$</td>
<td>$1_{L_3}$</td>
</tr>
<tr>
<td>$s$</td>
<td>$s$</td>
<td>$q$</td>
<td>$q$</td>
<td>$q$</td>
<td>$q$</td>
<td>$q$</td>
<td>$q$</td>
<td>$1_{L_3}$</td>
</tr>
<tr>
<td>$q$</td>
<td>$q$</td>
<td>$q$</td>
<td>$q$</td>
<td>$q$</td>
<td>$q$</td>
<td>$q$</td>
<td>$q$</td>
<td>$1_{L_3}$</td>
</tr>
<tr>
<td>$1_{L_3}$</td>
<td>$1_{L_3}$</td>
<td>$1_{L_3}$</td>
<td>$1_{L_3}$</td>
<td>$1_{L_3}$</td>
<td>$1_{L_3}$</td>
<td>$1_{L_3}$</td>
<td>$1_{L_3}$</td>
<td>$1_{L_3}$</td>
</tr>
</tbody>
</table>

Corollary 4.4. Let $(L, \leq, 0, 1)$ be a bounded lattice with $a, b \in L$ such that for all $x \in I_a$ it holds $x \lor a = x \lor a \lor b$ and $W$ be a t-conorm on $[0, a]$. Then, the function $S : L^2 \rightarrow L$ defined by

$$S(x, y) = \begin{cases} W(x, y) & \text{if } (x, y) \in (0, a]^2, \\ y \lor a & \text{if } (x, y) \in (0, a] \times I_a, \\ y \lor a & \text{if } (x, y) \in I_a \times (0, a], \\ x \lor y \lor a & \text{if } (x, y) \in I_a \times I_a, \\ x \lor y & \text{if } x = 0 \text{ or } y = 0, \\ x \lor y \lor b & \text{otherwise} \end{cases}$$

is a t-conorm on $L$.

We give next construction methods for t-conorms on complete lattices from Definition 4.4 and Definition 4.5.

Corollary 4.5. Let $(L, \leq, 0, 1)$ be a complete lattice with $a \in L$, $\uparrow : L \rightarrow L$ be defined in Definition 4.4 such that for all $x \in I_a$ it holds $x \lor a = \uparrow (x \lor a)$ and $W$ be a t-conorm on $[0, a]$. Then, the binary operation $S : L^2 \rightarrow L$ defined by

$$S(x, y) = \begin{cases} W(x, y) & \text{if } (x, y) \in (0, a]^2, \\ y \lor a & \text{if } (x, y) \in (0, a] \times I_a, \\ y \lor a & \text{if } (x, y) \in I_a \times (0, a], \\ x \lor y \lor a & \text{if } (x, y) \in I_a \times I_a, \\ x \lor y & \text{if } x = 0 \text{ or } y = 0, \\ \uparrow (x) \lor \uparrow (y) & \text{otherwise} \end{cases}$$

is a t-conorm on $L$.

We can give an example to illustrate Corollary 4.5.

Example 4.6. Consider the complete lattice $(I_{L_4} = \{0_{L_4}, m, r, a, k, c, d, 1_{L_4}\}, \leq, 0_{L_4}, 1_{L_4}\}$ in Figure 4. And we take the t-conorm $W(x, y) = x \lor y$ on $[0_{L_4}, a]$. It is clear that UC$(L_{I_{L_4}}) = \{0_{L_4}, m, a, 1_{L_4}\}$. So, we obtain $\uparrow (0_{L_4}) = 0_{L_4}$, $\uparrow (m) = m$, $\uparrow (r) = \uparrow (a) = \uparrow (k) = \uparrow (c) = \uparrow (d) = d$, and $\uparrow (1_{L_4}) = 1_{L_4}$. Since for all $x \in I_a$ it holds $x \lor a = \uparrow (x \lor a)$, $L_{I_{L_4}}$ satisfies the constraint of Corollary 4.5. That is, $k \lor a = d = \uparrow (d) = \uparrow (k \lor a)$ and $r \lor a = d = \uparrow (r \lor a)$. Then the t-conorm $S : L^2 \rightarrow L_4$ constructed via Corollary 4.5 is given by Table 6.

Remark 4.7. If we take $b = 0$ in Corollary 4.5, then we obtain corresponding t-conorm as follows constructed by Ertuğrul, Karaçal and Mesiar 13.

Corollary 4.8. Let $(L, \leq, 0, 1)$ be a bounded lattice and $W$ be a t-conorm on $[0, a]$. Then the function $S_1 : L^2 \rightarrow L$ is a t-conorm on $L$, where

$$S_1(x, y) = \begin{cases} W(x, y) & \text{if } (x, y) \in (0, a]^2, \\ x \lor y & \text{if } x = 0 \text{ or } y = 0, \\ x \lor y \lor a & \text{otherwise} \end{cases}$$
Figure 4: The lattice $L_4$

Table 6: The t-conorm $S$ on $L_4$

<table>
<thead>
<tr>
<th>$S$</th>
<th>$0_{L_4}$</th>
<th>$m$</th>
<th>$r$</th>
<th>$a$</th>
<th>$k$</th>
<th>$c$</th>
<th>$d$</th>
<th>$1_{L_4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0_{L_4}$</td>
<td>$0_{L_4}$</td>
<td>$m$</td>
<td>$r$</td>
<td>$a$</td>
<td>$k$</td>
<td>$c$</td>
<td>$d$</td>
<td>$1_{L_4}$</td>
</tr>
<tr>
<td>$m$</td>
<td>$m$</td>
<td>$m$</td>
<td>$d$</td>
<td>$d$</td>
<td>$a$</td>
<td>$d$</td>
<td>$d$</td>
<td>$1_{L_4}$</td>
</tr>
<tr>
<td>$r$</td>
<td>$r$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$1_{L_4}$</td>
</tr>
<tr>
<td>$a$</td>
<td>$a$</td>
<td>$a$</td>
<td>$d$</td>
<td>$a$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$1_{L_4}$</td>
</tr>
<tr>
<td>$k$</td>
<td>$k$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$1_{L_4}$</td>
</tr>
<tr>
<td>$c$</td>
<td>$c$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$1_{L_4}$</td>
</tr>
<tr>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$1_{L_4}$</td>
</tr>
<tr>
<td>$1_{L_4}$</td>
<td>$1_{L_4}$</td>
<td>$1_{L_4}$</td>
<td>$1_{L_4}$</td>
<td>$1_{L_4}$</td>
<td>$1_{L_4}$</td>
<td>$1_{L_4}$</td>
<td>$1_{L_4}$</td>
<td>$1_{L_4}$</td>
</tr>
</tbody>
</table>

Remark 4.9. If we take $b = 1$ in Corollary 4.4, then it must be $x \lor a = 1$ for all $x \in I_a$. So, we obtain corresponding t-conorm as follows constructed by Çaylı [8].

Corollary 4.10. Let $(L; \leq; 0; 1)$ be a bounded lattice and $a \in L \setminus \{0, 1\}$. If $W$ be a t-conorm on $[0, a]$, then the function $S_2 : L^2 \rightarrow L$ is a t-conorm on $L$, where

$$S_2(x, y) = \begin{cases} W(x, y) & \text{if } (x, y) \in (0, a]^2, \\ x \lor y & \text{if } x = 0 \text{ or } y = 0, \\ 1 & \text{otherwise}. \end{cases}$$

Remark 4.11. It should be noted that the t-conorms $S_1$ and $S_2$ in Corollary 4.8 and Corollary 4.10, respectively, are different from the t-conorm $S$ in Theorem 4.1. To show that this claim, we consider the bounded lattice $(L_4 = \{0_{L_4}, m, r, a, k, c, d, 1_{L_4}\}; \leq, 0_{L_4}, 1_{L_4})$ in Figure 4., we take the t-conorm $W(x, y) = x \lor y$ on $[0_{L_4}, a]$ and the closure operator $cl : L_4 \rightarrow L_4$ defined by $cl(0_{L_4}) = 0_{L_4}$, $cl(m) = m$, $cl(r) = cl(a) = cl(k) = cl(c) = cl(d) = d$ and $cl(1_{L_4}) = 1_{L_4}$. According to the Table 6, Table 7 and Table 8, it is clear that t-conorms $S, S_1$ and $S_2$ different from each other.

Table 7: The t-conorm $S_2$ on $L_4$

<table>
<thead>
<tr>
<th>$S$</th>
<th>$0_{L_4}$</th>
<th>$m$</th>
<th>$r$</th>
<th>$a$</th>
<th>$k$</th>
<th>$c$</th>
<th>$d$</th>
<th>$1_{L_4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0_{L_4}$</td>
<td>$0_{L_4}$</td>
<td>$m$</td>
<td>$r$</td>
<td>$a$</td>
<td>$k$</td>
<td>$c$</td>
<td>$d$</td>
<td>$1_{L_4}$</td>
</tr>
<tr>
<td>$m$</td>
<td>$m$</td>
<td>$m$</td>
<td>$d$</td>
<td>$d$</td>
<td>$a$</td>
<td>$d$</td>
<td>$d$</td>
<td>$1_{L_4}$</td>
</tr>
<tr>
<td>$r$</td>
<td>$r$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$1_{L_4}$</td>
</tr>
<tr>
<td>$a$</td>
<td>$a$</td>
<td>$a$</td>
<td>$d$</td>
<td>$a$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$1_{L_4}$</td>
</tr>
<tr>
<td>$k$</td>
<td>$k$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$1_{L_4}$</td>
</tr>
<tr>
<td>$c$</td>
<td>$c$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$1_{L_4}$</td>
</tr>
<tr>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$d$</td>
<td>$1_{L_4}$</td>
</tr>
<tr>
<td>$1_{L_4}$</td>
<td>$1_{L_4}$</td>
<td>$1_{L_4}$</td>
<td>$1_{L_4}$</td>
<td>$1_{L_4}$</td>
<td>$1_{L_4}$</td>
<td>$1_{L_4}$</td>
<td>$1_{L_4}$</td>
<td>$1_{L_4}$</td>
</tr>
</tbody>
</table>
5 Modified ordinal sum constructions of t-norms and t-conorms on bounded lattices

From [8] and [12], we know that new t-norms and t-conorms on bounded lattices can be obtained using recursion in Theorem 5.1, Theorem 5.2, and Theorem 5.3, respectively. In this section, based on the approaches of constructing t-norms and t-conorms by using interior and closure operators, respectively, proposed in Section 3 and Section 4, we show that it can not be obtained ordinal sum constructions of t-norms and t-conorms on bounded lattice L using recursion.

Theorem 5.1. [8] Let \((L, \leq, 0, 1)\) be a bounded lattice and \(\{a_0, a_1, a_2, \ldots, a_n\}\) be a finite chain in \(L\) such that \(1 = a_0 > a_1 > a_2 > \ldots > a_n = 0\). Let \(V : [a_1, 1]^2 \rightarrow [a_1, 1]\) be a t-norm. Then, the function \(T_n : L^2 \rightarrow L\) defined recursively as follows is a t-norm, where \(V = T_1\) and for \(i \in \{2, \ldots, n\}\), the function \(T_i : [a_i, 1]^2 \rightarrow [a_i, 1]\) is given by

\[
T_i(x, y) = \begin{cases} 
T_{i-1}(x, y) & \text{if } (x, y) \in [a_{i-1}, 1)^2, \\
\times y & \text{if } x = 1 \text{ or } y = 1, \\
a_i & \text{otherwise}.
\end{cases}
\]

Theorem 5.2. [12] Let \((L, \leq, 0, 1)\) be a bounded lattice and \(\{a_0, a_1, a_2, \ldots, a_n\}\) be a finite chain in \(L\) such that \(1 = a_0 > a_1 > a_2 > \ldots > a_n = 0\). Let \(V : [a_1, 1]^2 \rightarrow [a_1, 1]\) be a t-norm. Then, the function \(T_n : L^2 \rightarrow L\) defined recursively as follows is a t-norm, where \(V = T_1\) and for \(i \in \{2, \ldots, n\}\),

\[
T_i(x, y) = \begin{cases} 
T_{i-1}(x, y) & \text{if } (x, y) \in [a_{i-1}, 1)^2, \\
\times y & \text{if } x = 1 \text{ or } y = 1, \\
\times y \times a_{i-1} & \text{otherwise}.
\end{cases}
\]

Remark 5.3. Let \((L, \leq, 0, 1)\) be a bounded lattice and \(\{a_0, a_1, a_2, \ldots, a_n\}\) be a finite chain in \(L\) such that \(1 = a_0 > a_1 > a_2 > \ldots > a_n = 0\). Let \(\times a_i = \text{int}(x \times a_i)\) for all \(x \in I_{a_i}\), let \(V : [a_1, 1]^2 \rightarrow [a_1, 1]\) be a t-norm and \(\text{int} : L \rightarrow L\) be an interior operator. It should be noted that our construction method in Theorem 5.1 can not be obtained using recursion. Because, we can not obtain the binary operation \(T_i : [a_i, 1]^2 \rightarrow [a_i, 1]\) as follows, where \(T_1 = V\) and for \(i \in \{2, \ldots, n\}\),

\[
T_i(x, y) = \begin{cases} 
T_{i-1}(x, y) & \text{if } (x, y) \in [a_{i-1}, 1)^2, \\
\times a_{i-1} & \text{if } (x, y) \in [a_{i-1}, 1) \times I_{a_{i-1}}, \\
\times y \times a_{i-1} & \text{if } (x, y) \in I_{a_{i-1}} \times [a_{i-1}, 1), \\
\times y & \text{if } x = 1 \text{ or } y = 1, \\
\text{int}(x) \times \text{int}(y) & \text{otherwise}.
\end{cases}
\]

To illustrate this claim we shall give the following example:

Example 5.4. Consider the lattice \((L_5 = \{0_5, a_4, b, c, a_3, a_2, a_1, 1_{L_5}\}, \leq, 0_{L_5}, 1_{L_5})\) described in Figure 5 with the finite chain \(0_{L_5} < a_4 < a_3 < a_2 < a_1 < 1_{L_5}\) in \(L_5\). Then, the interior operator \(\text{int} : L_5 \rightarrow L_5\) defined by \(\text{int}(0_{L_5}) = 0_{L_5}\), \(\text{int}(a_4) = \text{int}(a_3) = \text{int}(a_2) = \text{int}(a_1) = \text{int}(c) = \text{int}(b) = a_4\), \(\text{int}(1_{L_5}) = 1_{L_5}\). It is clear that \(\times a_i = \text{int}(x \times a_i)\) for
all \( x \in I_n \). Define the t-norm \( V : [a_1, 1_{L_5}]^2 \rightarrow [a_1, 1_{L_5}] \) by \( V = T_\land \). Since \( \text{int}(a_1) \land \text{int}(a_2) = a_4 \notin [a_2, 1_{L_5}] \), we can not obtain the binary operation \( T_2 \) on \([a_2, 1_{L_5}]\). Since \( \text{int}(a_3) \land \text{int}(a_1) = a_4 \notin [a_3, 1_{L_5}] \), we can not obtain the binary operation \( T_3 \) on \([a_3, 1_{L_5}]\).

**Theorem 5.5.** Let \((L, \leq, 0, 1)\) be a bounded lattice and \(\{a_0, a_1, a_2, \cdots, a_n\}\) be a finite chain in \(L\) such that \(0 = a_0 < a_1 < a_2 < \cdots < a_n = 1\). Let \(W : [0, a_1]^2 \rightarrow [0, a_1]\) be a t-conorm. Then, the function \(S_n : L^2 \rightarrow L\) defined recursively as follows is a t-conorm, where \(S_1 = W\) and for \(i \in \{2, \cdots , n\}\), the binary function \(S_i : [0, a_i]^2 \rightarrow [0, a_i]\) is given by

\[
S_i(x, y) = \begin{cases} 
S_{i-1}(x, y) & \text{if } (x, y) \in (0, a_{i-1})^2, \\
x \lor y & \text{if } x = 0 \text{ or } y = 0, \\
 a_i & \text{otherwise}.
\end{cases}
\]  

**Theorem 5.6.** Let \((L, \leq, 0, 1)\) be a bounded lattice and \(\{a_0, a_1, a_2, \cdots, a_n\}\) be a finite chain in \(L\) such that \(0 = a_0 < a_1 < a_2 < \cdots < a_n = 1\). Let \(W : [0, a_1]^2 \rightarrow [0, a_1]\) be a t-conorm. Then, the function \(S_n : L^2 \rightarrow L\) defined recursively as follows is a t-conorm, where \(S_1 = W\) and for \(i \in \{2, \cdots , n\}\),

\[
S_i(x, y) = \begin{cases} 
S_{i-1}(x, y) & \text{if } (x, y) \in (0, a_{i-1})^2, \\
x \lor y & \text{if } x = 0 \text{ or } y = 0, \\
x \lor y \land a_{i-1} & \text{otherwise}.
\end{cases}
\]

**Remark 5.7.** Let \((L, \leq, 0, 1)\) be a bounded lattice and \(\{a_0, a_1, a_2, \cdots, a_n\}\) be a finite chain in \(L\) such that \(0 = a_0 < a_1 < a_2 < \cdots < a_n = 1\). Let \(x \lor a_i = cl(x \lor a_i)\) for all \(x \in I_n\), let \(W : [0, a_1]^2 \rightarrow [0, a_1]\) be a t-conorm and \(cl : L \rightarrow L\) be a closure operator. It should be noted that our construction method in Theorem 5.6.3 can not be obtained using recursion. Because we can not obtain the binary operation \(S_i : [0, a_i]^2 \rightarrow [0, a_i]\) as follows, where \(S_1 = W\) and for \(i \in \{2, \cdots , n\}\),

\[
S_i(x, y) = \begin{cases} 
S_{i-1}(x, y) & \text{if } (x, y) \in (0, a_{i-1})^2, \\
y \lor a_{i-1} & \text{if } (x, y) \in (0, a_{i-1}) \times I_{a_{i-1}}, \\
x \lor a_{i-1} & \text{if } (x, y) \in I_{a_{i-1}} \times (0, a_{i-1}), \\
x \lor y \land a_{i-1} & \text{if } (x, y) \in I_{a_{i-1}} \times I_{a_{i-1}}, \\
x \lor y & \text{if } x = 0 \text{ or } y = 0, \\
cl(x) \lor cl(y) & \text{otherwise}.
\end{cases}
\]

To illustrate this claim we shall give the following example

**Example 5.8.** Consider the lattice \((L_6 = \{0_{L_6}, a_1, a_2, a_3, m, n, a_4, 1_{L_6}\}, \leq, 0_{L_6}, 1_{L_6})\) described in Figure 6 with the finite chain \(0_{L_6} < a_1 < a_2 < a_3 < a_4 < 1_{L_6}\) in \(L_6\). Then, the closure operator \(cl : L_6 \rightarrow L_6\) defined by \(cl(0_{L_6}) = 0_{L_6}\), \(cl(m) = cl(n) = cl(a_1) = cl(a_2) = cl(a_3) = cl(a_4) = a_4\), \(cl(1_{L_6}) = 1_{L_6}\). It is clear that \(x \lor a_i = cl(x \lor a_i)\) for all \(x \in I_{a_i}\). Define the t-conorm \(W : [0_{L_6}, a_1]^2 \rightarrow [0_{L_6}, a_1]\) by \(W = S_\lor\). Since \(\text{int}(a_1) \lor \text{int}(a_2) = a_4 \notin [0_{L_6}, a_2]\), we can not obtain the binary operation \(S_2\) on \([0_{L_6}, a_2]\). Since \(\text{int}(a_3) \lor \text{int}(a_1) = a_4 \notin [0_{L_6}, a_3]\), we can not obtain the binary operation \(S_3\) on \([0_{L_6}, a_3]\).

---

**Figure 5:** The lattice \(L_5\)
6 Concluding remarks

In this paper, we have proposed the constructions of t-norms and t-conorms on bounded lattices with interior and closure operators, respectively. The main aim of this paper is to present a rather effective method to construct t-norms and t-conorms by using interior and closure operators on a bounded lattice, respectively. Also, using these methods, in Corollary 6.10 and Corollary 6.3, we obtain the methods proposed by Ertuğrul, Karaçal and Mesiar [15]. Also, in Corollary 6.8 and Corollary 6.11, we obtain the methods proposed by Çaylı [8]. Finally, we have shown that the new construction methods can not be generalized by induction to a modified ordinal sum for t-norms and t-conorms on arbitrary bounded lattice, respectively.

Acknowledgement

We are grateful to the anonymous reviewers and editors for their valuable comments, which helped to improve the original version of our manuscript greatly.

References


