Iranian Journal of Fuzzy Systems
Volume 19, Number 2, (2022), pp. 99-103
Original paper

Extended results of "Cores of fuzzy games and their convexity"

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Abstract

In the framework of fuzzy transferable-utility games, Wu [7] derived the coincidences among of the proper core and the dominance core by providing some suitable conditions. By extending the results proposed by Wu [7], we provide alternative relations to show that the proper core coincides with the dominance core under some necessary and sufficient conditions.

Keywords: The proper core, the dominance core, coincidence.

1 Introduction

A *fuzzy transferable-utility (TU) game* could be treated as a natural extension of a traditional TU game in which each agent has infinite various energy levels respectively. The *core* is an useful solution concept on game theory. The core concepts of fuzzy TU games have been investigated by Aubin [1, 2], Butnariu [3], Hwang and Liao [4], Liao and Chung [5], Wu [7, 8], and so on. Here we focus on the core concepts of the *proper core* and the *dominance core* on fuzzy TU games.

Wu [7] derived the coincidences among of the proper core and the dominance core by providing some suitable conditions. Wu [7] also showed that the dominance core is a convex subset of the Euclidean space under some mild conditions. Different from the results proposed by Wu [7], we show that the proper core coincides with the dominance core under alternative necessary and sufficient conditions.

2 Preliminaries

Let U be the universe of players. If $N \subseteq U$ is a set of players, then a *fuzzy coalition* is a vector $\alpha \in [0,1]^N$. The i-th coordinate α_i of α is called the participation level of player i in the fuzzy coalition α. For all $T \subseteq N$, let $|T|$ be the number of elements in T. Instead of $[0,1]^T$, we will write F^T for the set of fuzzy coalitions. A player-coalition $T \subseteq N$ corresponds in a canonical way to the fuzzy coalition $e^T(N) \in F^N$, which is the vector with $e_i^T(N) = 1$ if $i \in T$, and $e_i^T(N) = 0$ if $i \in N \setminus T$. The fuzzy coalition $e^T(N)$ corresponds to the situation where the players in T fully cooperate (i.e. with participation level 1) and the players outsides T are not involved at all (i.e. they have participation level 0). If no confusion can arise $e^T(N)$ will be denoted by e^T. Denote the zero vector in $\mathbb{R}^N$ by 0_N. The fuzzy coalition 0_N corresponds to the empty player-coalition.

A *fuzzy TU game* is a pair (N, v), where N is a non-empty and finite set of players and $v : F^N \to \mathbb{R}$ is a characteristic function with $v(0_N) = 0$. The map v assigns to each fuzzy coalition $\alpha = (\alpha_i)_{i\in N} \in F^N$ a number, telling what such a coalition can achieve in cooperation. Denote the class of all fuzzy TU games by Γ.

Let $(N, v) \in \Gamma$. A *payoff vector* of (N, v) is a function $x : N \to \mathbb{R}$. Then

Corresponding Author: L. Y. Chung
Received: August 2020; Revised: August 2021; Accepted: November 2021.
https://doi.org/10.22111/IJFS.2022.6790

- A payoff vector x of $(N,v)\in\Gamma$ is *efficient (EFF)* if $\sum\limits_{i\in N} x_i = v(e^N)$.
- A payoff vector x of $(N,v)\in\Gamma$ is *individually rational (IR)* if for all $i\in N$ and for all $j\in[0,1]$, $jx_i \geq v(je^{\{i\}})$.
- A payoff vector x of $(N,v)\in\Gamma$ is *weak individually rational (WIR)* if for all $i\in N$ and for all $j\in[0,1]$, $x_i\geq v(e^{\{i\}})$.

Moreover, x is an *imputation* of (N,v) if it is EFF and IR. x is a *weak imputation* of (N,v) if it is EFF and WIR. The set of *feasible payoff vectors* of (N,v) is denoted by $X^*(N,v)=\{x\in\mathbb{R}^N|\sum\limits_{i\in N}x_i\leq v(e^N)\}$, whereas $X(N,v)=\{x\in\mathbb{R}^N|\ x \text{ is EFF}\}$ is the set of *preimputations* of (N,v), the set of *imputations* of (N,v) is denoted by $I(N,v)$, and the set of weak imputations of (N,v) is denoted by $I^*(N,v)$.

Given $(N,v)\in\Gamma$, $x\in\mathbb{R}^N$, $\alpha\in F^N$ and $S\subseteq N$, we denote $A(\alpha)=\{i\in N|\alpha_i\neq 0\}$, $x_S\in\mathbb{R}^S$ to be the restriction of x to S, and $x(\alpha)=\sum_{i\in N}\alpha_i x_i$. Let $(N,v)\in\Gamma$, $x,y\in I^*(N,v)$ and $\alpha\in F^N$. We say that y *dominates* x *via* α, denoted by $y\ dom_\alpha\ x$, if $y(\alpha)\leq v(\alpha)$ and $y_i>x_i$ for all $i\in A(\alpha)$. We say that y *dominates* x if there exists $\alpha\in F^N$ such that $y\ dom_\alpha\ x$.

Definition 2.1. [6, 7, 8] *Three core concepts on fuzzy TU games are defined as follows.*

1. *The core $C(N,v)$ of $(N,v)\in\Gamma$ consists of all $x\in X(N,v)$ that satisfy for all $\alpha\in F^N$, $x(\alpha)\geq v(\alpha)$.*
2. *The proper core $PC(N,v)$ comprises all $x\in X(N,v)$ that satisfy $x(\alpha)\geq v(\alpha)$ for all $\alpha\in PF^N$, where $PF^N=\{\alpha\in F^N|A(\alpha)\neq N\}$.*
3. *The dominance core $DC(N,v)$ comprises all $x\in I^*(N,v)$ for which there exists no $y\in I^*(N,v)$ such that y dominates x.*

Remark 2.2. *Let $(N,v)\in\Gamma$, $\alpha\in F^N$ and $x,y\in I^*(N,v)$. If $y\ dom_\alpha\ x$, then $\alpha\in PF^N$. If $A(\alpha)=N$, then $v(e^N)=y(e^N)>x(e^N)=v(e^N)$, which contradicts $x,y\in I^*(N,v)$.*

3 Some coincidences

A fuzzy TU game (N,v) is said to be *normalized* if $v(e^{\{i\}})=0$ for all $i\in N$. The *normalization* of (N,v), denoted by $(N,\bar{v})$, is defined by

$$\bar{v}(\alpha)=v(\alpha)-\sum_{i\in A(\alpha)}\alpha_i\cdot v(e^{\{i\}})\ \text{ for all } \alpha\in F^N. \tag{1}$$

A game $(N,v)\in\Gamma$ is *singular* if (N,v) satisfies the following condition:

$$\text{If } x\in I^*(N,v)\setminus PC(N,v), \text{ then there exists } \alpha\in PF^N\setminus\{0_N\} \text{ such that } x(e_{A(\alpha)})<v(\alpha). \tag{2}$$

Denote that $\Gamma_C=\{(N,v)\in\Gamma|PC(N,v)\neq\emptyset\}$, and Γ_S is the collection of all singular fuzzy TU games.

Remark 3.1.

- *In Wu* [7]*, the normalization was defined as the name of the auxiliary fuzzy game.*
- *It is easy to see that if $(N,v)\in\Gamma$ is normalized, then $x_i\geq v(e^{\{i\}})=0$ for all $x\in I^*(N,v)$ and for all $i\in N$.*

Lemma 3.2. [6, 7, 8] *For all $(N,v)\in\Gamma$, $PC(N,v)\subseteq DC(N,v)$.*

In order to present the main results of this paper, two results due to Wu [7] are provided as follows.

- [7, Theorem 2.2] If $(N,v)\in\Gamma_S$ satisfies
$$\bar{v}(e^N)\geq\bar{v}(\alpha)\text{ for all }\alpha\in PF^N\setminus\{0_N\}, \tag{3}$$
then $PC(N,v)=DC(N,v)$.
- [7, Theorem 2.1] If $(N,v)\in\Gamma$ satisfies
$$v(e^N)\geq\frac{v(\alpha)}{\min\limits_{i\in A(\alpha)}\alpha_i}+\sum_{i\in N\setminus A(\alpha)}v(e^{\{i\}})\text{ for all }\alpha\in PF^N\setminus\{0_N\}, \tag{4}$$
then $PC(N,v)=DC(N,v)$.

The following lemma would provide the relations of the proper core (dominance core) among a fuzzy game and its normalization. This lemma plays key role under the proofs of the main results of this paper.

Lemma 3.3. *Let $(N, v) \in \Gamma$, $(N, \bar{v})$ be the normalization of (N, v) and $x \in \mathbb{R}^N$. For all $i \in N$, define $y \in \mathbb{R}^N$ to be that $y_i = x_i - v(e^{\{i\}})$. Then*

1. *$x \in PC(N, v)$ if and only if $y \in PC(N, \bar{v})$,*
2. *$x \in DC(N, v)$ if and only if $y \in DC(N, \bar{v})$,*

Proof. It is easy to complete these results by the definitions of the proper core, the dominance core and a normalization of a game. The proofs also could be found in the proofs of Wu [7]. □

Different from the proof techniques of Theorem 2.2 due to Wu [7], a sufficient condition is proposed to guarantee the coincidence among the proper core and the dominance core by Lemma 3.3.

Theorem 3.4. *If the normalization $(N, \bar{v})$ of $(N, v) \in \Gamma$ satisfies*

$$\bar{v}(e^N) \geq \frac{\bar{v}(\alpha)}{\min\limits_{i \in A(\alpha)} \alpha_i}, \tag{5}$$

for all $\alpha \in PF^N \setminus \{0_N\}$, then $PC(N, v) = DC(N, v)$.

Proof. By Lemma 3.3, it suffices to complete this proof for normalized games. Assume that $(N, v) \in \Gamma$ is normalized.

Now assume that $v(e^N) \geq \frac{v(\alpha)}{\min\limits_{i \in A(\alpha)} \alpha_i}$ for all $\alpha \in PF^N \setminus \{0_N\}$. By Lemma 3.2 and definition of the dominance core, it remains to show that $x \notin DC(N, v)$ for all $x \in I^*(N, v) \setminus PC(N, v)$.[1] Let $x \in I^*(N, v) \setminus PC(N, v)$. Since $x \in I^*(N, v) \setminus PC(N, v)$, there exists $\alpha \in PF^N$ with $1 < |A(\alpha)| < |N|$ such that $x(\alpha) < v(\alpha)$. Define $y \in \mathbb{R}^N$ as follows. For all $i \in N$,

$$y_i = \begin{cases} x_i + \frac{v(\alpha) - x(\alpha)}{|A(\alpha)|} & \text{if } i \in A(\alpha), \\ \frac{1}{|N \setminus A(\alpha)|} \cdot \left[v(e^N) - x(e^N_{A(\alpha)}) - v(\alpha) + x(\alpha) \right] & \text{if } i \notin A(\alpha). \end{cases}$$

By definition of y, it is easy to verify that $y(e^N) = v(e^N)$. Hence, $y \in X(N, v)$. Since $v(\alpha) > x(\alpha)$ and $x_i \geq 0$ for all $i \in N$,

$$\begin{aligned} y(\alpha) &= \sum_{i \in A(\alpha)} \alpha_i y_i \\ &= \sum_{i \in A(\alpha)} \alpha_i x_i + \sum_{i \in A(\alpha)} \alpha_i \cdot \frac{v(\alpha) - x(\alpha)}{|A(\alpha)|} \\ &\leq \sum_{i \in A(\alpha)} \alpha_i x_i + \sum_{i \in A(\alpha)} 1 \cdot \frac{v(\alpha) - x(\alpha)}{|A(\alpha)|} \\ &= x(\alpha) + v(\alpha) - x(\alpha) \\ &= v(\alpha). \end{aligned}$$

Based on Remark 3.1, $x_i \geq 0$ for all $i \in N$. Since $v(\alpha) > x(\alpha)$ and $x_i \geq 0$ for all $i \in N$, we have that $y_i > x_i \geq 0$ for all $i \in A(\alpha)$. Let $\delta_\alpha = \min_{i \in A(\alpha)} \alpha_i$. By definition of y,

$$\begin{aligned} x(e^N_{A(\alpha)}) + v(\alpha) - x(\alpha) &= \sum_{i \in A(\alpha)} y_i \\ &= \sum_{i \in A(\alpha)} \frac{1}{\alpha_i} \cdot \alpha_i \cdot y_i \\ &\leq \sum_{i \in A(\alpha)} \frac{1}{\delta_\alpha} \cdot \alpha_i \cdot y_i \\ &= \frac{1}{\delta_\alpha} \sum_{i \in A(\alpha)} \alpha_i \cdot \left[x_i + \frac{v(\alpha) - x(\alpha)}{|A(\alpha)|} \right] \\ &\leq \frac{1}{\delta_\alpha} \sum_{i \in A(\alpha)} \alpha_i \cdot \left[x_i + \frac{v(\alpha) - x(\alpha)}{\sum\limits_{k \in A(\alpha)} \alpha_k} \right] \\ &= \frac{1}{\delta_\alpha} \left[x(\alpha) + v(\alpha) - x(\alpha) \right] \\ &= \frac{1}{\delta_\alpha} \cdot v(\alpha). \end{aligned} \tag{6}$$

[1] The remaining proofs are also similar to the proofs in Wu [7].

Clearly,

$$v(e^N) - x(e^N_{A(\alpha)}) - v(\alpha) + x(\alpha) = v(e^N) - \frac{1}{\delta_\alpha} \cdot v(\alpha) + \frac{1}{\delta_\alpha} \cdot v(\alpha) - x(e^N_{A(\alpha)}) - v(\alpha) + x(\alpha). \tag{7}$$

Since $v(e^N) \geq \frac{v(\alpha)}{\min_{i \in A(\alpha)} \alpha_i}$ for all $\alpha \in PF^N \setminus \{0_N\}$, we have that $y_i \geq 0$ for all $i \notin A(\alpha)$ by definition of y and equations (6), (7).

Hence, $y_i \geq 0 = v(e^{\{i\}})$ for all $i \in N$. By $y \in X(N,v)$ and Remark 3.1, $y \in I^*(N,v)$. Since $y \in I^*(N,v)$, $y(\alpha) \leq v(\alpha)$ and $y_i > x_i$ for all $i \in A(\alpha)$, we have that $y \; dom_\alpha \; x$. Hence, $x \notin DC(N,v)$. □

Different from the Theorem 2.1 due to Wu [7], we show that the proper core coincides with the dominance core under two necessary and sufficient conditions.

Theorem 3.5. *Let $(N,v) \in \Gamma_S$, $DC(N,v) \neq \emptyset$. Then $PC(N,v) = DC(N,v)$ if and only if the normalization $(N,\bar{v})$ of (N,v) satisfies that*

$$\bar{v}(e^N) \geq \bar{v}(\alpha), \tag{8}$$

for all $\alpha \in PF^N \setminus \{0_N\}$.

Proof. By Lemma 3.3, it suffices to complete this proof for normalized games. Assume that $(N,v) \in \Gamma_S$ is normalized.

Assume that $DC(N,v) \neq \emptyset$, $PC(N,v) = DC(N,v)$ and $x \in PC(N,v)$. Based on Remark 3.1, $x_i \geq 0$ for all $i \in N$. Since $x \in PC(N,v)$ and $x_i \geq 0$ for all $i \in N$, we have that

$$v(e^N) = x(e^N) = \sum_{i \in N} x_i \geq \sum_{i \in N} \alpha_i \cdot x_i \geq x(\alpha) \geq v(\alpha).$$

for all $\alpha \in PF^N$. Now assume that for all $\alpha \in PF^N$, $v(e^N) \geq v(\alpha)$. By Lemma 3.2 and definition of DC, it remains to show that $x \notin DC(N,v)$ for all $x \in I^*(N,v) \setminus PC(N,v)$. Let $x \in I^*(N,v) \setminus PC(N,v)$. Since $(N,v) \in \Gamma_S$ and $x \notin PC(N,v)$, there exists $\alpha \in PF^N$ such that $v(\alpha) > x(e^N_{A(\alpha)})$. The remaining proof is similar to Theorem 3.4. □

Theorem 3.6. *For all $(N,v) \in \Gamma_C \cap \Gamma_S$, $PC(N,v) = DC(N,v)$.*

Proof. By Lemma 3.3, it suffices to complete this proof for normalized games. Assume that $(N,v) \in \Gamma_C \cap \Gamma_S$ is normalized. By Lemma 3.2, $\emptyset \neq PC(N,v) \subseteq DC(N,v)$ if $(N,v) \in \Gamma_C$. By the proof of Theorem 3.5, $PC(N,v) \neq \emptyset$ implies that $v(\alpha) \leq v(e^N)$ for all $\alpha \in PF^N$. By Theorem 3.5, the proof is completed. □

4 Conclusions

By building on the results due to Wu [7], we show that the proper core coincides with the dominance core under alternative necessary and sufficient conditions. One should compare our results with the results due to Wu [7]. Several major differences are as follows:

1. Based on Lemma 3.2 and equation (1), Theorem 2.2 due to Wu [7] shows that if a fuzzy TU game (N,v) satisfies equation (3) (or (8)), then the proper core coincides with the dominance core in (N,v). Based on equation (1), we firstly introduce Lemma 3.3, and further adopt the non-emptiness of the dominance core, Lemmas 3.2, 3.3 and equation (3) (or (8)) to present the same result (Theorem 3.5 of this paper). There are two major differences:
   - Lemma 3.3 do not appear in Wu [7].
   - Lemma 3.3 plays key role under the proof of Theorem 3.5 of this paper. Related techniques of the proofs among Wu [7] and this paper are also different.
2. Based on Lemma 3.2, equations (1) and (2), Wu [7] showed that if a fuzzy TU game (N,v) satisfies equation (4), then the proper core coincides with the dominance core in (N,v). Based on Lemma 3.2, equations (1) and (2), we show that a fuzzy TU game (N,v) satisfies equation (5) if and only if the proper core coincides with the dominance core in (N,v). There are two major differences:
   - Equations (4) and (5) are different.

- Based on equation (4), Wu [7] introduced a sufficient condition to guarantee the coincidence among the proper core and the dominance core. Different from Wu [7], we show that the proper core coincides with the dominance core under a sufficient and necessary conditions by applying equation (5). In particular, related sufficient conditions among Wu [7] and this paper are also different. For example, let $N = \{i, j, k\}$ and $(\frac{1}{2}, 0, \frac{1}{4}) \in PF^N \setminus \{0_N\}$.
  - Based on equation (4), the sufficient condition related to $(\frac{1}{2}, 0, \frac{1}{4})$ is $v(1,1,1) \geq 4v(\frac{1}{2}, 0, \frac{1}{4}) + v(0,1,0)$.
  - Based on equation (5), the sufficient condition related to $(\frac{1}{2}, 0, \frac{1}{4})$ is $v(1,1,1) \geq 4v(\frac{1}{2}, 0, \frac{1}{4}) - v(1,0,0) + 2v(0,1,0)$.

3. Based on the non-emptiness of the proper core and equations (2) simultaneously, we show that the proper core coincides with the dominance core. This result do not appear in Wu [7].

4. In future research, we will try to propose more general conditions for the coincidences among the proper core and the dominance core.

Acknowledgement

The author is very grateful to the Editor, the Associate Editor and anonymous referees for valuable comments which much improved the paper.

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