Iranian Journal of Fuzzy Systems

Volume 19, Number 2, (2022), pp. 169-186

Original paper

On the distributivity of T-power based implications

Z. Peng[1] and J. Pan[2]

[1,2] *College of Mathematics and Statistics, Yangtze Normal University, Chongqing, PR China*
[2] *Department of Mathematics and Statistics, University of Strathclyde, Glasgow, UK*

364886982@qq.com, jiazhu.pan@strath.ac.uk

Abstract

Due to the fact that Zadeh's quantifiers constitute the usual method to modify fuzzy propositions, the so-called family of T-power based implications was proposed. In this paper, the four basic distributive laws related to T-power based fuzzy implications and fuzzy logic operations (t-norms and t-conorms) are deeply studied. This study shows that two of the four distributive laws of the T-power based implications have a unique solution, while the other two have multiple solutions.

Keywords: T-power based implications, distributivity, t-norms, t-conorms.

1 Introduction

Due to fuzzy implications are the main operations in fuzzy logic, various fuzzy implications have been proposed. For example, the (S, N)-, R- and QL-implications are built by translating different classical logical formulae to the fuzzy context [4, 5]. The f- and g-implications are built from continuous additive generators of continuous Archimedean t-norms or t-conorms, respectively [21]. The probabilistic implications and probabilistic S-implications are built from copula functions [10]. The semicopula based implications are built from initial fuzzy implications and semicopula functions [2]. The fuzzy negation based implications are built from negation functions [15], etc.

In 2017, Massanet et al. noticed that a special property called invariance is required on a fuzzy implication when it is used in approximate reasoning. However, as most of the known fuzzy implications do not have this property, the so-called family of T-power based implications was proposed [13]. Most of the T-power based implications were found to satisfy the invariant property [14]. Nevertheless, there are no corresponding discussions on the distributive laws for the T-power based implications, although the distributive laws play a critical role in both theoretical and practical fields for fuzzy implications [7, 9]. On the other hand, there are many discussions on the distributive equations of fuzzy implications (detail see for [1, 3, 6, 8, 12, 16, 17, 18, 19, 20]). Therefore, as a supplement of this research topic from the theoretical point of view, it is necessary to investigate the distributive laws for the T-power implications.

The paper is organized as follows. In Section 2, some concepts and results are recalled. In Section 3, four distributive equations involving T-power based implications are analyzed. Finally, the paper ends with a section devoted to the conclusions.

2 Preliminaries

For convenience, in this section, the definitions and results to be used in the rest of the paper are outlined.

Definition 2.1. [4] *A function $I : [0,1]^2 \to [0,1]$ is called a fuzzy implication if it satisfies, for all x, x_1, x_2, y, y_1, $y_2 \in [0,1]$, the following conditions:*

if $x_1 < x_2$, then $I(x_1, y) \geq I(x_2, y)$, i.e., $I(\cdot, y)$ is decreasing, (I1)

Corresponding Author: Z. Peng

Received: January 2021; Revised: October 2021; Accepted: January 2022.

https://doi.org/10.22111/IJFS.2022.6797

if $y_1 < y_2$, then $I(x, y_1) \leq I(x, y_2)$, i.e., $I(x, \cdot)$ is increasing, (I2)
$I(0,0) = 1$, $I(1,1) = 1$, $I(1,0) = 0$. (I3)

The set of all fuzzy implications will be denoted by FI.

Definition 2.2. [4] *An operator $I : [0,1]^2 \to [0,1]$ is said to satisfy the ordering property, if $I(x,y) = 1 \Leftrightarrow x \leq y$ for all $x, y \in [0,1]$.* (OP)

Definition 2.3. [11] *An associative, commutative and increasing function $T : [0,1]^2 \to [0,1]$ is called a t-norm if it satisfies $T(x,1) = x$ for all $x \in [0,1]$.*

Example 2.4. [11] *The following are the three basic t-norms T_M, T_P, T_{LK}, given by, respectively:*

$$T_M(x,y) = min(x,y), \qquad T_P(x,y) = xy, \qquad T_{LK}(x,y) = max(x+y-1, 0).$$

Definition 2.5. [4] *A t-norm T is called*
- *continuous if it is continuous in both the arguments;*
- *strict, if it is continuous and strictly monotone;*
- *Archimedean, if for all $x, y \in (0,1)$ there exists an $n \in N$ such that $x_T^{(n)} < y$, where*

$$x_T^{(0)} = 1, \;\; x_T^{(1)} = x, \;\; x_T^{(n)} = T(x, x_T^{(n-1)}) \;\; for \;\; all \;\; n \geq 2.$$

- *nilpotent, if it is continuous and if each $x \in (0,1)$ is a nilpotent element of T , i.e., if there exists an $n \in N$ such that $x_T^{(n)} = 0$.*

Remark 2.6. [4] *If a t-norm T is strict or nilpotent, then it is Archimedean. Conversely, every continuous and Archimedean t-norm is strict or nilpotent.*

Theorem 2.7. [4] *For a function $T : [0,1]^2 \to [0,1]$ the following statements are equivalent:*
(i) T is a continuous Archimedean t-norm.
(ii) T has a continuous additive generator, i.e., there exists a continuous, strictly decreasing function $t : [0,1] \to [0,\infty]$ with $t(1) = 0$, which is uniquely determined up to a positive multiplicative constant, such that

$$T(x,y) = t^{-1}(min(t(x) + t(y), t(0))), \;\; x, \; y \in [0,1].$$

Remark 2.8. [4] *(i) T is a strict t-norm if and only if each continuous additive generator t of T satisfies $t(0) = \infty$.*
(ii) T is a nilpotent t-norm if and only if each continuous additive generator t of T satisfies $t(0) < \infty$.

Theorem 2.9. [11] *Let A be an index set and $(T_i)_{i \in A}$ a family of t-norms, let $\{(a_i, b_i)\}_{i \in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. Then the following function $T : [0,1]^2 \to [0,1]$ is a t-norm:*

$$T(x,y) = \begin{cases} a_i + (b_i - a_i) \cdot T_i(\frac{x - a_i}{b_i - a_i}, \frac{y - a_i}{b_i - a_i}), & \text{if } x, y \in [a_i, b_i], \\ min(x,y), & \text{otherwise.} \end{cases} \tag{1}$$

Definition 2.10. [11] *(i) A t-norm T is called an ordinal sum of t-norms, also known as the summands $< a_i, b_i, T_i >$, $i \in A$, if it is defined as (1). In this case we write $T = (< a_i, b_i, T_i >)_{i \in A}$, where A is an index set, $(T_i)_{i \in A}$ a family of t-norms, and $\{(a_i, b_i)\}_{i \in A}$ is a family of non-empty, pairwise disjoint open subintervals of $[0,1]$.*
(ii) $T = (< a_i, b_i, T_i >)_{i \in A}$ is trivial if $A = \{1\}$, $a_1 = 0$ and $b_1 = 1$.

Theorem 2.11. [4] *For a function $T : [0,1]^2 \to [0,1]$ the following statements are equivalent:*
(i) T is a continuous t-norm.
(ii) T is uniquely representable as an ordinal sum of continuous Archimedean t-norms, i.e, there exist a uniquely determined (finite or countably infinite) index set A, a family of uniquely determined pairwise disjoint open subintervals $\{(a_i, b_i)\}_{i \in A}$ of $[0,1]$ and a family of uniquely determined continuous Archimedean t-norms $(T_i)_{i \in A}$ such that

$$T(x,y) = \begin{cases} a_i + (b_i - a_i) \cdot T_i(\frac{x - a_i}{b_i - a_i}, \frac{y - a_i}{b_i - a_i}), & \text{if } x, y \in [a_i, b_i], \\ min(x,y), & \text{otherwise.} \end{cases}$$

Remark 2.12. *For a continuous t-norm T, if $T \neq T_M$, then it is either a continuous Archimedean t-norm or a non-trivial ordinal sum of continuous Archimedean t-norms.*

Definition 2.13. [4, 11] *(i) An associative, commutative and increasing function $S : [0,1]^2 \to [0,1]$ is called a t-conorm if it satisfies $S(x,0) = x$ for all $x \in [0,1]$.*

(ii) A t-conorm S is idempotent, if $S(x,x) = x$ for all $\in [0,1]$;

Example 2.14. *The following are four basic t-conorms S_M, S_{LK}, S_D, S_{nM} given by, respectively:*

$$S_M(x,y) = max(x,y), \qquad S_{LK}(x,y) = min(x+y,1),$$

$$S_D(x,y) = \begin{cases} 1, & \text{if } x,y \in (0,1], \\ max(x,y), & \text{otherwise,} \end{cases} \qquad S_{nM}(x,y) = \begin{cases} 1, & \text{if } x+y \geq 1, \\ max(x,y), & \text{otherwise.} \end{cases}$$

Definition 2.15. [11, 13] *Let T be a continuous t-norm. For each $x \in [0,1]$, n-th roots and rational powers of x with respect to T are defined by*

$$x_T^{(\frac{1}{n})} = sup\{z \in [0,1] | z_T^{(n)} \leq x\}, \quad x_T^{(\frac{m}{n})} = \left(x_T^{(\frac{1}{n})}\right)_T^{(m)},$$

where m, n are positive integers.

Definition 2.16. [13] *A binary operator $I : [0,1]^2 \to [0,1]$ is said to be a T-power based implication (power based implication for short) if there exists a continuous t-norm T such that*

$$I(x,y) = sup\{r \in [0,1] | y_T^{(r)} \geq x\}, \quad for \ all \ x,y \in [0,1]. \tag{2}$$

If I is a T-power based implication, then it will be denoted by I^T.

Proposition 2.17. [13] *Let T be a continuous t-norm and I^T its power based implication defined by (2).*

(i) If $T = T_M$, then $I^T(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{if } x > y, \end{cases}$ the Rescher implication I_{RS}.

(ii) If T is an Archimedean t-norm with additive generator t, then

$$I^T(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ \frac{t(x)}{t(y)}, & \text{if } x > y, \end{cases}$$

with the convention that $\frac{a}{\infty} = 0$ for all $a \in [0,1]$.

(iii) If T is an ordinal sum of t-norms of the form $T = (< a_i, b_i, T_i >)_{i \in A}$, where T_i is an Archimedean t-norm with additive generator t_i for all $i \in A$, then

$$I^T(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ \frac{t_i(\frac{x-a_i}{b_i-a_i})}{t_i(\frac{y-a_i}{b_i-a_i})}, & \text{if } x > y \text{ and } x,y \in [a_i,b_i], \\ 0, & \text{otherwise.} \end{cases}$$

3 Distributivity of the T-power based implications

The four distributive laws involving a fuzzy implication I are given as follows:

$$I(S(x,y),z) = T(I(x,z),I(y,z)), \tag{3}$$

$$I(T(x,y),z) = S(I(x,z),I(y,z)), \tag{4}$$

$$I(x,T_1(y,z)) = T_2(I(x,y),I(x,z)), \tag{5}$$

$$I(x,S_1(y,z)) = S_2(I(x,y),I(x,z)), \tag{6}$$

for all x, y, $z \in [0,1]$, where T, T_1, T_2 are t-norms, S, S_1, S_2 are t-conorms [1, 4, 8].

For the power based implication I^{T_M}, it is Rescher implication. The solutions of distributivity equations involving I^{T_M} are shown in Table 1, since its solutions are easily obtained. The complete proof of Table 1 is shown in Appendix A.

In the following, let us study the distributive laws of the T-power based implication I^T, where T is a continuous Archimedean t-norm, or a non-trivial ordinal sum of continuous Archimedean t-norms.

Table 1: Distributivity solutions of fuzzy implication I^{T_M}

Equation	Solution
$I^{T_M}(S(x,y),z) = T(I^{T_M}(x,z), I^{T_M}(y,z))$	$S = S_M$, any t-norm T
$I^{T_M}(T(x,y),z) = S(I^{T_M}(x,z), I^{T_M}(y,z))$	$T = T_M$, any t-conorm S
$I^{T_M}(x, T_1(y,z)) = T_2(I^{T_M}(x,y), I^{T_M}(x,z))$	$T_1 = T_M$, any t-norm T_2
$I^{T_M}(x, S_1(y,z)) = S_2(I^{T_M}(x,y), I^{T_M}(x,z))$	$S_1 = S_M$, any t-conorm S_2

3.1 On the equation $I(S(x,y),z) = T(I(x,z), I(y,z))$

Lemma 3.1. *Let a function $I : [0,1]^2 \to [0,1]$ satisfy (OP), T be a t-norm and S a t-conorm. If the triple (I, S, T) satisfies (3), then $S = S_M$.*

Proof. Assume that the triple (I, S, T) satisfies (3), then $I(S(x,y),z) = T(I(x,z), I(y,z))$ for all x, y, $z \in [0,1]$. Putting $x = y = z$, we get $I(S(x,x),x) = T(I(x,x), I(x,x)) = 1$ for all $x \in [0,1]$. Since I satisfies (OP), then $S(x,x) \le x$. Note that $S(x,x) \ge x$ for all $x \in [0,1]$. Then $S(x,x) = x$ for all $x \in [0,1]$, i.e., $S = S_M$. □

Theorem 3.2. *Let T be a continuous Archimedean t-norm (a non-trivial ordinal sum of continuous Archimedean t-norms, respectively) and I^T its power based implication, let T_1 be a t-norm and S a t-conorm. Then the following statements are equivalent:*

(i) The triple (I^T, S, T_1) satisfies (3).

(ii) $S = S_M$ and $T_1 = T_M$.

Proof. (i⇒ ii) Let the triple (I^T, S, T_1) satisfy (3). Since I^T satisfies (OP) ([13], Proposition 8), then $S = S_M$ by Lemma 3.1. Thus

$$I^T(\max(x,y),z) = T_1(I^T(x,z), I^T(y,z)) \text{ for all } x,\ y,\ z \in [0,1].$$

Let $x = y$. Then $I^T(x,z) = T_1(I^T(x,z), I^T(x,z))$ for all x, $z \in [0,1]$.

Case 1: T is a continuous Archimedean t-norm.

Let t be an additive generator of T, and let $x > z > 0$ in above equation, then

$$\frac{t(x)}{t(z)} = T_1\left(\frac{t(x)}{t(z)}, \frac{t(x)}{t(z)}\right).$$

Let $a = \frac{t(x)}{t(z)}$. Then $a \in [0,1)$ and $a = T_1(a,a)$. Hence $T_1 = T_M$.

Case 2: T is a non-trivial ordinal sum of continuous Archimedean t-norms.

Without loss of generality assume that $T = (< a_i, b_i, T_i >)_{i \in A}$, where A is an index set, T_i is a continuous Archimedean t-norm with additive generator t_i for all $i \in A$, and $\{(a_i, b_i)\}_{i \in A}$ is a family of non-empty, pairwise disjoint open subintervals of $[0,1]$.

Let x, $z \in [a_i, b_i]$ for some $i \in A$ with $x > z > a_i$. Then

$$\frac{t_i(\frac{x-a_i}{b_i-a_i})}{t_i(\frac{z-a_i}{b_i-a_i})} = T_1\left(\frac{t_i(\frac{x-a_i}{b_i-a_i})}{t_i(\frac{z-a_i}{b_i-a_i})}, \frac{t_i(\frac{x-a_i}{b_i-a_i})}{t_i(\frac{z-a_i}{b_i-a_i})}\right).$$

Let $m = \frac{t_i(\frac{x-a_i}{b_i-a_i})}{t_i(\frac{z-a_i}{b_i-a_i})}$. Then $m \in [0,1)$ and $m = T_1(m,m)$. Hence $T_1 = T_M$.

(ii⇒ i) Obvious. □

3.2 On the equation $I(T(x,y),z) = S(I(x,z), I(y,z))$

Theorem 3.3. *Let T be a continuous Archimedean t-norm (a non-trivial ordinal sum of continuous Archimedean t-norms, respectively) and I^T its power based implication, and let S be a t-conorm. Then the triple (I^T, T, S) satisfies (4) if and only if $S = S_{LK}$.*

Proof. **Case 1:** T is a continuous Archimedean t-norm.

(Necessity) Let the triple (I^T, T, S) satisfy (4). Suppose that $S \neq S_{LK}$, then there exist a, $b \in (0,1)$ such that

$$S(a,b) \neq \min(a+b,1). \tag{7}$$

Assume that t is an additive generator of T, then t is continuous, strictly decreasing ([4], Theorem 2.1.5). Thus there exist x_0, y_0, $z_0 \in (0,1)$ with $x_0 > z_0$, $y_0 > z_0$ such that

$$\frac{t(x_0)}{t(z_0)} = a \text{ and } \frac{t(y_0)}{t(z_0)} = b, \tag{8}$$

i.e., $I^T(x_0, z_0) = a$, $I^T(y_0, z_0) = b$.

If $a + b < 1$, i.e., $t(x_0) + t(y_0) < t(z_0)$, by (7) and (8) we get

$$S\left(I^T(x_0,z_0), I^T(y_0,z_0)\right) = S(a,b) \neq a+b = \frac{t(x_0)}{t(z_0)} + \frac{t(y_0)}{t(z_0)}. \tag{9}$$

However, by $t(z_0) < t(0)$, we get $t(x_0) + t(y_0) < t(0)$. Then

$$T(x_0,y_0) = t^{-1}(\min(t(x_0)+t(y_0),t(0))) = t^{-1}(t(x_0)+t(y_0)) > z_0.$$

Hence

$$I^T(T(x_0,y_0),z_0) = \frac{t(x_0)+t(y_0)}{t(z_0)} = a+b. \tag{10}$$

From (9), (10) we get $I^T(T(x_0,y_0),z_0) \neq S(I^T(x_0,z_0), I^T(y_0,z_0))$, this contradicts the fact that the triple (I^T, T, S) satisfies (4).

If $a + b \geq 1$, i.e., $t(x_0) + t(y_0) \geq t(z_0)$, by (7) we get

$$S\left(\frac{t(x_0)}{t(z_0)}, \frac{t(y_0)}{t(z_0)}\right) = S(a,b) \neq 1,$$

i.e., $S\left(I^T(x_0,z_0), I^T(y_0,z_0)\right) \neq 1$.

However, since $t^{-1}(t(0)) = 0 < z_0$, then $t^{-1}(\min(t(x_0)+t(y_0),t(0))) \leq z_0$, i.e., $T(x_0,y_0) \leq z_0$. Hence $I^T(T(x_0,y_0),z_0) = 1$. Thus $I^T(T(x_0,y_0),z_0) > S(I^T(x_0,z_0), I^T(y_0,z_0))$. A contradiction to the fact that the triple (I^T, T, S) satisfies (4).

(Sufficiency) Let $S = S_{LK}$. It suffices to prove that the triple (I^T, T, S) satisfies (4) for all x, y, $z \in [0,1]$ with $x > z$ and $y > z$.

If $T(x,y) > z$, i.e., $t^{-1}(\min(t(x)+t(y),t(0))) > z$, then $\min(t(x)+t(y),t(0)) < t(z)$. Note that $t(z) \leq t(0)$, then $t(x) + t(y) < t(z) \leq t(0)$. Thus

$$I^T(T(x,y),z) = \frac{t(T(x,y))}{t(z)} = \frac{\min(t(x)+t(y),t(0))}{t(z)} = \frac{t(x)+t(y)}{t(z)} = S_{LK}(I^T(x,z), I^T(y,z)).$$

If $T(x,y) \leq z$, i.e., $t^{-1}(\min(t(x)+t(y),t(0))) \leq z$, then

$$I^T(T(x,y),z) = 1 \text{ and } \min(t(x)+t(y),t(0)) \geq t(z).$$

Since $t(0) \geq t(z)$, then $t(x) + t(y) \geq t(z)$. Thus $\frac{t(x)}{t(z)} + \frac{t(y)}{t(z)} \geq 1$. Therefore,

$$S_{LK}(I^T(x,z), I^T(y,z)) = \min\left(\frac{t(x)}{t(z)} + \frac{t(y)}{t(z)}, 1\right) = 1.$$

Hence $I^T(T(x,y),z) = S_{LK}(I^T(x,z), I^T(y,z))$.

Thus we complete the proof in the case that T is a continuous Archimedean t-norm.

Case 2: T is a non-trivial ordinal sum of continuous Archimedean t-norms.

Without loss of generality assume that $T = (< a_i, b_i, T_i >)_{i \in A}$, where A is an index set, T_i is a continuous Archimedean t-norm for all $i \in A$, and $\{(a_i, b_i)\}_{i \in A}$ is a family of non-empty, pairwise disjoint open subintervals of $[0,1]$.

Let $x,\ y,\ z \in [0,1]$ with $x > z,\ y > z$. If there is not an $i \in A$ such that $x,\ y,\ z \in [a_i, b_i]$, then equation $I^T(T(x,y),z) = S(I^T(x,z), I^T(y,z))$ holds for any t-conorm S.

In fact, consider the following cases.

Case 2.1: for all $i \in A$, $z \notin [a_i, b_i]$. Obviously, $I^T(x,z) = 0$, and $I^T(y,z) = 0$.

If there exists a $k \in A$ such that $x,\ y \in [a_k, b_k]$, then

$$T(x,y) = a_k + (b_k - a_k) \cdot T_i(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}) \in [a_k, b_k].$$

Since $x > z,\ y > z$, then $z < a_k$. Thus $I^T(T(x,y),z) = 0$. If there is not a $k \in A$ such that $x,\ y \in [a_k, b_k]$, obviously, $T(x,y) = \min(x,y) > z$. Thus $I^T(T(x,y),z) = 0$. Hence, $I^T(T(x,y),z) = S(I^T(x,z), I^T(y,z))$ holds for any t-conorm S.

Case 2.2: there exists an $i \in A$ such that $z \in [a_i, b_i]$, $x \notin [a_i, b_i]$, $y \notin [a_i, b_i]$, and there is not a $k \in A$ such that x, $y \in [a_k, b_k]$. Then $T(x,y) = \min(x,y) > z$, and $T(x,y) \notin [a_i, b_i]$. Thus

$$I^T(T(x,y),z) = 0,\ \ I^T(x,z) = 0,\ \text{ and }\ I^T(y,z) = 0.$$

Hence $I^T(T(x,y),z) = S(I^T(x,z), I^T(y,z))$ holds for any t-conorm S.

Case 2.3: there exists an $i \in A$ such that $z \in [a_i, b_i]$, $x \notin [a_i, b_i]$, $y \notin [a_i, b_i]$, and there exists a $k \in A$ such that x, $y \in [a_k, b_k]$. Then

$$I^T(x,z) = 0,\ \ I^T(y,z) = 0,\ \text{ and }\ T(x,y) = a_k + (b_k - a_k) \cdot T_k(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}).$$

Since $x > z,\ y > z$, then $b_i \le a_k$.

If $b_i < a_k$, then $T(x,y) \notin [a_i, b_i]$. Thus, $I^T(T(x,y),z) = 0$.

If $b_i = a_k$, then $z < b_i$, since $z \in [a_i, b_i]$ and $z \notin [a_k, b_k]$. Note that $T(x,y) \ge a_k = b_i$. Obviously, $I^T(T(x,y),z) = 0$. The reason is that $T(x,y) \notin [a_i, b_i]$ when $T(x,y) > b_i$, and $I^T(T(x,y),z) = \frac{t_i(\frac{b_i-a_i}{b_i-a_i})}{t_i(\frac{z-a_i}{b_i-a_i})} = 0$ when $T(x,y) = b_i$.

Hence, equation $I^T(T(x,y),z) = S(I^T(x,z), I^T(y,z))$ holds for any t-conorm S.

Case 2.4: there exists an $i \in A$ such that $z,\ x \in [a_i, b_i]$, $y \notin [a_i, b_i]$. Then $I^T(y,z) = 0$. Since $y > z$, then $y > b_i \ge x$. Thus $T(x,y) = \min(x,y) = x$. Therefore,

$$I^T(T(x,y),z) = I^T(x,z).$$

Hence, equation $I^T(T(x,y),z) = S(I^T(x,z), I^T(y,z))$ holds for any t-conorm S.

Case 2.5: there exists an $i \in A$ such that $z,\ y \in [a_i, b_i]$, $x \notin [a_i, b_i]$. Similarly to Case 2.4, equation $I^T(T(x,y),z) = S(I^T(x,z), I^T(y,z))$ holds for any t-conorm S.

Hence, it suffices to consider $x,\ y,\ z \in [a_i, b_i]$ for some $i \in A$. The rest proof is similar to the proof of Case 1. □

To show the application of Theorem 3.3, an example is given.

Example 3.4. *Let T be a continuous Archimedean t-norm with additive generator $t(x) = 1 - x$, $x \in [0,1]$, then*

$$T = T_{LK},\ \text{and}\ \ I^T(x,y) = \begin{cases} 1, & \text{if } x \le y, \\ \frac{1-x}{1-y}, & \text{if } x > y. \end{cases}$$

If $x \le z$ or $y \le z$, then $I^T(T(x,y),z) = 1 = S_{LK}(I^T(x,z), I^T(y,z))$.

If $x > z$ and $y > z$, then

$$I^T(T(x,y),z) = \begin{cases} 1, & \text{if } x+y-1 \le z, \\ \frac{2-(x+y)}{1-z}, & \text{if } x+y-1 > z, \end{cases} = min\left(\frac{2-(x+y)}{1-z}, 1\right),$$

$$S_{LK}(I^T(x,z), I^T(y,z)) = S_{LK}\left(\frac{t(x)}{t(z)}, \frac{t(y)}{t(z)}\right) = min\left(\frac{2-(x+y)}{1-z}, 1\right).$$

Thus $I^T(T(x,y),z) = S_{LK}(I^T(x,z), I^T(y,z))$ for all $x,\ y,\ z \in [0,1]$. Hence the triple (I^T, T, S_{LK}) satisfies (4).

Remark 3.5. *Note that the triple (I, T_M, S_M) satisfies (4) for any fuzzy implication I. Therefore, equation (4) is also satisfied by the triple (I^T, T_M, S_M). This result indicates that there exist a t-norm T_1 different from T and a t-conorm S different from S_{LK}, such that the triple (I^T, T_1, S) satisfies (4).*

In the following, we study the t-norm T_1 different from T and the t-conorm S different from S_{LK} such that the triple (I^T, T_1, S) satisfies (4).

Lemma 3.6. *Let $\alpha \in (0, \infty)$ and $S : [0,1]^2 \to [0,1]$ be a function defined as*

$$S(x,y) = min\left((x^{\frac{1}{\alpha}} + y^{\frac{1}{\alpha}})^{\alpha}, 1\right), \ \ x, \ y \in [0,1],$$

then S is φ-conjugate with S_{LK}, i.e., S is a t-conorm.

Proof. Let $\varphi : [0,1] \to [0,1]$ be a function defined by

$$\varphi(x) = x^{\frac{1}{\alpha}}, \ x \in [0,1], \ \alpha > 0.$$

Obviously, φ is an automorphism. Consider the Lukasiewicz t-conorm S_{LK}, i.e.,

$$S_{LK}(x,y) = \min(x+y, 1), \ \ x, y \in [0,1].$$

Then, for all x, $y \in [0,1]$, we have

$$\varphi^{-1}(S_{LK}(\varphi(x), \varphi(y))) = \left(\min(x^{\frac{1}{\alpha}} + y^{\frac{1}{\alpha}}, 1)\right)^{\alpha} = \min\left((x^{\frac{1}{\alpha}} + y^{\frac{1}{\alpha}})^{\alpha}, 1\right) = S(x,y),$$

that is, S is φ-conjugate with S_{LK}. Therefore, S is a t-conorm. □

Proposition 3.7. *Let T be a continuous Archimedean t-norm with additive generator t and I^T its power based implication. Let T_1 be a continuous Archimedean t-norm with additive generator t_1 defined by*

$$t_1(x) = (k \cdot t(x))^{\frac{1}{\alpha}}, \ \ x \in [0,1],$$

and S be a t-conorm defined by $S(x,y) = min\left((x^{\frac{1}{\alpha}} + y^{\frac{1}{\alpha}})^{\alpha}, 1\right)$. Then the triple (I^T, T_1, S) satisfies (4), where k, α are constants, and $\alpha > 0$, $k > 0$.

Proof. Let x, y, $z \in [0,1]$. It suffices to prove that the triple (I^T, T_1, S) satisfies (4) for $x > z$ and $y > z$.

Since t_1 is an additive generator of T_1, then

$$T_1(x,y) = t_1^{-1}(\min(t_1(x) + t_1(y), t_1(0))), \ \ x, y \in [0,1].$$

If $T_1(x,y) \le z$, then $t_1(x) + t_1(y) \ge t_1(z)$, and $I^T(T_1(x,y), z) = 1$. From $t_1(x) + t_1(y) \ge t_1(z)$ we get

$$\frac{t_1(x)}{t_1(z)} + \frac{t_1(y)}{t_1(z)} \ge 1,$$

that is

$$\frac{t_1(t^{-1}(t(x)))}{t_1(t^{-1}(t(z)))} + \frac{t_1(t^{-1}(t(y)))}{t_1(t^{-1}(t(z)))} \ge 1. \tag{11}$$

From $t_1(x) = (k \cdot t(x))^{\frac{1}{\alpha}}$ we get $t_1(t^{-1}(x)) = (kx)^{\frac{1}{\alpha}}$, $x \in [0, t(0)]$. Then from (11) we have

$$\left(\frac{t(x)}{t(z)}\right)^{\frac{1}{\alpha}} + \left(\frac{t(y)}{t(z)}\right)^{\frac{1}{\alpha}} \ge 1.$$

Thus

$$S(I^T(x,z), I^T(y,z)) = S\left(\frac{t(x)}{t(z)}, \frac{t(y)}{t(z)}\right) = 1.$$

Therefore, $I^T(T_1(x,y), z) = 1 = S(I^T(x,z), I^T(y,z))$.

If $T_1(x,y) > z$, similarly, we obtain $\left(\frac{t(x)}{t(z)}\right)^{\frac{1}{\alpha}} + \left(\frac{t(y)}{t(z)}\right)^{\frac{1}{\alpha}} < 1$, then

$$S(I^T(x,z), I^T(y,z)) = S\left(\frac{t(x)}{t(z)}, \frac{t(y)}{t(z)}\right) = \left(\left(\frac{t(x)}{t(z)}\right)^{\frac{1}{\alpha}} + \left(\frac{t(y)}{t(z)}\right)^{\frac{1}{\alpha}}\right)^{\alpha}.$$

On the other hand, from $T_1(x,y) > z$ we obtain $\min(t_1(x)+t_1(y), t_1(0)) < t_1(z)$. Since $t(z) \le t_1(0)$, then $t_1(x)+t_1(y) < t_1(z) \le t_1(0)$. Thus

$$I^T(T_1(x,y),z) = \frac{t(T_1(x,y))}{t(z)} = \frac{t(t_1^{-1}(t_1(x)+t_1(y)))}{t(z)} = \frac{1}{k}\cdot\frac{(t_1(x)+t_1(y))^\alpha}{t(z)}$$
$$= \frac{1}{k}\cdot\left(\frac{(k\cdot t(x))^{\frac{1}{\alpha}}+(k\cdot t(y))^{\frac{1}{\alpha}}}{t(z)^{\frac{1}{\alpha}}}\right)^\alpha = \left(\frac{t(x)^{\frac{1}{\alpha}}+t(y)^{\frac{1}{\alpha}}}{t(z)^{\frac{1}{\alpha}}}\right)^\alpha = \left(\left(\frac{t(x)}{t(z)}\right)^{\frac{1}{\alpha}}+\left(\frac{t(y)}{t(z)}\right)^{\frac{1}{\alpha}}\right)^\alpha.$$

Thus $I^T(T_1(x,y),z) = S(I^T(x,z), I^T(y,z))$. From the above discussion it is easy to see that the triple (I^T, T_1, S) satisfies (4). □

Similarly, we have the following result for the case that T is a non-trivial ordinal sum of continuous Archimedean t-norms.

Proposition 3.8. *Let A be an index set and $\{(a_i,b_i)\}_{i\in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. Let $T = (< a_i, b_i, T_i >)_{i\in A}$ be a non-trivial ordinal sum of Archimedean t-norms and I^T its power based implication, where T_i is a continuous Archimedean t-norm with additive generator t_i for all $i \in A$. Let $T_1 = (< a_i, b_i, T_{1i} >)_{i\in A}$ be an ordinal sum of Archimedean t-norms, where T_{1i} is a continuous Archimedean t-norm with additive generator t_{1i} defined as*

$$t_{1i}(x) = (k\cdot t_i(x))^{\frac{1}{\alpha}},\ \ x\in[0,1],\ i\in A.$$

Let S be a t-conorm defined as

$$S(x,y) = min\left((x^{\frac{1}{\alpha}}+y^{\frac{1}{\alpha}})^\alpha, 1\right).$$

Then the triple (I^T, T_1, S) satisfies (4), where k, α are constants with $\alpha > 0$, $k > 0$.

Proof. Let $x, y, z \in [0,1]$ with $x > z$, $y > z$. Analogues to the proof in case 2 of Theorem 3.3, if there is not an $i \in A$ such that $x, y, z \in [a_i, b_i]$, then $I^T(T_1(x,y),z) = S(I^T(x,z), I^T(y,z))$ holds for any t-conorm S.

Hence, it suffices to consider $x, y, z \in [a_i, b_i]$ for some $i \in A$. The rest proof is similar to the proof of Proposition 3.7. □

3.3 On the equation $I(x, T_1(y,z)) = T_2(I(x,y), I(x,z))$

Lemma 3.9. *Let a function $I : [0,1]^2 \to [0,1]$ satisfy (OP), and let T_1, T_2 be t-norms. If the triple (I, T_1, T_2) satisfies (5), then $T_1 = T_M$.*

Proof. Assume that the triple (I, T_1, T_2) satisfies (5), i.e.,

$$I(x, T_1(y,z)) = T_2(I(x,y), I(x,z)) \text{ for all } x,y,z\in[0,1].$$

Taking $x = y = z$, then

$$I(x, T_1(x,x)) = T_2(I(x,x), I(x,x)) \text{ for all } x\in[0,1].$$

Since I satisfies (OP), then $I(x, T_1(x,x)) = 1$. Hence $x \le T_1(x,x)$ for all $x \in [0,1]$. As $T_1(x,x) \le x$ for all $x \in [0,1]$, then $T_1(x,x) = x$ for all $x \in [0,1]$. Thus $T = T_M$. □

Theorem 3.10. *Let T be a continuous Archimedean t-norm (a non-trivial ordinal sum of continuous Archimedean t-norms, respectively) and I^T its power based implication, and let T_1, T_2 be t-norms. Then the following statements are equivalent:*

(i) The triple (I^T, T_1, T_2) satisfies (5).

(ii) $T_1 = T_2 = T_M$.

Proof. (i ⇒ ii) Let the triple (I^T, T_1, T_2) satisfy (5). Since I^T satisfies (OP), then $T_1 = T_M$ by Lemma 3.9. Thus, for all $x, y, z \in [0,1]$, we get

$$I^T(x, \min(y,z)) = T_2(I^T(x,y), I^T(x,z)).$$

Taking $y = z$, then

$$I^T(x,y) = T_2(I^T(x,y), I^T(x,y)).$$

Case 1: T is a continuous Archimedean t-norm.

Consider $x > y > 0$. Let t be an additive generator of T, and let $I^T(x,y) = a$, then $a = \frac{t(x)}{t(y)}$. Thus $a \in [0,1)$ by the continuity of T. Therefore,

$$a = T_2(a,a) \text{ for all } a \in [0,1),$$

i.e., $T_2 = T_M$.

Case 2: T is a non-trivial ordinal sum of continuous Archimedean t-norms.

Without loss of generality assume that $T = (< a_i, b_i, T_i >)_{i\in A}$, where A is an index set and T_i is a continuous Archimedean t-norm with additive generator t_i for all $i \in A$, and $\{(a_i,b_i)\}_{i\in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$.

Let $x, y \in [a_i, b_i]$ for some $i \in A$ with $x > y > a_i$. Then

$$\frac{t_i(\frac{x-a_i}{b_i-a_i})}{t_i(\frac{y-a_i}{b_i-a_i})} = T_2\left(\frac{t_i(\frac{x-a_i}{b_i-a_i})}{t_i(\frac{y-a_i}{b_i-a_i})}, \frac{t_i(\frac{x-a_i}{b_i-a_i})}{t_i(\frac{y-a_i}{b_i-a_i})}\right).$$

Let $m = \frac{t_i(\frac{x-a_i}{b_i-a_i})}{t_i(\frac{y-a_i}{b_i-a_i})}$. Then $m \in [0,1)$ and $m = T_2(m,m)$. Hence $T_2 = T_M$.

(ii $\Rightarrow$ i) Obvious. □

3.4 On the equation $I(x, S_1(y,z)) = S_2(I(x,y), I(x,z))$

Lemma 3.11. [4] *For a function $I : [0,1]^2 \to [0,1]$ the following statements are equivalent:*

(i) I is increasing in the second variable, i.e., I satisfies (I2).

(ii) I satisfies $I(x, \max(y,z)) = \max(I(x,y), I(x,z))$ for all $x,y,z \in [0,1]$, i.e., the triple (I, S_M, S_M) satisfies (6).

Remark 3.12. (i) The t-conorm S_2 such that the triple (I, S_M, S_2) satisfies (6) may not be unique. To see this consider the Rescher implication I_{RS}, i.e., I^{T_M}. It is easy to see that the triple (I_{RS}, S_M, S_2) satisfies (6) for any t-conorm S_2 from Table 1.

(ii) It is easy to see that the pair (S_M, S_M) is a solution of equation (6) involving I_T.

Lemma 3.13. *Let $I \in FI$ satisfy one of the following conditions:*

(i) For some x, the function $I_x(y)$ defined by $I_x(y) = I(x,y)$, $y \in [0,1]$ is onto [0,1].

(ii) For some y, the function $I_y(x)$ defined by $I_y(x) = I(x,y)$, $x \in [0,1]$ is onto [0,1].

If the triple (I, S_M, S_2) satisfies (6), then $S_2 = S_M$,

Proof. Assume that the triple (I, S_M, S_2) satisfies (6), i.e.,

$$I(x, \max(y,z)) = S_2(I(x,y), I(x,z)) \text{ for all } x,y,z \in [0,1].$$

Taking $y = z$, then

$$I(x,y) = S_2(I(x,y), I(x,y)) \text{ for all } x,y \in [0,1]. \tag{12}$$

For condition (i): the function $I_x(y)$ defined by $I_x(y) = I(x,y)$, $y \in [0,1]$ is onto [0,1] for some x. Taking $p = I_x(y)$, then $p = S_2(p,p)$ for all $p \in [0,1]$. Therefore, $S_2 = S_M$.

For the condition (ii): for some y, the function $I_y(x)$ defined by $I_y(x) = I(x,y), x \in [0,1]$ is onto [0,1]. Similarly, taking $p = I_y(x)$ in (12), then $p = S_2(p,p)$ for all $p \in [0,1]$, thus $S_2 = S_M$. □

Lemma 3.14. *Let $I \in FI$ satisfy one of the following conditions:*

(i) For some x, the function $I_x(y)$ defined by $I_x(y) = I(x,y)$ is a strictly increasing function.

(ii) I satisfies (OP).

If the triple (I, S_1, S_M) satisfies (6), then $S_1 = S_M$.

Proof. Assume that the triple (I, S_1, S_M) satisfies (6), i.e.,

$$I(x, S_1(y,z)) = \max(I(x,y), I(x,z)) \text{ for all } x,y,z \in [0,1].$$

Taking $y = z$, then $I(x, S_1(y,y)) = I(x,y)$ for all $x,y \in [0,1]$, i.e.,

$$I_x(S_1(y,y)) = I_x(y) \text{ for all } y \in [0,1].$$

For condition (i): for some x, the function $I_x(y)$ is a strictly increasing function. Then $S_1(y,y)) = y$ for all $y \in [0,1]$. Therefore $S_1 = S_M$.

For condition (ii): I satisfies (OP). Suppose that $S_1 \neq S_M$, then there exists a $y \in (0,1)$ such that $S_1(y,y) > y$. Hence, there exists an $x \in (0,1)$ such that $S_1(y,y) > x > y$, then $I(x, S_1(y,y)) = 1 > I(x,y)$ by (OP). A contradiction to $I(x, S_1(y,y)) = I(x,y)$. □

Proposition 3.15. *Let T be a continuous Archimedean t-norm (a non-trivial ordinal sum of continuous Archimedean t-norms, respectively) and I^T its power based implication. If the triple (I^T, S_1, S_2) satisfies (6), then $S_1 = S_M \Leftrightarrow S_2 = S_M$.*

Proof. ($S_1 = S_M \Rightarrow S_2 = S_M$)

Case 1: T is a continuous Archimedean t-norm. Suppose that t is an additive generator of T. Let $x \geq y$, fix $y \in (0,1)$. Since t is a continuous function with $t(1) = 0$, then $I_y(x) = \frac{t(x)}{t(y)}$ is onto $[0,1]$. Hence $S_2 = S_M$ by Lemma 3.13.

Case 2: T is a non-trivial ordinal sum of continuous Archimedean t-norms.

Without loss of generality assume that $T = (< a_i, b_i, T_i >)_{i \in A}$, where A is an index set, T_i is a continuous Archimedean t-norm with additive generator t_i for all $i \in A$, and $\{(a_i, b_i)\}_{i\in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$.

Taking x, $y \in [a_i, b_i]$ with $x \geq y > a_i$. Fix y, then the following function

$$I_y(x) = \frac{t_i(\frac{x-a_i}{b_i-a_i})}{t_i(\frac{y-a_i}{b_i-a_i})},\ x \in [y, b_i],$$

is onto $[0,1]$. Therefore $S_2 = S_M$ by Lemma 3.13.

($S_2 = S_M \Rightarrow S_1 = S_M$) Since I^T satisfies (OP), then $S_2 = S_M \Rightarrow S_1 = S_M$ by Lemma 3.14. □

Theorem 3.16. *Let T be a nilpotent, continuous t-norm and I^T its power based implication, then the triple (I^T, S_1, S_2) satisfies (6) if and only if $S_1 = S_M$, $S_2 = S_M$.*

Proof. (Necessity) Let the triple (I^T, S_1, S_2) satisfy (6), i.e,

$$I^T(x, S_1(y,z)) = S_2(I^T(x,y), I^T(x,z)). \tag{13}$$

for all x, y, $z \in [0,1]$.

Suppose that t is an additive generator of T. Taking $y = 0$, $z = 0$ in (13), then

$$\frac{t(x)}{t(0)} = S_2\left(\frac{t(x)}{t(0)}, \frac{t(x)}{t(0)}\right) \text{ for all } x \in [0,1].$$

Let $p = \frac{t(x)}{t(0)}$, then $p = S_2(p,p)$ for all $p \in [0,1]$. Hence $S_2 = S_M$. Therefore, $S_1 = S_M$ by Lemma 3.14 (ii).

(Sufficiency) Obvious. □

Proposition 3.17. *Let A be an index set and $(T_i)_{i\in A}$ a family of continuous Archimedean t-norms, let $(a_i, b_i)_{i\in A}$ be a family of non-empty, pairwise disjoint open subintervals of [0,1]. Let T be a non-trivial ordinal sum of continuous Archimedean t-norms with the form $(< a_i, b_i, T_i >)_{i\in A}$ and I^T its power based implication, let S_1, S_2 be t-conorms. If there exists an $i \in A$ such that $a_i = 0$ and T_i is a nilpotent t-norm, or a_i is an idempotent point of S_1 and T_i is a nilpotent t-norm, then the following statements are equivalent:*

(i) The triple (I^T, S_1, S_2) satisfies (6).

(ii) $S_1 = S_M$, $S_2 = S_M$.

Proof. Taking $y = z = a_i$, and $x \in [a_i, b_i]$. The rest proof is similar to the proof of Theorem 3.16. □

Problem 3.18. For the power based implication I^T generated from a strict t-norm T, does the fact that the triple (I^T, S_1, S_2) satisfies (6) if and only if $S_1 = S_2 = S_M$ is true ?

Unfortunately, the answer is negative. To see this consider the following example.

Example 3.19. *Let T be a strict t-norm with additive generator $t(x) = \frac{1}{x} - 1$, $x \in [0,1]$ and I^T its power based implication, i.e.,*

$$I^T(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ \frac{y(1-x)}{x(1-y)}, & \text{otherwise} \end{cases}$$

with the understanding $\frac{0}{0} = 1$. Let S_1 be a t-conorm defined as following:

$$S_1(x,y) = \frac{x+y-2xy}{1-xy}, \ \ x,y \in [0,1],$$

with the understanding $\frac{0}{0} = 1$. Let S_2 be the t-conorm S_{LK}, i.e.,

$$S_2(x,y) = min(x+y,1), \ \ x,y \in [0,1].$$

For x, y, $z \in [0,1]$ with $x > y$, $x > z$.
Case 1: $x = 1$. Obviously, $I^T(x, S_1(y,z)) = 0 = S_2(0,0) = S_2(I^T(x,y), I^T(x,z))$.
Case 2: $y = 0$ or $z = 0$. Obviously, $I^T(x, S_1(y,z)) = S_2(I^T(x,y), I^T(x,z))$.
Case 3: x, y, $z \in (0,1)$. If $x > S_1(y,z)$, i.e., $x > \frac{y+z-2yz}{1-yz}$, then

$$\begin{aligned} I^T(x, S_1(y,z)) &= \frac{t(x)}{t(S_1(y,z))} = t(x) \cdot \frac{S_1(y,z)}{1-S_1(y,z)} = t(x) \cdot \frac{y+z-2yz}{1-y-z+yz} \\ &= t(x) \cdot \frac{(y-yz)+(z-yz)}{(1-y)(1-z)} = t(x) \cdot \left(\frac{y}{1-y} + \frac{z}{1-z}\right) = t(x) \cdot \left(\frac{1}{t(y)} + \frac{1}{t(z)}\right). \end{aligned}$$

On the other hand, since

$$\begin{aligned} x > \frac{y+z-2yz}{1-yz} &\Leftrightarrow \frac{1}{x} < \frac{1-yz}{y+z-2yz} \\ &\Leftrightarrow \frac{1}{x} - 1 < \frac{1-y-z+yz}{y+z-2yz} \\ &\Leftrightarrow (\frac{1}{x} - 1)\frac{y+z-2yz}{1-y-z+yz} < 1 \\ &\Leftrightarrow (\frac{1}{x} - 1)\frac{(y-yz)+(z-yz)}{(1-y)(1-z)} < 1 \\ &\Leftrightarrow (\frac{1}{x} - 1)(\frac{y}{1-y} + \frac{z}{1-z}) < 1 \\ &\Leftrightarrow (\frac{1}{x} - 1)\left(\frac{1}{\frac{1}{y}-1} + \frac{1}{\frac{1}{z}-1}\right) < 1 \\ &\Leftrightarrow \frac{\frac{1}{x}-1}{\frac{1}{y}-1} + \frac{\frac{1}{x}-1}{\frac{1}{z}-1} < 1 \\ &\Leftrightarrow \frac{t(x)}{t(y)} + \frac{t(x)}{t(z)} < 1. \end{aligned}$$

Then

$$S_2(I^T(x,y), I^T(x,z)) = min\left(\frac{t(x)}{t(y)} + \frac{t(x)}{t(z)}, 1\right) = \frac{t(x)}{t(y)} + \frac{t(x)}{t(z)}.$$

Hence $I^T(x, S_1(y,z)) = S_2(I^T(x,y), I^T(x,z))$.

If $x \leq S_1(y,z)$, i.e., $x \leq \frac{y+z-2yz}{1-yz}$, then $I^T(x, S_1(y,z)) = 1$. Note that

$$x \leq \frac{y+z-2yz}{1-yz} \Leftrightarrow \frac{t(x)}{t(y)} + \frac{t(x)}{t(z)} \geq 1.$$

Then $S_2(I^T(x,y), I^T(x,z)) = 1$. Thus $I^T(x, S_1(y,z)) = S_2(I^T(x,y), I^T(x,z))$.

From the above discussion, we get that the triple (I^T, S_1, S_2) satisfies (6).

Obviously, the solution (S_1, S_2) of equation (6) involving I^T may not be unique when T is a strict t-norm. Moreover, we can be sure that $S_2 \neq S_D$ (S_{nM}, respectively). See the following remark.

Remark 3.20. *(i) Let T be a continuous Archimedean t-norm. If the triple (I^T, S_1, S_2) satisfies (6), then $S_2 \neq S_D$.*

Actually, suppose that $S_2 = S_D$, then $S_1 \neq S_M$ by Proposition 3.15. Hence there exists a $y_0 \in (0,1)$ such that $1 > S_1(y_0, y_0) > y_0$.

Consider an $x_0 \in [0,1]$ such that $1 > x_0 > S_1(y_0, y_0)$, we get

$$I^T(x_0, S_1(y_0, y_0)) < 1, \ \ I^T(x_0, y_0) \in (0,1).$$

Hence $S_2(I^T(x_0, y_0), I^T(x_0, y_0)) = 1$, a contradiction to

$$I^T(x_0, S_1(y_0, y_0)) = S_2(I^T(x_0, y_0), I^T(x_0, y_0)).$$

(ii) For a power based implication I^T ($T \neq T_M$), if the triple (I^T, S_1, S_2) satisfies (6), then $S_2 \neq S_{nM}$.

Actually, suppose that $S_2 = S_{nM}$, then $S_1 \neq S_M$ by Proposition 3.15. Hence, there exists a $y_0 \in (0,1)$ such that $1 > S_1(y_0, y_0) > y_0$.

Case 1: *T is a continuous Archimedean t-norm.*

Assume that t is an additive generator of T. Consider an $x_0 \in (0,1)$ such that

$$1 > x_0 > max\left(S_1(y_0, y_0), t^{-1}(\frac{1}{2}t(y_0))\right),$$

then $\frac{t(x_0)}{t(y_0)} < \frac{1}{2}$. Thus

$$I^T(x_0, S_1(y_0, y_0)) = \frac{t(x_0)}{t(S_1(y_0, y_0))} > \frac{t(x_0)}{t(y_0)} = S_2(I^T(x_0, y_0), I^T(x_0, y_0)),$$

a contradiction to $I^T(x_0, S_1(y_0, y_0)) = S_2(I^T(x_0, y_0), I^T(x_0, y_0))$.

Case 2: *T is a non-trivial ordinal sum t-norms.*

Without loss of generality assume that $T = (< a_i, b_i, T_i >)_{i \in A}$, where A is an index set, T_i is a continuous Archimedean t-norm with additive generator t_i for all $i \in A$, and $(a_i, b_i)_{i \in A}$ be a family of non-empty, pairwise disjoint open subintervals of [0,1].

Case 2.1: *$y_0 \notin [a_i, b_i]$ for all $i \in A$. Consider an $x_0 \in (y_0, S_1(y_0, y_0))$, then*

$$I^T(x_0, S_1(y_0, y_0)) = 1 > 0 = S_2(I^T(x_0, y_0), I^T(x_0, y_0)),$$

a contradiction to $I^T(x_0, S_1(y_0, y_0)) = S_2(I^T(x_0, y_0), I^T(x_0, y_0))$.

Case 2.2: *$y_0 \in [a_i, b_i]$ for an $i \in A$.*

If $S_1(y_0, y_0) > b_i$, consider an $x_0 \in (b_i, S_1(y_0, y_0))$, then

$$I^T(x_0, S_1(y_0, y_0)) = 1 > 0 = S_2(I^T(x_0, y_0), I^T(x_0, y_0)),$$

a contradiction to $I^T(x_0, S_1(y_0, y_0)) = S_2(I^T(x_0, y_0), I^T(x_0, y_0))$.

If $S_1(y_0, y_0) = b_i$, consider an $x_0 \in [a_i, b_i]$ such that

$$b_i > x_0 > a_i + (b_i - a_i) \cdot t^{-1}\left(\frac{1}{2}t(\frac{y_0 - a_i}{b_i - a_i})\right),$$

then $\frac{t(\frac{x_0 - a_i}{b_i - a_i})}{t(\frac{y_0 - a_i}{b_i - a_i})} < \frac{1}{2}$. Thus

$$I^T(x_0, S_1(y_0, y_0)) = 1 > \frac{t(\frac{x_0 - a_i}{b_i - a_i})}{t(\frac{y_0 - a_i}{b_i - a_i})} = S_2\left(I^T(x_0, y_0), I^T(x_0, y_0)\right),$$

a contradiction to $I^T(x_0, S_1(y_0, y_0)) = S_2(I^T(x_0, y_0), I^T(x_0, y_0))$.

If $S_1(y_0, y_0) < b_i$, consider an $x_0 \in [a_i, b_i]$ such that

$$b_i > x_0 > max\left(S_1(y_0, y_0), a_i + (b_i - a_i) \cdot t^{-1}\left(\frac{1}{2}t(\frac{y_0 - a_i}{b_i - a_i})\right)\right),$$

then $\frac{t(\frac{x_0-a_i}{b_i-a_i})}{t(\frac{y_0-a_i}{b_i-a_i})} < \frac{1}{2}$. *Thus*

$$I^T(x_0, S_1(y_0, y_0)) = \frac{t(\frac{x_0-a_i}{b_i-a_i})}{t(\frac{S_1(y_0,y_0)-a_i}{b_i-a_i})} > \frac{t(\frac{x_0-a_i}{b_i-a_i})}{t(\frac{y_0-a_i}{b_i-a_i})} = S_2\left(I^T(x_0, y_0), I^T(x_0, y_0)\right),$$

a contradiction to $I^T(x_0, S_1(y_0, y_0)) = S_2(I^T(x_0, y_0), I^T(x_0, y_0))$.

Therefore, $S_2 \neq S_{nM}$.

In the following, we give a result on the solution of equation (6) involving I^T when T is a strict t-norm.

Proposition 3.21. *Let* T *be a strict t-norm and* I^T *its power based implication, and let* S_1, S_2 *be t-conorms. If the triple* (I^T, S_1, S_2) *satisfies (6), then* S_1 *is either idempotent or* $S_1(y, y) > y$ *for all* $y \in (0, 1)$.

Proof. Let t be an additive generator of T. If there exists a $y_0 \in (0, 1)$ such that $S_1(y_0, y_0) = y_0$, then from the triple (I^T, S_1, S_2) satisfies (6) we get that for all $x \in [y_0, 1]$,

$$\frac{t(x)}{t(y_0)} = S_2\left(\frac{t(x)}{t(y_0)}, \frac{t(x)}{t(y_0)}\right).$$

Let $p = \frac{t(x)}{t(y_0)}$, $x \in [y_0, 1]$. Then $S_2(p, p) = p$ for all $p \in [0, 1]$. Hence $S_2 = S_M$, thus $S_1 = S_M$ by Proposition 3.15.

If there is not a $y_0 \in (0, 1)$ such that $S_1(y_0, y_0) = y_0$, obviously, $S_1(y, y) > y$ for all $y \in (0, 1)$. □

Proposition 3.22. *Let* T *be a strict t-norm with additive generator* t *and* I^T *its power based implication, let* S_2 *be the following t-conorm:*

$$S_2(x, y) = min\left((x^{\frac{1}{\alpha}} + y^{\frac{1}{\alpha}})^{\alpha}, 1\right), \ \ x, y \in [0, 1], \ \ \alpha > 0.$$

Then there exists a t-conorm S_1 *with the following additive generator*

$$s_1(x) = t(x)^{-\frac{1}{\alpha}}, \ \ x \in [0, 1], \ \ \alpha > 0,$$

such that the triple (I^T, S_1, S_2) *satisfies (6).*

Proof. Since T is strict, then t is continuous, strictly decreasing, with $t(0) = \infty$ and $t(1) = 0$. Thus the function $s_1 : [0, 1] \to [0, \infty]$ defined by

$$s_1(x) = t(x)^{-\frac{1}{\alpha}}, \ \ x \in [0, 1], \ \ \alpha > 0,$$

is continuous, strictly increasing, with $s_1(0) = 0$ and $s_1(1) = \infty$. Therefore,

$$S_1(x, y) = s_1^{-1}(s_1(x) + s_1(y)) = t^{-1}\left(\left(t(x)^{-\frac{1}{\alpha}} + t(y)^{-\frac{1}{\alpha}}\right)^{-\alpha}\right),$$

is a strict t-conorm by Theorem 2.2.6 in [4].

Let x, y, $z \in [0, 1]$ with $x > y$ and $x > z$.

Case 1: $x = 1$, or $y = 0$, or $z = 0$. Obviously, $I^T(x, S_1(y, z)) = S_2(I^T(x, y), I^T(x, z))$.

Case 2: x, y, $z \in (0, 1)$. If $x > S_1(y, z)$, then

$$I^T(x, S_1(y, z)) = \frac{t(x)}{t(S_1(y, z))} = t(x) \cdot \left(t(y)^{-\frac{1}{\alpha}} + t(z)^{-\frac{1}{\alpha}}\right)^{\alpha} = \left(\left(\frac{t(x)}{t(y)}\right)^{\frac{1}{\alpha}} + \left(\frac{t(x)}{t(z)}\right)^{\frac{1}{\alpha}}\right)^{\alpha}.$$

On the other hand, note that $x > S_1(y, z) \Leftrightarrow \left(\left(\frac{t(x)}{t(y)}\right)^{\frac{1}{\alpha}} + \left(\frac{t(x)}{t(z)}\right)^{\frac{1}{\alpha}}\right)^{\alpha} < 1$. Then

$$S_2\left(I^T(x, y), I^T(x, z)\right) = \min\left(\left(\left(\frac{t(x)}{t(y)}\right)^{\frac{1}{\alpha}} + \left(\frac{t(x)}{t(z)}\right)^{\frac{1}{\alpha}}\right)^{\alpha}, 1\right) = \left(\left(\frac{t(x)}{t(y)}\right)^{\frac{1}{\alpha}} + \left(\frac{t(x)}{t(z)}\right)^{\frac{1}{\alpha}}\right)^{\alpha},$$

thus, we get

$$I^T(x, S_1(y, z)) = S_2(I^T(x, y), I^T(x, z)).$$

If $x \leq S_1(y, z)$, note that $x \leq S_1(y, z) \Leftrightarrow \left(\left(\frac{t(x)}{t(y)}\right)^{\frac{1}{\alpha}} + \left(\frac{t(x)}{t(z)}\right)^{\frac{1}{\alpha}}\right)^{\alpha} \geq 1$, then

$$I^T(x, S_1(y, z)) = 1 = S_2(I^T(x, y), I^T(x, z)).$$

From the above discussion, the triple (I^T, S_1, S_2) satisfies (6). □

Next, we give a result on the solution of equation (6) involving I^T when $T = (< a_i, b_i, T_i >)_{i\in A}$, where T_i is a strict t-norm for all $i \in A$.

Proposition 3.23. *Let T be a non-trivial ordinal sum of t-norms with the form $(< a_i, b_i, T_i >)_{i\in A}$ and I^T its power based implication, where A is an index set, $(T_i)_{i\in A}$ is a family of strict t-norms, and $(a_i, b_i)_{i\in A}$ be a family of non-empty, pairwise disjoint open subintervals of [0,1]. Let S_2 be the following t-conorm:*

$$S_2(x,y) = min\left((x^{\frac{1}{\alpha}} + y^{\frac{1}{\alpha}})^{\alpha}, 1\right), \ \ x, y \in [0,1], \ \ \alpha > 0.$$

Then there exists a t-conorm S_1 with the following form:

$$S_1 = (< a_i, b_i, S_{1i} >)_{i\in A},$$

such that the triple (I^T, S_1, S_2) satisfies (6), where S_{1i} is a t-conorm with additive generator $s_{1i}(x) = t_i(x)^{-\frac{1}{\alpha}}$, $x \in [0,1]$, and t_i is an additive generator of T_i for all $i \in A$.

Proof. It is easy to see that, for every $i \in A$, the following function

$$s_{1i}(x) = t_i(x)^{-\frac{1}{\alpha}}, x \in [0,1]$$

is strictly increasing, continuous, with $s_{1i}(0) = 0$ and $s_{1i}(1) = \infty$. Therefore,

$$S_{1i}(x,y) = s_{1i}^{-1}(s_{1i}(x) + s_{1i}(y)) = t_i^{-1}\left(\left(t_i(x)^{-\frac{1}{\alpha}} + t_i(y)^{-\frac{1}{\alpha}}\right)^{-\alpha}\right), \ \ x, y \in [0,1]$$

is a t-conorm by Theorem 2.2.6 in [4]. Obviously, for $x < 1$ and $y < 1$, we have

$$S_{1i}(x,y) < 1. \tag{14}$$

In fact, suppose that $x < 1$ and $y < 1$, then $s_{1i}(x) < \infty$, $s_{1i}(y) < \infty$. Thus $s_{1i}(x) + s_{1i}(y) < \infty$. Therefore, $s_{1i}^{-1}(s_{1i}(x) + s_{1i}(y)) < 1$, i.e., $S_{1i}(x,y) < 1$.

Let S_1 be a function defined by

$$S_1(x,y) = \begin{cases} a_i + (b_i - a_i) \cdot S_{1i}(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}), & \text{if } x, y \in [a_i, b_i], \\ \max(x,y), & \text{otherwise.} \end{cases} \tag{15}$$

Then S_1 is a non-trivial ordinal sum of t-conorms by Corollary 3.58 in [11], i.e., $S_1 = (< a_i, b_i, S_{1i} >)_{i\in A}$. Obviously, if $x < a_i$ and $y < a_i$ for some $i \in A$, then we have

$$S_1(x,y) < a_i. \tag{16}$$

In fact, let $x < a_i$, $y < a_i$ for some $i \in A$. If there exists a $k \in A$ such that x, $y \in [a_k, b_k]$ $(k \neq i)$, then $b_k \leq a_i$. For $b_k < a_i$, we get

$$S_1(x,y) = a_k + (b_k - a_k) \cdot S_{1k}(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}) \leq a_k + (b_k - a_k) = b_k < a_i.$$

For $b_k = a_i$, since $x < a_i$ and $y < a_i$, i.e., $x < b_k$ and $y < b_k$, then

$$\frac{x-a_k}{b_k-a_k} < 1, \ \ \frac{y-a_k}{b_k-a_k} < 1.$$

Hence, by (14) we get

$$S_{1k}(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}) < 1.$$

Thus,

$$S_1(x,y) = a_k + (b_k - a_k) \cdot S_{1k}(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}) < a_k + (b_k - a_k) = b_k = a_i.$$

If there is not a $k \in A$ such that x, $y \in [a_k, b_k]$, then $S_1(x,y) = \max(x,y) < a_i$.

In the following, we prove that the triple (I^T, S_1, S_2) satisfies (6).

Let x, y, $z \in [0,1]$ with $x > y$ and $x > z$.

Case 1: for every $i \in A$, $x \notin [a_i, b_i]$, $y \notin [a_i, b_i]$ and $z \notin [a_i, b_i]$. Then

$$I^T(x, S_1(y, z)) = I^T(x, \max(y, z)) = 0 = S_2(I^T(x, y), I^T(x, z)).$$

Case 2: there exists an $i \in A$, such that $x \notin [a_i, b_i]$, $y \notin [a_i, b_i]$ and $z \in [a_i, b_i]$. If $y \geq z$, then

$$I^T(x, S_1(y, z)) = I^T(x, \max(y, z)) = I^T(x, y) = S_2(I^T(x, y), 0) = S_2(I^T(x, y), I^T(x, z)).$$

If $y < z$, then $I^T(x, y) = 0$ by $I^T(x, y) \leq I^T(x, z) = 0$. Thus

$$\begin{aligned} I^T(x, S_1(y, z)) &= I^T(x, \max(y, z)) \\ &= I^T(x, z) \\ &= 0 \\ &= S_2(0, 0) \\ &= S_2(I^T(x, y), I^T(x, z)). \end{aligned}$$

Case 3: there exists an $i \in A$, such that $x \notin [a_i, b_i]$, $y \in [a_i, b_i]$ and $z \notin [a_i, b_i]$. The rest of the proof is similarly to Case 2.

Case 4: there exists an $i \in A$, such that $x \in [a_i, b_i]$, $y \notin [a_i, b_i]$ and $z \notin [a_i, b_i]$. Since $x > y$ and $x > z$, then $y < a_i$ and $z < a_i$. Thus $S_1(y, z) < a_i$ by (16). Therefore,

$$I^T(x, S_1(y, z)) = 0 = S_2(I^T(x, y), I^T(x, z)).$$

Case 5: there exists an $i \in A$, such that $x \notin [a_i, b_i]$, $y \in [a_i, b_i]$ and $z \in [a_i, b_i]$. It is easy to see that

$$I^T(x, S_1(y, z)) = 0 = S_2(I^T(x, y), I^T(x, z)).$$

Case 6: there exists an $i \in A$, such that $x \in [a_i, b_i]$, $y \in [a_i, b_i]$ and $z \notin [a_i, b_i]$. Since, $x > z$, then $z < a_i$. Thus

$$I^T(x, S_1(y, z)) = I^T(x, y) = S_2(I^T(x, y), 0) = S_2(I^T(x, y), I^T(x, z)).$$

Case 7: there exists an $i \in A$, such that $x \in [a_i, b_i]$, $y \notin [a_i, b_i]$ and $z \in [a_i, b_i]$. Similar to Case 6.

Case 8: there exists an $i \in A$, such that x, y, $z \in [a_i, b_i]$. The rest of the proof is analogue to the proof of Proposition 3.22. □

Table 2 summarizes the distributivity solutions of the power based implication I^T. Here, T is a continuous Archimedean t-norm, or a non-trivial ordinal sum of continuous Archimedean t-norms.

Table 2: Distributivity solutions of the power based implication I^T ($T \neq T_M$)

Equation	Universal solution	Other solution
$I^T(S(x, y), z) = T_1(I^T(x, z), I^T(y, z))$	$S = S_M, T_1 = T_M$	None
$I^T(T_1(x, y), z) = S(I^T(x, z), I^T(y, z))$	$T_1 = T_M$, $S = S_M$	$T_1 = T$, $S = S_{LK}$ and $T_1 = T_1^*$, $S = S^*$, etc.
$I^T(x, T_1(y, z)) = T_2(I^T(x, y), I^T(x, z))$	$T_1 = T_M, T_2 = T_M$	None
$I^T(x, S_1(y, z)) = S_2(I^T(x, y), I^T(x, z))$	$S_1 = S_M, S_2 = S_M$	T is nilpotent: None T is $T^\star$: None T is strict: $S_1 = S_1^\star$, $S_2 = S^*$, etc. T is $T^{\star\star}$: $S_1 = S_1^{\star\star}$, $S_2 = S^*$, etc.

Note (i) T_1^* has an additive generator $t_1(x) = (k \cdot t(x))^{\frac{1}{\alpha}}$, $x \in [0, 1]$ when T has a continuous additive generator t, or $T_1^* = (< a_i, b_i, T_{1i} >)_{i \in A}$ when $T = (< a_i, b_i, T_i >)_{i \in A}$, where T_{1i} has an additive generator $t_{1i}(x) = (k \cdot t_i(x))^{\frac{1}{\alpha}}$, $x \in [0, 1]$, t_i is a continuous additive generator of T_i, $i \in A$, $k > 0$, $\alpha > 0$.
(ii) $S^*(x, y) = \min\left((x^{\frac{1}{\alpha}} + y^{\frac{1}{\alpha}})^{\alpha}, 1\right)$, x, $y \in [0, 1]$, where $\alpha > 0$.

(iii) $T^\star = (< a_i, b_i, T_i >)_{i\in A}$. There exists an $i \in A$ such that $a_i = 0$, and T_i is nilpotent, or a_i is an idempotent point of S_1 and T_i is nilpotent.
(iv) $S_1^\star(x,y) = t^{-1}\left(\left(t(x)^{-\frac{1}{\alpha}} + t(y)^{-\frac{1}{\alpha}}\right)^{-\alpha}\right)$, $x,\, y \in [0,1]$, $\alpha > 0$, where t is an additive generator of T.
(v) $T^{\star\star} = (< a_i, b_i, T_i >)_{i\in A}$, where $(T_i)_{i\in A}$ is a family of strict t-norms.
(vi) $S^{\star\star} = (< a_i, b_i, S_{1i} >)_{i\in A}$, where $S_{1i}(x,y) = t_i^{-1}\left(\left(t_i(x)^{-\frac{1}{\alpha}} + t_i(y)^{-\frac{1}{\alpha}}\right)^{-\alpha}\right)$, $x, y \in [0,1]$, t_i is an additive generator of T_i in $T^{\star\star}$, $i \in A$.

4 Conclusions

In this paper, four distributivity equations of T-power based implications are deeply studied respectively. This study shows that the equations (3) and (5) have a unique solution, while the the equations (4) and (6) have multiple solutions. This study has a certain significance for the application of T-power based implication in rule reduction. However, it is difficult to find all solutions for equations (4) and (6), this is a problem to be solved in the future.

Acknowledgement

The authors express their sincere thanks to the editors and reviewers for their most valuable comments and suggestions in improving this paper greatly.

Appendix A: The distributivity laws of implication I^{T_M}.

(1) Let T be a t-norm, and S a t-conorm. Then the triple $(I^{T_M}, S,\, T)$ satisfies (3) if and only if $S = S_M$.

Proof.(Necessity) Let the triple $(I^{T_M}, S,\, T)$ satisfy (3). Then, for all $x,\, y,\, z \in [0,1]$, we get

$$I^{T_M}(S(x,y),z) = T(I^{T_M}(x,z), I^{T_M}(y,z)).$$

Putting $x = y = z$, then $I^{T_M}(S(x,x),x) = T(I^{T_M}(x,x), I^{T_M}(x,x)) = 1$. Since I^{T_M} satisfies (OP), then $S(x,x) \le x$. Since $S(x,x) \ge x$, thus $S(x,x) = x$ for all $x \in [0,1]$. Hence $S = S_M$.

(Sufficiency) Let $S = S_M$. It suffice to prove that

$$I^{T_M}(S(x,y),z) = T(I^{T_M}(x,z), I^{T_M}(y,z)) \tag{17}$$

for all $x,\, y,\, z \in [0,1]$.

If $x \le y \le z$, then $I^{T_M}(S(x,y),z) = I^{T_M}(y,z) = 1$, $T(I^{T_M}(x,z), I^{T_M}(y,z)) = T(1,1) = 1$. Thus equation (17) holds.

If $x \le z < y$, then $I^{T_M}(S(x,y),z) = I^{T_M}(y,z) = 0$, $T(I^{T_M}(x,z), I^{T_M}(y,z)) = T(1,0) = 0$. Thus equation (17) holds.

If $z < x \le y$, then $I^{T_M}(S(x,y),z) = I^{T_M}(y,z) = 0$, $T(I^{T_M}(x,z), I^{T_M}(y,z)) = T(0,0) = 0$. Thus equation (17) holds.

If $x > y \ge z$, then $I^{T_M}(S(x,y),z) = I^{T_M}(x,z) = 0$, $T(I^{T_M}(x,z), I^{T_M}(y,z)) = T(0, I^{T_M}(y,z)) = 0$. Thus equation (17) holds.

If $x > z > y$, then $I^{T_M}(S(x,y),z) = I^{T_M}(x,z) = 0$, $T(I^{T_M}(x,z), I^{T_M}(y,z)) = T(0, I^{T_M}(y,z)) = 0$. Thus equation (17) holds.

If $z \ge x > y$, then $I^{T_M}(S(x,y),z) = I^{T_M}(x,z) = 1$, $T(I^{T_M}(x,z), I^{T_M}(y,z)) = T(1,1) = 1$. Thus the equation (17) holds.

From the above discussion, equation (17) holds for all $x,\, y,\, z \in [0,1]$.

(2) Let T be a t-norm, and S a t-conorm. Then the triple $(I^{T_M}, T,\, S)$ satisfies (4) if and only if $T = T_M$.

Proof. (Necessity) Let the triple $(I^{T_M}, T,\, S)$ satisfy (4), i.e.,

$$I^{T_M}(T(x,y),z) = S(I^{T_M}(x,z), I^{T_M}(y,z)), \text{ for all } x,\ y,\ z \in [0,1].$$

Assume that $T \ne T_M$, then there exists an $x_0 \in (0,1)$ such that $T(x_0,x_0) < x_0$. Taking $z_0 \in (0,1)$ such that $T(x_0,x_0) < z_0 < x_0$. Thus

$$I^{T_M}(T(x_0,x_0),z_0) = 1 > 0 = S(I^{T_M}(x_0,z_0), I^{T_M}(x_0,z_0)).$$

A contradiction to the triple $(I^{T_M}, T,\, S)$ satisfies (4).

(Sufficiency) Let $T = T_M$, and $x,\ y,\ z \in [0,1]$. If $x \le z$ or $y \le z$, then $T(x,y) = T_M(x,y) \le z$. Thus

$$I^{T_M}(T(x,y),z) = 1 = S(I^{T_M}(x,z), I^{T_M}(y,z)).$$

If $x > z$ and $y > z$, then $T(x,y) = T_M(x,y) > z$. Thus

$$I^{T_M}(T(x,y),z) = 0 = S(0,0) = S(I^{T_M}(x,z), I^{T_M}(y,z)).$$

From the above discussion, we get that the triple $(I^{T_M}, T,\ S)$ satisfies (4).

(3) Let $T_1,\ T_2$ be t-norms. Then the triple $(I^{T_M}, T_1,\ T_2)$ satisfies (5) if and only if $T_1 = T_M$.

Proof. (Necessity) Let the triple $(I^{T_M}, T_1,\ T_2)$ satisfy (5). Then

$$I^{T_M}(x, T_1(y,z)) = T_2(I^{T_M}(x,y), I^{T_M}(x,z)) \text{ for all } x,\ y,\ z \in [0,1].$$

Taking $x = y = z$. Then $I^{T_M}(x, T_1(x,x)) = T_2(I^{T_M}(x,x), I^{T_M}(x,x)) = 1$. Since I^{T_M} satisfies (OP), then $x \le T_1(x,x)$ for all $x \in [0,1]$. Thus $T_1(x,x) = x$, i.e., $T_1 = T_M$.

(Sufficiency) Let $T_1 = T_M$. If $x > y$ or $x > z$, then $x > T_1(y,z)$. Thus

$$I^{T_M}(x, T_1(y,z)) = 0 = T_2(I^{T_M}(x,y), I^{T_M}(x,z)).$$

If $x \le y$ and $x \le z$, then $x \le T_M(y,z) = T_1(y,z)$. Thus

$$I^{T_M}(x, T_1(y,z)) = 1 = T_2(I^{T_M}(x,y), I^{T_M}(x,z)).$$

From the above discussion, we get $I^{T_M}(x, T_1(y,z)) = T_2(I^{T_M}(x,y), I^{T_M}(x,z))$ for all $x,\ y,\ z \in [0,1]$, i.e., the triple $(I^{T_M}, T_1,\ T_2)$ satisfies (5).

(4) Let $S_1,\ S_2$ be t-conorms. Then the triple $(I^{T_M}, S_1,\ S_2)$ satisfies (6) if and only if $S_1 = S_M$.

Proof. (Necessity) Let the triple $(I^{T_M}, S_1,\ S_2)$ satisfy (6), then

$$I^{T_M}(x, S_1(y,z)) = S_2(I^{T_M}(x,y), I^{T_M}(x,z)) \text{ for all } x,\ y,\ z \in [0,1].$$

Assume that $S_1 \ne S_M$, then there exists a $y_0 \in (0,1)$ such that $y_0 < S_1(y_0,y_0)$. Taking $x_0 \in (0,1)$ such that $y_0 < x_0 < S_1(y_0,y_0)$. Thus

$$I^{T_M}(x_0, S_1(y_0,y_0)) = 1 > 0 = S_2(0,0) = S_2(I^{T_M}(x_0,y_0), I^{T_M}(x_0,y_0)).$$

A contradiction to the triple $(I^{T_M}, S_1,\ S_2)$ satisfies (6).

(Sufficiency) Let $S_1 = S_M$, and $x,\ y,\ z \in [0,1]$. If $x \le y$ or $x \le z$, then $x \le S_1(y,z)$. Thus

$$I^{T_M}(x, S_1(y,z)) = 1 = S_2(I^{T_M}(x,y), I^{T_M}(x,z)).$$

If $x > y$ and $x > z$, then $x > S_M(y,z) = S_1(y,z)$. Thus

$$I^{T_M}(x, S_1(y,z)) = 0 = S_2(0,0) = S_2(I^{T_M}(x,y), I^{T_M}(x,z)).$$

From the above discussion, it is easy to see that the triple $(I^{T_M}, S_1,\ S_2)$ satisfies (6)

References

[1] M. Baczyński, *On the distributivity of fuzzy implications over continuous and Archimedean triangular conorms*, Fuzzy Sets and Systems, **161** (2010), 1406-1419.

[2] M. Baczyński, P. Grzegorzewski, R. Mesiar, P. Helbin, W. Niemyska, *Fuzzy implications based on semicopulas*, Fuzzy Sets and Systems, **323** (2017), 138-151.

[3] M. Baczyński, B. Jayaram, *On the characterization of (S, N)-implications*, Fuzzy Sets and Systems, **158**(15) (2007), 1713-1727.

[4] M. Baczyński, B. Jayaram, *Fuzzy implications*, Studies in Fuzziness and Soft Computing, **231**, Springer, Berlin, Heidelberg, 2008.

[5] M. Baczyński, B. Jayaram, *(S, N)- and R-implications: A state-of-the-art survey*, Fuzzy Sets and Systems, **159**(14) (2008), 1836-1859.

[6] M. Baczyński, B. Jayaram, *On the distributivity of fuzzy implications over nilpotent or strict triangular conorms*, IEEE Transactions Fuzzy Systems, **17** (2009), 590-603.

[7] J. Balasubramaniam, *Rule reduction for efficient inferencing in similarity based reasoning*, International Journal of Approximate Reasoning, **48** (2008), 156-173

[8] J. Balasubramaniam, C. J. M. Rao, *On the distributivity of implication operators over T and S norms*, IEEE Transactions Fuzzy Systems, **12** (2004), 194-198.

[9] W. E. Combs, J. E. Andrews, *Combinatorial rule explosion eliminated by a fuzzy rule, configuration*, IEEE Transactions on Fuzzy Systems, **6** (1998), 1-11.

[10] P. Grzegorzewski, *Probabilistic implications*, Fuzzy Sets and Systems, **226** (2013), 53-66.

[11] E. P. Klement, R. Mesiar, E. Pap, *Triangular norms*, Kluwer, Dordrecht, 2000.

[12] J. Lu, B. Zhao, *Distributivity of a class of ordinal sum implications over t-norms and t-conorms*, Fuzzy Sets and Systems, **378** (2020), 103-124.

[13] S. Massanet, J. Recasens, J. Torrens, *Fuzzy implication functions based on powers of continuous t-norms*, International Journal of Approximate Reasoning, **83** (2017), 265-279.

[14] S. Massanet, J. Recasens, J. Torrens, *Corrigendum to "Fuzzy implication functions based on powers of continuous t-norms"*, [Int. J. Approx. Reason. 83 (2017) 265-279], International Journal of Approximate Reasoning, **104** (2019), 144-147.

[15] Z. Peng, *A new family of (A, N)-implications: Construction and properties*, Iranian Journal of Fuzzy Systems, **17**(2) (2020), 129-145.

[16] Z. Peng, *The study on semicopula based implications*, Kybernetika, **56**(4) (2020), 662-694.

[17] Y. Su, W. W. Zong, H. W. Liu, *On distributivity equations for uninorms over semi-t-operators*, Fuzzy Sets and Systems, **299** (2016), 41-65.

[18] Y. Su, W. W. Zong, H. W. Liu, *Distributivity of the ordinal sum implications over t-norms and t-conorms*, IEEE Transactions Fuzzy Systems, **24** (2016), 827-840.

[19] E. Trillas, C. Alsina, *On the law $[(p \wedge q) \to r] \equiv [(p \to r) \vee (q \to r)]$ in fuzzy logic*, IEEE Transactions Fuzzy Systems, **10**(1) (2002), 84-88.

[20] A. Xie, H. Liu, F. Zhang, C. Li, *On the distributivity of fuzzy implications over continuous Archimedean t-conorms and continuous t-conorms given as ordinal sums*, Fuzzy Sets and Systems, **205** (2012), 76-100.

[21] R. R. Yager, *On some new classes of implication operators and their role in approximate reasoning*, Information Sciences, **167** (2004), 193-216.