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Characterizations of L-order L-convex spaces

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Abstract

In this paper, the concepts of L-enclosed L-order space, L-order L-concave space, L-internal L-order space and L-order L-convex filter are introduced. The main results are: (1) the categories of L-order L-convex spaces, L-enclosed L-order spaces, L-order L-concave spaces and L-internal L-order spaces are isomorphic; (2) the category of L-order convergence spaces based on L-order L-convex filters is topological; (3) there is a Galois correspondence between the category of L-order convergence spaces and that of L-order L-convex spaces.

Keywords: L-order L-convexity, L-enclosed L-order, L-order L-convex filter, L-order L-concavity, L-order convergence structure.

1 Introduction

With the development of fuzzy mathematics, the order structure, as the basic mathematical structure, has been endowed with fuzzy sets. This kind of order structures is usually called L-orders (or fuzzy orders). Now, L-orders have been widely used in various branches of mathematics and computer sciences. For a nonempty set X, the L-powerset L^X endows with an intrinsic L-order called an inclusion L-order, denoted by sub_X. Making use of the inclusion L-order, Bělohlávek [1] defined and studied an L-closure system on a set X as a subset $\mathcal{T} \subseteq L^X$ satisfying $\bigwedge_{B\in\mathcal{T}} sub_X(A,B) \to B \in \mathcal{T}$, $\forall A \in L^X$. where L is a complete residuated lattice (or a commutative unital quantale). From the view of L-order, $(\mathcal{T}, sub_X)$ is an L-ordered set and $\mathcal{T}$ is an L-closure system if and only if for any $\mathcal{A} \in L^{\mathcal{T}}$, $\bigcap \mathcal{A} = \bigwedge_{A\in\mathcal{T}} \mathcal{A}(A) \to A \in \mathcal{T}$. This implies that $(\mathcal{T}, sub_X)$ is a complete L-lattice and so for any $\mathcal{A} \in L^{\mathcal{T}}$, $\sqcup\mathcal{A}$ exists. Unfortunately, it doesn't have to be equal to $\bigcup \mathcal{A} = \bigvee_{A\in\mathcal{T}} \mathcal{A}(A) * A$ and thus it is not the generalization of set union in crisp case. In [28], Su et al. proposed the notion of algebraic L-closure system, which partially makes up for the above regret. Moreover, they proved that categories of algebraic L-closure system spaces and algebraic L-lattices are dual.

As a special kind of L-closure system, L-convex structure (or L-convexity) proposed by Rosa [23] and Maruyama [14] is an important fuzzy structure. Based on L-convex structures, Jin et al. [8] and Pang et al. [19] introduced several subcategories of L-convex spaces and discussed their categorical relationships with classical convex structures. Afterwards, Pang et al. [15, 21] provided several characterizations of L-convex spaces and investigated their categorical relations. More research on the theory of L-convex structure can be found in [12, 20, 22, 33, 34, 40]. In 2014, Shi and Xiu [25] proposed a M-fuzzifying convex structure different from an L-convex structure, and then extended it to an (L, M)-fuzzy convex structure [26]. To date, the theory of fuzzy convex structures has captured more and more attention and has been extensively studied by many authors with fruitful results (see [16, 17, 18, 31, 32]).

L-powersets are the main research object of fuzzy convex structures and endow with the inclusion L-order. A natural problem is whether to construct a fuzzy convex structure by using the inclusion L-order instead of classical pointwise order. Wang and Shi answered the question in [30]. They used the inclusion L-order to construct a kind of fuzzy convex structure called strong L-convexity and discussed some basic properties of it. Different from the above fuzzy convex structures focusing on the closeness of crisp subsets of the L-powerset, the strong L-convexity focuses on the closeness

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of L-subsets of the L-powerset on the basis of the inclusion L-order. Therefore, it can be regarded as a new fuzzification form of convex structure. In order to highlight the characteristic of L-order, a strongly L-convexity will be called an L-order L-convexity in this paper. Following this viewpoint, we focus on the following three aspects to further discuss the L-order L-convexity:

(1) At present, the theoretical research of fuzzy convex structure is very rich, but its application examples are rarely involved. Therefore, we try to present some application examples on fuzzy convex structures;

(2) How to characterize the L-order L-convexity as in the situation of L-topology?

(3) As we all know, convergence theory is one of the core contents of classical convex structure, many concepts and properties can be described by it. In 2020, Pang [17] and Xiu et al. [32] initiated convergence research on M-fuzzifying convex space and L-convex space, respectively. In particular, they established the relationship between fuzzy convex spaces and their corresponding convergent spaces from the perspective of categories. At the same time, fuzzy convergent structures were established on the characterization space of these fuzzy convex structures [32, 38, 39]. These research results of Pang et al. [17, 32, 33, 38, 39] not only enrich and improve fuzzy convergence structures, but also have distinct theoretical significance for fuzzy convex space theory. Inspired by these works, we try to construct L-order L-convex filters by using directed L-subsets of L-powersets and thus establish the L-order convergence structures of L-order L-convex spaces.

This paper is organized as follows: In Section 2, some basic notions and conclusions be used throughout this paper are listed. Moreover, some examples of L-order L-convexity are presented. In Section 3, we propose the notion of L-enclosed L-order and present some elementary properties. In Section 4, we introduce a dual concept of L-order L-convex space, which is called an L-order L-concave space. Then we consider the corresponding L-interior L-order. In Section 5, we introduce the notion of L-order L-convex filter and investigate categorical relationships among L-order convergence spaces, L-order L-concave spaces and L-order L-convex spaces. Finally, some conclusions are reached in Section 6.

2 Preliminaries

For convenience of the reader, in this section, some basic concepts are reviewed. In this paper, L always denotes a complete residuated lattice. A complete residuated lattice [3] is an algebraic structure $(L, \wedge, \vee, *, \to, 0, 1)$ such that (1) $(L, *, 1)$ is a commutative monoid; (2) $(L, \wedge, \vee, 0, 1)$ is a complete lattice with the least element 0 and the greatest element 1, i.e., $*$ is commutative, associative, and $a * 1 = a$ holds for all $a \in L$; (3) $*$ and $\to$ form an adjoint pair, i.e., $a * b \leq c \Leftrightarrow a \leq b \to c$ for all $a, b, c \in L$. Such an algebraic structure is significant in fuzzy logic (see [2, 3]).

L^X denotes the set of all L-subsets of X. An L-subset A of X is called nonempty if $\bigvee_{x\in X} A(x) = 1$. For $a \in L$ and $A, B \in L^X$, we write L-subsets a_X, $a*A$, $a \to A$ and $A \subseteq B$ by $a_X(x) = a$, $(a*A)(x) = a*A(x)$, $(a \to A)(x) = a \to A(x)$ and $A(x) \leq B(x)$ for all $x \in X$.

2.1 L-ordered sets

Definition 2.1. [4, 5] *Let X be a set and $e_X : X \times X \longrightarrow L$ a mapping. Consider the following conditions:*
(1) $e_X(x, x) = 1$ *for all* $x \in X$;
(2) $e_X(x, y) * e_X(y, z) \leq e_X(x, z)$ *for all* $x, y, z \in X$;
(3) $e_X(x, y) = e_X(y, x) = 1$ *implies* $x = y$ *for all* $x, y \in X$.
e_X is called an L-preorder if it satisfies (1) and (2), and the pair (X, e_X) is called an L-preordered set. e_X is called an L-order if it satisfies (1), (2) and (3), and the pair (X, e_X) is called an L-ordered set.

Example 2.2. *A labelled transition systems [24] $(S; \to_S; \Lambda)$ is made up of a set S of states, a transition relation $\to_S \subseteq S \times \Lambda \times S$, and a set Λ of labels. As always, $s \xrightarrow{l}_S s'$ is used to denote $(s, l, s') \in \to_S$. A relation $R \subseteq S \times S$ is called a simulation relation if it satisfies*

$$(s, t) \in R,\ l \in \Lambda,\ s' \in S, s \xrightarrow{l}_S s' \Rightarrow \exists t' \in S,\ t \xrightarrow{l}_S t',\ (s', t') \in R.$$

Define the following sequence of binary relations on S: $\leq_0 = S \times S$,

$$\leq_{n+1} = \{(s, t) \in S \times S :\ \forall l \in \Lambda,\ \forall s' \in S, s \xrightarrow{l}_S s',\ \exists t' \in S,\ t \xrightarrow{l}_S t',\ (s', t') \in \leq_n\}.$$

$(s, t) \in \leq_n$ can be understood as the first n successive migration steps from state s can always be simulated by the migration steps from state t. From this, we define an L-preorder $e_S : S \times S \longrightarrow \mathbb{N} \bigcup \{\infty\}$ as $e_S(s, t) = \bigvee\{n : (s, t) \in \leq_n\}$

for all $s, t \in S$, where e_S describes the degree to which the migration step starting from state s is simulated by the migration step starting from state t. Obviously, the larger the value of $e_S(s,t)$, the higher the degree to which t simulates s. By Theorem 1.4.4 in [36], *for a labelled transition systems with finite images and $s, t \in S$, $e_S(s,t) = \infty$ iff there is a simulation relation R such that $(s,t) \in R$.*

Definition 2.3. [11] *In an L-ordered set (X, e_X), an L-subset D of X is called a directed (resp., down-directed) L-subset of X if $\bigvee_{x\in X} D(x) = 1$ and for all $x_1, x_2 \in X$, $D(x_1) * D(x_2) \leq \bigvee_{x\in X} D(x) * e_X(x_1, x) * e_X(x_2, x)$ (resp., $D(x_1) * D(x_2) \leq \bigvee_{x\in X} D(x) * e_X(x, x_1) * e_X(x, x_2)$). The set of all directed (resp., down-directed) L-subsets of X is denoted by $\mathrm{D}_L(X)$ (resp., $\mathrm{CD}_L(X)$).*

Definition 2.4. [35, 37] *Let (X, e_X) be an L-ordered set. An element $x_0 \in X$ is called a join (resp., meet) of A, in symbols $x_0 = \sqcup A$ (resp., $x_0 = \sqcap A$), iff for any $y \in X$,*

$$\bigwedge_{x\in X} A(x) \to e_X(x,y) = e_X(x_0, y) \ (\textit{resp.}, \bigwedge_{x\in X} A(x) \to e_X(y,x) = e_X(y, x_0)).$$

For a nonempty set X and $A, B \in L^X$, the subsethood degree of A in B [2] is defined by $sub_X(A,B) = \bigwedge_{x\in X} A(x) \to B(x)$ and $sub_X : L^X \times L^X \longrightarrow L$ is called an inclusion L-order on L^X. Moreover, (L^X, sub_X) is an L-ordered set.

An L-ordered set (X, e_X) is a complete L-lattice iff $\sqcup A$ (or $\sqcap A$) exist for all $A \in L^X$. For $Y \subseteq X$, let $e_Y(x,y) = e_X(x,y)$ for all $x, y \in Y$, then (Y, e_Y) is an L-ordered set and it is called a complete L-sublattice of (X, e_X) if $\sqcup B$ and $\sqcap B$ exist for all $B \in L^Y$. In fact, it is easy to check that (Y, e_Y) is a complete L-sublattice of (X, e_X) iff $\sqcup i^{\to}(B), \sqcap i^{\to}(B) \in Y$ for all $B \in L^Y$, where $i : Y \longrightarrow X$ is an inclusion mapping defined by $i(x) = x$.

Example 2.5. (1) *(L, e_L) is complete and for all $A \in L^L$, $\sqcup A = \bigvee_{a\in L} A(a) * a$, $\sqcap A = \bigwedge_{a\in L} A(a) \to a$, where $e_L : L \times L \longrightarrow L$ is defined by $e_L(a,b) = a \to b$.*
(2) *For any $\mathcal{A} \in L^{L^X}$, $\bigvee_{A\in L^X} \mathcal{A}(A) * A$ (resp., $\bigwedge_{A\in L^X} \mathcal{A}(A) \to A$) is usually denoted by $\bigcup \mathcal{A}$ (resp., $\bigcap \mathcal{A}$), which is the generalization of set union (resp., set intersection) in crisp case. (L^X, sub_X) is complete and for all $\mathcal{A} \in L^{L^X}$, $\sqcup \mathcal{A} = \bigcup \mathcal{A}$ and $\sqcap \mathcal{A} = \bigcap \mathcal{A}$. However, for any complete L-sublattice $(\mathcal{T}, sub_X)$ of (L^X, sub_X) and any $\mathcal{B} \in L^{\mathcal{T}}$, $\sqcup \mathcal{B} = \bigcup \mathcal{B}$ and $\sqcap \mathcal{B} = \bigcap \mathcal{B}$ are not necessarily true.*

A mapping $f : X \to Y$ between (X, e_X) and (Y, e_Y) is called L-order preserving if $e_X(x,y) \leq e_Y(f(x), f(y)), \forall x, y \in X$. Given a mapping $f : X \to Y$ between two sets X and Y, as usual, define $f^{\to} : L^X \to L^Y$ and $f^{\leftarrow} : L^Y \to L^X$ by $f^{\to}(A)(y) = \bigvee_{f(x)=y} A(x)$, $f^{\leftarrow}(B)(x) = B(f(x))$, $\forall\, y \in Y$, $\forall\, x \in X$. Then for any L-order preserving $f : X \longrightarrow Y$, $f^{\to}(D) \in \mathrm{D}_L(Y)$ (resp., $f^{\to}(D) \in \mathrm{CD}_L(Y)$) for all $D \in \mathrm{D}_L(X)$ (resp., $D \in \mathrm{CD}_L(X)$).

2.2 Some examples of L-order L-convexity

Definition 2.6. [30] *An L-order L-convexity (or a strong L-convexity) on a set X is a subset $\mathcal{C}$ of L^X which satisfies the following conditions:*
(LOLC1) *$0_X \in \mathcal{C}$;*
(LOLC2) *$\sqcap i^{\to}(\mathcal{A}) \in \mathcal{C}$ for any $\mathcal{A} \in L^{\mathcal{C}}$;*
(LOLC3) *$\sqcup i^{\to}(\mathcal{D}) \in \mathcal{C}$ for any $\mathcal{D} \in \mathrm{D}_L(\mathcal{C})$.*
In this case, the pair $(X, \mathcal{C})$ is called an L-order L-convex space.

Obviously, every L-order L-convex space is a special case of L-convex space.

Remark 2.7. (1) *For any $\mathcal{A} \in L^{\mathcal{C}}$, $i^{\to}(\mathcal{A}) \in L^{L^X}$ and then $\sqcap i^{\to}(\mathcal{A})$ exists and $\sqcap i^{\to}(\mathcal{A}) = \bigwedge_{A\in\mathcal{C}} \mathcal{A}(A) \to A$. Similarly, $\sqcup i^{\to}(\mathcal{A}) = \bigvee_{A\in\mathcal{C}} \mathcal{A}(A) * A$. Hence* (LOLC2) *can be represented as $\bigcap \mathcal{A} \in \mathcal{C}$ for any $\mathcal{A} \in L^{\mathcal{C}}$ and* (LOLC3) *can be represented as $\bigcup \mathcal{D} \in \mathcal{C}$ for any $\mathcal{D} \in \mathrm{D}_L(\mathcal{C})$.*
(2) *An algebraic L-closure system defined in* [28] *is a subset $\mathcal{C}$ of L^X satisfying* (LOLC2) *and* (LOLC3)*, thus it is an extension of L-order L-convexity.*
(3) *An Alexandrov L-topology defined in* [10] *is a special case of L-order L-convexity.*

Example 2.8. (1) *For every L-ordered set (X, e_X), the set $\mathcal{L}$ of all lower L-subsets of X is an L-order L-convexity and $e_X(x,y) = \bigwedge_{A\in\mathcal{L}} A(y) \to A(x)$.*
(2) *The usual ordered convexity* [29] *on a partially ordered set $(X, \leq)$ can be defined in terms of ordering as follows: a set $A \subseteq X$ is an order convex set iff $x, y \in A$ and $x \leq z \leq y$ then $z \in A$. In the same way, an L-convexity* [13] *on an L-ordered set is defined as follows: for an L-ordered set (X, e_X), $A \in L^X$ is an L-ordered L-convex set provided $e_X(x,z) * e_X(z,y) * A(x) * A(y) \leq A(z)$ for all $x, y, z \in X$. Obviously, every lower L-subset of X is an L-ordered L-convex sets. Let $\mathcal{C}_{e_X}$ denote the set of all L-ordered L-convex sets. Then $(X, \mathcal{C}_{e_X})$ is an L-order L-convex space when L is a frame.*

Example 2.9. *Let* $(L = \{0, 0.5, 1\}, \vee, \wedge, *, \rightarrow, 0, 1)$ *be a complete residuated lattice as* $x * y = max\{0, x + y - 1\}$, $x \rightarrow y = min\{1 - x + y, 1\}$, $X = \{x_1, x_2, x_3, x_4\}$ *be a set of four goods, and* $A \in L^X$ *be a referee, to estimate the degree of goodness. We denote* A *by a vector of dimension 4, where the* i*th coordinate is* $A(x_i)$ *for* $i = 1, 2, 3, 4$*. Then* $\mathcal{C} = \{A_1 = (0, 0, 0, 0), A_2 = (0, 0, 0, 1), A_3 = (0, 0.5, 0.5, 1), A_4 = (0, 0.5, 1, 1), A_5 = (0, 1, 0.5, 1), A_6 = (0, 1, 1, 1), A_7 = (1, 1, 1, 1)\}$ *is an* L*-order* L*-convexity. Noted that in economic theory it is often assumed that consumer preferences are convex, we can extend convex preferences to the setting of* L*-order* L*-convex spaces. Specific as follows: an* L*-order* e_X *is called an* L*-convex preference on an* L*-order* L*-convex space* $(X, \mathcal{C})$ *if for every* $x \in X$, $e_X(x, \_) \in \mathcal{C}$*. Now, we define an* L*-order* $e_X : X \times X \longrightarrow L$ *as follows:*

$$(e_{ij}) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0.5 & 1 \\ 0 & 0.5 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $e_{ij} = e_X(x_i, x_j)$ *for* $i, j = 1, 2, 3, 4$ *can be explained as the degree of the* i*th item* x_i *better than the* j*th item* x_j *in some way. Obviously,* $e_X(x_1, \_) = (1, 1, 1, 1), e_X(x_2, \_) = (0, 1, 0.5, 1), e_X(x_3, \_) = (0, 0.5, 1, 1), e_X(x_4, \_) = (0, 0, 0, 1) \in \mathcal{C}$ *and so* e_X *is an* L*-convex preference. Moreover, it easily follows that every* $A_i \in \mathcal{C}$ *is an* L*-order* L*-convex set and* $\sqcup A_i$ *exists. For* $A_i \in \mathcal{C}$*, we can explain* $\sqcup A_i$ *as the best item in the mind of the referee* A_i*.*

Example 2.10. *Let* $(S; \rightarrow_S; \Lambda)$ *be a labelled transition systems with finite images. It is easy to check that* $R = \bigcap_{n \in \mathbb{N}} \leq_n \subseteq S \times S$ *is a simulation relation. By Example 2.8 (1), we know that* e_S *can be completely characterized by the* L*-ordered* L*-convexity* $\mathcal{L}$*, i.e.,* $e_S(s, t) = \bigwedge_{A \in \mathcal{L}} A(t) \rightarrow A(s)$*. Clearly,* A *means that if the migration step starting from state* t *belongs to* A*, all migration steps simulated by the migration step starting from state* t *belong to* A*. From Example 2.2, for any* $s, t \in S$, $(s, t) \in \bigcap_{n \in \mathbb{N}} \leq_n$ *iff* $A(t) \leq A(s)$ *for any* $A \in \mathcal{L}$*.*

Definition 2.11. *[30] An algebraic* L*-ordered closure operator on* X *is a mapping* $co : L^X \longrightarrow L^X$ *which satisfies:*
(ALOC1) $co(0_X) = 0_X$*;*
(ALOC2) $A \leq co(A)$*;*
(ALOC3) $sub_X(A, B) \leq sub_X(co(A), co(B))$*;*
(ALOC4) $co(co(A)) = co(A)$*;*
(ALOC5) $co(\sqcup \mathcal{D}) = \sqcup co^{\rightarrow}(\mathcal{D})$ *for any* $\mathcal{D} \in \mathcal{D}(L^X)$*.*
For an algebraic L*-ordered closure operator* co *on* X*, the pair* (X, co) *is called an algebraic* L*-ordered closure operator space.*

Remark 2.12. *Every algebraic* L*-ordered closure operator is an algebraic* L*-closure operator defined in [28], that is, a mapping* $co : L^X \longrightarrow L^X$ *satisfying (ALOC2)-(ALOC5).*

For a mapping $f : X \longrightarrow Y$ between L-order L-convex spaces $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$, it is called convexity-preserving (L-CP for short) if $f^{\leftarrow}(B) \in \mathcal{C}_X$ for any $B \in \mathcal{C}_Y$. The category of L-order L-convex spaces and L-CP mappings is denoted by L-**OLCVS**.

3 L-enclosed L-ordered spaces

Different from the L-topological derived enclosed relation [31] and the L-enclosed relation [39], we construct a new L-relation on the L-powerset by means of directed L-subsets of the L-powerset, which is called an L-enclosed L-order. Then the relationship between it and the L-ordered L-convexity is established.

Definition 3.1. *A binary* L*-relation* $\preceq$ *on* L^X *is called an* L*-enclosed* L*-order and the pair* $(X, \preceq)$ *is called an* L*-enclosed* L*-order space, if* $\preceq$ *satisfies:*
(LELO1) $\preceq(0_X, 0_X) = 1$*;*
(LELO2) $\preceq(A, B) \leq sub_X(A, B)$*;*
(LELO3) $\preceq(A, \sqcap \mathcal{B}) = \bigwedge_{B \in L^X} \mathcal{B}(B) \rightarrow \preceq(A, B)$ *for any* $\mathcal{B} \in L^{L^X}$*;*
(LELO4) $\preceq(\sqcup \mathcal{D}, A) = \bigwedge_{D \in L^X} \mathcal{D}(D) \rightarrow \preceq(D, A)$ *for any* $\mathcal{D} \in \mathrm{D}_L(L^X)$*;*
(LELO5) $\preceq(A, B) = \bigvee_{C \in L^X} \preceq(A, C) * \preceq(C, B)$*.*

Remark 3.2. (1) *It is easy to check that (LELO4) implies (ELR4) and (ELR6) in the definition of* L*-enclosed relation* [39] *and so the* L*-enclosed* L*-order is a special case of the* L*-enclosed relation.*
(2) *From* (LELO3) *and* (LELO4) *we can easily obtain the following results: for any* $A, B, C \in L^X$, $\preceq(A, B) * sub_X(B, C) \leq \preceq(A, C)$, $sub_X(A, B) * \preceq(B, C) \leq \preceq(A, C)$*, and* $\preceq(A, C) * \preceq(C, B) \leq \preceq(A, B)$*.*

Theorem 3.3. *Let $(X, \preceq)$ be an L-enclosed L-order space. Define $co_{\preceq} : L^X \longrightarrow L^X$ by*

$$co_{\preceq}(A)(x) = \bigwedge_{D \in L^X} \preceq (A, D) \to D(x),\ \forall\ A \in L^X,\ \forall\ x \in X.$$

Then
(1) *for any $A \in L^X$, $\preceq (A, co_{\preceq}(A)) = 1$;*
(2) *for any $A, B \in L^X$, $sub_X(co_{\preceq}(A), B) = \preceq (A, B)$;*
(3) *$co_{\preceq}$ is an algebraic L-ordered closure operator which naturally induces an L-order L-convexity denoted by $\mathcal{C}_{\preceq}$.*

Proof. (1) $\forall A \in L^X$, let $\mathcal{B}(B) = \preceq (A, B), \forall B \in L^X$, then $\mathcal{B} \in L^{L^X}$ and $\sqcap \mathcal{B} = co_{\preceq}(A)$. From this we obtain

$$\preceq (A, co_{\preceq}(A)) = \preceq (A, \sqcap \mathcal{B}) = \bigwedge_{B \in L^X} \mathcal{B}(B) \to \preceq (A, B) = \bigwedge_{B \in L^X} \preceq (A, B) \to \preceq (A, B) = 1.$$

(2) By (1), we have $sub_X(co_{\preceq}(A), B) = \preceq (A, co_{\preceq}(A)) * sub_X(co_{\preceq}(A), B) \leq \preceq (A, B)$. On the other hand,

$$sub_X(co_{\preceq}(A), B) = \bigwedge_{x \in X} co_{\preceq}(A)(x) \to B(x) \geq \bigwedge_{x \in X} (\preceq (A, B) \to B(x)) \to B(x) \geq \preceq (A, B).$$

Therefore, $sub_X(co_{\preceq}(A), B) = \preceq (A, B)$.

(3) (ALOC1) $co_{\preceq}(0_X) = 0_X$ from

$$0 \leq co_{\preceq}(0_X)(x) = \bigwedge_{B \in L^X} \preceq (0_X, B) \to B(x) \leq \preceq (0_X, 0_X) \to 0_X(x) = 0,\ \forall x \in X.$$

(ALOC2) For any $A \in L^X$, $co_{\preceq}(A) \geq A$ because for any $x \in X$,

$$co_{\preceq}(A)(x) = \bigwedge_{D \in L^X} \preceq (A, D) \to D(x) \geq \bigwedge_{D \in L^X} sub_X(A, D) \to D(x) \geq A(x).$$

(ALOC3) For any $A, B \in L^X$,

$$\begin{aligned} sub_X(co_{\preceq}(A), co_{\preceq}(B)) &= \bigwedge_{x \in X} [(\bigwedge_{D \in L^X} \preceq(A, D) \to D(x)) \to (\bigwedge_{C \in L^X} \preceq(B, C) \to C(x))] \\ &\geq \bigwedge_{x \in X} \bigwedge_{C \in L^X} (\preceq (A, C) \to C(x)) \to (\preceq (B, C) \to C(x)) \\ &\geq \bigwedge_{C \in L^X} \preceq (B, C) \to \preceq (A, C) \geq sub_X(A, B). \end{aligned}$$

(ALOC4) $\forall A \in L^X$, we have $co_{\preceq}(co_{\preceq}(A)) \geq co_{\preceq}(A)$ and $\forall x \in X$,

$$\begin{aligned} co_{\preceq}(A)(x) &= \bigwedge_{D, C \in L^X} \preceq(A, D) * \preceq(D, C) \to C(x) \\ &= \bigwedge_{D \in L^X} \preceq(A, D) \to co_{\preceq}(D)(x) \\ &= \bigwedge_{D \in L^X} sub_X(co_{\preceq}(A), D) \to co_{\preceq}(D)(x) \\ &\geq \bigwedge_{D \in L^X} sub_X(co_{\preceq}(co_{\preceq}(A)), co_{\preceq}(D)) \to co_{\preceq}(D)(x) \\ &\geq co_{\preceq}(co_{\preceq}(A))(x). \end{aligned}$$

Therefore, $co_{\preceq}(co_{\preceq}(A)) = co_{\preceq}(A)$.

(ALOC5) For any $\mathcal{D} \in \mathrm{D}_L(L^X)$ and any $B \in L^X$,

$$\begin{aligned} sub_X(\sqcup co_{\preceq}^{\to}(\mathcal{D}), B) &= \bigwedge_{A \in L^X} \mathcal{D}(A) \to sub_X(co_{\preceq}(A), B) = \bigwedge_{A \in L^X} \mathcal{D}(A) \to \preceq (A, B) \\ &= \preceq(\sqcup \mathcal{D}, B) = sub_X(co_{\preceq}(\sqcup \mathcal{D}), B). \end{aligned}$$

This implies that $\sqcup co_{\preceq}^{\to}(\mathcal{D}) = co_{\preceq}(\sqcup \mathcal{D})$. $\square$

The following theorem shows that we can obtain an L-enclosed L-order from an L-order L-convexity.

Theorem 3.4. *Let $(X,\mathcal{C})$ be an L-order L-convex space. For all $A,B\in L^X$, define $\preceq_{\mathcal{C}}(A,B)=sub_X(co(A),B)$. Then $\preceq_{\mathcal{C}}$ is an L-enclosed L-order.*

Proof. (LELO1) $\preceq_{\mathcal{C}}(0_X,0_X)=sub_X(co(0_X),0_X)=sub_X(0_X,0_X)=1$.
(LELO2) $\preceq_{\mathcal{C}}(A,B)=sub_X(co(A),B)\le sub_X(A,B)$.
(LELO3) For any $\mathcal{B}\in L^{L^X}$ and any $A\in L^X$,

$$\preceq_{\mathcal{C}}(A,\sqcap\mathcal{B})=sub_X(co(A),\sqcap\mathcal{B})=\bigwedge_{B\in L^X}\mathcal{B}(B)\to sub_X(co(A),B)=\bigwedge_{B\in L^X}\mathcal{B}(B)\to\preceq_{\mathcal{C}}(A,B).$$

(LELO4) For any $\mathcal{D}\in \mathrm{D}_L(L^X)$ and any $C\in L^X$,

$$\begin{aligned}\preceq_{\mathcal{C}}(\sqcup\mathcal{D},C)&=sub_X(co(\sqcup\mathcal{D}),C)=sub_X(\sqcup co^{\rightarrow}(\mathcal{D}),C)\\&=\bigwedge_{B\in L^X}\mathcal{D}(B)\to sub_X(co(B),C)=\bigwedge_{B\in L^X}\mathcal{D}(B)\to\preceq_{\mathcal{C}}(B,C).\end{aligned}$$

(LELO5) For any $A,B\in L^X$,

$$\begin{aligned}\bigvee_{C\in L^X}\preceq_{\mathcal{C}}(A,C)*\preceq_{\mathcal{C}}(C,B)&=\bigvee_{C\in L^X}sub_X(co(A),C)*sub_X(co(C),B)\\&=sub_X(co(A),B)=\preceq_{\mathcal{C}}(A,B).\end{aligned}$$

□

Corollary 3.5. *Let $\preceq$ be an L-enclosed L-order on L^X and $\mathcal{C}_{\preceq}$ its induced an L-order L-convexity. Then $A\in\mathcal{C}_{\preceq}\Leftrightarrow\ \preceq(A,A)=1$ for any $A\in L^X$.*

Proof. Let $A\in\mathcal{C}_{\preceq}$, then $\preceq(A,A)=\preceq(A,co_{\preceq}(A))=1$. On the contrary, suppose that $\preceq(A,A)=1$, then $co_{\preceq}(A)(x)\le\ \preceq(A,A)\to A(x)=A(x)$ and so $A\in\mathcal{C}_{\preceq}$. □

From Theorems 3.3 and 3.4 the following result is obtained immediately.

Theorem 3.6. *Let $\preceq$ be an L-enclosed L-order and $\mathcal{C}$ an L-order L-convexity on (L^X,sub_X). Then $\preceq_{\mathcal{C}_{\preceq}}=\ \preceq$ and $\mathcal{C}_{\preceq_{\mathcal{C}}}=\mathcal{C}$.*

Definition 3.7. *Let $(X,\preceq_X)$ and $(Y,\preceq_Y)$ be L-enclosed L-order spaces. A mapping $f:X\longrightarrow Y$ is called an L-enclosed L-order dual-preserving mapping (L-ELODP mapping for short), if $\preceq_Y(A,B)\le\ \preceq_X(f^{\leftarrow}(A),f^{\leftarrow}(B))$, $\forall A,B\in L^Y$.*

By the definition of L-ELODP mapping, the following result is easy to check.

Proposition 3.8. *Let $f:(X,\preceq_X)\longrightarrow(Y,\preceq_Y)$, $g:(Y,\preceq_Y)\longrightarrow(Z,\preceq_Z)$ be L-ELODP mappings. Then $g\circ f:X\longrightarrow Z$ is an L-ELODP mapping.*

By the above proposition, it is easy to check that all L-enclosed L-order spaces and all L-ELODP mappings form a category denoted by L-**ELOS**.

Proposition 3.9. (1) *For an L-ELODP mapping $f:(X,\preceq_X)\longrightarrow(Y,\preceq_Y)$, $f:(X,\mathcal{C}_{\preceq_X})\longrightarrow(Y,\mathcal{C}_{\preceq_Y})$ is an L-CP mapping.*
(2) *For an L-CP mapping $f:(X,\mathcal{C}_X)\longrightarrow(Y,\mathcal{C}_Y)$, $f:(X,\preceq_{\mathcal{C}_X})\longrightarrow(Y,\preceq_{\mathcal{C}_Y})$ is an L-ELODP mapping.*

Proof. (1) It is suffices to show that $\forall B\in\mathcal{C}_{\preceq_Y}$, $f^{\leftarrow}(B)\in\mathcal{C}_{\preceq_X}$. In fact, $\forall x\in X$,

$$\begin{aligned}co_{\preceq_X}(f^{\leftarrow}(B))(x)&=\bigwedge_{A\in L^X}\preceq_X(f^{\leftarrow}(B),A)\to A\le\bigwedge_{C\in L^Y}\preceq_X(f^{\leftarrow}(B),f^{\leftarrow}(C))\to f^{\leftarrow}(C)\\&\le\bigwedge_{C\in L^Y}\preceq_Y(B,C)\to C(f(x))=co_{\preceq_Y}(B)(f(x))=B(f(x))=f^{\leftarrow}(B)(x).\end{aligned}$$

This implies $f^{\leftarrow}(B)\in\mathcal{C}_{\preceq_X}$, as desired.
(2) Indeed, for any $A,B\in L^Y$,

$$\begin{aligned}\preceq_{\mathcal{C}_Y}(A,B)&=sub_Y(co_Y(A),B)\le sub_X(f^{\leftarrow}(co_Y(A)),f^{\leftarrow}(B))\\&\le sub_X(co_X(f^{\leftarrow}(A)),f^{\leftarrow}(B))=\preceq_{\mathcal{C}_X}(f^{\leftarrow}(A),f^{\leftarrow}(B)).\end{aligned}$$

This implies that $f:(X,\preceq_{\mathcal{C}_X})\longrightarrow(Y,\preceq_{\mathcal{C}_Y})$ is an L-ELODP mapping. □

By Theorem 3.6 and Proposition 3.9, the following theorem is directly obtained.

Theorem 3.10. *L-**OLCVS** is isomorphic to L-**ELOS**.*

4 L-order L-concave spaces

In this section, we are about to introduce the notion of L-order L-concavity as a dual notion of L-order L-convexity on a set.

Definition 4.1. *An L-order L-concavity τ on X is a subsets of L^X which satisfies:*
(LOLCA1) $1_X \in \tau$;
(LOLCA2) $\forall \mathcal{A} \in L^\tau$, $\sqcup i^{\rightarrow}(\mathcal{A}) \in \tau$;
(LOLCA3) $\forall \mathcal{D} \in \mathrm{CD}_L(\tau)$, $\sqcap i^{\rightarrow}(\mathcal{D}) \in \tau$.
In this case, the pair (X, τ) is called an L-order L-concave space.

Obviously, every L-order L-concave space is an L-concave space defined in [21] and so it is a special case of L-concave spaces.

Remark 4.2. (1) *$\tau \subseteq L^X$ is an L-order L-concavity iff $\bigcup \mathcal{A} \in \tau$ and $\bigcap \mathcal{D} \in \tau$ for all $\mathcal{A} \in L^\tau$ and all $\mathcal{D} \in \mathrm{CD}_L(\tau)$.*
(2) *For an L-order L-concavity τ, (τ, sub_X) is a complete L-lattice, moreover, $\sqcup \mathcal{A} = \bigcup \mathcal{A}$ and $\sqcap \mathcal{D} = \bigcap \mathcal{D}$ for all $\mathcal{A} \in L^\tau$ and all $\mathcal{D} \in \mathrm{CD}_L(\tau)$.*

A mapping $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ is called L-concavity-preserving (L-CAP for short) if $f^{\leftarrow}(B) \in \tau_X$ for any $B \in \tau_Y$. Then all L-order L-concave spaces and all L-CAP mappings form a category denoted by L-**OLCAS**.

Similar to the proofs of Proposition 4.1 in [30] and Theorem 3.9 in [28], we can check the following theorem.

Theorem 4.3. *Let (X, τ) be an L-order L-concave space. An L-order L-co-hull operator $ca : L^X \longrightarrow L^X$ of τ is defined by $ca(A) = \bigvee_{D \in \tau} sub_X(D, A) * D(x)$ for all $A \in L^X$. Then it satisfies the following conditions:*
(LOLCHO1) $ca(1_X) = 1_X$;
(LOLCHO2) $ca(A) \leq A$ *for all* $A \in L^X$;
(LOLCHO3) $sub_X(A, B) \leq sub_X(ca(A), ca(B))$ *for all* $A, B \in L^X$;
(LOLCHO4) $ca(ca(A)) = ca(A)$ *for all* $A \in L^X$;
(LOLCHO5) $ca(\sqcap \mathcal{D}) = \sqcap ca^{\rightarrow}(\mathcal{D})$ *for all* $\mathcal{D} \in \mathrm{CD}_L(L^X)$.
Conversely, if $ca : L^X \longrightarrow L^X$ satisfies (LOLCHO1)-(LOLCHO5), *the $\tau_{ca} = \{A \in L^X : ca(A) = A\}$ is an L-order L-concavity. Moreover, $\tau_{ca_\tau} = \tau$ and $ca_{\tau_{ca}} = ca$.*

A mapping $f : X \longrightarrow Y$ between L-order L-co-hull spaces (X, ca_X) and (Y, ca_Y) is continuous if $f^{\rightarrow}(ca_X(A)) \leq ca_Y(f^{\rightarrow}(A))$ for any $A \in L^X$. The category of L-order L-co-hull spaces and continuous mappings is denoted by L-**OLCHS**. Then from Theorem 4.3, we have the following result.

Theorem 4.4. *L-**OLCAS** is isomorphic to L-**OLCHS**.*

When L is equipped with an order-reversing involution operator $\neg$, it is easy to check that (1) for any L-order L-convexity $\mathcal{T}$, $\neg\mathcal{T} = \{\neg A : A \in \mathcal{T}\}$ is an L-order L-concavity; (2) for any L-order L-concavity τ, $\neg\tau = \{\neg A : A \in \tau\}$ is an L-order L-convexity.

From this we can obtain the following functors between L-**OLCAS** and L-**OLCVS**: $\mathbb{H}$: L-**OLCVS**$\longrightarrow$ L-**OLCAS** defined by $\mathbb{H}(X, \mathcal{T}_X) = (X, \neg\mathcal{T}_X)$, $\mathbb{H}(f) = f$ and $\mathbb{K}$: L-**OLCAS**$\longrightarrow$ L-**OLCVS** defined by $\mathbb{K}(X, \tau_X) = (X, \neg\tau_X)$, $\mathbb{K}(f) = f$.

Then we can immediately come to the following conclusion:

Theorem 4.5. *L-**OLCAS** is isomorphic to L-**OLCVS** when L is equipped with an order-reversing involution operator $\neg$.*

Definition 4.6. *A binary L-relation $\preceq$ on L^X is called an L-internal L-order and the pair $(X, \preceq)$ is called an L-internal L-order space, if $\preceq$ satisfies:*
(LILO1) $\preceq (1_X, 1_X) = 1$;
(LILO2) $\preceq (A, B) \leq sub_X(A, B)$;
(LILO3) $\preceq (\sqcup \mathcal{B}, A) = \bigwedge_{B \in L^X} \mathcal{B}(B) \rightarrow \preceq (B, A)$ *for any* $\mathcal{B} \in L^{L^X}$;
(LILO4) $\preceq (A, \sqcap \mathcal{D}) = \bigwedge_{D \in L^X} \mathcal{D}(D) \rightarrow \preceq (A, D)$ *for any* $\mathcal{D} \in \mathrm{CD}_L(L^X)$;
(LILO5) $\preceq (A, B) = \bigvee_{C \in L^X} \preceq (A, C) * \preceq (C, B)$.

Let $(X, \preceq_X)$ and $(Y, \preceq_Y)$ be L-internal L-order spaces. A mapping $f : X \longrightarrow Y$ is called an L-internal L-order dual-preserving mapping (L-ILODP mapping for short), if $\preceq_Y (A, B) \leq \preceq_X (f^{\leftarrow}(A), f^{\leftarrow}(B))$ for any $A, B \in L^Y$. The category of L-internal L-order spaces and L-ILODP mappings is denoted by L-**ILOS**.

When L is equipped with an order-reversing involution operator $\neg$, L-internal L-orders and L-enclosed L-orders are dual concepts, since a binary L-relation $\preceq$ on X is an L-internal L-order iff the binary L-relation $\preceq^c$ on X, defined by $\preceq^c (A, B) = \preceq (\neg B, \neg A)$, is an L-enclosed L-order.

Through the conventional derivation we can get the following conclusions:

Theorem 4.7. *Let $(X, \preceq)$ be an L-internal L-order space. Define $ca_{\preceq} : L^X \longrightarrow L^X$ by $\forall\ A \in L^X$, $\forall\ x \in X$, $ca_{\preceq}(A)(x) = \bigvee_{D \in L^X} \preceq (D, A) * D(x)$. Then $ca_{\preceq}$ is an L-order L-co-hull which naturally induces an L-order L-concavity denoted by $\tau_{\preceq}$. Conversely, let (X, τ) be an L-order L-concave space. Define a binary L-relation $\preceq_\tau$ on X by $\preceq_\tau (A, B) = sub_X(A, ca(B))$. Then $\preceq_\tau$ is an L-internal L-order. In addition, $\preceq_{\tau_{\preceq}} = \preceq$ and $\tau_{\preceq_\tau} = \tau$.*

Proposition 4.8. (1) *For an L-ILODP mapping $f : (X, \preceq_X) \longrightarrow (Y, \preceq_Y)$, $f : (X, \tau_{\preceq_X}) \longrightarrow (Y, \tau_{\preceq_Y})$ is an L-CAP mapping.*
(2) *For an L-CAP mapping $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$, $f : (X, \preceq_{\tau_X}) \longrightarrow (Y, \preceq_{\tau_Y})$ is an L-ILODP mapping.*

Theorem 4.9. *L-**OLCAS** is isomorphic to L-**ILOS**.*

5 L-order convergence structures

In this section, based on the inclusion L-order, we introduce the notion of L-order L-convex filter by means of L-subsets of L-powersets. This is different from the fuzzy filters constructed by using crisp subsets of L-powersets in [6, 17, 32, 33]. Thus, the L-order convergence structure established by the L-order L-convex filters is also different from the convergence structures defined in [6, 17, 32, 33]. Moreover, we investigate the categorical relationships among L-order convergence spaces, L-order L-concave spaces and L-order L-convex spaces.

Definition 5.1. *A mapping $\mathcal{F} : L^X \longrightarrow L$ is called an L-order L-convex filter on X if it satisfies:*
(LOCF1) $\mathcal{F}(0_X) = 0$, $\mathcal{F}(1_X) = 1$;
(LOCF2) $sub_X(A, B) \leq \mathcal{F}(A) \to \mathcal{F}(B)$, $\forall A, B \in L^X$;
(LOCF3) $\mathcal{F}(\sqcap \mathcal{D}) = sub_{L^X}(\mathcal{D}, \mathcal{F})$, $\forall \mathcal{D} \in \mathrm{CD}_L(L^X)$, *where* $sub_{L^X}(\mathcal{D}, \mathcal{F}) = \bigwedge_{A \in L^X} \mathcal{D}(A) \to \mathcal{F}(A)$.

Remark 5.2. (1) *It is easy to check that an L-order L-convex filter is an extension of an Alexandrov L-filter defined in* [9]. *By Proposition 3.2 in* [27], *we have that $\mathcal{F} : L^X \longrightarrow L$ is an Alexandrov L-filter iff it is both an L-order L-convex filter and an L-ordered filter defined in* [7].
(2) (LOCF2) *is equivalent to $a * \mathcal{F}(A) \leq \mathcal{F}(a * A)$ and $A \leq B \Rightarrow \mathcal{F}(A) \leq \mathcal{F}(B)$ for all $A, B \in L^X$ and $a \in L$. Moreover, it is easy to check that for any $\mathcal{A} \in L^{L^X}$, $\mathcal{F}(\bigcup \mathcal{A}) \geq \bigvee_{A \in L^X} \mathcal{A}(A) * \mathcal{F}(A)$.*
(3) (LOCF3) *implies $\mathcal{F}(\bigwedge_{i \in I} A_i) = \bigwedge_{i \in I} \mathcal{F}(A_i)$ for each down-directed family $\{A_i\ :\ i \in I\} \subseteq L^X$, and $\mathcal{F}(a \to A) = a \to \mathcal{F}(A)$ for any $a \in L$ and any $A \in L^X$.*

The family of all L-order L-convex filters on X is denoted by $\mathcal{OF}_L(X)$ and then $(\mathcal{OF}_L(X), sub_{L^X})$ is an L-ordered set. Moreover, we have the following result:

Proposition 5.3. *Let X be a set and $\mathbb{F}$ a nonempty L-subset of $\mathcal{OF}_L(X)$. Then $\sqcap \mathbb{F}$ exists and $\sqcap \mathbb{F} = \bigcap \mathbb{F} \in \mathcal{OF}_L(X)$.*

Proof. We only need to show that $\bigcap \mathbb{F} \in \mathcal{OF}_L(X)$. In fact,

(LOCF1) $(\bigcap \mathbb{F})(1_X) = \bigwedge_{\mathcal{F} \in \mathcal{OF}_L(X)} \mathbb{F}(\mathcal{F}) \to \mathcal{F}(1_X) = 1$ and

$$(\bigcap \mathbb{F})(0_X) = \bigwedge_{\mathcal{F} \in \mathcal{OF}_L(X)} \mathbb{F}(\mathcal{F}) \to \mathcal{F}(0_X) = (\bigvee_{\mathcal{F} \in \mathcal{OF}_L(X)} \mathbb{F}(\mathcal{F})) \to 0 = 1 \to 0 = 0.$$

(LOCF2) For any L-subset A, B of X, we have

$$(\bigcap \mathbb{F})(A) \to (\bigcap \mathbb{F})(B) = (\bigwedge_{\mathcal{F} \in \mathcal{OF}_L(X)} \mathbb{F}(\mathcal{F}) \to \mathcal{F}(A)) \to (\bigwedge_{\mathcal{F} \in \mathcal{OF}_L(X)} \mathbb{F}(\mathcal{F}) \to \mathcal{F}(B))$$

$$\geq \bigwedge_{\mathcal{F} \in \mathcal{OF}_L(X)} (\mathbb{F}(\mathcal{F}) \to \mathcal{F}(A)) \to (\mathbb{F}(\mathcal{F}) \to \mathcal{F}(B)) \geq \mathcal{F}(A) \to \mathcal{F}(B) \geq sub_X(A, B).$$

(LOCF3) For any down-directed L-subset $\mathcal{D}$ of L^X, we have

$$(\bigcap \mathbb{F})(\sqcap \mathcal{D}) = \bigwedge_{\mathcal{F}\in \mathcal{OF}_L(X)} \mathbb{F}(\mathcal{F}) \to \mathcal{F}(\sqcap \mathcal{D}) = \bigwedge_{A\in L^X} \mathcal{D}(A) \to (\bigwedge_{\mathcal{F}\in \mathcal{OF}_L(X)} \mathbb{F}(\mathcal{F}) \to \mathcal{F}(A))$$
$$= \bigwedge_{A\in L^X} \mathcal{D}(A) \to (\bigcap \mathbb{F})(A) = sub_{L^X}(\mathcal{D}, \bigcap \mathbb{F}).$$

These implies that $\bigcap \mathbb{F} \in \mathcal{OF}_L(X)$ and so $\sqcap \mathbb{F} = \bigcap \mathbb{F} \in \mathcal{OF}_L(X)$. □

Example 5.4. *It is easy to check that the following mappings are L-order L-convex filters on X.*
(1) *For any $x \in X$, define $[x] : L^X \longrightarrow L$ by $[x](A) = A(x)$ for any $A \in L^X$.*
(2) *A mapping $\mathcal{F}_0 : L^X \longrightarrow L$ by $\mathcal{F}_0(A) = \bigwedge_{x\in X} A(x)$.*

Example 5.5. *For an L-order L-concave space (X, τ) and $x \in X$, the mapping $\mathcal{F}_x^{ca_\tau} : L^X \longrightarrow L$ defined by $\mathcal{F}_x^{ca_\tau}(A) = ca_\tau(A)(x)$ for any $A \in L^X$ is an L-order L-convex filter.*

For a mapping $f : X \longrightarrow Y$ and $\mathcal{F} \in \mathcal{OF}_L(X)$, define $f^{\Rightarrow}(\mathcal{F}) : L^Y \longrightarrow L$ as $f^{\Rightarrow}(\mathcal{F})(B) = \mathcal{F}(f^{\leftarrow}(B))$ for any $B \in L^Y$.

Proposition 5.6. *Let $f : X \longrightarrow Y$ be a mapping and $\mathcal{F} \in \mathcal{OF}_L(X)$. Then $f^{\Rightarrow}(\mathcal{F}) \in \mathcal{OF}_L(Y)$.*

Proof. (LOCF1) holds from $f^{\Rightarrow}(\mathcal{F})(0_X) = \mathcal{F}(f^{\leftarrow}(0_X)) = 0$ and $f^{\Rightarrow}(\mathcal{F})(1_X) = \mathcal{F}(f^{\leftarrow}(1_X)) = 1$.
(LOCF2) For any $B_1, B_2 \in L^Y$,

$$f^{\Rightarrow}(\mathcal{F})(B_1) \to f^{\Rightarrow}(\mathcal{F})(B_2) = \mathcal{F}(f^{\leftarrow}(B_1)) \to \mathcal{F}(f^{\leftarrow}(B_2))$$
$$\geq sub_X(f^{\leftarrow}(B_1), f^{\leftarrow}(B_2)) \geq sub_Y(B_1, B_2).$$

(LOCF3) For any down-directed L-subset $\mathcal{D}$ of L^Y, define $\mathcal{A} : L^X \longrightarrow L$ as

$$\mathcal{A}(A) = \begin{cases} \mathcal{D}(B) & A = f^{\leftarrow}(B), \\ 0 & otherwise. \end{cases}$$

Then $\mathcal{A} \in \mathrm{CD}_L(L^X)$ from $\bigvee_{A\in L^X} \mathcal{A}(A) = \bigvee_{B\in L^Y} \mathcal{D}(B) = 1$ and $\forall B_1, B_2 \in L^Y$,

$$\mathcal{A}(f^{\leftarrow}(B_1)) * \mathcal{A}(f^{\leftarrow}(B_2)) = \mathcal{D}(B_1) * \mathcal{D}(B_2) \leq \bigvee_{B\in L^Y} \mathcal{D}(B) * sub_Y(B, B_1) * sub_Y(B, B_2)$$
$$= \bigvee_{B\in L^Y} \mathcal{A}(f^{\leftarrow}(B)) * sub_Y(B, B_1) * sub_Y(B, B_2)$$
$$\leq \bigvee_{B\in L^Y} \mathcal{A}(f^{\leftarrow}(B)) * sub_X(f^{\leftarrow}(B), f^{\leftarrow}(B_1)) * sub_X(f^{\leftarrow}(B), f^{\leftarrow}(B_2))$$
$$\leq \bigvee_{A\in L^X} \mathcal{A}(A) * sub_X(A, f^{\leftarrow}(B_1)) * sub_X(A, f^{\leftarrow}(B_2)).$$

Thus we have

$$f^{\Rightarrow}(\mathcal{F})(\sqcap \mathcal{D}) = \mathcal{F}(f^{\leftarrow}(\sqcap \mathcal{D})) = \mathcal{F}(\bigwedge_{B\in L^Y} \mathcal{D}(B) \to f^{\leftarrow}(B)) = \mathcal{F}(\bigwedge_{B\in L^Y} \mathcal{A}(f^{\leftarrow}(B)) \to f^{\leftarrow}(B))$$
$$= \mathcal{F}(\bigwedge_{A\in L^X} \mathcal{A}(A) \to A) = \mathcal{F}(\sqcap \mathcal{A}) = \bigwedge_{A\in L^X} \mathcal{A}(A) \to \mathcal{F}(A)$$
$$= \bigwedge_{B\in L^Y} \mathcal{D}(B) \to f^{\Rightarrow}(\mathcal{F})(B).$$

□

Definition 5.7. *An L-order convergence structure on X is a mapping $lim : \mathcal{OF}_L(X) \longrightarrow L^X$ which satisfies:*
(LOCS1) $lim([x])(x) = 1$;
(LOSC2) $sub_{L^X}(\mathcal{F}, \mathcal{G}) \leq sub_X(lim(\mathcal{F}), lim(\mathcal{G}))$.
In this case, the pair $(X, \lim)$ is called an L-order convergence space.

Clearly, this convergence structure is a stratified L-ordered convergence structure defined in [6] when L-order L-convex filters in the definition are replaced by stratified L-filters.

Proposition 5.8. *Let (X,τ) be an L-order L-concave space. Define a mapping $lim_\tau : \mathcal{OF}_L(X) \longrightarrow L^X$ as*

$$\forall \mathcal{F} \in \mathcal{OF}_L(X),\ \forall x \in X,\ lim_\tau(\mathcal{F})(x) = \bigwedge_{A\in L^X} ca(A)(x) \longrightarrow \mathcal{F}(A).$$

Then lim_τ is an L-order convergence structure on X.

Proof. (LOCS1) $lim_{\mathcal{T}}([x])(x) = \bigwedge_{A\in L^X} ca(A)(x) \to [x](A) = 1$.

(LOCS2) For any $\mathcal{F},\mathcal{G} \in \mathcal{OF}_L(X)$,

$$\begin{aligned} sub_X(lim_\tau(\mathcal{F}), lim_\tau(\mathcal{G})) &= \bigwedge_{x\in X} \Big(\bigwedge_{A\in L^X} ca(A)(x) \to \mathcal{F}(A)\Big) \to \Big(\bigwedge_{B\in L^X} ca(B)(x) \to \mathcal{G}(B)\Big) \\ &\geq \bigwedge_{x\in X} \bigwedge_{B\in L^X} (ca(B)(x) \to \mathcal{F}(B)) \to (ca(B)(x) \to \mathcal{G}(B)) \\ &\geq \bigwedge_{B\in L^X} \mathcal{F}(B) \to \mathcal{G}(B) = sub_{L^X}(\mathcal{F},\mathcal{G}). \end{aligned}$$

□

Definition 5.9. *A mapping $f : (X, lim_X) \longrightarrow (Y, lim_Y)$ between L-order convergence spaces is called L-order convergence-preserving (L-OCP) for short) provided that*

$$\forall \mathcal{F} \in \mathcal{OF}_L(X),\ \forall x \in X,\ lim_X(\mathcal{F})(x) \leq lim_Y(f^{\Rightarrow}(\mathcal{F}))(f(x)).$$

It is easy to check that all L-order convergence spaces as objects and all L-order convergence-preserving mappings as morphisms form a category, called a category of L-order convergence spaces and denoted by L-**OCS**.

Definition 5.10. *For a nonempty set X, let $Fib_L(X)$ denote the fibre*

$$\{(X, lim)\ :\ lim \text{ is an } L - \text{order convergence structure on } X\}$$

of X. For L-order convergence spaces (X, lim_1) and (X, lim_2), we say (X, lim_1) is finer than (X, lim_2), denoted by $(X, lim_1) \leq (X, lim_2)$, if the identity mapping $id_X : (X, lim_1) \longrightarrow (X, lim_2)$ is an L-OCP. *In this case, we also write $lim_1 \leq lim_2$.*

Clearly, $(Fib_L(X), \leq)$ is a poset, which implies that L-**OCS** is amnestic.

Lemma 5.11. *Let X be a nonempty set, $((X_j, lim_j))_{j\in J}$ a family of L-order convergence spaces and $f_j : X \longrightarrow X_j$ a mapping for each $j \in J$. Then there exists an L-order convergence structure lim_{fin} on X such that for any L-order convergence space (Y, lim_Y), a mapping $g : (Y, lim_Y) \longrightarrow (X, lim_{fin})$ is an* L-OCP *iff for any $j \in J$, $f_j \circ g : (Y, lim_Y) \longrightarrow (X_j, lim_j)$ is an* L-OCP.

Proof. First, for any $\mathcal{F} \in \mathcal{OF}_L(X)$ and any $x \in X$, define lim_{fin} by

$$lim_{fin}(\mathcal{F})(x) = \bigwedge_{j\in J} lim_j(f_j^{\Rightarrow}(\mathcal{F}))(f_j(x)).$$

Then $lim_{fin}([x])(x) = 1$ since $f_j^{\Rightarrow}([x])(B_j) = B_j(f_j(x)) = [f_j(x)](B_j)$ for each $j \in J$ and each $B_j \in L^{X_j}$. Moreover, for any $\mathcal{F},\mathcal{G} \in \mathcal{OF}_L(X)$,

$$\begin{aligned} sub_X(lim_{fin}(\mathcal{F}), lim_{fin}(\mathcal{G})) &\geq \bigwedge_{x\in X} \bigwedge_{j\in J} lim_j(f_j^{\Rightarrow}(\mathcal{F}))(f_j(x)) \to lim_j(f_j^{\Rightarrow}(\mathcal{G}))(f_j(x)) \\ &\geq \bigwedge_{j\in J} sub_Y(lim_j(f_j^{\Rightarrow}(\mathcal{F})), lim_j(f_j^{\Rightarrow}(\mathcal{G}))) \geq sub_{L^X}(\mathcal{F},\mathcal{G}). \end{aligned}$$

Hence lim_{fin} is an L-order convergence structure on X. Second, if $g : (Y, lim_Y) \longrightarrow (X, lim_{fin})$ is an L-OCP, then $f_j \circ g : (Y, lim_Y) \longrightarrow (X_j, lim_j)$ is an L-OCP for all $j \in J$. Conversely, suppose $f_j \circ g$ is an L-OCP for each $j \in J$. Then $\forall \mathcal{G} \in \mathcal{OF}_L(Y)$,

$$\begin{aligned} lim_Y(\mathcal{G})(y) &\leq \bigwedge_{j\in J} lim_j((f_j \circ g)^{\Rightarrow}(\mathcal{G}))((f_j \circ g)(y)) \\ &= \bigwedge_{j\in J} lim_j(f_j^{\Rightarrow}(g^{\Rightarrow}(\mathcal{G})))(f_j(g(y))) = lim_{fin}(g^{\Rightarrow}(\mathcal{G}))(g(y)). \end{aligned}$$

This implies that $g : (Y, lim_Y) \longrightarrow (X, lim_{fin})$ is an L-OCP, as desired. □

From this lemma we can get the following conclusion directly.

Theorem 5.12. *The category L-**OCS** is topological over set in the sense that it is initial and fibre small.*

Proposition 5.13. *Let (X, lim) be an L-order convergence space and $x \in X$. Define a mapping $\Phi^x_{lim} : \mathcal{OF}_L(X) \longrightarrow L$ as $\Phi^x_{lim}(\mathcal{F}) = lim(\mathcal{F})(x)$. Then*

$$\tau_{lim} = \{A \in L^X \ : \ \sqcap \Phi^x_{lim}(A) = A(x), \forall x \in X\},$$

is an L-order L-concavity on X.

Proof. We first show $\sqcap\Phi^x_{lim}(A) = \bigwedge_{\mathcal{F}\in\mathcal{OF}_L(X)} lim(\mathcal{F})(x) \to \mathcal{F}(A)$, $\forall A \in L^X$. In fact, $\forall x \in X$, $\bigvee_{\mathcal{F}\in\mathcal{OF}_L(X)} \Phi^x_{lim}(\mathcal{F})(x) \geq \Phi^x_{lim}([x])(x) = 1$. Thus $\sqcap\Phi^x_{lim}$ exists and

$$\forall A \in L^X, \sqcap\Phi^x_{lim}(A) = \bigcap \Phi^x_{lim} = \bigwedge_{\mathcal{F}\in\mathcal{OF}_L(X)} lim(\mathcal{F})(x) \to \mathcal{F}(A) \in \mathcal{OF}_L(X),$$

by Proposition 5.3.

Next, we show that $\tau_{lim} = \{A \in L^X \ : \ \sqcap \Phi^x_{lim}(A) = A(x), \forall x \in X\}$ is an L-order L-concavity on X. Clearly, for any $A \in L^X$, $\sqcap\Phi^x_{lim}(A) \leq A(x)$.

(LOLCA1) for any $x \in X$,

$$\begin{aligned} \sqcap\Phi^x_{lim}(0_X) &= \bigwedge_{\mathcal{F}\in\mathcal{OF}_L(X)} lim(\mathcal{F})(x) \to \mathcal{F}(0_X) \\ &= (\bigvee_{\mathcal{F}\in\mathcal{OF}_L(X)} lim(\mathcal{F})(x)) \to 0 = 1 \to 0 = 0 = 0_X(x). \end{aligned}$$

and $\sqcap\Phi^x_{lim}(1_X) = 1$ is obvious.

(LOLCA2) For any $\mathcal{A} \in L^{\tau_{lim}}$ and any $x \in X$,

$$(\bigcup \mathcal{A})(x) = \bigvee_{A\in\tau_{lim}} \mathcal{A}(A) * A(x) = \bigvee_{A\in\tau_{lim}} \mathcal{A}(A) * \sqcap\Phi^x_{lim}(A) \leq \sqcap\Phi^x_{lim}(\bigcup \mathcal{A}) \leq (\bigcup \mathcal{A})(x)$$

and so $\bigcup \mathcal{A} \in \tau_{lim}$.

(LOLCA3) For any down-directed L-subset $\mathcal{D}$ of τ_{lim} and any $x \in X$,

$$(\bigcap \mathcal{D})(x) = \bigwedge_{A\in\tau_{lim}} \mathcal{D}(A) \to A(x) = \bigwedge_{A\in\tau_{lim}} \mathcal{D}(A) \to \sqcap\Phi^x_{lim}(A) = \sqcap\Phi^x_{lim}(\bigcap \mathcal{D})$$

and so $\bigcap \mathcal{D} \in \tau_{lim}$. □

Corollary 5.14. *Let L be equipped with an order-reversing involution operator $\neg$ and (X, lim) an L-order convergence space. $\mathcal{T}_{lim} = \{\neg A \in L^X \ : \ \sqcap \Phi^x_{lim}(A) = A(x), \forall x \in X\}$ is an L-order L-convexity on X.*

Theorem 5.15. *Let τ an L-order L-concavity on X and lim an L-order convergence structure on X. Then $\tau_{lim_\tau} = \tau$ and $lim_{\tau_{lim}} \geq lim$.*

Proof. Step 1. Show that $\tau_{lim_\tau} = \tau$. In fact, for any $A \in L^X$ and any $x \in X$,

$$\sqcap\Phi^x_{lim_\tau}(A) \leq \Phi^x_{lim_\tau}(\mathcal{F}^{ca_\tau}_x) \to \mathcal{F}^{ca_\tau}_x(A) = ca_\tau(A)(x),$$

and

$$\begin{aligned}\sqcap\Phi^x_{lim_\tau}(A) &= \bigwedge_{\mathcal{F}\in\mathcal{OF}_L(X)} (\bigwedge_{B\in L^X} ca_\tau(B)(x) \to \mathcal{F}(B)) \to \mathcal{F}(A)\\ &\geq \bigwedge_{\mathcal{F}\in\mathcal{OF}_L(X)} (ca_\tau(A)(x) \to \mathcal{F}(A)) \to \mathcal{F}(A) \geq ca_\tau(A)(x).\end{aligned}$$

Then $\sqcap\Phi^x_{lim_\tau}(A) = ca_\tau(A)(x)$ and so $\tau_{lim_\tau} = \tau$.

Step 2. Show that $lim_{\tau_{lim}} \geq lim$. For any $x \in X$ and $\mathcal{F} \in \mathcal{OF}_L(X)$,

$$\begin{aligned}lim_{\tau_{lim}}(\mathcal{F})(x) &= \bigwedge_{A\in L^X} ca_{\tau_{lim}}(A)(x) \to \mathcal{F}(A)\\ &= \bigwedge_{A\in L^X} \bigwedge_{B\in\tau_{lim}} sub(B,A) * B(x) \to \mathcal{F}(A)\\ &= \bigwedge_{A\in L^X} \bigwedge_{B\in\tau_{lim}} sub(B,A) * \sqcap\Phi^x_{lim}(B) \to \mathcal{F}(A)\\ &\geq \bigwedge_{A\in L^X} \bigwedge_{B\in\tau_{lim}} sub(B,A) * (lim(\mathcal{F})(x) \to \mathcal{F}(B)) \to \mathcal{F}(A)\\ &\geq \bigwedge_{A\in L^X} \bigwedge_{B\in\tau_{lim}} (\mathcal{F}(B) \to \mathcal{F}(A)) * (lim(\mathcal{F})(x) \to \mathcal{F}(B)) \to \mathcal{F}(A)\\ &\geq \bigwedge_{A\in L^X} (\lim(\mathcal{F})(x) \to \mathcal{F}(A)) \to \mathcal{F}(A) \geq \lim(\mathcal{F})(x).\end{aligned}$$

□

Proposition 5.16. (1) *For an L-OCP mapping $f : (X, lim_X) \longrightarrow (Y, lim_Y)$, $f : (X, \tau_{lim_X}) \longrightarrow (Y, \tau_{lim_Y})$ is an L-CAP mapping.*
(2) *For an L-CAP mapping $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$, $f : (X, lim_{\tau_X}) \longrightarrow (Y, lim_{\tau_Y})$ is an L-OCP mapping.*

Proof. (1) For any $B \in \tau_{lim_Y}$ and any $x \in X$,

$$\begin{aligned}\sqcap\Phi^x_{lim}(f^\leftarrow(B)) &= \bigwedge_{\mathcal{F}\in\mathcal{OF}_L(X)} lim_X(\mathcal{F})(x) \to \mathcal{F}(f^\leftarrow(B))\\ &\geq \bigwedge_{\mathcal{F}\in\mathcal{OF}_L(X)} lim_Y(f^\Rightarrow(\mathcal{F}))(f(x)) \to f^\Rightarrow(\mathcal{F})(B)\\ &\geq \bigwedge_{\mathcal{G}\in\mathcal{OF}_L(Y)} lim_Y(\mathcal{G})(f(x)) \to \mathcal{G}(B)\\ &= \sqcap\Phi^{f(x)}_{lim_Y}(B) = B(f(x)) = f^\leftarrow(B)(x)\end{aligned}$$

and so $f^\leftarrow(B) \in \tau_{lim}$.

(2) For any $\mathcal{F} \in \mathcal{OF}_L(X)$ and any $x \in X$,

$$\begin{aligned}lim_{\tau_Y}(f^\Rightarrow(\mathcal{F}))(f(x) &= \bigwedge_{B\in L^Y} ca_{\tau_Y}(B)(f(x)) \to f^\Rightarrow(\mathcal{F})(B)\\ &= \bigwedge_{B\in L^Y} (\bigvee_{C\in\tau_Y} sub_Y(C,B) * C(f(x))) \to f^\Rightarrow(\mathcal{F})(B)\\ &\geq \bigwedge_{B\in L^Y} (\bigvee_{C\in\tau_Y} sub_X(f^\leftarrow(C), f^\leftarrow(B)) * f^\leftarrow(C)(x)) \to f^\Rightarrow(\mathcal{F})(B)\\ &\geq \bigwedge_{B\in L^Y} (\bigvee_{D\in\tau_X} sub_X(D, f^\leftarrow(B)) * D(x)) \to \mathcal{F}(f^\leftarrow(B))\\ &= \bigwedge_{B\in L^Y} ca_{\tau_X}(f^\leftarrow(B))(x) \to \mathcal{F}(f^\leftarrow(B)) \geq lim_{\tau_X}(\mathcal{F})(x).\end{aligned}$$

□

Then we can obtain the following functors between L-**OLCAS** and L-**OCS**:
$\mathbb{F}$: L-**OCS**$\longrightarrow$ L-**OLCAS** defined by $\mathbb{F}(X, lim_X) = (X, \tau_{lim_X})$, $\mathbb{F}(f) = f$ and $\mathbb{G}$: L-**OLCAS**$\longrightarrow$ L-**OCS** defined by $\mathbb{G}(X, \tau_X) = (X, lim_{\tau_X})$, $\mathbb{G}(f) = f$.

By Proposition 5.13 and Theorem 5.15, we have the following result.

Theorem 5.17. *The pair* $(\mathbb{F}, \mathbb{G})$ *is a Galois correspondence between* L-**OCS** *and* L-**OLCAS**.

Corollary 5.18. *Let* L *be equipped with an order-reversing involution operator* $\neg$. *Then the pair* $(\mathbb{K} \circ \mathbb{F}, \mathbb{G} \circ \mathbb{H})$ *is a Galois correspondence between* L-**OCS** *and* L-**OLCVS**.

6 Conclusions

In this paper, we provided several characterizations of L-order L-convexities from a categorical viewpoint, including L-enclosed L-orders and L-order L-concavities. In the case of L with an order-reversing involution operator, all these resulting categories are isomorphic. Furthermore, we constructed Galois correspondences among the categories of L-order L-convex spaces, L-order L-concave spaces and L-order convergence spaces.

Inspired by the examples of economic models that consider classical convex spaces and convex preferences, we presented an example of fuzzy convex preference on fuzzy convex spaces. It may be an interesting direction of future studies to consider fuzzy convex preference in different types of fuzzy convex structures.

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