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# Statistical testing quality and its Monte Carlo simulation based on fuzzy specification limits

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**Abstract**

This paper presents two approaches for testing quality to make a decision based on the extended process capability indices. Common methods in measuring quality of the manufactured product have widely focused on the precise specification limits, but in this study the lower and upper specification limits are considered as non-precise/fuzzy sets. Based on a general statistical approach using an extended process capability index, the purpose of this study is estimating a critical value to determine whether the process meets the customer requirements. Moreover, a simulation approach to analyze the manufacturing process capability has been suggested for testing quality based on fuzzy specifications by normal data. Meanwhile, this paper discusses how well the Monte Carlo simulation approach can be used for non-normal data. Finally, the real application of the proposed methods is investigated in a real case study.

*Keywords:* Quality control, process capability indices, fuzzy specification limits, testing hypotheses, Monte Carlo simulation.

## 1 Introduction and literature review

The process capability index (PCI) is a numerical measure for instating the relationships between the manufacturing specifications and the actual performance of process. In general, process capability indices, such as $C_p$, $C_{pk}$ and $C_{pm}$, have been widely utilized based on a normal distribution of process output in the manufacturing industry. The PCI $C_p = \frac{USL-LSL}{6\sigma}$ has been suggested by Juran [9] in the manufacturing industry to provide a numerical of the potential ability to meet customer requirements, where $LSL$ is the lower specification limit, $USL$ is the upper specification limit, and $\sigma$ is the process standard deviation. Actually, the index $C_p$ is an important engineering decision tool when the quality improvement programs chiefly focuses on decreasing the variation of process. Generally, the process mean and the process standard deviation are unknown and so a random sample is required to estimate the unknown parameters. If we take a random sample of size $n$, then, $\mu$ and $\sigma$ can be estimated by the sample mean $\bar{X} = \sum_{i=1}^{n} X_i/n$ and the sample standard deviation $S_{n-1} = \sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2/n-1}$, to acquire the natural estimator $\widehat{C_p} = \frac{USL-LSL}{6S_{n-1}}$. Also, when the manufacturing process is normally distributed, the index $C_p$ presents an exact quantity of the manufacturing process yield [35].

The main assumption for analyzing the PCI $C_p$ is that its usual interpretation is based on the normal distribution of process output [16]. If probability distribution of a process characteristic is non-normal, using the traditional PCIs such as $C_p$ often leads to erroneous interpretation of process capability [14]. Essentially, the index $C_p$ is useful when the process mean, $\mu$, is located in the center of the specification limits (SLs) [31], that is, $\mu = (USL + LSL)/2$.

Some studies on fuzzy PCIs have been conducted, e.g. see [1, 2, 3, 10, 11, 12, 13, 24, 35, 36]. The first question which gets up is that why is there a necessity to measure fuzzy PCIs? Fuzzy logic can be applied to manage the uncertainties. These uncertainties could exist in the SLs, which are being used in order to measure the capability of

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processes. Sometimes the SLs cannot be indicated by crisp numbers. Kaya and Kahraman [13] defined the SLs by fuzzy numbers.

Two basic definitions for providing fuzzy concepts are introduced as follows: (i) let R be the set of all real numbers and $F(\mathrm{R}) = \{A | A : \mathrm{R} \to [0,1], A \text{ is a continuous function}\}$ be the set of all fuzzy sets on R; and (ii) the $\alpha$-cut of $\tilde{A} \in F(\mathrm{R})$ is the crisp set given by $\tilde{A}_\alpha = \left\{x | \tilde{A}(x) \geq \alpha\right\}$, for any $\alpha \in [0,1]$ [17]. Yongting [37] introduced the concept of fuzzy quality. The motivations of fuzzy quality were mentioned in [28]. The construction of membership functions of the PCI was introduced in [27] when the SLs are triangular fuzzy numbers. The results of Parchami et al. [27] were revised for the general case, where the SLs are L-R fuzzy intervals in [15]. Parchami et al. [25] redefined some concepts about fuzzy confidence interval and they indicated the consistency property of the fuzzy confidence interval in [24]. Moreover, they discussed the problem of measuring the manufacturing process capability based on introducing two operations of summation and subtraction on fuzzy SLs [26]. The PCIs were generalized by considering a meaningful meter where vagueness was involved into both specification limits and data [29]. Abbasi Ganji and Sadeghpour Gildeh [2] developed fuzzy approaches for measuring the process capability in simple linear profiles for the situations in which lower and upper specification limits are fuzzy. Sadeghpour Gildeh [33] suggested a general multivariate PCI based on fuzzy SLs. Additionally, the capability of a fuzzy process was tested in [22] when SLs were triangular fuzzy sets.

Parchami [18] compared two R packages "FPV" and "Fuzzy.p.value" for testing hypotheses in a fuzzy environment using a fuzzy $p$-value and he focused on calculating the fuzzy $p$-value based on fuzzy data using the generalization principle in [19]. Arefi [4] utilized a method to test the hypotheses for the mean of the normal distribution with known/unknown variance based on the fuzzy data. Parchami et al. [21] applied fuzzy hypotheses testing instead of classical hypotheses testing based on the fuzzy $p$-value method. Testing the capability of a manufacturing process on the basis of Yongtings index was proposed in [20]. Tsai and Chen [35] extended the applications of the PCI $C_p$ in a fuzzy environment, with a methodology for testing the index $C_p$ based on fuzzy numbers. Grau constructed a critical value to test whether a process is capable in the presence of measurement errors [8].

The extended PCIs such as $C_{\tilde{p}}$, $C_{\widetilde{pk}}$ and $C_{\widetilde{pm}}$ were defined in [23] to measure the capability of a normal process when SLs are non-precise. This paper involves two proposed approaches to test the capability of the extended PCI $C_{\tilde{p}}$. A comparison of both proposed testing approaches with conventional PCI $C_p$ is investigated in this paper. The proposed approaches are denoted as the statistical fuzzy quality test(FQT) and the simulated FQT, which estimate/simulate the critical value for testing the manufacturing process capability based on the extended PCI $C_{\tilde{p}}$.

Herein, the statistical FQT for analyzing the manufacturing process based on fuzzy specification limits is proposed in the first place. This approach is provided for determining the critical value to the capability test of the extended PCI $C_{\tilde{p}}$, and in the second place, the simulated FQT can be applied for testing the relevant PCI, especially where the statistical distribution of the process capability estimator cannot be specified. By comparing two approaches, one can observe that the power function of both approaches are same. Therefore, as a matter of fact, one can conclude that the simulated FQT is a good procedure for finding the critical value in testing more complex PCIs. Of particular significance is our work on presenting an Algorithm based on the Monte Carlo simulation method to simulate the critical value, the $p$-value and the probability of type $II$ error for evaluating the performance of $C_{\tilde{p}}$. The contribution of this paper is presenting a motivation to determine the critical value for testing hypotheses on more complex PCIs by the simulated FQT.

The rest of this paper is organized as follows. Section 2 briefly reviews preliminary concepts about the extended process capability index $C_{\tilde{p}}$. Section 3 introduces a new unbiased estimator for $C_{\tilde{p}}$. Section 4 suggests the statistical FQT to evaluate the performance of the extended PCI $C_{\tilde{p}}$ and mentions the Pearn et al.'s methodology for testing the conventional PCI $C_p$. Section 5 proposes the Monte Carlo simulation technique for testing fuzzy quality to assess the performance of $C_{\tilde{p}}$. Section 6 generalizes the simulated FQT based on non-normal process. Section 7 presents a case study to illustrate the application of the proposed approaches. Section 8 provides a comparison study between two proposed approaches in crisp and fuzzy quality tests. Moreover, the advantages of using them are discussed. The final section is conclusions and future researches.

## 2 The extended process capability index $C_{\tilde{p}}$

This section reviews a short history of the extended process capability indices. Herein, the study of analyzing the production quality focuses on the vague SLs. If vagueness is involved into SLs, we face a new process and the usual PCIs like $C_p$ are not suitable for measuring the capability of it. To evaluate the manufacturing process in such vague situation, the extended process capability index

$$C_{\tilde{p}} = \frac{\widetilde{USL} \ominus \widetilde{LSL}}{6\sigma}, \tag{1}$$

is proposed in [23], where $\widetilde{LSL}, \widetilde{USL} \in F(\mathrm{R})$ are linear fuzzy SLs with membership functions

$$\widetilde{LSL}(x) = \begin{cases} 0 & \text{if} \quad x \leq l_0, \\ \frac{x-l_0}{l_1-l_0} & \text{if} \quad l_0 < x < l_1, \\ 1 & \text{if} \quad l_1 \leq x, \end{cases} \tag{2}$$

$$\widetilde{USL}(x) = \begin{cases} 1 & \text{if} \quad x \leq u_1, \\ \frac{x-u_0}{u_1-u_0} & \text{if} \quad u_1 < x < u_0, \\ 0 & \text{if} \quad u_0 \leq x, \end{cases} \tag{3}$$

and $\widetilde{USL} \ominus \widetilde{LSL} = \int_0^1 g(\alpha)(u_\alpha - l_\alpha)d\alpha$, such that $g(\alpha)$ is a non-decreasing function on $[0,1]$ with the conditions $g(0) = 0$ and $\int_0^1 g(\alpha)d\alpha = 1$ (see [23] for the motivation of definition).

**Notation 2.1.** *Let $\widetilde{LSL}$ and $\widetilde{USL}$ be the introduced lower and upper linear fuzzy SLs in Eqs.* (2) *and* (3)*. If $g(\alpha) = (j+1)\alpha^j$, for any $j \in \mathrm{R}^+$, then the extended PCI $C_{\tilde{p}}$ is converted to* [26].

$$C_{\tilde{p}} = \frac{(j+1)(u_1 - l_1) + (u_0 - l_0)}{6(j+2)\sigma}. \tag{4}$$

**Remark 2.2.** *If the linear fuzzy SLs $\widetilde{LSL}$ and $\widetilde{USL}$ in Eqs.* (2) *and* (3) *reduce to crisp numbers $LSL$ and $USL$ (i.e., they are indicator functions $I_{\{x|x \geq LSL\}}$ and $I_{\{x|x \leq USL\}}$), hence $l_1 = l_0 = LSL$ and $u_1 = u_0 = USL$, and then the introduced extended PCI in Eq.* (4) *coincides with the Juran's index $C_p$*

$$C_{\tilde{p}} = \frac{(j+1)(u_1 - l_1) + (u_0 - l_0)}{6(j+2)\sigma} = \frac{(j+2)(u_1 - l_1)}{6(j+2)\sigma} = \frac{USL - LSL}{6\sigma} = C_p.$$

# 3 Unbiased estimator

In order to find an unbiased estimation of the PCI $C_{\tilde{p}}$, let us take a random sample of size $n$, $X_1, X_2, \ldots, X_n$, and the estimator $S_{n-1}$ is considered to estimate the unknown parameter $\sigma$. Under the presented assumptions in Notation 2.1, by substituting $S_{n-1}$ in Eq. (4), the statistical estimation of $C_{\tilde{p}}$ for any $j \in \mathrm{R}^+$ can be written as

$$\widehat{C_{\tilde{p}}} = \frac{(j+1)(u_1-l_1)+(u_0-l_0)}{6(j+2)S_{n-1}}. \tag{5}$$

**Theorem 3.1.** *Under the given assumptions in Notation 2.1,*

$$\widehat{C}_{\tilde{p},u} = b_{n-1}\frac{(j+1)(u_1 - l_1) + (u_0 - l_0)}{6(j+2)S_{n-1}}, \tag{6}$$

*is an unbiased estimator for $C_{\tilde{p}}$ in which $b_{n-1} = \sqrt{\frac{2}{n-1}}\frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})}$.*

*Proof.* It is obvious that the statistic $\frac{(n-1)S_{n-1}^2}{\sigma^2}$ has chi-squared distribution with $n-1$ degrees of freedom and therefore, the mathematical expectation of $\widehat{C}_{\tilde{p}}$ is

$$E(\widehat{C}_{\tilde{p}}) = E\left[\frac{(j+1)(u_1 - l_1) + (u_0 - l_0)}{6(j+2)S_{n-1}}\right] = \frac{(j+1)(u_1 - l_1) + (u_0 - l_0)}{6(j+2)}E\left[\frac{1}{S_{n-1}}\right] = \frac{1}{b_{n-1}}C_{\tilde{p}}.$$

It satisfies $E(b_{n-1}\widehat{C}_{\tilde{p}}) = C_{\tilde{p}}$ and hence an unbiased estimator of $C_{\tilde{p}}$, denoted by $\widehat{C}_{\tilde{p},u}$, is $\widehat{C}_{\tilde{p},u} = b_{n-1}\frac{(j+1)(u_1-l_1)+(u_0-l_0)}{6(j+2)S_{n-1}}$. □

**Remark 3.2.** *Note that the index $C_{\tilde{p}}$ is useful when the process mean is located at the center of fuzzy tolerance, that is, $\mu = M$ in which $M = (\widetilde{USL} \oplus \widetilde{LSL})/2$ (see more details about the summation of $\widetilde{USL}$ and $\widetilde{LSL}$ in* [26]*). To cover this hard condition, $j \in \mathrm{R}^+$ is selected in this study such that $\mu = M$, which is equivalent to*

$$\mu = \frac{\widetilde{USL} \oplus \widetilde{LSL}}{2} = \frac{1}{2j+4}[(j+1)(u_1 + l_1) + (u_0 + l_0)]. \tag{7}$$

*Now, if the parameter $\mu$ is estimated by the sample mean $\hat{\mu} = \bar{X} = \sum_{i=1}^n X_i/n$, then $j$ can be determined by*

$$j = \frac{u_1 + l_1 + u_0 + l_0 - 4\bar{X}}{2\bar{X} - u_1 - l_1}, \tag{8}$$

*to cover the condition of using $C_{\tilde{p}}$, such that $u_1 + l_1 + u_0 + l_0 > 4\bar{X}$.*

# 4 Statistical fuzzy quality test

Testing the manufacturing process capability is a general statistical approach for analyzing the process performance. In this section, a statistical approach is proposed for the practitioners to make a reliable decision in evaluating process by the index $C_{\tilde{p}}$. The purpose of this section is to estimate the critical value for testing the process capability based on the linear fuzzy SLs. The following theorem leads to make a decision about how well the process meets the fuzzy specifications.

**Theorem 4.1.** *Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with the normal distribution $N(\mu, \sigma^2)$. In testing the statistical hypotheses*

$$\begin{cases} H_0 : C_{\tilde{p}} \leq c_0 & \text{(fuzzy process is not capable)}, \\ H_1 : C_{\tilde{p}} > c_0 & \text{(fuzzy process is capable)}, \end{cases} \tag{9}$$

*at the given significance level $\alpha$, the critical value is*

$$c = b_{n-1} c_0 \sqrt{\frac{n-1}{\chi^2_{n-1,\alpha}}}, \tag{10}$$

*in which $c_0 \in \mathrm{R}^+$ is the standard minimal criterion for $C_{\tilde{p}}$, $\chi^2_{n-1,\alpha}$ is the $\alpha$th quantile of the chi-square distribution with $n-1$ degrees of freedom and $b_{n-1}$ is expressed in Theorem 3.1.*

*Proof.* The probability of type $I$ error in testing the hypotheses (9) can be computed as follows

$$\begin{aligned} \alpha &= \sup_{H_0} Pr\left(\widehat{C}_{\tilde{p},u} > c\right) = Pr\left(\widehat{C}_{\tilde{p},u} > c | C_{\tilde{p}} = c_0\right) \\ &= Pr\left(b_{n-1}\widehat{C_{\tilde{p}}} > c | C_{\tilde{p}} = c_0\right) = Pr\left(b_{n-1}\frac{\widetilde{USL} \ominus \widetilde{LSL}}{6S_{n-1}} > c | C_{\tilde{p}} = c_0\right) \\ &= Pr\left(b_{n-1}C_{\tilde{p}}(\frac{\sigma}{S_{n-1}}) > c | C_{\tilde{p}} = c_0\right) = Pr\left(\frac{b_{n-1}c_0}{c} > \frac{S_{n-1}}{\sigma}\right) \\ &= Pr\left(\chi^2_{n-1} < \frac{(n-1)b^2_{n-1}c_0^2}{c^2}\right), \end{aligned}$$

and so $c = b_{n-1}c_0\sqrt{\frac{n-1}{\chi^2_{n-1,\alpha}}}$. □

Numerical values of the statistical critical values, which are computed in Table 1 by Theorem 4.1 for $c_0 = 1.00, 1.33, 1.50$, $n = 10(5)240$ and different significance levels $\alpha = 0.01, 0.025, 0.05, 0.1$. To decide if a given fuzzy process meets the customer requirement, we first specify the value of $c_0$ and the significance level $\alpha$, then we find the appropriate critical value in Table 1 based on given values of the $\alpha$, $c_0$, and the considered sample size $n$. Regarding to the proposed test procedure, the *p-value* of the statistical FQT using capability index $C_{\tilde{p}}$ is equal to

$$p\text{-}value = Pr\left(\widehat{C}_{\tilde{p},u} > \widehat{c}_{\tilde{p},u} | C_{\tilde{p}} = c_0\right) = Pr\left(\chi^2_{n-1} < \frac{(n-1)b^2_{n-1}c_0^2}{\widehat{c}^2_{\tilde{p},u}}\right), \tag{11}$$

where $\widehat{c}_{\tilde{p},u}$ is the observed value of capability index $C_{\tilde{p}}$ based on $x_1, \ldots, x_n$ from Eq. (6). Also, the power function of the statistical FQT for every $C_{\tilde{p}} > 0$ is

$$\Pi(C_{\tilde{p}}) = Pr\left(\chi^2_{n-1} < \frac{(n-1)b^2_{n-1}{C_{\tilde{p}}}^2}{c^2}\right), \tag{12}$$

where $c$ is the critical value on the basis of the statistical FQT. Therefore, the probability of type $II$ error at the point $c^*_{\tilde{p}} > c_0$ is equal to

$$\beta\left(c^*_{\tilde{p}}\right) = 1 - \Pi\left(c^*_{\tilde{p}}\right) = 1 - Pr\left(\chi^2_{n-1} < \frac{(n-1)b^2_{n-1}{c^*_{\tilde{p}}}^2}{c^2}\right), \tag{13}$$

Table 1: Critical values on the basis of the statistical FQT for $c_0 = 1.00, 1.33, 1.50$, $n = 10(5)240$ at different significance levels $\alpha = 0.01, 0.025, 0.05, 0.1$.

| | $c_0 = 1.00$ | | | | $c_0 = 1.33$ | | | | $c_0 = 1.50$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | 0.01 | 0.025 | 0.05 | 0.1 | 0.01 | 0.025 | 0.05 | 0.1 | 0.01 | 0.025 | 0.05 | 0.1 |
| 10 | 1.897 | 1.668 | 1.504 | 1.343 | 2.524 | 2.219 | 2.000 | 1.786 | 2.846 | 2.503 | 2.255 | 2.014 |
| 15 | 1.638 | 1.491 | 1.380 | 1.267 | 2.179 | 1.983 | 1.835 | 1.685 | 2.458 | 2.236 | 2.070 | 1.901 |
| 20 | 1.514 | 1.402 | 1.315 | 1.226 | 2.014 | 1.865 | 1.750 | 1.630 | 2.272 | 2.103 | 1.973 | 1.839 |
| 25 | 1.440 | 1.347 | 1.275 | 1.199 | 1.915 | 1.792 | 1.695 | 1.594 | 2.160 | 2.021 | 1.912 | 1.798 |
| 30 | 1.389 | 1.309 | 1.246 | 1.180 | 1.847 | 1.741 | 1.658 | 1.569 | 2.083 | 1.964 | 1.869 | 1.769 |
| 35 | 1.352 | 1.281 | 1.225 | 1.165 | 1.798 | 1.704 | 1.629 | 1.549 | 2.028 | 1.922 | 1.837 | 1.747 |
| 40 | 1.323 | 1.259 | 1.208 | 1.153 | 1.760 | 1.675 | 1.607 | 1.534 | 1.985 | 1.889 | 1.812 | 1.730 |
| 45 | 1.300 | 1.242 | 1.195 | 1.144 | 1.729 | 1.651 | 1.589 | 1.521 | 1.950 | 1.862 | 1.792 | 1.716 |
| 50 | 1.281 | 1.227 | 1.183 | 1.136 | 1.704 | 1.632 | 1.574 | 1.511 | 1.922 | 1.840 | 1.775 | 1.704 |
| 55 | 1.265 | 1.215 | 1.174 | 1.129 | 1.683 | 1.615 | 1.561 | 1.502 | 1.898 | 1.822 | 1.760 | 1.694 |
| 60 | 1.252 | 1.204 | 1.165 | 1.123 | 1.665 | 1.601 | 1.550 | 1.494 | 1.878 | 1.806 | 1.748 | 1.685 |
| 65 | 1.240 | 1.195 | 1.158 | 1.118 | 1.649 | 1.589 | 1.540 | 1.487 | 1.860 | 1.792 | 1.737 | 1.677 |
| 70 | 1.230 | 1.187 | 1.152 | 1.114 | 1.636 | 1.578 | 1.532 | 1.481 | 1.845 | 1.780 | 1.728 | 1.670 |
| 75 | 1.221 | 1.180 | 1.146 | 1.109 | 1.623 | 1.569 | 1.524 | 1.476 | 1.831 | 1.769 | 1.719 | 1.664 |
| 80 | 1.212 | 1.173 | 1.141 | 1.106 | 1.613 | 1.560 | 1.518 | 1.471 | 1.819 | 1.760 | 1.712 | 1.659 |
| 85 | 1.205 | 1.167 | 1.136 | 1.102 | 1.603 | 1.553 | 1.512 | 1.466 | 1.808 | 1.751 | 1.705 | 1.654 |
| 90 | 1.198 | 1.162 | 1.132 | 1.099 | 1.594 | 1.546 | 1.506 | 1.462 | 1.798 | 1.743 | 1.698 | 1.649 |
| 95 | 1.192 | 1.157 | 1.128 | 1.097 | 1.586 | 1.539 | 1.501 | 1.459 | 1.788 | 1.736 | 1.693 | 1.645 |
| 100 | 1.187 | 1.153 | 1.125 | 1.094 | 1.578 | 1.533 | 1.496 | 1.455 | 1.780 | 1.729 | 1.687 | 1.641 |
| 105 | 1.182 | 1.149 | 1.122 | 1.092 | 1.572 | 1.528 | 1.492 | 1.452 | 1.772 | 1.723 | 1.683 | 1.638 |
| 110 | 1.177 | 1.145 | 1.119 | 1.090 | 1.565 | 1.523 | 1.488 | 1.449 | 1.765 | 1.717 | 1.678 | 1.634 |
| 115 | 1.172 | 1.141 | 1.116 | 1.087 | 1.559 | 1.518 | 1.484 | 1.446 | 1.759 | 1.712 | 1.674 | 1.631 |
| 120 | 1.168 | 1.138 | 1.113 | 1.086 | 1.554 | 1.514 | 1.481 | 1.444 | 1.753 | 1.707 | 1.670 | 1.628 |
| 125 | 1.165 | 1.135 | 1.111 | 1.084 | 1.549 | 1.510 | 1.477 | 1.441 | 1.747 | 1.703 | 1.666 | 1.626 |
| 130 | 1.161 | 1.132 | 1.108 | 1.082 | 1.544 | 1.506 | 1.474 | 1.439 | 1.741 | 1.698 | 1.663 | 1.623 |
| 135 | 1.158 | 1.130 | 1.106 | 1.080 | 1.540 | 1.502 | 1.471 | 1.437 | 1.736 | 1.694 | 1.659 | 1.621 |
| 140 | 1.154 | 1.127 | 1.104 | 1.079 | 1.535 | 1.499 | 1.469 | 1.435 | 1.732 | 1.691 | 1.656 | 1.618 |
| 145 | 1.151 | 1.125 | 1.102 | 1.078 | 1.531 | 1.496 | 1.466 | 1.433 | 1.727 | 1.687 | 1.654 | 1.616 |
| 150 | 1.149 | 1.122 | 1.101 | 1.076 | 1.528 | 1.493 | 1.464 | 1.431 | 1.723 | 1.684 | 1.651 | 1.614 |
| 155 | 1.146 | 1.120 | 1.099 | 1.075 | 1.524 | 1.490 | 1.461 | 1.430 | 1.719 | 1.680 | 1.648 | 1.612 |
| 160 | 1.143 | 1.118 | 1.097 | 1.074 | 1.521 | 1.487 | 1.459 | 1.428 | 1.715 | 1.677 | 1.646 | 1.611 |
| 165 | 1.141 | 1.116 | 1.096 | 1.073 | 1.517 | 1.485 | 1.457 | 1.426 | 1.711 | 1.674 | 1.643 | 1.609 |
| 170 | 1.139 | 1.114 | 1.094 | 1.071 | 1.514 | 1.482 | 1.455 | 1.425 | 1.708 | 1.671 | 1.641 | 1.607 |
| 175 | 1.136 | 1.113 | 1.093 | 1.070 | 1.511 | 1.480 | 1.453 | 1.424 | 1.705 | 1.669 | 1.639 | 1.606 |
| 180 | 1.134 | 1.111 | 1.091 | 1.069 | 1.509 | 1.477 | 1.451 | 1.422 | 1.702 | 1.666 | 1.637 | 1.604 |
| 185 | 1.132 | 1.109 | 1.090 | 1.068 | 1.506 | 1.475 | 1.450 | 1.421 | 1.699 | 1.664 | 1.635 | 1.603 |
| 190 | 1.130 | 1.108 | 1.089 | 1.067 | 1.503 | 1.473 | 1.448 | 1.420 | 1.696 | 1.661 | 1.633 | 1.601 |
| 195 | 1.129 | 1.106 | 1.087 | 1.067 | 1.501 | 1.471 | 1.446 | 1.419 | 1.693 | 1.659 | 1.631 | 1.600 |
| 200 | 1.127 | 1.105 | 1.086 | 1.066 | 1.499 | 1.469 | 1.445 | 1.417 | 1.690 | 1.657 | 1.629 | 1.599 |
| 205 | 1.125 | 1.103 | 1.085 | 1.065 | 1.496 | 1.467 | 1.443 | 1.416 | 1.688 | 1.655 | 1.628 | 1.597 |
| 210 | 1.123 | 1.102 | 1.084 | 1.064 | 1.494 | 1.466 | 1.442 | 1.415 | 1.685 | 1.653 | 1.626 | 1.596 |
| 215 | 1.122 | 1.101 | 1.083 | 1.063 | 1.492 | 1.464 | 1.441 | 1.414 | 1.683 | 1.651 | 1.625 | 1.595 |
| 220 | 1.120 | 1.100 | 1.082 | 1.063 | 1.490 | 1.462 | 1.439 | 1.413 | 1.681 | 1.649 | 1.623 | 1.594 |
| 225 | 1.119 | 1.098 | 1.081 | 1.062 | 1.488 | 1.461 | 1.438 | 1.412 | 1.678 | 1.648 | 1.622 | 1.593 |
| 230 | 1.118 | 1.097 | 1.080 | 1.061 | 1.486 | 1.459 | 1.437 | 1.411 | 1.676 | 1.646 | 1.620 | 1.592 |
| 235 | 1.116 | 1.096 | 1.079 | 1.060 | 1.484 | 1.458 | 1.435 | 1.410 | 1.674 | 1.644 | 1.619 | 1.591 |
| 240 | 1.115 | 1.095 | 1.078 | 1.060 | 1.483 | 1.456 | 1.434 | 1.410 | 1.672 | 1.643 | 1.618 | 1.590 |

where $b_{n-1}$ is expressed in Theorem 3.1.

**The main decision rule in statistical FQT.** In order to make a decision based on the proposed statistical FQT, if the estimated value $\widehat{c}_{\tilde{p},u}$ is greater than the critical value $c$ (i.e., $\widehat{c}_{\tilde{p},u} > c$), then the null hypothesis is rejected at the significance level $\alpha$. Hence, one can conclude that the fuzzy process meets the capability requirement, this implies that, the fuzzy process is capable; otherwise it is incapable.

For the desired quality condition with $c_0 = 1.00$, $\alpha = 0.01$ and various sample sizes $n = 75(25)175$, one can compute the appropriate critical value $c$ from Eq. (10) and the power functions of the statistical FQT can be drawn in Figure 1, which shows the larger sample size can increase the power of the statistical FQT.

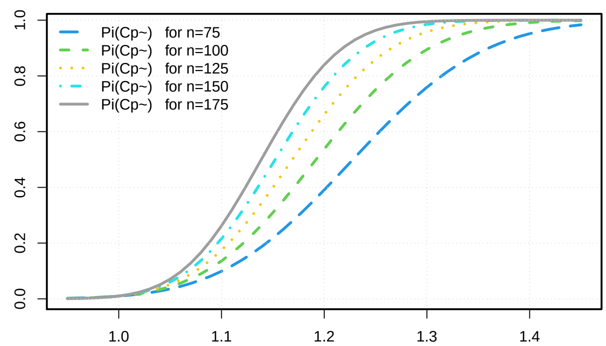


Figure 1: Plots of the power function of the statistical FQT $\Pi(C_{\tilde{p}})$ for the computed critical values based on the given values of $\alpha = 0.01$, $c_0 = 1.00$, and various sample sizes $n = 75(25)175$.

**Remark 4.2.** *Pearn et al. investigated traditional testing quality based on Juran's index $C_p$ in [32] for normal data. Their work is extended here by testing fuzzy quality when SLs are membership function rather than crisp numbers LSL and USL. In other words, when the fuzzy membership functions $\widetilde{USL}$ and $\widetilde{LSL}$ reduce to the indicator function $I_{\{x|x\geq LSL\}}$ and $I_{\{x|x\leq USL\}}$, such that $l_1 = l_0 = LSL$ and $u_1 = u_0 = USL$, and so considering Remark 2.2, all components of the decision-making in testing fuzzy quality - such as critical value, p-value and the power function of statistical FQT - coincide with the relevant components of decision-making process in Pearn et al.'s traditional testing quality by $C_p$. This obvious results can be easily proved regarding to Remark 2.2 and the replacement of index $C_p$ by the extended index $C_{\tilde{p}}$ throughout the statistical FQT.*

# 5 Simulated fuzzy quality test

A Monte Carlo simulation approach is proposed in this section to simulate the critical value $c$. For the implementation of hypothesis testing theory using the unbiased estimator of $C_{\tilde{p}}$, the probability of type $I$ error in testing the hypotheses (9) is equal to $\alpha = Pr\left(\widehat{C}_{\tilde{p},u} > c | C_{\tilde{p}} = c_0\right)$. Therefore,

$$1 - \alpha = Pr\left(\widehat{C}_{\tilde{p},u} \leq c | C_{\tilde{p}} = c_0\right), \tag{14}$$

and $c$ is the $(1-\alpha)$th quantile of $\widehat{C}_{\tilde{p},u}$ distribution such that $C_{\tilde{p}} = c_0$. Moreover, the presented *p-value* in Eq. (11) can be written as

$$p\text{-}value \quad = \quad E\left[I\left(\widehat{C}_{\tilde{p},u} > \widehat{c}_{\tilde{p},u} | C_{\tilde{p}} = c_0\right)\right], \tag{15}$$

where $\widehat{c}_{\tilde{p},u}$ is the observed value of capability index $C_{\tilde{p}}$ based on $x_1, \ldots, x_n$ from Eq. (6) and $I(A)$ is the indicator function of event/set $A$. Similarly, the probability of type *II* error at the point $c^*_{\tilde{p}}$ in Eq. (13) can be written as

$$\beta(c^*_{\tilde{p}}) \quad = \quad E\left[I\left(\widehat{C}_{\tilde{p},u} \leq c \mid C_{\tilde{p}} = c^*_{\tilde{p}}\right)\right]. \tag{16}$$

A step-by-step operating method for the simulated FQT is presented in the following algorithm at the given significance level $\alpha$.

**Algorithm 1.**

**Step 1:** Estimate the unknown parameter $\mu$ based on the observed data sample $x_1, x_2, \ldots, x_n$, by the sample mean $\hat{\mu} = \bar{x}$. Then, calculate $j$ referring to Remark 3.2 as

$$j = \frac{u_1 + l_1 + u_0 + l_0 - 4\bar{x}}{2\bar{x} - u_1 - l_1},$$

which guarantees the process mean is located at the center of fuzzy tolerance.

**Step 2:** Calculate the unbiased estimated capability index $\hat{c}_{\tilde{p},u}$ based on the observed random sample $x_1, x_2, \ldots, x_n$ by Eq. (6).

**Step 3:** For the specified value of $j$, compute the unknown value of root $\sigma_0$ from the equation $C_{\tilde{p}} = c_0$ by

$$\sigma_0 = \frac{(j+1)(u_1 - l_1) + (u_0 - l_0)}{6(j+2)c_0}. \tag{17}$$

**Step 4:** Consider a suitable sequence $\{n_1, n_2, \ldots, n_s\}$ to cover sample sizes in the implementation of simulation.

**Step 5:** Follow the bellow (a)-(e) parts for all $n_i \in \{n_1, n_2, \ldots, n_s\}$,

(a) simulate $k = 10^4$ random samples with size $n_i$ from the normal distribution $N(\hat{\mu}, \sigma_0^2)$;

(b) compute the estimated indices $\hat{c}_{\tilde{p},u}^{[1]}, \ldots, \hat{c}_{\tilde{p},u}^{[k]}$, using Eq. (6) based on the linear fuzzy SLs for each simulated sample in Part (a);

(c) considering Eq. (14), the critical value for $k$ simulated samples in Part (a) is the $(1-\alpha)$th quantile of $\widehat{C}_{\tilde{p},u}$ distribution. That is,

$$c_i = \hat{c}_{\tilde{p},u}^{(k(1-\alpha))}, i = 1, ..., s, \tag{18}$$

where $\hat{c}_{\tilde{p},u}^{(1)} < \hat{c}_{\tilde{p},u}^{(2)} < \cdots < \hat{c}_{\tilde{p},u}^{(k)}$ are the ordered unbiased estimators of $C_{\tilde{p}}$ in Part (b);

(d) simulate the $p$-value by

$$p\text{-}value_i = \overline{I\left(\widehat{C}_{\tilde{p},u} > \hat{c}_{\tilde{p},u} \mid C_{\tilde{p}} = c_0\right)} = \frac{1}{k}\sum_{r=1}^{k} I\left(\hat{c}_{\tilde{p},u}^{[r]} > \hat{c}_{\tilde{p},u} \mid \sigma = \sigma_0\right),\ i = 1, \ldots, s, \tag{19}$$

in which the simulated capability indices are denoted by $\hat{c}_{\tilde{p},u}^{[1]}, \ldots, \hat{c}_{\tilde{p},u}^{[k]}$ and $\sigma_0$ is the computable root of the equation $C_{\tilde{p}} = c_0$ from Step 3. Regarding to the strong law of large numbers, it must be mentioned that the Monte Carlo estimator $p$-value$_i$ almost surely converges to Eq. (15), for each iteration, as $k \to \infty$;

(e) for any arbitrary point $c_{\tilde{p}}^* > c_0$, the simulated probability of type $II$ error at $c_{\tilde{p}}^*$ is

$$\begin{aligned} \beta(c_{\tilde{p}}^*)_i &= Pr\left(\widehat{C}_{\tilde{p},u} \le c_i \mid C_{\tilde{p}} = c_{\tilde{p}}^*\right) = \overline{I\left(\widehat{C}_{\tilde{p},u} \le c_i \mid C_{\tilde{p}} = c_{\tilde{p}}^*\right)} \\ &= \frac{1}{k}\sum_{r=1}^{k} I\left(\widehat{C}_{\tilde{p},u}^{[r]} \le c_i \mid \sigma = \sigma_0^*\right),\ i = 1, \ldots, s, \end{aligned} \tag{20}$$

in which $\sigma_0^*$ is the computable root of the equation $C_{\tilde{p}} = c_{\tilde{p}}^*$ and the simulated indices based on $\sigma_0^*$ are denoted by $\hat{c}_{\tilde{p},u}^{[1]}, \ldots, \hat{c}_{\tilde{p},u}^{[k]}$. Note that the Monte Carlo estimator $\beta(c_{\tilde{p}}^*)_i$ almost surely converges to Eq. (16), as $k \to \infty$.

**Step 6:** The Monte Carlo critical value in testing fuzzy quality is equal to the total average of $s$ simulated critical values in iterations of Part (c) from Step 5, which can be formulated as

$$c_s = \frac{1}{s}\sum_{i=1}^{s} c_i. \tag{21}$$

**Step 7: (*p*-value)** The Monte Carlo $p$-value in testing fuzzy quality is equal to the average of $s$ calculated $p$-values in iterations of Part (d) from Step 5, that is,

$$\hat{p}\text{-}value = \frac{1}{s}\sum_{i=1}^{s} p\text{-}value_i. \tag{22}$$

**Step 8: (Probability of type *II* error)** The Monte Carlo probability of type $II$ error at the point $c^*_{\tilde{p}}$ in testing fuzzy quality, for any arbitrary point $c^*_{\tilde{p}} > c_0$, is simulated by the total average of $s$ calculated probability of type $II$ errors in iterations of Part (e) from Step 5, that is,

$$\hat{\beta}(c^*_{\tilde{p}}) = \frac{1}{s}\sum_{i=1}^{s} \beta(c^*_{\tilde{p}})_i. \tag{23}$$

**Step 9: (Decision rule in Simulated FQT)** Lastly, at the significance level $\alpha$, if $\widehat{c}_{\tilde{p},u} > c_s$, the fuzzy process is capable; otherwise it is incapable.

**Remark 5.1.** *Note that the power function of the simulated FQT based on the Monte Carlo simulation approach for every $C_{\tilde{p}} > 0$ is equal to*

$$\Pi_s(C_{\tilde{p}}) = 1 - \hat{\beta}(C_{\tilde{p}}). \tag{24}$$

**Remark 5.2.** *It is remarkable that, considering the crisp SLs LSL and USL for performing Algorithm 1, the PCI $C_{\tilde{p}}$ coincides with the conventional PCI $C_p$ by referring Remark 2.2. Then the root $\sigma_0$ in Eq. (17) can be computed on the basis of the conventional PCI $C_p$ in Step 3 of Algorithm 1 as follows:*

$$\sigma_0 = \frac{(j+1)(u_1 - l_1) + (u_0 - l_0)}{6(j+2)c_0} = \frac{(j+2)(u_1 - l_1)}{6(j+2)c_0} = \frac{USL - LSL}{6c_0}. \tag{25}$$

*Therefore, all components of the decision-making fuzzy process based on the Monte Carlo simulation approach (such as the critical value, the p-value and the power function of the simulated FQT) coincide with the components of the decision-making process based on the conventional PCI $C_p$ in the crisp quality test.*

# 6 Generalization of simulated FQT based on non-normal process

Generally, the indices $C_p$ and $C_{\tilde{p}}$ are applied based on the normal distribution of process output and using these indices for processes that are not normally distributed often lead to erroneous interpretation of process capability. For non-normal distribution of process output, Clements [5] proposed a new method for calculating estimators of $C_p$ and $C_{pk}$. Furthermore, Pearn and Kotz [30] applied Clements's approach to obtain estimators of the indices $C_{pm}$ and $C_{pmk}$. The proposed index $C_{\tilde{p}}$ can be generalized based on the Clements's approach for non-normal as follows in which $\xi_p$, is the $p$th percentile of the probability distribution (for $p \in [0,1]$):

$$C'_{\tilde{p}} = \frac{(j+1)(u_1 - l_1) + (u_0 - l_0)}{(j+2)(\xi_{0.99865} - \xi_{0.00135})}. \tag{26}$$

**Remark 6.1.** *Note that Algorithm 1 is presented for normal distribution of process output. Under non-normality assumption, by consideration the following points one can generalize Algorithm 1:*

(1) *The generalization idea is used for two-parameter family of distributions - e.g. weibull, gamma and the generalized exponential distribution - by considering two-parameter statistical model $\{f(x;\theta_1,\theta_2) : (\theta_1,\theta_2) \in \Theta \subseteq \mathbf{R}^2\}$.*

(2) *Similar to Remark 3.2, one can obtain a suitable $j \in \mathrm{R}^+$ based on $\hat{\xi}_{0.5} = M$ by the following formula:*

$$j = \frac{u_1 + l_1 + u_0 + l_0 - 4\hat{\xi}_{0.5}}{2\hat{\xi}_{0.5} - u_1 - l_1}, \tag{27}$$

*in which $\hat{\xi}_{0.5}$ is the median of sample data.*

(3) *The unbiased estimator for $C'_{\tilde{p}}$ can be calculated by*

$$\widehat{C}'_{\tilde{p},u} = b_{n-1}\frac{(j+1)(u_1-l_1)+(u_0-l_0)}{(j+2)(\hat{\xi}_{0.99865}-\hat{\xi}_{0.00135})}, \tag{28}$$

*where $b_{n-1}$ is expressed in Theorem 3.1, and $\hat{\xi}_p$ is pth sample percentile.*

(4) *To simulate random data in Part (a) from Step 5, the simulation of parameters $\theta_1$ and $\theta_2$ is necessary in each iteration of the algorithm. The estimations $\theta_1$ and $\theta_2$ can be considered as the roots of the following system of two non-linear equations*

$$\begin{cases} F^{-1}(0.5;\theta_1,\theta_2) = \frac{1}{2j+4}[(j+1)(u_1+l_1)+(u_0+l_0)], \\ F^{-1}(0.99865;\theta_1,\theta_2) - F^{-1}(0.00135;\theta_1,\theta_2) = \frac{(j+1)(u_1-l_1)+(u_0-l_0)}{(j+2)c_0}. \end{cases} \tag{29}$$

*It must be mentioned that the first equation in system (29) is based on equation $\xi_{0.5} = M$. Also, the second equation in system (29) is obtained from $C'_{\tilde{p}} = c_0$ which is equivalent to*

$$\xi_{0.99865} - \xi_{0.00135} = \frac{(j+1)(u_1-l_1)+(u_0-l_0)}{(j+2)c_0}.$$

(5) *Considering Eq. (28), the estimated indices for all $n_i \in \{n_1, n_2, \ldots, n_s\}$ in Part (b) from Step 5 can be calculated by*

$$\widehat{c}'^{[r]}_{\tilde{p},u} = b_{n_i-1}\frac{(j+1)(u_1-l_1)+(u_0-l_0)}{(j+2)(\hat{\xi}_{0.99865}-\hat{\xi}_{0.00135})}; \quad r = 1,\ldots,k.$$

A case study for testing fuzzy quality is investigated in Section 7 to illustrate the proposed approaches based on normal data.

# 7 A case study in the automobile manufacturing industry

Inside diameters of piston rings were measured in an automobile engine production line. Twenty-five samples of size five were collected from the inside diameter measurements (in terms of a millimeter) [16]. To illustrate two proposed approaches, the extended PCI $C_{\tilde{p}}$ is tested to assess the manufacturing process capability for the inside diameter measurement of the produced piston rings at the significance level 0.01. Figure 2 shows details of a piston ring in the automobile engine. It should be noted that the piston rings data are available by qcc package in R software [34].

Figure 2: The graph of a piston ring in the automobile engine.

In this work, by considering the specified linear fuzzy SLs $\widetilde{LSL}$ and $\widetilde{USL}$ with membership functions

$$\widetilde{LSL}(x) = \begin{cases} 0 & \text{if} \quad x \le 73.95, \\ \frac{x-73.95}{0.03} & \text{if} \quad 73.95 < x < 73.98, \\ 1 & \text{if} \quad 73.98 \le x, \end{cases} \tag{30}$$

and

$$\widetilde{USL}(x) = \begin{cases} 1 & \text{if} \quad x \le 74.02, \\ \frac{74.05-x}{0.03} & \text{if} \quad 74.02 < x < 74.05, \\ 0 & \text{if} \quad 74.05 \le x, \end{cases} \tag{31}$$

we intend to test $H_0 : C_{\tilde{p}} \le 1$, against $H_1 : C_{\tilde{p}} > 1$, based on the observed random sample $x_1, x_2, \ldots, x_{125}$. The membership functions of the linear upper fuzzy SL, $\widetilde{USL}$ , and the linear lower fuzzy SL, $\widetilde{LSL}$, are shown in Figure 3. Likewise, the details of the Figure 3 are as follows. The blue signs show the observed inside diameter measurements of 125 piston rings and also the degrees of conformity for each observation are denoted at the left side of Figure 3 with red signs. On this figure, the solid line indicates the membership function of $\widetilde{USL}$, and the dashed line indicates the membership function of $\widetilde{LSL}$.

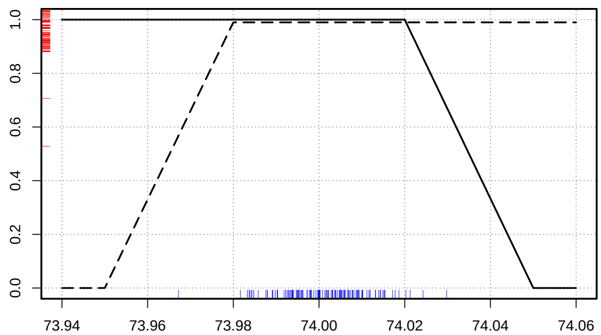


Figure 3: Membership functions of the specified linear fuzzy SLs.

To examine data and specify data distribution for each data set, three usual methods can be considered namely, the Cullen and Frey graph [6], the probability density function plot, and the goodness of fit test. The histogram of the recorded data with the fitted density function is shown in Figure 4. The Cullen and Frey graph is also called the

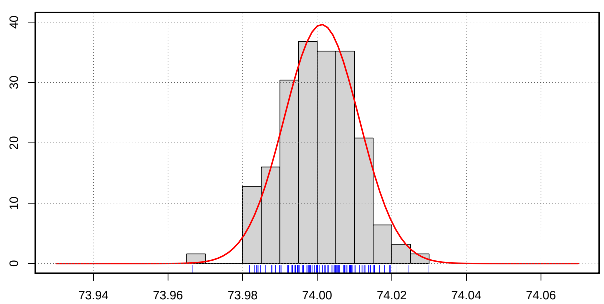


Figure 4: Kernel density estimation and histogram plot for 125 observed inside diameters.

skewness-kurtosis graph. It proposes the selection of the best fit for an unknown distribution according to skewness level and kurtosis. On this graph, values for common distributions are shown to help in the model selection. Figure 5 shows the distribution of observation has a skewness of zero, which also normal, uniform, and logistic distribution have zero skewness, but the kurtosis of observation model is close to the normal distribution. Thus, by examining the skewness and kurtosis graph of the observed inside diameters of piston rings in Figure 5, the normal distribution model is suitable to fit the data. Also, for this normal distribution model, the maximum likelihood estimations of $\mu$ and $\sigma^2$ are 74.001 and 0.0001, respectively. So, the goodness-of-fit statistic of Anderson-Darling is 0.191 with $p-value = 0.8958$ [7]. Since we consider the sample mean $\hat{\mu} = \bar{x} = 74.001$ to estimate the unknown parameter $\mu$, therefore, $j$ can be computed by $j = \frac{u_1+l_1+u_0+l_0-4\bar{x}}{2\bar{x}-u_1-l_1} = 0.1701$, and the estimated capability index $\hat{c}_{\tilde{p},u}$ based on the observed random sample $x_1, x_2, \ldots, x_{125}$, can be calculated by

$$\hat{c}_{\tilde{p},u} = \frac{b_{n-1}[(0.1701+1)(74.02-73.98)+(74.05-73.95)]}{6s_{n-1}(0.1701+2)} = 1.113,$$

where $b_{n-1} = 0.99$ and $s_{n-1} = 0.01007$. Regarding to Theorem 4.1 the critical value for testing fuzzy quality can be estimated using the statistical approach, as mentioned in Section 4, and so we can obtain the appropriate critical value $c = 1.165$ from Table 1 at the significance level $\alpha = 0.01$.

In order to use the simulation approach to determine the critical value for testing fuzzy quality, we consider various sample sizes 65(5)200 for implementing the simulation in Step 4 from Algorithm 1. By considering the membership

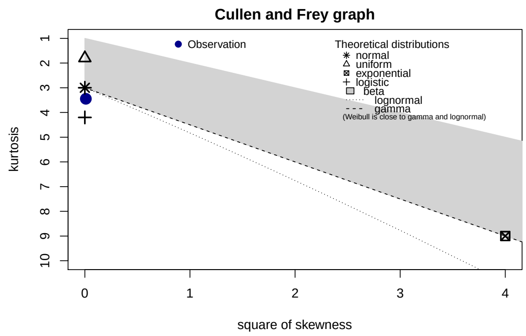


Figure 5: The skewness and kurtosis graph of the observed inside diameters.

functions of the linear fuzzy SLs in Eqs. (30) and (31), the unknown root $\sigma_0$ is computed by Eq. (17). For each $n_i$ (28 cases), 10000 independent random samples are generated from the normal distribution $N(\hat{\mu}, \sigma_0^2)$. After ordering 10000 estimated capability indices, the 9900th capability index is considered as the critical value for each 28 cases. See Figure 6, for the histogram of 28 simulated critical values in iterations of Step 5 from Algorithm 1. The total average of 28 obtained critical values, as given in Eq. (21), is equal to $c_s = 1.167$ at the significance level $\alpha = 0.01$.

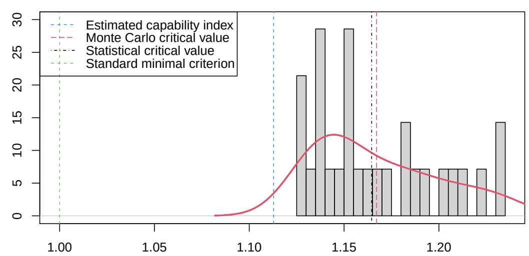


Figure 6: Histogram of simulated critical values in iterations of the Monte Carlo FQT with its kernel density estimation based on the specified linear fuzzy SLs.

As a result, the estimated value $\widehat{c}_{\tilde{p},u} = 1.113$ is not greater than the critical value which is determined using both approaches (with/without simulation). This implies that, the fuzzy process is incapable at the significance level 0.01. Hence, with regard to the specified linear fuzzy SLs, one cannot conclude that this process meets the capability requirement.

Moreover, the Monte Carlo critical value ($c_s$), the statistical critical value ($c$), the observed capability index ($\hat{c}_{\tilde{p},u}$), and the standard minimal criterion ($c_0$) are shown in Figure 6 by different vertical lines.

# 8 Discussion on the proposed approaches: Comparison with crisp quality test

In this section, we are going to compare several quality tests for inside diameters of piston rings on the basis of indices $C_p$ and $C_{\tilde{p}}$ at different significance levels. It must be clarified that the critical value, the probability of type *II* error and the $p$-value for the statistical QT by $C_p$ - on the basis of the unbiased estimator $b_{n-1}\widehat{C_p}$ - are compared with the statistical FQT and the simulated FQT in Table 2. Therefore, six following quality tests are considered with $c_0 = 1.00$ in Table 2 at different significance levels for inside diameters of piston rings:

(1) testing the crisp quality by $C_p$, on the basis of the Pearn et al.s procedure by considering the crisp SLs $LSL = 73.98$ and $USL = 74.02$,

(2) testing the fuzzy quality by $C_{\tilde{p}}$, on the basis of the statistical approach using the considered SLs $I_{\{x|x\geq 73.98\}}$ and $I_{\{x|x\leq 74.02\}}$,

(3) testing the trapezoidal fuzzy quality by $C_{\tilde{p}}$, on the basis of the statistical approach by considering the fuzzy SLs in Eqs. (30) and (31),

(4) testing the crisp quality by $C_p$, on the basis of the Monte Carlo simulation approach by considering the crisp SLs $LSL = 73.98$ and $USL = 74.02$,

(5) testing the fuzzy quality by $C_{\tilde{p}}$, on the basis of Monte Carlo simulation approach using the considered SLs $I_{\{x|x\geq 73.98\}}$ and $I_{\{x|x\leq 74.02\}}$, lastly,

(6) testing the trapezoidal fuzzy quality by $C_{\tilde{p}}$, on the basis of Monte Carlo simulation approach by considering the fuzzy SLs in Eqs. (30) and (31).

For the desired quality condition with $c_0 = 1.00$, $\alpha = 0.01$ and the sample size $n = 125$, the power functions of three following quality tests are drawn in Figure 7, which shows equivalent results for the crisp quality test:
(i) statistical FQT,
(ii) simulated FQT, and
(iii) Pearn et al.'s statistical QT.

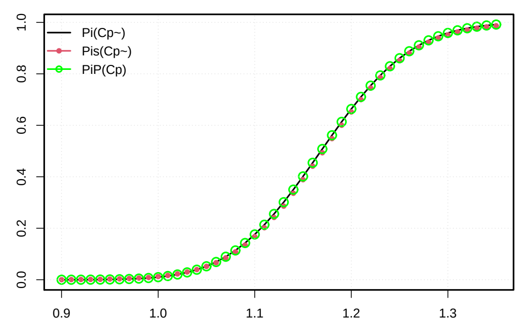


Figure 7: Plots of the power functions of three quality tests involves $\Pi(C_{\tilde{p}}), \Pi_s(C_{\tilde{p}})$ and $\Pi_P(C_p)$ for the computed critical values based on the given values of $\alpha = 0.01$, $c_0 = 1.00$, and the sample size $n = 125$.

It must be noted that, the power function of Pearn et al.'s statistical QT is shown by $\Pi_P(C_p)$ in Figure 7.

Although, Pearn et al.'s procedure [32] provides an accurate theoretical quality test based on the conventional $C_p$, but it is important to pay attention to the following points:

(1) The theoretical Pearn et al.'s procedure [32] on the theory of testing hypotheses attempts to find the critical value based on crisp specifications but our approaches for determining the critical value to test the index $C_{\tilde{p}}$ are proposed based on fuzzy specification limits.

(2) The Pearn et al.'s procedure is applicable only for normal data, but the proposed Monte Carlo simulation technique can be applied for non-normal quality characteristic too. This class of distributions involves a lot of useful distribution such as normal, log-normal, weibull, gamma and the generalized exponential distribution which are used in common industrial processes.

Also, according to the above discussion to confirm the obtained results, three following quality tests were compared in Table 3 with various common values of the standard minimal criterion $c_0$ at the significance level 0.05, for inside diameters of piston rings:

(1) testing the crisp quality by conventional PCI $C_p$, based on the crisp SLs $LSL = 73.967$ and $USL = 74.035$, and making reliable decisions on the basis of the Pearn et al.'s critical value,

(2) testing the fuzzy quality by $C_{\tilde{p}}$, based on the considered SLs $I_{\{x|x\geq 73.967\}}$ and $I_{\{x|x\leq 74.035\}}$, and making reliable decisions on the basis of the statistical critical value $c$, which can be computed by Eq. (10),

(3) testing the fuzzy quality by $C_{\tilde{p}}$, based on the considered SLs $I_{\{x|x\geq 73.967\}}$ and $I_{\{x|x\leq 74.035\}}$, and making reliable decisions on the basis of the Monte Carlo critical value $c_s$, which can be calculated by Eq. (21).

Finally, the results of comparing between both proposed approaches for the considered fuzzy SLs in Eqs. (30) and (31), based on common values of the standard minimal criterion $c_0$ at the significance level 0.05, for inside diameters of piston rings are summarized in Table 4.

Table 2: Results of comparing the crisp quality tests with fuzzy quality tests based on the standard minimal criterion $c_0 = 1.00$, at different significance levels.

| | | $\alpha = 0.010$ | $\alpha = 0.025$ | $\alpha = 0.050$ | $\alpha = 0.100$ |
|---|---|---|---|---|---|
| | Quality test based on $c_0 = 1.00$ and the crisp SLs $LSL = 73.98$ and $USL = 74.02$ | | | | |
| Statistical QT | Critical value for $C_p$ | 1.165 | 1.135 | 1.111 | 1.084 |
| | Decision based on $b_{n-1}\hat{c}_p = 0.658$ | Not reject $H_0$ | Not reject $H_0$ | Not reject $H_0$ | Not reject $H_0$ |
| | Process | Incapable | Incapable | Incapable | Incapable |
| | *p-value* | 1.000 | 1.000 | 1.000 | 1.000 |
| | Probability of type *II* error | $\beta(1.340) = 0.012$ | $\beta(1.310) = 0.031$ | $\beta(1.290) = 0.054$ | $\beta(1.260) = 0.112$ |
| | Fuzzy Quality test based on $c_0 = 1.00$ and the considered SLs $I_{\{x\vert x\geq 73.98\}}$ and $I_{\{x\vert x\leq 74.02\}}$ | | | | |
| Statistical FQT | Statistical critical value for $C_{\tilde{p}}$ | 1.165 | 1.135 | 1.111 | 1.084 |
| | Decision based on $\hat{c}_{\tilde{p},u} = 0.658$ | Not reject $H_0$ | Not reject $H_0$ | Not reject $H_0$ | Not reject $H_0$ |
| | Fuzzy Process | Incapable | Incapable | Incapable | Incapable |
| | *p-value* | 1.000 | 1.000 | 1.000 | 1.000 |
| | Probability of type *II* error | $\beta(1.340) = 0.012$ | $\beta(1.310) = 0.031$ | $\beta(1.290) = 0.054$ | $\beta(1.260) = 0.112$ |
| | Fuzzy Quality test based on $c_0 = 1.00$ and the considered fuzzy SLs in Eqs. (30) and (31) | | | | |
| Statistical FQT | Statistical critical value for $C_{\tilde{p}}$ | 1.165 | 1.135 | 1.111 | 1.084 |
| | Decision based on $\hat{c}_{\tilde{p},u} = 1.113$ | Not reject $H_0$ | Not reject $H_0$ | Reject $H_0$ | Reject $H_0$ |
| | Fuzzy Process | Incapable | Incapable | Capable | Capable |
| | *p-value* | 0.047 | 0.047 | 0.047 | 0.047 |
| | Probability of type *II* error | $\beta(1.340) = 0.012$ | $\beta(1.310) = 0.031$ | $\beta(1.290) = 0.054$ | $\beta(1.260) = 0.112$ |
| | Quality test based on $c_0 = 1.00$ and the crisp SLs $LSL = 73.98$ and $USL = 74.02$ | | | | |
| Simulated QT | Monte Carlo critical value for $C_p$ | 1.167 | 1.137 | 1.112 | 1.085 |
| | Decision based on $b_{n-1}\hat{c}_p = 0.658$ | Not reject $H_0$ | Not reject $H_0$ | Not reject $H_0$ | Not reject $H_0$ |
| | Process | Incapable | Incapable | Incapable | Incapable |
| | $\hat{p}$*-value* | 1.000 | 1.000 | 1.000 | 1.000 |
| | Probability of type *II* error | $\hat{\beta}(1.350) = 0.014$ | $\hat{\beta}(1.290) = 0.027$ | $\hat{\beta}(1.230) = 0.060$ | $\hat{\beta}(1.170) = 0.124$ |
| | Fuzzy Quality test based on $c_0 = 1.00$ and the considered SLs $I_{\{x\vert x\geq 73.98\}}$ and $I_{\{x\vert x\leq 74.02\}}$ | | | | |
| Simulated FQT | Monte Carlo critical value for $C_p$ | 1.167 | 1.137 | 1.112 | 1.085 |
| | Decision based on $\hat{c}_{\tilde{p},u} = 0.658$ | Not reject $H_0$ | Not reject $H_0$ | Not reject $H_0$ | Not reject $H_0$ |
| | Process | Incapable | Incapable | Incapable | Incapable |
| | $\hat{p}$*-value* | 1.000 | 1.000 | 1.000 | 1.000 |
| | Probability of type *II* error | $\hat{\beta}(1.350) = 0.014$ | $\hat{\beta}(1.290) = 0.027$ | $\hat{\beta}(1.230) = 0.060$ | $\hat{\beta}(1.170) = 0.124$ |
| | Fuzzy Quality test based on $c_0 = 1.00$ and the considered fuzzy SLs in Eqs. (30) and (31) | | | | |
| Simulated FQT | Monte Carlo critical value for $C_p$ | 1.167 | 1.137 | 1.112 | 1.085 |
| | Decision based on $\hat{c}_{\tilde{p},u} = 1.113$ | Not reject $H_0$ | Not reject $H_0$ | Reject $H_0$ | Reject $H_0$ |
| | Process | Incapable | Incapable | Capable | Capable |
| | $\hat{p}$*-value* | 0.050 | 0.050 | 0.050 | 0.050 |
| | Probability of type *II* error | $\hat{\beta}(1.350) = 0.014$ | $\hat{\beta}(1.290) = 0.027$ | $\hat{\beta}(1.230) = 0.060$ | $\hat{\beta}(1.170) = 0.124$ |

Table 3: Comparison between the statistical QT (based on $b_{n-1}\hat{c}_p = 1.119$), the statistical FQT and the simulated FQT (based on $\hat{c}_{\tilde{p},u} = 1.119$), for making decision based on common values of $c_0$ at the significance level 0.05.

| | Statistical QT | | | Statistical FQT | | | Simulated FQT | | |
|---|---|---|---|---|---|---|---|---|---|
| $c_0$ | $c$ | Decision rule | Process | $c$ | Decision rule | Fuzzy process | $c_s$ | Decision rule | Fuzzy process |
| 0.55 | 0.611 | Reject $H_0$ | Capable | 0.611 | Reject $H_0$ | Capable | 0.612 | Reject $H_0$ | Capable |
| 0.67 | 0.744 | Reject $H_0$ | Capable | 0.744 | Reject $H_0$ | Capable | 0.745 | Reject $H_0$ | Capable |
| 0.75 | 0.833 | Reject $H_0$ | Capable | 0.833 | Reject $H_0$ | Capable | 0.834 | Reject $H_0$ | Capable |
| 1.00 | 1.111 | Reject $H_0$ | Capable | 1.111 | Reject $H_0$ | Capable | 1.112 | Reject $H_0$ | Capable |
| 1.33 | 1.477 | Not reject $H_0$ | Incapable | 1.477 | Not reject $H_0$ | Incapable | 1.479 | Not reject $H_0$ | Incapable |
| 1.50 | 1.666 | Not reject $H_0$ | Incapable | 1.666 | Not reject $H_0$ | Incapable | 1.669 | Not reject $H_0$ | Incapable |
| 1.67 | 1.855 | Not reject $H_0$ | Incapable | 1.855 | Not reject $H_0$ | Incapable | 1.857 | Not reject $H_0$ | Incapable |
| 2.00 | 2.222 | Not reject $H_0$ | Incapable | 2.222 | Not reject $H_0$ | Incapable | 2.225 | Not reject $H_0$ | Incapable |

Table 4: The comparison of proposed fuzzy quality tests (based on $\hat{c}_{\tilde{p},u} = 1.113$), with different critical values for making decision based on common values of $c_0$ at the significance level 0.05.

| | Statistical FQT | | | Simulated FQT | | |
|---|---|---|---|---|---|---|
| $c_0$ | $c$ | Decision rule | Fuzzy process | $c_s$ | Decision rule | Fuzzy process |
| 0.55 | 0.611 | Reject $H_0$ | Capable | 0.612 | Reject $H_0$ | Capable |
| 0.67 | 0.744 | Reject $H_0$ | Capable | 0.745 | Reject $H_0$ | Capable |
| 0.75 | 0.833 | Reject $H_0$ | Capable | 0.834 | Reject $H_0$ | Capable |
| 1.00 | 1.111 | Reject $H_0$ | Capable | 1.112 | Reject $H_0$ | Capable |
| 1.33 | 1.477 | Not reject $H_0$ | Incapable | 1.479 | Not reject $H_0$ | Incapable |
| 1.50 | 1.666 | Not reject $H_0$ | Incapable | 1.669 | Not reject $H_0$ | Incapable |
| 1.67 | 1.855 | Not reject $H_0$ | Incapable | 1.857 | Not reject $H_0$ | Incapable |
| 2.00 | 2.222 | Not reject $H_0$ | Incapable | 2.225 | Not reject $H_0$ | Incapable |

# 9 Conclusions and future works

**Conclusions**

Fuzzy quality evaluation of the capability of manufacturing processes was investigated in this paper using one of the main extended process capability indices. This investigation attempted to propose the general statistical approach for testing fuzzy quality to analyze the process performance. The main aim of this paper was to estimate the critical value in testing the capability of a normal process based on the linear fuzzy specification limits. Also, the simulated approach has been focused on the FQT to simulate the critical value, the $p$-value and the probability of type $II$ error for testing hypotheses on $C_{\tilde{p}}$. Meanwhile, this paper discussed how well the Monte Carlo simulation approach can be used for non-normal data. Finally, a case study has been provided to illustrate the performance of the proposed approaches in the automobile manufacturing industry.

**Future works**

Regarding to the contribution of this paper for presenting the motivation to determine critical value for making decision in testing hypotheses based on more complex PCIs, discussions on other PCIs using the proposed Monte Carlo simulation approach are some potential topics for future works. When the quality of the products depends on more than one characteristic, multivariate process capability indices are applied. Since this study focuses on the FQT of a single process, the FQT of multivariate PCIs is another potential topic for future research.

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