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Original paper

Monadic algebras of an involutive monoidal t-norm based logic

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Abstract

The main goal of this paper is to study universal and existential quantifiers on involutive monoidal t-norm based algebras, which are algebraic semantics for the logic of involutive left-continuous t-norms and their residua, and the resulting class of algebras will be called monadic IMTL-algebras. First we study some of their related algebraic properties and prove that the variety of monadic IMTL-algebras is the equivalent algebraic semantics of monadic predicate fuzzy logic $\mathbf{mMTL}_\forall$, which is equivalent to the modal fuzzy logic **S5**(**IMTL**), and show the completeness for $\mathbf{IMTL}_\forall$ via functional monadic IMTL-algebras. Moreover we start a systematic study of monadic algebraic structures that related to the monadic IMTL-algebras, some of which constitute the monadic MTL-algebras, monadic WNM-algebras, monadic NM-algebras, monadic BL-algebras, monadic MV-algebras and monadic Boolean algebras. Finally we give some representations of monadic IMTL-algebras. In particular, we character representable and directly indecomposable monadic IMTL-algebras by monadic filters.

Keywords: Mathematical fuzzy logic, monadic predicate fuzzy logic, monadic IMTL-algebra, functional monadic IMTL-algebra, representation.

1 Introduction

Fuzzy logic takes the advantage of the classical logic to handle uncertain information and fuzzy information. In recent decades, various logical algebras have been proposed as the semantical systems of fuzzy logic, for example, MV-algebras, BL-algebras, Gödel algebras and MTL-algebras. Among these logical algebras, MTL-algebras are the most significant structures, since the others are all particular cases of MTL-algebras. In fact, MTL-algebras contain all algebras induced by left continuous t-norm and their residua [13]. As a very important subclass of MTL-algebras, IMTL-algebras are MTL-algebras satisfying the involutivity law $\neg\neg x = x$, are the corresponding algebraic structures of involutive monoidal t-norm based logic, **IMTL** for short, which was introduced in [7] in order to give the propositional logic corresponding to involutive left-continuous t-norms. **IMTL** can also be seen as a weaker logic of the Łukasiewicz infinite valued propositional calculus, failing to satisfy the divisibility condition of the strong conjunction, the same way as **MTL** can be obtained from Hájek's Basic Fuzzy Logic, **BL** for short, a general framework in which tautologies of continuous t-norm and their residua, by dropping this divisibility condition [9].

Monadic Boolean algebra $(L, \exists)$, in the sense of Halmos [11], is a Boolean algebra equipped with a closure operator $\exists$, which abstracts algebraic properties of the standard existential quantifier "for some". The name "monadic" comes from the connection with predicate logics for languages having one placed predicates and a single quantifier. After then, monadic MV-algebras, the algebraic counterpart of monadic Łukasiewicz logic, were introduced and studied in [17]. Monadic BL-algebras, monadic residuated lattices, monadic residuated ℓ-monoids, monadic bounded hoops, monadic NM-algebras and monadic pseudo equality algebras were introduced and investigated in [4, 8, 14, 15, 18, 19]. It is noted that both MV-algebras and basic algebras satisfy De Morgan and double negation laws, in the definition of the

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corresponding monadic algebras, it is possible to use only one of the existential and universal quantifiers as primitive, the other being definable as the dual of the one defined. However, definitions of monadic Heyting algebras, monadic BL-algebras, monadic residuated lattices and monadic bounded hoops require the introduction of both kinds of quantifiers simultaneously, because these quantifiers are not mutually interdefinable.

The main focus of existing research about monadic structures is on algebras of fuzzy logics based on continuous t-norms. However, there is few research about monadic structures on algebras of fuzzy logics based on left continuous t-norms except for NM-algebras [18]. Therefore, it is necessary to study monadic MTL-algebras and their subclasses for treating a variant of the concept of quantifiers on algebras of fuzzy logics based on left continuous t-norms within the framework of universal algebras. In this paper, we aim to generalize monadic MV-algebras and monadic NM-algebras, and provide a unified algebraic foundations for monadic predicates fuzzy logic, which is based on left-continuous t-norms with an involutive negation and their residua. Moreover, in the related papers on monadic MV-algebras, considering the compatibility of ideals and existential quantifier, the existential quantifier $\exists$ together with the axioms were used to define monadic MV-algebras. But for IMTL-algebras, just like NM-algebras, using the universal quantifier as primitive one is more natural and convenient because filters are more compatible with universal quantifiers. Hence we introduce monadic IMTL-algebras as IMTL-algebras equipped with universal quantifiers, and generalize some related results of monadic MV-algebras and monadic NM-algebras. These are the motivation for us to introduce and study monadic IMTL-algebras.

This paper is organized as follows: In Section 2, we review some basic definitions and results about IMTL-algebras and some of their related algebraic structures. In Section 3, we introduce the notion of monadic IMTL-algebras and obtain some of their related algebraic properties. In Section 4, we prove that the variety of monadic IMTL-algebras is the equivalent algebraic semantics of monadic predicate fuzzy logic $\mathbf{mMTL}_\forall$. In Section 5, we study some monadic algebras related to monadic IMTL-algebras and discuss the relationships between them. In Section 6, we give some representations of monadic IMTL-algebras by monadic filters.

2 Preliminaries

In this section, we recall some results about MTL-algebras and their related algebraic structures.

Definition 2.1. [7] *An algebraic structure $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ is called an MTL-algebra if it satisfies the following conditions:*
(1) $(L, \wedge, \vee, 0, 1)$ *is a bounded lattice,*
(2) $(L, \odot, 1)$ *is a commutative monoid,*
(3) $x \odot y \leq z$ *if and only if* $x \leq y \rightarrow z$,
(4) $(x \rightarrow y) \vee (y \rightarrow x) = 1$, *for any* $x, y, z \in L$.

In what follows, by L we denote the universe of an MTL-algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$.
For any $x \in L$ and a natural number n, we define

$$\neg x = x \rightarrow 0,\ \neg\neg x = \neg(\neg x),\ x \oplus y = \neg(\neg x \odot \neg y),\ x^0 = 1 \text{ and } x^n = x^{n-1} \odot x \text{ for } n \geq 1.$$

An MTL-algebra is called a BL-algebra if it satisfies the following equation:

$$\text{(DIV)} \quad x \wedge y = x \odot (x \rightarrow y).$$

An MTL-algebra is called an IMTL-algebra if it satisfies the following equation:

$$\text{(INV)} \quad \neg\neg x = x.$$

An IMTL-algebra is called an NM-algebra if it satisfies the following equation:

$$\text{(WNM)} \quad \neg(x \odot y) \vee ((x \wedge y) \rightarrow (x \odot y)) = 1.$$

An MTL-algebra is called a Boolean algebra if it satisfies the following equation:

$$\text{(EM)} \quad \neg x \vee x = 1.$$

The classes of IMTL-algebras forms a variety, which is a proper subvariety of the variety of MTL-algebras. Hence, the notions of IMTL-equation, chain, homomorphism, quotient, subalgebra, subdirect product and direct product for IMTL-algebras are just the particular cases of the corresponding universal algebraic notions [23].

Proposition 2.2. [7, 21] *In any IMTL-algebra L, the following hold: for all $x, y, z \in L$,*
(1) $x \leq y$ *if and only if* $x \to y = 1$,
(2) $x \leq y \to x$,
(3) $x \leq y$ *implies* $y \to z \leq x \to z$,
(4) $x \leq y$ *implies* $z \to x \leq z \to y$,
(5) $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$.

Let L be an IMTL-algebra. A nonempty subset F of L is called a filter if it satisfies: (1) $1 \in F$; (2) $x \in F$ and $x \to y \in F$ imply $y \in F$. A filter F of L is called a proper filter if $F \neq L$. Unless otherwise explicitly stated, filters are assumed to be proper. A proper filter F of L is called a maximal filter if it is not contained in any proper filter of L. A proper filter F of L is called a prime filter if for any $x, y \in L, x \vee y \in F$, implies $x \in F$ or $y \in F$. A prime filter F is said to be minimal if it is a minimal element in the set of prime filters of L ordered by inclusion [7, 20].

Moreover, we denote the filter generated by a nonempty subset X of L, by $\langle X \rangle$. Clearly, we have

$$\langle X \rangle = \{x \in L | x \geq x_1 \odot x_2 \odot \cdots \odot x_n, \text{ for some } n \in N \text{ and some } x_i \in X\}.$$

In particular, the principal filter generated by an element $x \in L$ is $\langle x \rangle = \{y \in L | y \geq x^n\}$. If F is a filter and $x \in L$, then $\langle F \cup \{x\} \rangle = \{y \in L | y \geq f \odot x^n$, for some $f \in F\}$. We denote by $F[L]$ the set of all filers of L and obtain that $(F[L], \subseteq)$ forms a complete lattice [2, 12].

Let L be an IMTL-algebra and $\theta_1, \theta_2 \in Con[L]$ such that $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$. Then we say that θ_1 and θ_2 are permutable. An IMTL-algebra L is congruence-permutable if every pair of congruences permutes . Also, every congruence θ on L is a factor congruence if there is a congruence $\neg\theta$ such that

$$\theta \cap \neg\theta = [0]_\theta,\ \theta \vee \neg\theta = [1]_\theta,\ \theta \circ \neg\theta = \neg\theta \circ \theta.$$

Moreover, if $(\theta, \neg\theta)$ is a pair of factor congruence on L, then

$$L \cong L/\theta \times L/\neg\theta.$$

As a consequence, every IMTL-algebra L is directly indecomposable if and only if the only factor congruence of algebras L are $[0]_\theta$ and $[1]_\theta$ [6].

Theorem 2.3. [7] *Let L be an IMTL-algebra and P be a proper filter of L. Then the following statements are equivalent:*
(1) *P is a minimal prime,*
(2) $P = \cup\{a^\perp | a \notin P\}$, *where* $a^\perp = \{x \in L | a \vee x = 1\}$.

Definition 2.4. [3] *An IMTL-algebra L is called representable if L is isomorphic to a subdirect product of linearly ordered IMTL-algebras.*

Theorem 2.5. [7] *Every IMTL-algebra is representable.*

Theorem 2.6. [7] *Let L be an IMTL-algebra. Then the following statements are equivalent:*
(1) *L is a representable IMTL-algebra,*
(2) *there exists a set $\mathcal{P}$ of prime filters such that* $\cap\mathcal{P} = \{1\}$.

Definition 2.7. [3] *An IMTL-algebra L is called simple if it has exactly two filters: $\{1\}$ and L.*

3 Monadic IMTL-algebras

In this section, we introduce the notion of monadic IMTL-algebras and study some of their basic algebraic properties.

Definition 3.1. *An algebra $(L, \wedge, \vee, \odot, \to, \forall, 0, 1)$ of type $(2, 2, 2, 2, 1, 0, 0)$ is called a monadic IMTL-algebra if $(L, \wedge, \vee, \odot, \to, 0, 1)$ is an IMTL-algebra and $\forall : L \to L$ is a unary operation satisfying the following identities:*
($\forall$1) $\forall(x) \to x = 1$,
($\forall$2) $\forall(\forall x \to y) = \forall x \to \forall y$,
($\forall$3) $\forall(x \vee \forall y) = \forall x \vee \forall y$,
($\forall$4) $\forall(\neg x \to x) = \neg\forall x \to \forall x$.

A monadic IMTL-algebra $(L, \wedge, \vee, \odot, \to, \forall, 0, 1)$ will be denoted simply by $(L, \forall)$. The class of monadic IMTL-algebras will be denoted by $\mathbb{MIMTL}$. Clearly, in light of the above axiomatization, the $\mathbb{MIMTL}$ forms a variety.

Remark 3.2. *IMTL-algebra is an involutive De Morgan algebra and hence we can define a unary operation* $\exists x = \neg\forall\neg x$ *corresponding to a universal quantifier* $\forall$. *Then in any monadic IMTL-algebra hold the following identities which are dual to* $(\forall 1) - (\forall 4)$:

$(\exists 1)$ $x \to \exists x = 1$,
$(\exists 2)$ $\exists(\neg\exists x \odot y) = \neg\exists x \odot \exists y$,
$(\exists 3)$ $\exists(x \wedge \exists y) = \exists x \wedge \exists y$,
$(\exists 4)$ $\exists(x \odot x) = \exists x \odot \exists x$.

Proof. If $\forall$ is a universal quantifier on L, then we can prove that $\exists x = \neg\forall\neg x$ satisfies the identities $(\exists 1), (\exists 2), (\exists 3)$ and $(\exists 4)$.

$(\exists 1)$ Applying $(\forall 1)$, we have $\exists(x) = \neg\forall\neg(x) \geq \neg\neg x = x$.

$(\exists 2)$ Applying $(\forall 2)$, we have

$$\begin{aligned} \forall(\forall x \to y) &= \forall x \to \forall y \\ &\Leftrightarrow \neg\exists\neg(\neg\exists\neg x \to y) = \neg\exists\neg x \to \neg\exists\neg y \\ &\Leftrightarrow \neg\exists\neg(\exists\neg x \oplus y) = \exists\neg x \oplus \neg\exists\neg y \\ &\Leftrightarrow \exists(\neg\exists\neg x \odot \neg y) = \neg\exists\neg x \odot \exists\neg y \\ &\Leftrightarrow \exists(\neg\exists x \odot y) = \neg\exists x \odot \exists y. \end{aligned}$$

$(\exists 3)$ Applying $(\forall 3)$, we have

$$\begin{aligned} \forall(x \vee \forall y) &= \forall x \vee \forall y \\ &\Leftrightarrow \neg\exists\neg(x \vee \neg\exists\neg y) = \neg\exists\neg x \vee \neg\exists\neg y \\ &\Leftrightarrow \neg\exists(\neg x \wedge \exists\neg y) = \neg(\exists\neg x \wedge \exists\neg y) \\ &\Leftrightarrow \exists(\neg x \wedge \exists\neg y) = \exists\neg x \wedge \exists\neg y \\ &\Leftrightarrow \exists(x \wedge \exists y) = \exists x \wedge \exists y. \end{aligned}$$

$(\exists 4)$ Applying $(\forall 4)$, we have

$$\begin{aligned} \forall(\neg x \to x) &= \neg\forall x \to \forall x \\ &\Leftrightarrow \neg\exists\neg(\neg x \to x) = \exists\neg x \to \neg\exists\neg x \\ &\Leftrightarrow \exists(\neg x \odot \neg x) = \neg(\exists\neg x \to \neg\exists\neg x) \\ &\Leftrightarrow \exists(\neg x \odot \neg x) = (\exists\neg x \odot \exists\neg x) \\ &\Leftrightarrow \exists(x \odot x) = \exists x \odot \exists x. \end{aligned}$$

The proof of the converse assertions is similar via $\forall x = \neg\exists\neg x$. □

Example 3.3. *Let* L *be an IMTL-algebra. Then* (L, id_L) *is a monadic IMTL-algebra, that is, every IMTL-algebra can be seen as a monadic IMTL-algebra.*

Example 3.4. *Let* L *be a* 2*-valued Boolean algebra. Now, we define* $\forall$ *as follows:*

$$\forall x = \begin{cases} 1, & x = 1 \\ 0, & x \neq 1, \end{cases} \qquad \exists x = \begin{cases} 0, & x = 0 \\ 1, & x \neq 0 \end{cases}$$

Then $(L, \forall)$ *is a monadic Boolean algebra and hence a monadic IMTL-algebra.*

Example 3.5. *Let* $L = \{0, a, b, c, d, 1\}$, *where* $0 \leq a, b \leq c, d \leq 1$. *Define operations* $\odot$ *and* $\to$ *on* L *as follows:*

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	a	0	a
b	0	0	b	0	b	b
c	0	a	0	c	a	c
d	0	0	b	a	b	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	1	1
b	c	c	1	c	1	1
c	b	d	b	1	d	1
d	a	c	d	c	1	1
1	0	a	b	c	d	1

Then $(\{0, a, b, c, d, 1\}, \wedge, \vee, \odot, \to, 0, 1)$ *is an IMTL-algebra. Now, we define* $\forall$ *as follows:*

$$\forall x = \begin{cases} 1, & x = b, d, 1 \\ 0, & x = 0, a, c \end{cases}$$

It is easily checked that $(L, \forall)$ *is a monadic IMTL-algebra. Also, by* $\exists x = \neg\forall\neg x$*, we have*

$$\exists x = \begin{cases} 1, & x = b, d, 1 \\ 0, & x = 0, a, c. \end{cases}$$

Proposition 3.6. *Let* $(L, \forall)$ *be a monadic IMTL-algebra. Then the following properties hold: for any* $x, y \in L$*,*

(1) $\forall 0 = 0$,
(2) $\forall 1 = 1$,
(3) $\forall\forall x = \forall x$,
(4) $x \leq y$ *implies* $\forall x \leq \forall y$,
(5) $\forall(x \to y) \leq \forall x \to \forall y$, *especially,* $\forall\neg x \leq \neg\forall x$,
(6) $\forall x \leq y$ *if and only if* $\forall x \leq \forall y$,
(7) $\forall(\forall x \to \forall y) = \forall x \to \forall y$,
(8) $\forall\neg\forall x = \neg\forall x$,
(9) $\forall(x \wedge y) = \forall x \wedge \forall y$,
(10) $\forall(x \odot y) \geq \forall x \odot \forall y$,
(11) $\forall(\forall x \odot \forall y) = \forall x \odot \forall y$,
(12) $\forall L = L_\forall$, *where* $L_\forall = \{x \in L | \forall x = x\}$,
(13) $\forall L$ *is a subalgebra of* L,
(14) $\neg\exists x = \forall\neg x$,
(15) $\neg\forall x = \exists\neg x$,
(16) $\forall(\exists x \vee y) = \exists x \vee \forall y$,
(17) $\forall(x \to \forall y) = \exists x \to \forall y$,
(18) $\forall(\exists x \to y) = \exists x \to \forall y$.

Proof. The proof of (1)-(13) are similar to that of Proposition 3.7 in [19] and (14)-(18) are similar to that of Proposition 3.1 in [14]. □

4 Completeness for monadic fuzzy predicate logic $\mathbf{IMTL}_\forall$ via functional monadic IMTL-algebras

In this section, we prove that the monadic predicate fuzzy logic $\mathbf{mMTL}_\forall$ is equivalent to the S5-like modal fuzzy logic **S5**(**IMTL**), and then show that the variety of monadic IMTL-algebras is the equivalent algebraic semantics of **S5**(**IMTL**), while these results are proved as in [18] under the extra condition of WNM, we can prove them without this extra condition. Moreover we prove that completeness for $\mathbf{IMTL}_\forall$ follows from algebraic representation results, namely, functional representations of finitely subdirectly irreducible algebras.

Hájek proved in [10] that monadic fragment in one variable (without constants) of the first-order logic $\mathbf{BL}_\forall$, that is, monadic basic predicate logic $\mathbf{mBL}_\forall$ is equivalent to S5-like modal fuzzy logic **S5**(**BL**), which is a logic **BL** together with the following axioms (β is a propositional combination of formulas beginning by $\Box$ and $\Diamond$)

($\Box$1) $\Box\alpha \Rightarrow \alpha$,
($\Diamond$1) $\alpha \Rightarrow \Diamond\alpha$,
($\Box$2) $\Box(\beta \Rightarrow \alpha) \Rightarrow (\beta \Rightarrow \Box\alpha)$,
($\Diamond$2) $\Box(\alpha \Rightarrow \beta) \Rightarrow (\Diamond\alpha \Rightarrow \beta)$,
($\Box$3) $\Box(\beta \sqcup \alpha) \Rightarrow (\beta \sqcup \Box\alpha)$,
($\Diamond$3) $\Diamond(\alpha \& \alpha) \equiv \Diamond\alpha \& \Diamond\alpha$,

closed under Modus Ponens **MP**: α, $\alpha \Rightarrow \beta \vdash \beta$ and Necessitation Rule **Nec** : $\alpha / \Box\alpha$.

Motivated by the results due to Hájek in [10], we define $S5$-like modal fuzzy logic **S5**(**IMTL**), which is a logic **IMTL** together with the axioms (β is a propositional combination of formulas beginning by ($\Box$1), ($\Box$2), ($\Box$3) and

$$(\Box 4)\ \Box({\sim}\alpha \Rightarrow \alpha) \equiv {\sim}\Box\alpha \Rightarrow \Box\alpha,$$

closed under Modus Ponens **MP**: $\alpha, \alpha \Rightarrow \beta \vdash \beta$ and Necessitation Rule **Nec** : $\alpha/\Box\alpha$.
It is easily verified that the sets

$$\{(\Box 1), (\Diamond 1), (\Box 2), (\Diamond 2), (\Box 3), (\Diamond 3)\},$$
$$\{(\Box 1), (\Box 2), (\Box 3), (\Box 4)\},$$

for the logic **IMTL** are equivalent, where $\Box = \sim\Diamond\sim$. Along the same line as that in ([10], Theorems 1, 2), we can prove that modal fuzzy logic **S5**(**IMTL**) is equivalent to monadic fragment $\mathbf{mIMTL}_\forall$ of fuzzy predicate logic $\mathbf{IMTL}_\forall$, which contains only unary predicates and just one object variable x (without object constants). The propositional variable p_i is associated with the unary predicate $P_i(x)$ and the modalities $\Box$, correspond to the quantifiers $(\forall x)$. For this reason, and to continue the algebraic tradition of naming monadic the algebraic semantics of monadic fragments of several logics (Boolean, intuitionistic, Łukasiewicz, etc.), we opted to call the algebras corresponding to the logic **S5**(**IMTL**) monadic IMTL-algebras. However, in this section we will work in the language of the modal fuzzy logic **S5**(**IMTL**) instead of in the monadic fuzzy language of $\mathbf{mIMTL}_\forall$ and show that monadic IMTL-algebras are the equivalent algebraic semantics of the logic **S5**(**IMTL**).

In order to obtain the main result of this section, we first study m-relatively complete subalgebras with respect to monadic IMTL-algebras. In particular, we characterize those subalgebras of a given IMTL-algebra that may be the range of the universal quantifier $\forall$. As a consequence of this characterization, we build the most important examples of monadic IMTL-algebras, which will be called functional monadic IMTL-algebras.

Given an IMTL-algebra L, we say that a subalgebra $S \leq L$ is m-relatively complete if the following conditions hold:

(S1) For every $a \in L$, the subset $\{s \in S \mid s \leq a\}$ has a greatest element and $\{s \in S \mid s \geq a\}$ has a least element.

(S2) For every $a \in L$ and $s_1, s_2 \in S$ such that $s_1 \leq s_2 \vee a$, there exists $s_3 \in S$ such that $s_1 \leq s_2 \vee s_3$ and $s_3 \leq a$.

(S3) For every $a \in L$ and $c_1 \in S$ such that $a \odot a \leq c_1$, there exists $c_2 \in S$ such that $a \leq c_2$ and $c_2 \odot c_2 \leq c_1$.

Theorem 4.1. *Given an IMTL-algebra L and an m-relatively complete subalgebra S. If we define on L the operation*

$$\forall a = \max\{s \in S \mid s \leq a\},$$

then $(L, \forall)$ is a monadic IMTL-algebra such that $\forall L = S$. Conversely, if $(L, \forall)$ is a monadic IMTL-algebra, then $\forall L$ is an m-relatively complete subalgebra of L.

Proof. ($\forall$1) From the definition of $\forall x$, it is clear that $\forall x \leq x$. Thus $\forall x \to x = 1$.

($\forall$2) From $\forall y \leq y$, we get $\forall x \to \forall y \leq \forall x \to y$, In addition, if $s \in S$ and $s \leq \forall x \to y$, then $s \odot \forall x \leq y$. Thus $s \odot \forall x \leq \forall y$ and $s \leq \forall x \to \forall y$. Hence we have shown that $\forall(\forall x \to y) = \forall x \to \forall y$.

($\forall$3) Since $\forall y \leq y$, $\forall x \vee \forall y \leq \forall x \vee y$. If $s \in S$ and $s \leq \forall x \vee y$, by condition (S2), there exists $s' \in S$ such that $s \leq \forall x \vee s'$ and $s' \leq y$. Then $s' \leq \forall y$ and $s \leq \forall x \vee \forall y$. Thus, we have shown that $\forall(\forall x \vee y) = \forall x \vee \forall y$.

($\forall$4) Since $\forall x \leq x$, and $\neg\forall x \geq \neg x$, it follows that $\neg x \to x \geq \neg\forall x \to \forall x$. The $\neg x \to x \in \{s \in S : s \leq \neg\forall x \to \forall x\}$. Let us see that $\neg x \to x = \max\{s \in S : s \leq \neg\forall x \to \forall x\}$. Indeed, if $s \in S$ and $s \leq \neg x \to x$, then $\neg x \leq s \to x$. Then, by definition of $\forall x$, $\neg\forall x \leq s \to \forall x$. Thus $s \leq \neg\forall x \to \forall x$. This shows that $\forall(\neg x \to y) = \neg\forall x \to \forall x$.

Conversely, let $(L, \forall)$ be a monadic IMTL-algebra. Then we have that $\forall L$ is a subalgebra of L. Now, let us show that conditions (S1)-(S3) hold.

(S1) Using the basic properties of monadic IMTL-algebra, we have that if $s \leq x$ for some $s \in \forall L$, then $s = \forall s \leq \forall x \leq x$. Thus $\forall x = \max\{s \in \forall L | s \leq x\}$.

(S2) Assume $s_1 \leq s_2 \vee x$ for some $s_1, s_2 \in \forall L$, $x \in L$. Then, using the basic properties of monadic IMTL-algebras, we have $s_1 \leq s_2 \vee \forall x$ and $\forall x \leq x$.

(S3) If $a \odot a \leq c$ for some $c \in \forall L$ and $a \in L$, then $\exists a \odot \exists a \leq c$ and $a \leq \exists a$, $\exists a \in \forall L$.

This shows that $\forall L$ is an m-relatively complete subalgebra of L. $\Box$

The most important example of monadic IMTL-algebras built according to the previous theorem.

Example 4.2. *Considering a linearly ordered IMTL-algebra L and a nonempty set X. We restrict our attention to those elements $f \in L^X$ such that $\inf\{f(x) \mid x \in X\}$ exist in L. We denote by M the subset of L^X of these safe elements. For every $f \in M$, we define $\forall_f(x) = \inf\{f(y) \mid y \in X\}$, and $\forall_f$ is a constant map.*

Remark 4.3. *Let G be a subalgebra of L^X contained in M such that for every $f \in G$, $\forall_f \in G$. We will prove that G has a natural structure of monadic IMTL-algebra. Let S be the subset of constant maps of L^X. Then we prove that $G \cap S$ is an m-relatively complete subalgebra of G. Indeed, since G and S are subalgebra of L^X, it is clear that $G \cap S$ is a subalgebra of G.*

(S1) *If $f \in G$, then $\forall_f \in G$, so $\max\{s \in G \cap S | s \leq f\} = \forall_f \in G$.*

(S2) Since $G \cap S$ is totally ordered, we may check condition (s_2). If $1 = s \vee f$ for some $f \in G$ and $s \in G \cap S$. Putting $s(x) = s_0 \in L, x \in X$. Then $s_0 \vee f(x) = 1$ for every $x \in X$. As L is linearly ordered, either $s_0 = 1$ or $f(x) = 1$ for every $x \in X$.

(S3) If $f \odot f \leq c$ for some $f \in G$ and $c \in G \cap S$, then $f(x) \odot f(x) \leq c_0$ for any $x \in X$. Moreover, $f(x) \odot y \leq c_0$ for any $x, y \in X$, since

$$f(x) \odot f(y) \leq (f(x) \vee f(y))^2 = f(x)^2 \vee f(y)^2 \leq c_0.$$

Hence $f(x) \leq f(y) \to c_0$ for a fixed $y \in X$ and any $x \in X$. Thus $\exists_f(x) \leq f(y) \to c_0$, where $\exists_f = \sup\{f(y)|y \in X\}$. Now, $f(y) \leq \exists_f(x) \to c_0$ for any $y \in Y$. Then $\exists_f(x) \leq \exists_f(x) \to c_0$ and $\exists_f(x) \odot \exists_f(x) \leq c_0$, which concludes the proof that $G \cap S$ is an m-relatively complete subalgebra of G.

Then it follows from Theorem 4.1 that $(G, \forall_f)$ is a monadic IMTL-algebra. Monadic IMTL-algebras of this form are called functional monadic IMTL-algebras. *Also, if L is $|X|$-complete, then $S = L^X$ and $(L^X, \forall_f)$ is a functional monadic IMTL-algebra.*

Then we prove that the main result of this section is based on functional monadic IMTL-algebras. We first recall that the general semantics of **S5**(**IMTL**) is given by Kripke models.

Definition 4.4. *A* Kripke model *for* **S5**(**IMTL**) *is a triple $K = (X, e, L)$ where X is a nonempty set of worlds, L is a linearly ordered IMTL-algebra and $e : Prop \times X \to L$ is an evaluation map, Prop being the set of propositional variables. The evaluation map extends to any formula:*
(e_1) $e(0, x) = 0$, $e(1, x) = 1$,
(e_2) $e(\varphi \sqcap \psi, x) = e(\varphi, x) \wedge e(\psi, x)$,
(e_3) $e(\varphi \sqcup \psi, x) = e(\varphi, x) \vee e(\psi, x)$,
(e_4) $e(\varphi \& \psi, x) = e(\varphi, x) \odot e(\psi, x)$,
(e_5) $e(\varphi \Rightarrow \psi, x) = e(\varphi, x) \to e(\psi, x)$,
(e_6) $e(\Box\varphi, x) = \inf\{e(\varphi, y) : y \in X\}$.
We can also define the truth degree $\|\varphi\|_K, x$ of a formula φ in K at the world x. This is done recursively on the structure of φ. For propositional variables $p \in Prop$ we have that $\|p\|_K, x = e(w, p)$. The definition of the truth value is then extended for the logical connectives of the involutive monoidal t-norm based logic in the usual way, and for the modal connectives by

$$\|\Box\varphi\|_{K,x} = \inf\nolimits_{x' \in X} \|\varphi\|_{K,x'}.$$

Analogously by Hájek was only interested in the tautologies of this logic, albeit implicitly, a global consequence relation $\models_{\mathbf{S5(IMTL)}}$. It is noted that $e(\Box\varphi, x)$ may be undefined. We say that the Kripke model K is safe *if $e(\Box\varphi, x)$ is defined for every formula φ. We write $K \models \varphi$ if $e(\varphi, x) = 1$ for every $x \in X$. K is a model of a set of formulas Γ if $K \models \varphi$ for every $\varphi \in \Gamma$.*

Already noted that **S5**(**IMTL**) *is actually equivalent to* $\mathbf{mIMTL}_\forall$ *because there is a natural correspondence between formulas of both logics, between corresponding models and between the corresponding truth degrees. Thus, since the latter is finitary (being a fragment of a finitary logic), the consequence relation* **S5**(**IMTL**) *is also finitary.*

Theorem 4.5. *The modal logic* **S5**(**IMTL**) *is strongly complete with respect to its general semantics, that is, the following statements are equivalent for every set of formulas $\Gamma \cup \{\varphi\}$:*
(1) $\Gamma \vdash \varphi$,
(2) $K \models \varphi$ *for every safe model K of Γ.*

Proof. The proof is similar to that of Theorem 2 in [10]. □

Consider a safe Kriple model $K = (X, e, L)$, we can turn the map $e : Prop \times X \to L$ into a map $\overline{e} : Prop \to L^X$ given by the relation $\overline{e}(p)(x) = e(p, x)$. Since K is safe, $\overline{e}$ extends to formulas in the following way:
(e_1) $\overline{e}(x) = 0$, $\overline{e}(1) = 1$,
(e_2) $\overline{e}(\varphi \sqcap \psi) = \overline{e}(\varphi) \wedge \overline{e}(\psi)$,
(e_3) $\overline{e}(\varphi \sqcup \psi) = \overline{e}(\varphi) \vee \overline{e}(\psi)$,
(e_4) $\overline{e}(\varphi \& \psi) = \overline{e}(\varphi) \odot \overline{e}(\psi)$,
(e_5) $\overline{e}(\varphi \Rightarrow \psi) = \overline{e}(\varphi) \to \overline{e}(\psi)$,
(e_6) $\overline{e}(\Box\varphi) = \forall_f \overline{e}(\varphi)$.
Thus it is clear that $\{\overline{e}(\varphi) : \varphi \text{ formula}\} \subseteq L^X$ is the universe of a functional monadic IMTL-algebra in Example 4.2.

Then we have the following result similar to Theorem 4.1.

Theorem 4.6. *The following statements are equivalent for every set of formulas* $\Gamma \cup \{\varphi\}$*:*

(1) $\Gamma \vdash \varphi$,

(2) $\overline{e}(\varphi) = 1$ *for every* $\overline{e} : Prop \to L$*, where* $(L, \forall)$ *is any functional monadic IMTL-algebra and* $\overline{e}(\gamma) = 1$ *for every* $\gamma \in \Gamma$.

Proof. The proof is similar to that of Theorem 4.3 in [4]. □

Theorem 4.7. *The variety* $\mathbb{MIMTL}$ *of monadic IMTL-algebras is the equivalent algebraic semantics for the logic* **S5(IMTL)**.

Proof. It is enough to show that the next two conditions for every set for formulas $\Gamma \cup \{\alpha, \beta\}$:

(1) $\Gamma \vdash \alpha$ if and only if $\{\gamma \equiv 1 | \gamma \in \Gamma\} \vdash_{\mathbb{MIMTL}} \alpha \equiv 1$,

(2) $\alpha \equiv \beta \models_{\mathbb{MIMTL}} (\alpha \Rightarrow \beta) \sqcap (\beta \Rightarrow \alpha) \equiv 1$.

Condition (2) is trivially verified. We only show the condition (1). For the forward implication, if $\Gamma \vdash \alpha$, there exists a proof of α from Γ and the axioms of **S5(IMTL)** by successive application of the reference rules **MP** and **Nec**. Thus, it is enough to show that the equation $\alpha \equiv 1$ is valid in $\mathbb{MIMTL}$ for every axiom α of **S5(IMTL)** and that the inference rules preserve validity. The former statement follows from the basic properties of monadic IMTL-algebras. The preservation of **MP** and **Nec** are also easily verified. □

Thus, from the general theory of Algebraic Logic, we get the next result.

Corollary 4.8. *There is a one-one correspondence between axiomatic extensions of* **S5(IMTL)** *and subvarieties of* $\mathbb{MIMTL}$.

We also give a Hilbert-style syntactic calculus in the language of **S5(IMTL)** whose consequence we denote by $\vdash_{\mathbf{S5(IMTL)}}$. The axioms of this calculus are the instantiations of the axioms schemata **IMTL** for formulas in the language of **S5(IMTL)**, plus the following axioms:

(M1) $\Box\alpha \Rightarrow \alpha$,

(M2) $\Box(\Box\alpha \Rightarrow \beta) \equiv \Box\alpha \Rightarrow \Box\beta$,

(M3) $\Box(\alpha \sqcup \Box\beta) \equiv \Box\alpha \sqcup \Box\beta$,

(M4)' $\Box({\sim}\alpha \Rightarrow \alpha) \equiv {\sim}\Box\alpha \Rightarrow \Box\alpha$,

and closed under **MP**: α, $\alpha \Rightarrow \beta \vdash \beta$ and **Nec** : $\alpha/\Box\alpha$.

It is proved in [5] that $\vdash_{\mathbf{S5(C)}} = \models_{\mathbf{S5(C)}}$ for every axiomatic extension **C** of **BL**, thus proving the completeness of the calculus for **S5(C)**. Notice that the main focus of existing research about this important problem is only on continuous t-norm based fuzzy logic case. However, there is few research report about this result on the left continuous t-norm based fuzzy logic. Then we aim to prove this using algebraic models, and this paper proves some partial results towards this goal.

Theorem 4.9. *The following statements are equivalent for every set of formulas* $\Gamma \cup \{\varphi\}$*:*

(1) *for any formula* φ *and any set of formulas* Γ*, we have*

$$\Gamma \vdash_{\mathbf{S5(IMTL)}} \varphi \textit{ if and only if } \Gamma \models_{\mathbf{S5(IMTL)}} \varphi$$

(2) *the variety* $\mathbb{IMTL}$ *is generated by the functional monadic IMTL-algebras*

Proof. We prove first that (2) implies (1). The soundness and implication is straightforward and does not depend on the hypothesis. Assume $\Gamma \models_{\mathbf{S5(IMTL)}} \varphi$ and since $\varphi \models_{\mathbf{S5(IMTL)}}$ is finitary, we may also assume that Γ is finite. By way of contradiction, suppose $\Gamma \nvdash_{\mathbf{S5(IMTL)}} \varphi$. Then $\Gamma \not\models$ **S5(IMTL)**φ (this is the consequence relation associated to the variety of $\mathbb{MIMTL}$), which is equivalent to saying that $\mathbb{MIMTL}$ does not satisfy the quasi-equation

$$\gamma_1 \approx \& \ldots \& \gamma_n \approx 1 \Rightarrow \varphi \approx 1,$$

where $\{\gamma_1 \ldots \gamma_n\} = \Gamma$. By hypothesis, there is a functional algebra $(M, \forall_f) \in \mathbb{MIMTL}$ with $M \leq L^X$ and L is an IMTL-algebra, and a valuation h into $(M, \forall_f)$ such that $h(\Gamma) \subseteq \{1\}$ but $h(\varphi) \neq 1$. Consider the structure $K = (X, e, L)$ with $e(x, p) = h(p)(x)$ for $x \in X$ and $p \in Prop$. For every $\gamma \in \Gamma$ and $x \in X$ we have that $\|\gamma\|_K, x = h(\gamma)(x) = 1$, but $\|\gamma\|_K, x = h(\gamma)(x) \neq 1$ for some $x \in X$, which is a contradiction.

Conversely, we now prove that (1) implies (2). Assume $\mathbb{MIMTL}$ is not generated by its functional members. Thus, there is an identity

$$\gamma_1 \approx 1\& \ldots \& \gamma_n \approx 1 \Rightarrow \varphi \approx 1,$$

that is on every functional monadic IMTL-algebra, but is it not true in $\mathbb{IMTL}$. From the properties of algebraizable logics, we get that $\{\gamma_1 \dots \gamma_n\} \nvdash_{\mathbf{S5(IMTL)}} \varphi$. However, we claim that $\{\gamma_1 \dots \gamma_n\} \models \mathbf{S5(IMTL)}\varphi$. Indeed, assume $K = (X, e, L)$ is a model of $\{\gamma_1 \dots \gamma_n\}$, where L is a linearly ordered IMTL-algebra. Then $\|\gamma_i\|_K, x = h(\gamma_i)(x) = 1$ for any $x \in X$ and $1 \le i \le n$. Let Fm be the set of propositional formulas in the language of $\mathbf{S5(IMTL)}$. For each $\psi \in Fm$, we define $f_\psi : X \to L$ such that $f_\psi(x) = \|\psi\|_K, x$ for every $x \in X$. Consider $B = \{f_\psi | \psi \in Fm\}$. Then $B \subseteq L^X$ and, in addition, B is a subuniverse of the IMTL-algebra L^X. Moreover, it is straightforward to check that $\forall_f \psi = f\Box\psi$. Then $(L, \forall_f)$ is a functional monadic IMTL-algebra. Consider now the interpretation $e' : Fm \to L$ given by $e'(\psi) = f_\psi$ for any $\psi \in Fm$. Then $e'(\gamma_i) = 1$ for $1 \le i \le n$. By hypothesis, $e'(\varphi) = 1$, that is, $f_\varphi = 1$, so $\|\varphi\|_K, x = 1$ for any $x \in X$. This completes the proof that $\{\gamma_1 \dots \gamma_n\} \vdash_{\mathbf{S5(IMTL)}} \varphi$. □

5 Some monadic algebraic structures related to monadic IMTL-algebras

In this section, we review some monadic algebraic structures related to monadic IMTL-algebras, for instance monadic MTL-algebras, monadic WNM-algebras, monadic NM-algebras, monadic BL-algebras, monadic MV-algebras, monadic Boolean algebras and discuss the relationships between them.

The authors [18] introduced the monadic NM-algebra as an NM-algebra and in addition $\forall$ satisfies the conditions ($\forall$1),($\forall$2),($\forall$3) and

$$(\forall 5) \quad \forall(\neg x \to \forall y) = \neg\forall x \to \forall y.$$

However, there's a small flaw in our definition, that is, in [1], it is proof that the next formula

$$(\exists x)(\alpha(x)\&\alpha(x))) \equiv (\exists x)\alpha(x)\&(\exists x)\alpha(x),$$

is a theorem in $\mathbf{NM}_\forall$, and hence it also belongs to $\mathbf{mNM}_\forall$. However, the corresponding algebraic equation

$$(\exists 4) \quad \exists(x \odot x) = \exists x \odot \exists x,$$

is not true in any monadic NM-algebra, please see the following example.

Example 5.1. [18] *Let $L = [0, 1]$ be a unit interval. Defining $\wedge, \vee, \odot, \to$ as follows:*

$$x \wedge y = \min\{x, y\}, \quad x \vee y = \max\{x, y\}, \quad \neg x = 1 - x,$$

$$x \odot y = \begin{cases} 0, & x \le \neg y, \\ x \wedge y, & x > \neg y, \end{cases} \qquad x \to y = \begin{cases} 1, & x \le y, \\ \neg x \vee y, & x > y. \end{cases}$$

Then $(L, \wedge, \vee, \to, \odot, 0, 1)$ is an NM-algebra, which is called the standard NM-algebra. Now we define two unary operations $\forall : L \to L$ and $\exists : L \to L$ as in Example 3.4. It is easily checked that $\forall$ satisfies the axioms ($\forall$1) − ($\forall$4). But the corresponding existential quantifier $\exists$ does not satisfies the above formula, since

$$\exists(\tfrac{1}{2} \odot \tfrac{1}{2}) = \exists 0 = 0 \neq 1 = \exists\tfrac{1}{2} \odot \exists\tfrac{1}{2}.$$

Then we introduced MNM-algebras, which are the revised version of monadic NM-algebras.

Definition 5.2. *An algebra $(L, \wedge, \vee, \odot, \to, \forall, 0, 1)$ is said to be an MNM-algebra if $(L, \wedge, \vee, \odot, \to, 0, 1)$ is an NM-algebra, in addition $\forall$ satisfies the identities: ($\forall$1) − ($\forall$4). The variety of MNM-algebras is denoted by $\mathbb{MNM}$.*

Remark 5.3. (1) *Since* ($\forall$4) *and* ($\exists$4) *are dual via $\exists = \neg\forall\neg$, Definition* 5.2 *can solve the above drawback.*

(2) *In order to separate two notions of monadic NM-algebra, we called the revised monadic NM-algebra in the present paper by MMN-algebra.*

Diego Castaño et.al introduced the monadic BL-algebras and prove that it is the equivalent algebraic semantics for the logic $\mathbf{S5(BL)}$ [4]. Recalled that an algebra $(L, \wedge, \vee, \odot, \to, \forall, \exists, 0, 1)$ is called a monadic BL-algebra if $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra and in addition $\forall$ and $\exists$ satisfies the following identities:($\forall$1), ($\forall$2), ($\exists$4) and

$$(\forall 6) \quad \forall(x \to \forall y) = \exists x \to \forall y,$$
$$(\forall 7) \quad \forall(\exists x \vee y) = \exists x \vee \forall y.$$

The variety of monadic BL-algebras is denoted by $\mathbb{MBL}$.

Theorem 5.4. *Let L be an IMTL-algebra, $\forall : L \to L$ and $\exists : L \to L$ be two mappings. Then*

$$H=\{(\forall 1),(\forall 2),(\forall 3),(\forall 4)\}, \text{ and } W=\{(\forall 1),(\forall 2),(\forall 7),(\forall 8),(\exists 4)\},$$

are equivalent.

Proof. H⇒W:

($\exists$4) The proof is similar to Remark 3.2.

($\forall$7) Since $x \leq \exists x$ and $\exists x \to \forall y \leq x \to \forall y$, by Proposition 3.6(3) and (19), we have

$$\exists x \to \forall y = \exists x \to \forall\forall y = \forall(\exists x \to \forall y) \leq \forall(x \to \forall y).$$

On the other hand, we have

$$\forall(x \to \forall y) = \forall(\neg\forall y \to \neg x) \leq \forall\neg\forall y \to \forall\neg x = \neg\forall y \to \neg\exists x = \exists x \to \forall y.$$

($\forall$8) The proof is similar to Proposition 3.6(17).

W⇒H:

($\forall$3) By Proposition 3.6(12) and ($\forall$8), we have

$$\forall(\forall x \vee y) = \forall(\exists\forall x \vee y) = \exists\forall x \vee \forall y = \forall x \vee \forall y.$$

($\forall$4) The proof is similar to Remark 3.2. □

Remark 5.5. (1) *Theorem* 5.4 *gives us an ideal of introducing the variety* $\mathbb{MMTL}$ *of monadic MTL-algebras, in which every monadic MTL-algebra is an MTL-algebra satisfying the axioms* ($\forall$1), ($\forall$2), ($\forall$6), ($\forall$7), ($\exists$4).

(2) *MNM-algebra and monadic IMTL-algebra have the same axioms, and hence the identity of WNM has no effect on this axioms, which implies that monadic WNM-algebra and monadic MTL-algebra should have the same axioms.*

Moreover monadic MV-algebras were introduced and studied by Rutledge [17] as an algebra $(L, \oplus, \odot, *, \forall, 0, 1)$ satisfies the following identities:($\forall$1),

$$(\forall 8)\ \ \forall(x \wedge y) = \forall(x) \wedge \forall(y),$$
$$(\forall 9)\ \ \forall\neg\forall x = \neg\forall x,$$
$$(\forall 10)\ \ \forall(\forall x \odot \forall y) = \forall x \odot \forall y,$$
$$(\forall 11)\ \ \forall(x \odot x) = \forall x \odot \forall x,$$
$$(\forall 12)\ \ \forall(x \oplus x) = \forall x \oplus \forall x.$$

The variety of monadic MV-algebras is denoted by $\mathbb{MMV}$.

Theorem 5.6. *Let L be an MV-algebra and $\forall$ a mapping on L. Then $(L, \forall)$ is a monadic MV-algebra if and only if it satisfies* ($\forall$1),($\forall$2),($\forall$4).

Proof. In any MV-algebra L, $x \vee y = (x \to y) \to y$ for any $x, y \in L$. By ($\forall$2), we have

$$\begin{aligned}\forall(x \vee \forall y) &= \forall((\forall y \to x) \to x)\\ &\leq \forall(\forall y \to x) \to \forall x\\ &= (\forall y \to \forall x) \to \forall x\\ &= \forall x \vee \forall y.\end{aligned}$$

Conversely, by ($\forall$1) and ($\forall$2), we have

$$\forall x \vee \forall y \leq \forall(x \vee \forall y),$$

which implies ($\forall$3) hold. □

The next result shows that the variety $\mathbb{MMV}$ is term-equivalent to the subvariety of $\mathbb{MIMTL}$ determined by the divisibility condition.

Theorem 5.7. *The subvariety of* $\mathbb{MIMTL}$ *determined by the divisibility condition is term-equivalent to the variety* $\mathbb{MMV}$.

Proof. It follows from Theorems 4.4 and Theorem 5.5 in [4]. □

Monadic (Boolean) algebras were introduced and studied by Halmos [11], as an algebraic model for classical predicate logics for languages having one placed predicates and a single quantifier. An algebra $(B, \wedge, \vee, \neg, \exists, 0, 1)$ is said to be a *monadic Boolean algebra* if $(B, \wedge, \vee, \neg, 0, 1)$ is a Boolean algebra and in addition $\exists$ satisfies the following identities: $(\exists 1), (\exists 3)$ and

$$(\exists 5) \quad \exists 0 = 0.$$

The variety of monadic Boolean algebras is denoted by $\mathbb{MBA}$.

Theorem 5.8. *The subvariety of* $\mathbb{MIMTL}$ *determined by the idempotency condition is term-equivalent to the variety* $\mathbb{MBA}$*.*

Proof. Let $(L, \wedge, \vee, \rightarrow, \odot, \forall, 0, 1)$ be a monadic IMTL-algebra that satisfies the idempotency condition. Now, we define $(L, \wedge, \vee, \neg, \exists, 0, 1)$, where $\neg x = x \rightarrow 0$, $\forall x = \neg \exists \neg x$ and prove that it is a monadic Boolean algebra. Indeed, $(\exists 1), (\exists 3)$ are easily followed from Remark 3.2, and further by Proposition 3.6(2), we have

$$\exists 0 = \neg \forall \neg 0 = \neg \forall 1 = \neg 1 = 0.$$

Thus $(L, \wedge, \vee, \neg, \exists, 0, 1)$ is a monadic Boolean algebra. □

Remark 5.9. *The following two figures show that the relationship between monadic algebras of left continuous t-norm based fuzzy logics completely maintains the relationship between corresponding logical algebras.*

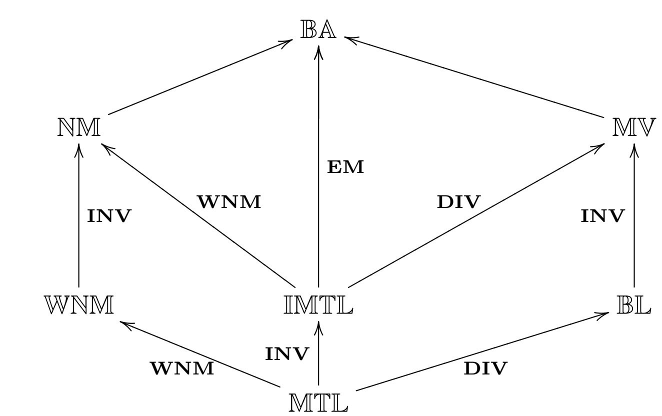

Figure 1. *Relationships between algebras of left continuous t-norm based fuzzy logics.*

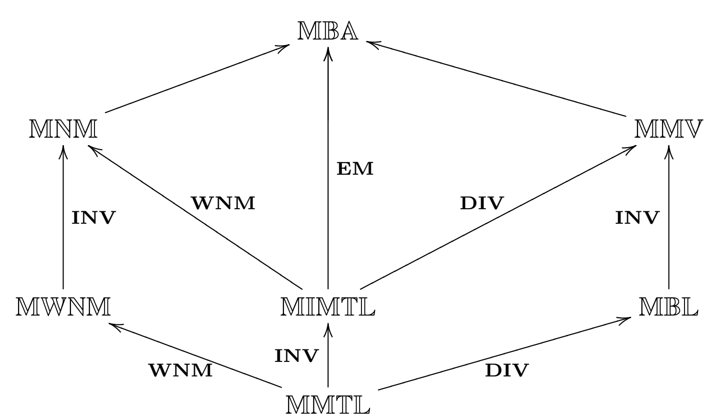

Figure 2. *Relationships between monadic algebras of left continuous t-norm based fuzzy logics.*

6 Some representations of monadic IMTL-algebras

In this section, we give some representations of monadic IMTL-algebras, while these results are proved as in [18] under the extra condition of WNM, we can also prove them without this extra condition. In particular, we characterize representable and directly indecomposable monadic IMTL-algebras, which are important members of $\mathbb{IMTL}$.

Given a monadic IMTL-algebra $(L, \forall)$, a filter F is called a monadic filter of $(L, \forall)$ if it is closed under $\forall$. For any nonempty subset X of L, we denote by $\langle X\rangle_\forall$ the monadic filter of $(L, \forall)$ generated by X, that is, $\langle X\rangle_\forall$ is the smallest monadic filter of $(L, \forall)$ containing X.

$$\langle X\rangle_\forall = \{x \in L | x \geq \forall x_1 \odot \forall x_2 \odot \cdots \odot \forall x_n, x_i \in X, n \geq 1\}.$$

In particular, $\langle a\rangle_\forall = \{x \in L | x \geq (\forall a)^n, n \geq 1\}$. Also, if F is a monadic filter of $(L, \forall)$ and $x \notin F$, then we put

$$\langle F, x\rangle_\forall := \langle F \cup \{x\}\rangle_\forall = \{y \in L | y \geq f \odot (\forall x)^n, f \in F\} = F \vee [\forall x).$$

We denote the set of all monadic filters of $(L, \forall)$ by $MF([L, \forall])$ and characterize the congruences of each monadic IMTL-algebras $(L, \forall)$ by means of monadic filters in the standard way.

Theorem 6.1. *Let $(L, \forall)$ be a monadic IMTL-algebra. Then the lattice of monadic congruences is isomorphic to the set of monadic filters. Indeed, let $f : Con([L, \forall]) \to MF([L, \forall])$ be defined by: if $\equiv$ is a monadic congruence, then $f(\equiv)$ is the monadic filter $F_\equiv = \{a \in L | a \equiv 1\}$. The function f is an isomorphism such that if F is a monadic filter, then $f^{-1}(F)$ is a monadic congruence $\equiv_F$ defined by $a \equiv_F b$ if and only if $a \to b, b \to a \in F$.*

Proposition 6.2. *Let $(L, \forall)$ be a monadic IMTL-algebra and F be a monadic filter of $(L, \forall)$. Then $(L/F, \forall_F)$ is a monadic IMTL-algebra, where $\forall_F([x]) = [\forall x]$, for any $x \in L$.*

The next result shows that $\mathbb{IMTL}$ has the congruence extension property.

Theorem 6.3. *The variety $\mathbb{IMTL}$ has the congruence extension property.*

Proof. Let $(L, \forall)$ be a monadic IMTL-algebra and $(L', \forall)$ a subalgebra of $(L, \forall)$. If F is a monadic filter of $(L', \forall)$ and consider $F' = \langle F\rangle_\forall$, then the monadic filter is generated by F in $(L, \forall)$. Now, we will prove $F = F' \cap L'$. If $b \in F' \cap L'$, then there exist $x_1, x_2, \cdots, x_n \in F$ such that $\forall x_1 \odot, \cdots, \odot \forall x_n \leq b$. Since F is a monadic filter of $(L, \forall)$, we have $\forall x_i \in F$ for any $i \in I$. Hence $\forall x_1 \odot, \cdots, \odot \forall x_n \in F$. In addition, if $b \in L'$, by F upward closed, then $b \in F$, which implies $F' \cap L' \subseteq F$. □

Now, we give some characterizations of representable monadic IMTL-algebras.

Definition 6.4. *A monadic IMTL-algebra is called* representable *if it is a subdirect product of a system of linearly ordered monadic IMTL-algebras.*

Theorem 6.5. *Let $(L, \forall)$ be a monadic IMTL-algebra. Then the following statements are equivalent: for any $x, y \in L$,*
(1) *$(L, \forall)$ is representable,*
(2) $\forall(x \to y) \vee (y \to x) = 1$,
(3) *$x \vee y = 1$ implies $x \vee \forall y = 1$,*
(4) *any minimal prime filter is a monadic filter of $(L, \forall)$.*

Proof. (1) ⇒ (2) If $(L, \forall)$ is representable, then an equation holds in a representable monadic IMTL-algebra if and only if it holds in the linearly ordered monadic IMTL-algebras. Thus, we only need to prove that $\forall(x \to y) \vee (y \to x) = 1$ holds in any linearly ordered monadic IMTL-algebra. In fact, if $x \leq y$, then $\forall(x \to y) = 1$, and hence $\forall(x \to y) \vee (y \to x) = 1$. Moreover, if $y \leq x$, then $y \to x = 1$, and hence $\forall(x \to y) \vee (y \to x) = 1$. Therefore $\forall(x \to y) \vee (y \to x) = 1$.

(2) ⇒ (3) If $x \vee y = 1$, then $x \to y = y$, $y \to x = x$, and hence $x \vee \forall y = \forall(x \to y) \vee (y \to x) = 1$. Thus, $x \vee y = 1$ implies $x \vee \forall y = 1$.

(3) ⇒ (4) We will prove that any minimal prime filter F is a monadic filter. If $x, y \in F$, then there exists $z \in L$ such that $z \notin F$ and $z \wedge x = 1$. Since F is prime filter and $z \notin F$, by $z \vee \forall x = 1 \in F$, we have $\forall x \in F$. Similarly, we have $\forall y \in F$. Hence $\forall x \odot \forall y \in F$. Further by Proposition 3.6(10), we have $\forall(x \odot y) \in F$, that is, F is a monadic filter of $(L, \forall)$.

(4) ⇒ (1) Let $(L, \forall)$ be a monadic IMTL-algebra and $\mathcal{F}$ be the set of all the minimal prime filters of MTL-algebra L. From Theorem 2.4, we obtain any MTL-algebra L is a subdirect product of the family $\{L/\sim_F | F \in \mathcal{F}\}$, and let $\imath : L \to \Pi_{F \in \mathcal{F}} L/F$ be the corresponding representation. By hypothesis, any $F \in \mathcal{F}$ is a monadic filter of $(L, \forall)$, so by Proposition 6.2, we obtain that $(L/F, \forall_F)$ is a monadic IMTL-algebra. It is straightforward that $\imath$ is a presentation of $(L, \forall)$ as a subdirect product of the family $\{L/\sim_F | F \in \mathcal{F}\}$. □

Revaz introduced the strong universal quantifier and the result class of algebras called strong monadic algebras [16], which is a universal quantifier $\forall$ satisfies the following condition:

$$(*)\ \forall(x \vee y) = \forall x \vee \forall y.$$

and proved that every strong monadic NM-algebra is representable [19]. Indeed, strong and representable monadic IMTL-algebras are equivalent, see the following theorem.

Theorem 6.6. *Let $(L, \forall)$ be a monadic IMTL-algebra. Then the following statements are equivalent:*
(1) $(L, \forall)$ *is representable,*
(2) $(L, \forall)$ *is strong.*

Proof. (1) $\Rightarrow$ (2) If $(L, \forall)$ is representable, then an equation holds in a general monadic IMTL-algebra if and only if it holds in the linearly ordered monadic IMTL-algebras. Hence $\forall(x \vee y) = \forall(x) \vee \forall(y)$ holds in all monadic IMTL-algebras.
(2) $\Rightarrow$ (1) If $(L, \forall)$ is a strong monadic IMTL-algebra, then

$$1 = \forall(x \to y) \vee \forall(y \to x) \leq \forall(x \to y) \vee (y \to x).$$

Hence $\forall(x \to y) \vee (y \to x) = 1$, by Theorem 6.5(2), $(L, \forall)$ is representable. □

Remark 6.7. *Theorem* 6.6 *shows that strong monadic IMTL-algebras are not a new class of monadic IMTL-algebras but coincide with representable monadic IMTL-algebras.*

Then we characterize directly indecomposable monadic IMTL-algebra in the following.

Proposition 6.8. *Every monadic IMTL-algebra is congruence distributive and congruence permutable*

Proof. It follows from $MF([L, \forall])$ that it is a distributive lattice and Theorem 6.1 that $Cong([L, \forall])$ is a complete distributive lattice. Also, for any $\theta_1, \theta_2 \in Cong([L, \forall, \exists])$, since $\theta_1 \circ \theta_2 = \theta_1 \vee \theta_2$ and the operation $\vee$ is commutative, we have $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$. □

Theorem 6.9. *Let $(L, \forall)$ be a monadic IMTL-algebra and $a \in B(L)$. Then the monadic congruences $\theta_a, \theta_{\neg a}$ form a pair of factor monadic congruence on $(L, \forall)$.*

Proof. First, since $\langle a \rangle_\forall$ and $\langle \neg a \rangle_\forall$ are monadic filters of $(L, \forall)$, by Theorem 6.1, θ_a, and $\theta_{\neg a}$ are monadic congruences on $(L, \forall)$.
Then it follows from Proposition 6.8 that every monadic IMTL-algebra is congruence permutable, and hence monadic congruences θ_a and $\theta_{\neg a}$ are permutable.
Finally, we shall prove that $\theta_a \cap \theta_{\neg a} = [0]_\theta$, $\theta_a \vee \theta_{\neg a} = [1]_\theta$. Notice that

$$\begin{aligned} x &\in \theta_a \cap \theta_{\neg a} \\ &\Leftrightarrow x \geq (\forall a)^n, x \geq (\forall \neg a)^m \\ &\Leftrightarrow x \geq (\forall a)^n \vee (\forall \neg a)^m \geq (\forall a \vee \neg \forall a)^{nm} \\ &= \{1\}, \end{aligned}$$

we have $\theta_a \cap \theta_{\neg a} = \{1\} = [0]_\theta$.
Also, since $\langle a \rangle_\forall \vee \langle \neg a \rangle_\forall = \langle \{\{x \in L | x \geq (\forall a)^n\} \cup \{x \in L | x \geq \neg \forall a\} \rangle_\forall$, we have

$$\begin{aligned} \langle a \rangle_\forall \vee \langle \neg a \rangle_\forall &= \{x \in L | x \geq (\forall a)^n \odot (\forall \neg a)^m\} \\ &= \{x \in L | x \geq 0\} \\ &= L, \end{aligned}$$

which implies $\theta_a \vee \theta_{\neg a} = L = [1]_\theta$. □

As a consequence, we have the following important result.

Theorem 6.10. *Let $(L, \forall)$ be a monadic IMTL-algebra. Then the following statements are equivalent:*
(1) $(L, \forall)$ *is directly indecomposable,*
(2) *monadic Boolean algebra $(B(L), \exists)$ is simple, where $B(L) = \{x \mid x \vee \neg x = 1\}$.*

Proof. As we have already mentioned, every monadic IMTL-algebra $(L, \forall)$ is directly indecomposable if and only if the only factor monadic congruences on $(L, \forall)$ are $[0]_\theta$ and $[1]_\theta$. By Theorem 6.9, the number of pairs of factor monadic congruences coincide with the number of elements of $(B(L), \exists)$. Thus, $(L, \forall)$ is directly indecomposable if and only if $(B(L), \exists)$ has only two elements, that is, $B(L) = \{0, 1\}$, which implies that $(B(L), \exists)$ is simple. □

The next result will give a representation of monadic IMTL-algebras.

Proposition 6.11. *Let $(L, \forall)$ be a monadic IMTL-algebra and $a \in L_\forall$. Then*

$$([a), \wedge, \vee, \odot_a, \to, \forall, a, 1),$$

is a monadic IMTL-algebra, where $x \odot_a y = (x \odot y) \vee a$ for any $x, y \in [a)$.

Proof. $(\forall 1) - (\forall 4)$ are immediately follows from Definition 3.1. □

Theorem 6.12. *Every monadic IMTL-algebra can be embedded into a family of monadic IMTL-algebras $\{[\forall z)|z \in L\}$.*

Proof. Let us consider the family $\{(a]|a \in L\}$. Since $(a \wedge b] = (a] \cap (b]$, the family $\{(a]|a \in L\}$ has the finite intersection property. Thus, there exists a maximal filter F in the Boolean algebra $\mathcal{P}(\mathcal{L})$ of subsets of L, containing all the members of the family. Let

$$\psi : (L, \forall) \to (\textstyle\prod_{z \in L} [\forall z))/F,$$

be defined by

$$\psi(a) = (a \vee \forall z)_{z \in L}/F.$$

So, given $a, b \in L$,

$$\psi(a) = \psi(b) \text{ if and only if } \{z \in L|\forall z \vee a = \forall z \vee b\} \in F.$$

Then, we will prove that ψ is injective, that is, $\psi(\forall a) = \forall(\psi(a))$, $\psi(a \odot b) = \psi(a) \odot \psi(b)$, $\psi(a \wedge b) = \psi(a) \wedge \psi(b)$, $\psi(a \to b) = \psi(a) \to \psi(b)$.

(1) For any $a \in L$, we have $\forall z \vee \forall a = \forall(a \vee \forall z) = \forall(a \vee \forall z)$. Thus,

$$\begin{aligned} \forall(\psi(a)) &= \forall((a \vee \forall z)_{z \in L}/F) = \forall((a \vee \forall z)_{z \in L})/F \\ &= (\forall(a \vee \forall z))_{z \in L}/F = (\forall a \vee \forall z)_{z \in L}/F \\ &= \psi(\forall a). \end{aligned}$$

(2) If $z \le a \odot b$, then $\forall z \vee (a \odot b) = a \odot b = (\forall z \vee a) \odot (\forall z \vee b)$, which implies

$$(a \odot b] \subseteq \{z \in L|\forall z \vee (a \odot b) = (\forall z \vee a) \odot (\forall z \vee b)\}.$$

Since $(a \odot b] \in F$, we have

$$\{z \in L|\forall z \vee (a \odot b) = (\forall z \vee a) \odot (\forall z \vee b)\} \in F.$$

Thus, $\psi(a \odot b) = \psi(a) \odot \psi(b)$.

(3) For any $a, b \in L$, we have $(a \wedge b) \vee \forall z = (a \vee \forall z) \wedge (b \vee \forall z)$. Thus,

$$\begin{aligned} \psi(a \wedge b) &= \psi(((a \wedge b) \vee \forall z)_{z \in L}/F) = \psi((a \vee \forall z) \wedge (b \vee \forall z))_{z \in L})/F \\ &= \psi(a \vee \forall z))_{z \in L}/F \wedge \psi(b \vee \forall z))_{z \in L}/F \\ &= \psi(a) \wedge \psi(b). \end{aligned}$$

(4) Notice that $\psi(a \to b) = \psi(a) \to \psi(b)$ if and only if

$$\{z \in L|\forall z \vee (a \to b) = (\forall z \vee a) \to (\forall z \vee b)\} \in F.$$

In particular, if $c \in L$ such that $c \le a$ and $c \le b$, then

$$(c] \subseteq \{z \in L|\forall z \vee (a \to b) = (\forall z \vee a) \to (\forall z \vee b)\} \in F.$$

Indeed, if $z \in (c]$, then $\forall z \le z \le c \le a, b$. So, $\forall z \le a \to b$ and consequently $\forall z \vee (a \to b) = a \to b = (\forall z \vee a) \to (\forall z \vee b)$. Since $(c] \in F$, we have $\{z \in L|\forall z \vee (a \to b) = (\forall z \vee a) \to (\forall z \vee b)\} \in F$.

Finally, let us consider $a, b \in L$, such that $\psi(a) = \psi(b)$. Since $(a] \in F$, we have $(a] \cap \{z \in L|\forall z \vee a = \forall z \vee b\} \in F$. In particular, this intersection is not empty. Let $\omega \in (a] \cap \{z \in L|\forall z \vee a = \forall z \vee b\}$. Then, $a = \forall \omega \vee a = \forall \omega \vee b$ and $b \le a$. Analogously, considering $(b] \in F$ we obtain that $a \le b$. So, we prove that $a = b$. □

7 Conclusions

In this paper, we introduce the variety of monadic IMTL-algebras and prove that the completeness for monadic predicate fuzzy logic $\mathbf{IMTL}_\forall$ via functional monadic IMTL-algebras. Then we discuss the relationships between monadic IMTL-algebras and other monadic algebraic structures. Finally we give some representations of monadic IMTL-algebras. Recently, Zhang has studied fuzzy quantifiers and their integral semantics, and obtained some interesting and meaningful results on monadic Boolean algebras (Section 6.3, [22]). Hence the relationship between integral semantics and algebraic semantics of fuzzy quantifiers is worth studying in the future.

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