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States on weak pseudo EMV-algebras. II. Representations of states

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Abstract

Recently in [7, 8], new algebras, called weak pseudo EMV-algebras, wPEMV-algebras for short, were introduced generalizing pseudo MV-algebras, generalized Boolean algebras and pseudo EMV-algebras. For these algebras a top element is not assumed a priori. For this class of algebras, we define a state as a finitely additive mapping from a wPEMV-algebra into the real interval $[0,1]$ which preserves a partial addition of two non-interactive elements and attaining the value 1 in some element. It can happen that some commutative wPEMV-algebras are stateless, e.g. cancellative ones. The paper is divided into two parts. Part I deals with basic properties of states and state-morphisms which are wPEMV-homomorphisms from a wPEMV-algebra into the real interval $[0,1]$ endowed as a commutative wPEMV-algebra. We show that there is a one-to-one correspondence between the set of state-morphisms and the set of maximal and normal ideals having a special property.
In Part II, we present an analogue of the Krein-Mil'man theorem applied to the set of states. We characterize the space of the state-morphisms of a wPEMV-algebra without top element as a Hausdorff locally compact space in the weak topology of states and we present its Alexandroff's one-point compactification. Moreover, we give an integral representation of any (finitely additive) state by a unique regular Borel σ-additive probability measure.

Keywords: Pseudo MV-algebra, pseudo EMV-algebra, wPEMV-algebra, generalized Boolean algebra, state, state-morphism, extremal state, pre-state, maximal and normal ideal, weak convergence, simplex, integral representation.

Introduction

In this paper, we continue with the study from [3], where we have introduced states and state-morphisms on weak pseudo EMV-algebras defined in [7, 8]. Motivation for the study of states and basic properties of states can be found in [3] together with the necessary notions and definitions. Sections, theorems, propositions, lemmas, examples, and equations are numbered in continuation of [3].

5 Some topological properties of the set of state-morphisms

In the section, we describe the Krein–Mil'man theorem for states and we present a one-point compactification of state-morphisms on wPEMV-algebras without top element. In addition, we describe a homeomorphism between the set $\mathrm{MaxN}_0(M)$ in the hull-kernel topology and the set of state-morphisms in the weak topology. Moreover, we show that the space of state-morphisms of a proper wPEMV-algebra, i.e. without top element, is always locally compact and its one-point compactification is homeomorphic with the set of state-morphisms on the representing wPEMV-algebra with top element.

The Krein-Mil'man theorem says that when we have a compact convex set X, then every its point is a limit of a net of convex combinations of extreme points of X, [9, Thm 5.17]. Whence, applying this result to pseudo MV-algebras, or equivalently to wPEMV-algebras M with top element, we have that every state on M is a weak limit of a net of

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convex combinations of state-morphisms. However, if M has no top element, then $\mathcal{S}(M)$ is not necessarily compact in the weak topology (e.g. if M is an associated wPEMV-algebra without top element then $\mathcal{S}(M)$ is not compact, see Corollary 4.8), so that we cannot apply the Krein-Mil'man theorem. However, we can establish the following result for all wPEMV-algebras. Its proof is identical to the proof of [2, Thm 4.1].

Theorem 5.1. *Let M be a proper wPEMV-algebra. Then*

$$\mathcal{S}(M) \subseteq (\mathrm{Con}(\mathcal{SM}(M)))^{-_M}, \tag{1}$$

where Con *and* $^{-_M}$ *denote the convex hull and the closure, respectively, in the weak topology of states on M.*

We note that if M has no top element, it can happen that the inclusion is proper. Indeed, take into account M from Example 4.1, it induces the restriction property. Then for the sequence of its state-morphisms $(s_n)_n$, we have $\lim_n s_n(A) = 0$ for each finite subset A of $\mathbb{N}$, and this limit is not a state on M.

On the other hand, if $\mathcal{S}(M)$ is a compact (i.e. the state space is a Bauer simplex (see Theorem 6.3 below)), and in (1) we have equality even in some cases when M is without top element as we can see in the following examples.

Now, we illustrate situations in Theorem 4.7 and Theorem 4.10 showing that if a wPEMV-algebra M is not associated, then it can happen that M is without top element but $\mathcal{S}(M)$ and $\mathcal{SM}(M)$ are compact.

Example 5.2. *Let M_1 and M_2 be linearly ordered wPEMV-algebras such that M_1 is bounded and M_2 is not bounded, e.g. the conic wPEMV-algebra $\mathbb{Z}^+$. The first one has a unique maximal and normal ideal I, M_2 has a unique maximal ideal $\{0\}$. The state space $\mathcal{S}(M_1)$ is a singleton whereas $\mathcal{S}(M_2) = \emptyset$. Define $M = M_1 \times M_2$. M is not an associated wPEMV-algebra. It has two maximal and normal ideals $I_1 = I \times M_2$ and $I_2 = M_1 \times \{0\}$, and only I_1 gives a state-morphism s such that* $\mathrm{Ker}(s) = I_1$*. Therefore, $\mathcal{S}(M)$ and $\mathcal{SM}(M)$ are singletons, so that they are compact in the weak topology, but M is without top element, but in* (1) *we have the equality.*

Moreover, M has the representing wPEMV-algebra of the form $N = M_1 \times \Gamma_a(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (1,0))$. It has two maximal ideals $I_1 \times \Gamma_a(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (1,0))$ and $M_1 \times \mathbb{Z}^+$. The first one describes a state-morphism being the extension of s and the second one describes s_∞.

Example 5.2 shows also that if, for a wPEMV-algebra M, we have in (1) equality and there is an element $x \in M$ such that $s(x) = 1$ for each state s on M, then it does not entail necessarily that M is with top element.

There is a natural question. If in (1) we have the equality for a wPEMV-algebra M, when is M with top element? Something more we can say if M is an associated wPEMV-algebra with at most countably many state-morphisms.

Proposition 5.3. *Let an associated wPEMV-algebra M such that every its maximal ideal is normal have at most countably many state-morphisms. If in* (1) *we have the equality, then M is with top element.*

Proof. Let $s_n \in K$, where K is an at most countable non-empty set of state-morphisms. It is possible to show that $s_0 = \sum_n \lambda_n s_n$ a state on M. There is an element $x \in M$ such that $s_0(x) = 1$. Since M is associated, there is an idempotent element $a \in M$, $x \le a$, so that $s_0(a) = 1$ and $s_n(a) = 1$ for each $n \in K$. We show that a is a top element of M. Define the interval $[0, a]$, it is an ideal of M, we assert that $[0, a] = M$. If not, then it is a proper ideal of M and due to [5, Prop 5.4-5.5], there is a maximal and normal ideal I of M containing $[0, a]$. Theorem 3.10 entails there is a unique state-morphism s on M such that $I = \mathrm{Ker}(s)$ which implies $s(a) = 1$ as well as $s = 0$, a contradiction. Whence, a is a top element of M. ☐

Using Example 5.2, we see that if M is not associated but every state s admits an idempotent element a such that $s(a) = 1$, and M has at most countably many state-morphisms such that (1) holds, then M has no top element.

Another example of a non-associated wPEMV-algebra satisfying the conditions of the last paragraph will be present in Example 6.5 below.

As we have seen in Example 5.2, even if a non-associated wPEMV-algebra M has a unique state, so that in (1) we have equality, this M has no top element.

The proof of the following proposition will follow the basic steps for an analogous statement [6, Prop 8.20] which was established for state-morphisms on pseudo EMV-algebras. Here we improved it for all wPEMV-algebras where not necessarily every element is dominated by some idempotent.

Proposition 5.4. *Let M be a proper wPEMV-algebra and let X be a non-empty subspace of state-morphisms on M that is closed in the weak topology of state-morphisms. Let t be a state-morphism such that $t \notin X$. There exists an $a \in M$ such that $t(a) > 1/2$ while $s(a) < 1/2$ for all $s \in X$. Moreover, the element $a \in M$ can be chosen such that $t(a) = 1$ and $s(a) = 0$ for each $s \in X$.*

Proof. According to Lemma 4.4, M induces the restriction property. The proof is divided into three parts.

(1) Let t be a state-morphism such that $t \notin X$. We assert that there exists an element $a \in M$ such that $t(a) > 1/2$ while $s(a) < 1/2$ for all $s \in X$.

In fact, set $A = \{a \in M : t(a) > 1/2\}$, and for all $a \in A$, let

$$W(a) := \{s \in \mathcal{SM}(M)\colon\ s(a) < 1/2\},$$

which is an open subset of $\mathcal{SM}(M)$. We note that $A \neq \emptyset$ and A is downward directed and closed under $\oplus$.

We assert that these open subsets cover X. Consider any $s \in X$. Since $\mathrm{Ker}(s)$ and $\mathrm{Ker}(t)$ are non-comparable subsets of M, there exists $x \in \mathrm{Ker}(t) \setminus \mathrm{Ker}(s)$. Hence $t(x) = 0$ and $s(x) > 0$.

Define a state $\mu = (s+t)/2$ on M. Then $\mathrm{Ker}(\mu) = \mathrm{Ker}(s) \cap \mathrm{Ker}(t)$ and let $\hat\mu$ be a state on the bounded quotient wPEMV-algebra $M/\mathrm{Ker}(\mu)$ defined by $\hat\mu(z/\mathrm{Ker}(\mu)) = \mu(z)$ $(z \in M)$, see Proposition 3.1(iii). There is an element $b \in M$ such that $b/\mathrm{Ker}(\mu)$ is a top element of $M/\mathrm{Ker}(\mu)$, so that $x/\mathrm{Ker}(\mu) \le b/\mathrm{Ker}(\mu)$ and $\hat\mu(b/\mathrm{Ker}(\mu)) = 1$. Therefore, $t(b) = 1 = s(b)$. There are integers $n, k \ge 1$ such that $s(n.x) > 1/2$ and $s(k.x) = k.s(x) = 1$. Moreover, $(k.x)/\mathrm{Ker}(\mu) \le b/\mathrm{Ker}(\mu)$. Because t is a state-morphism, we have $t(n.x) = 0$. Putting $a = b \ominus (n.x)$, we have $t(a) = t(b) - t(b \wedge (n.x)) = 1 > 1/2$ and $s(a) < 1/2$. Therefore $\{W(a)\colon a \in A\}$ is an open covering of X.

(i) If M is with top element, the space $\mathcal{SM}(M)$ is compact and Hausdorff, so that X is compact, and $X \subseteq W(a_1) \cup \cdots \cup W(a_n)$ for some $a_1, \ldots, a_n \in A$.

(ii) If M has no top element, embed M into a representing pseudo EMV-algebra N as its maximal and normal ideal. It is clear that $\mathcal{SM}(N)$ is a compact set in the weak topology of state-morphisms on N. The mapping $\phi : \mathcal{SM}(M) \to \mathcal{SM}(N)$ given by $\phi(s) = \tilde s$, where $\tilde s$ is defined through (4.1), is by Proposition 4.3 injective and continuous.

We assert the set $\phi(X) \cup \{s_\infty\}$ is a compact subset of $\mathcal{SM}(N)$. Indeed, let $(s_\alpha)_\alpha$ be a net of state-morphisms from $\phi(X) \cup \{s_\infty\}$. Since $\mathcal{SM}(N)$ is compact, there is a subnet $(s_{\alpha_\beta})_\beta$ of the net $(s_\alpha)_\alpha$ converging weakly to a state-morphism s on N. If $s = s_\infty$, $s \in \phi(X) \cup \{s_\infty\}$. If $s \neq s_\infty$, there is a state-morphism $s_0 \in \mathcal{SM}(M)$ such $s = \tilde s_0$. Then there is β_0 such that for each $\beta > \beta_0$, $s_{\alpha_\beta} \in X$. Therefore, $s_0 \in X$ and $s = \phi(s_0) \in \phi(X) \cup \{s_\infty\}$. We note that $\tilde t \notin \phi(X) \cup \{s_\infty\}$.

For each $a \in A$, let $\widetilde W(a) := \{s \in \mathcal{SM}(N)\colon s(a) < 1/2\}$. Then $\tilde t(a) = t(a) > 1/2$ and $0 = s_\infty(a) < 1/2$, so that $s_\infty \in \widetilde W(a)$ for each $a \in A$. Therefore, $\{\widetilde W(a)\colon a \in A\}$ is an open covering of the compact set $\phi(X) \cup \{s_\infty\}$. There are $a_1, \ldots, a_n \in A$ such that $\phi(X) \cup \{s_\infty\} \subseteq \widetilde W(a_1) \cup \cdots \cup \widetilde W(a_n)$, consequently, $X \subseteq W(a_1) \cup \cdots \cup W(a_n)$. Put $a = a_1 \wedge \cdots \wedge a_n$. Then $a \in A$ and for each $s \in X$, we have $s(a) \le s(a_i) < 1/2$ for $i = 1, \ldots, n$, which proves $X \subseteq W(a)$, i.e., $s(a) < 1/2$ for all $s \in X$.

(2) By the first part of the present proof, there exists an element $a \in M$ such that $t(a) > 1/2$ while $s(a) < 1/2$ for all $s \in X$. In addition, if we set $\mu = (s+t)/2$, according to part (1), there is an element b of M such that $b/\mathrm{Ker}(\mu)$ is an idempotent element of $M/\mathrm{Ker}(\mu)$ such that $b/\mathrm{Ker}(\mu)$ is a top element of $M/\mathrm{Ker}(\mu)$. Then $\mu(b \oplus b) = \mu(b) = 1$ implying $s(b \oplus b) = s(b) = 1$ and $t(a \oplus a) = t(a) = 1$ as well as $\mu(a) = \mu(a \wedge b)$ and $s(a) = s(a \wedge b)$, $t(a \oplus a) = t(a)$.

Check $t(a \wedge (b \ominus a)) = t(b \ominus a)$ and $t(a \odot (b \ominus (a \wedge (b \ominus a)))) = t(a) - t(a \wedge (b \ominus a)) = t(a) - t(b \ominus a) = 2t(a) - 1 > 0$.

Now, let s be an arbitrary element of X. If $s(a) = 0$, then $s(a \odot (b \ominus (a \wedge (b \ominus a)))) = 0$. If $s(a) > 0$, we have, $s(a \wedge (b \ominus a)) = s(a)$, so that $s(a \odot (b \ominus (a \wedge (b \ominus a)))) = s(a) - s(a \wedge (b \ominus a)) = 0$. In any case, the element $a_0 = a \odot (b \ominus (a \wedge (b \ominus a)))$ is an element of $\bigcap\{\mathrm{Ker}(s)\colon s \in X\}$ for which $t(a \odot (b \ominus (a \wedge (b \ominus a)))) > 0$.

(3) From (1) and (2), we conclude that $s(a_0) = 0$ for each $s \in X$ and $t(a_0) > 0$. There is an integer $r > 0$ such that $t(r.a_0) = r.t(a_0) = 1$ and $s(r.a_0) = 0$ for each $s \in X$. Therefore, the element $a_r = r.a_0$ satisfies the last statement of the proposition. ☐

In what follows, we show that the space of state-morphisms $\mathcal{SM}(M)$ and the set $\mathrm{MaxN}_0(M)$ of maximal and normal ideals I of a wPEMV-algebra M such that M/I is a bounded wPEMV-algebra are homeomorphic. Therefore, we introduce the hull kernel-topology of normal and maximal ideals of a wPEMV-algebra M in the standard way. For every $a \in M$, we put

$$M(a) := \{I \in \mathrm{MaxN}_0(M)\colon a \notin I\}.$$

Then (i) $M(0) = \emptyset$, $M(a) \subseteq M(b)$ whenever $a \le b$, $M(a \wedge b) = M(a) \cap M(b)$, $a, b \in M$, $M(a \vee b) = M(a) \cup M(b)$, $a, b \in M$, and $\{M(a)\colon a \in M\}$ is the base of the so-called *hull-kernel topology* on $\mathrm{MaxN}_0(M)$. It is possible to show that the hull-kernel topology defines a Hausdorff topology such that the closed subspaces of $\mathrm{MaxN}_0(M)$ are exactly of the form

$$C = C(J) := \{I \in \mathrm{MaxN}_0(M)\colon I \supseteq J\}, \tag{2}$$

where J is an ideal of M. Similarly, every open set O is of the form

$$O = O(J) := \{I \in \mathrm{MaxN}_0(M)\colon I \not\supseteq J\}. \tag{3}$$

Theorem 5.5. *Let M be a wPEMV-algebra. The mapping $\kappa : \mathcal{SM}(M) \to \mathrm{MaxN}_0(M)$ given by $\kappa(s) = \mathrm{Ker}(s)$, $s \in \mathcal{SM}(M)$, is a homeomorphism.*

Proof. If M is stateless, then $\mathcal{SM}(M) = \emptyset$ as well as $\mathrm{MaxN}_0(M) = \emptyset$, see Theorem 3.6. Thus let M have at least one state, consequently, M has at least one state-morphism.

Due to Theorem 3.6, the mapping κ is a bijection. Let $C(I)$ be any closed subspace of $\mathrm{MaxN}_0(M)$. Then

$$\kappa^{-1}(C(I)) = \{s \in \mathcal{SM}(M) \colon s(x) = 0 \text{ for all } x \in I\},$$

which is a closed subset of $\mathcal{SM}(M)$. Therefore, κ is continuous.

Given a non-empty subset X of $\mathcal{SM}(M)$, we set

$$\mathrm{Ker}(X) := \bigcap \{\mathrm{Ker}(s) \colon s \in X\}.$$

Then $\mathrm{Ker}(X)$ is a normal ideal of M. If, in addition, X is a closed subset of $\mathcal{SM}(M)$, we assert

$$\kappa(X) = C(\mathrm{Ker}(X)). \tag{4}$$

The inclusion $\kappa(X) \subseteq C(\mathrm{Ker}(X))$ is evident. By Proposition 5.4, if $t \notin X$, there is an element $a \in M$ such that $s(a) = 0$ for each $s \in X$ and $t(a) = 1$. Consequently, $t \notin X$ implies $\kappa(t) \notin C(\mathrm{Ker}(X))$, and $C(\mathrm{Ker}(X)) \subseteq \kappa(X)$. Finally, we conclude κ is a homeomorphism. ☐

The local compactness of $\mathcal{SM}(M)$ from Theorem 4.5 can be strengthened as follows.

Theorem 5.6. *Let M be a wPEMV-algebra and let, given $s \in \mathcal{S}(M)$, there be an idempotent element $a \in M$ with $s(a) = 1$. Then $\mathcal{SM}(M)$ is either an empty set or is a locally compact non-empty Hausdorff space in the weak topology of state-morphisms such that $S(a)$ is a compact set for each $a \in \mathcal{I}(M)$.*

Proof. The proof uses the hypotheses of the theorem and is practically the same as that of the analogous statement of Theorem 4.5 to show that $S(a)$ is compact for any idempotent $a \in M$. After that, if s is a state, there is an idempotent element $a \in M$ such that $s(a) = 1$. Then $s \in S(a)$, and $S(a)$ is a compact neighborhood of s. ☐

The following result generalizes both Theorem 4.5 and the latter one. Its proof uses different and more general techniques than those used in these theorems.

Theorem 5.7. *Let M be a proper wPEMV-algebra and let N be its representing wPEMV-algebra. The space $\mathcal{SM}(N)$ is the one-point compactification of the space $\mathcal{SM}(M)$.*

Moreover, $\mathcal{SM}(M)$ is locally compact.

Proof. If M has no states, then $\mathcal{SM}(M) = \emptyset$ and the one-point compactification of $\mathcal{SM}(M)$ is trivially $\mathcal{SM}(N) = \{s_\infty\}$.

Thus, let M has at least one state-morphism. Applying the Alexandroff theorem, see [10, Thm 4.21], there is the one-point compactification of $\mathcal{SM}(M)$. We show that the one-point compactification of $\mathcal{SM}(M)$ is the space $\mathcal{SM}(N)$. Without loss of generality, we can assume M is a maximal and normal ideal of N.

We proceed in five steps. We note that s_∞ is a unique two-valued state-morphism on N such that $\mathrm{Ker}(s_\infty) = M$, see (4.2).

We underline that the mapping $\phi : \mathcal{SM}(M) \to \mathcal{SM}(N)$, defined by $\phi(s) = \tilde{s}$, is injective and continuous.

(1) If O_N is an open set of $\mathcal{SM}(N)$ such that $s_\infty \notin O_N$, then $O_N = \phi(O)$ for some open subset O of $\mathcal{SM}(M)$.

In the next four cases, we show that if O_N is an open set in $\mathcal{SM}(N)$ containing s_∞, then the unique set $X' \subseteq \mathcal{SM}(M)$ such that $\phi(X') = X := \mathcal{SM}(N) \setminus O_N$ is a closed and compact set of $\mathcal{SM}(M)$.

(2) Now, take an open set O_N containing s_∞ and $O_N = S_N(x)_{u,v} := \{s \in \mathcal{SM}(N) \colon u < s(x) < v\}$, where $x \in M$ and u, v are real numbers with $u < v$. We define also open sets $S(x)_{u,v} = \{s \in \mathcal{SM}(M) \colon u < s(x) < v\}$. Since $s_\infty(x) = 0$, $u < 0 < v$ and we have $S_N(x)_{u,v} = \{s_\infty\} \cup \{\tilde{s} \colon s \in \mathcal{SM}(M), s(x) < v\} = \{s_\infty\} \cup \phi(\{s \in \mathcal{SM}(M) \colon s(x) < v\})$. If $X := \mathcal{SM}(N) \setminus S_N(x)_{u,v}$, then $X = \phi(\mathcal{SM}(M)) \setminus (\{s_\infty\} \cup \phi(\{s \in \mathcal{SM}(M) \colon s(x) < v\})) = \phi(\{s \in \mathcal{SM}(M) \colon s(x) \ge v\})$. If $v > 1$, then $X' = \emptyset$ which is a compact set. Assume $v \le 1$, we claim X' is a closed and compact set in $\mathcal{SM}(M)$. Let $(s_\alpha)_\alpha$ be a net of state-morphisms from X'. Then $(\tilde{s}_\alpha)_\alpha$ is a net in $X = \{\tilde{s} \colon s \in X'\} = \{\tilde{s} \colon s(x) \ge v\} = \mathcal{SM}(N) \setminus O_N(x)_{u,v}$. Since the set X is closed, it is a compact set of $\mathcal{SM}(N)$, there is a subnet $(\tilde{s}_{\alpha_\beta})_\beta$ of the net $(\tilde{s}_\alpha)_\alpha$ converging weakly to a state-morphism $s_0(z) = \lim_\beta s_{\alpha_\beta}(z)$, $z \in N$. Since $s_0(x) \ge v > 0$, $s_0 \ne s_\infty$. Whence $(s_{\alpha_\beta})_\beta$ is a weakly converging subnet of $(s_\alpha)_\alpha$. Therefore, the restriction s_0^M of s_0 onto M is a state-morphism on M, $s_0^M(y) = \lim_\beta s_{\alpha_\beta}(y)$, $y \in M$, and the closedness of X' implies $s_0^M \in X'$ entailing X' is a compact set.

(3) Now let $s_\infty \in O_N = S_N(x)_{u,v}$, where $x \in M$ and u, v are real numbers with $u < v$ and $x = \rho_1(x_0)$, where $x_0 \in M$. Since $s_\infty(x) = 1$, we have $u < 1 < v$. Then $S_N(x)_{u,v} = \{s_\infty\} \cup \{\tilde{s}\colon s \in \mathcal{SM}(M), u < \tilde{s}(x)\} = \{s_\infty\} \cup \phi(\{s \in \mathcal{SM}(M)\colon s(x_0) < 1-u\})$. Consequently, $X := \mathcal{SM}(N) \setminus S_N(x)_{u,v} = \phi(\mathcal{SM}(M)) \setminus (\{s_\infty\} \cup \phi(\{s \in \mathcal{SM}(M)\colon s(x_0) < 1-u\})) = \phi(\mathcal{SM}(M) \setminus \{s \in \mathcal{SM}(M)\colon s(x_0) < 1-u\}) = \phi(\{s \in \mathcal{SM}(M)\colon s(x_0) \geq 1-u\})$ and we have $X' = \{s \in \mathcal{SM}(M)\colon s(x_0) \geq 1-u\}$. As in part (2), we can show that X' is closed and compact in $\mathcal{SM}(M)$.

(4) Let $s_\infty \in O_N = \bigcap_{i=1}^n S_N(x_i)_{u_i,v_i}$, where $u_i \in N$, $u_i < v_i$ and $s_\infty \in S_N(x_i)_{u_i,v_i}$ for each $i = 1, \dots, n$. Then $S_N(x_i)_{u_i,v_i} = \{s_\infty\} \cup \phi(S(x'_i)_{u'_i,v'_i})$ where if $x_i \in M$, then $x'_i = x_i$ and $u'_i = u_i$, $v'_i = v_i$ and if $x_i \in N \setminus M$, then $x'_i = \rho_1(x_i)$ and $u'_i = 1 - v_i$, $v'_i = 1 - u_i$.

Hence, $\phi(\mathcal{SM}(M)) \setminus \bigcap_{i=1}^n S_N(x_i)_{u_i,v_i} = \phi(\mathcal{SM}(M)) \setminus (\{s_\infty\} \cup \phi(\bigcap_{i=1}^n S(x'_i)_{u'_i,v'_i})) = \phi(\bigcup_{i=1}^n (\mathcal{SM}(M) \setminus S(x'_i)_{u'_i,v'_i}))$, so that $X' = \bigcup_{i=1}^n (\mathcal{SM}(M) \setminus S(x'_i)_{u'_i,v'_i})$ is a compact set of $\mathcal{SM}(M)$ in view of (3).

(5) $O_N = \bigcup_\alpha O_\alpha^N$, where each O_α^N is the set of the form (4). Then $O_\alpha^N = \{s_\infty\} \cup \phi(O_\alpha)$ if $s_\infty \in O_\alpha^N$, otherwise $O_\alpha^N = O_\alpha$, where O_α is an open set in $\mathcal{SM}(M)$.

Then $\phi(\mathcal{SM}(M) \setminus \bigcup_\alpha O_\alpha^N) = \phi(\mathcal{SM}(M) \setminus \bigcup_\alpha O_\alpha)$, where O_α is a subset of $\mathcal{SM}(M)$ such that $O_\alpha^N = \phi(O_\alpha)$. Whence, $\phi(\mathcal{SM}(M) \setminus \bigcup_\alpha O_\alpha) = \phi(\bigcap_\alpha (\mathcal{SM}(M) \setminus O_\alpha)) \subseteq \phi(\mathcal{SM}(M) \setminus O_{\alpha_0})$, where α_0 is an index α such that $s_\infty \in O_{\alpha_0}^N$. The set $\mathcal{SM}(M) \setminus O_{\alpha_0}$ is by (4) a compact set of $\mathcal{SM}(M)$, consequently, $X' = \bigcap_\alpha (\mathcal{SM}(M) \setminus O_\alpha)$ is a compact set, too.

Therefore, $\mathcal{SM}(N)$ is the Alexandroff one-point compactification of $\mathcal{SM}(M)$.

Moreover, $\mathcal{SM}(N)$ is a compact Hausdorff space, so that due to [10, Thm 4.21], this implies $\mathcal{SM}(M)$ is a Hausdorff and locally compact topological space. □

We note that if a wPEMV-algebra M is with top element, then $\mathcal{SM}(M)$ is compact and its one-point compactification is not the space $\mathcal{SM}(N)$.

If M is an associated wPEMV-algebra, then $\mathcal{SM}(M)$ is compact iff M is with top element. In Example 5.2, we have seen that there are proper non-associated wPEMV-algebras such that $\mathcal{SM}(M)$ is a compact set. Then the singleton $\{s_\infty\}$ is both an open and closed set in $\mathcal{SM}(N)$.

6 Simplices and integral representation of states

In the present section, we show that the state space of any wPEMV-algebra is a simplex. For states on EMV-algebras this was proved in [4] and for states on pseudo EMV-algebra in [6]. In these papers this was proved using the fact that every element $x \in M$ is dominated by some idempotent element. In wPEMV-algebras, this is not true in general. Moreover, we show an integral representation of any state on M using a unique regular probability measure on the Borel σ-algebra space generated by state-morphisms. This generalizes the result by [11, 12] originally established for states on MV-algebras.

To prove that, we introduce the following notions. A mapping $m : M \to \mathbb{R}$ is said to be a *signed measure* if $m(x+y) = m(x) + m(y)$. If a signed measure m is positive, then m is said to be a *measure*, and if m is a difference of two measures, m is said to be a *Jordan signed measure*. Whence a pre-state is a measure with values in the interval $[0,1]$. We denote by $\mathcal{J}(M)$ the set of Jordan signed measures. If for a measure m, there is an element $a \in M$ such that $m(a) = \sup\{m(x)\colon x \in M\}$, m is said to be a *strong measure*. Every state s is a strong measure on M and if $r \in [0, \infty)$, then sr is a strong measure. Conversely, if m is a not-vanishing strong measure, then $\frac{1}{r}m$, where $r = \sup\{m(x)\colon x \in M\}$, $r > 0$, is a state on M. Difference of two strong measures is said to be a *strong Jordan signed measure*. We denote by $\mathcal{J}_s(M)$ the set of strong Jordan signed measures on M.

We define a partial order $\leq^+$ on the set of Jordan signed measurez by $m_1 \leq^+ m_2$ if $m_1(x) \leq m_2(x)$ for each $x \in M$.

Remark 6.1. In [4, Sec 6], it was proved that $\mathcal{J}(M)$ is a Dedekind complete ℓ-group with respect to $\leq^+$ such that if given $m_1, \dots, m_n \in \mathcal{J}(M)$, we have

$$\left(\bigvee_{i=1}^n m_i\right)(x) = \sup\{m_1(x_1) + \cdots + m_n(x_n)\colon x = x_1 + \cdots + x_n,\ x_1, \dots, x_n \in M\},$$
$$\left(\bigwedge_{i=1}^n m_i\right)(x) = \inf\{m_1(x_1) + \cdots + m_n(x_n)\colon x = x_1 + \cdots + x_n,\ x_1, \dots, x_n \in M\},$$

for all $x \in M$, see [4, Thm 6.2–6.3]. We have to underline this result was established for any EMV-algebra M. However, in their proof, it was not used the fact that each $x \in M$ is dominated by some idempotent element of M, consequently, it holds also for wEMV-algebras.

In what follows, we show that the space of strong Jordan signed measures is a Riesz space (= vector lattice).

Theorem 6.2. *Let M be a wPEMV-algebra. Then $\mathcal{J}_s(M)$ is a Riesz space.*

Proof. If M m has no state, then $\mathcal{J}_s(M)=\{o\}$, where $o(x)=x$, $x\in M$. Thus let M have at least one state. Then $\mathcal{J}_s(M)$ has infinitely many elements. We define $I=\bigcap\{\mathrm{Ker}(s)\colon s\in\mathcal{S}(M)\}$. Then I is a normal ideal of M and let us define the quotient M/I. Then $x/I=y/I$ iff $s(x)=s(x\wedge y)=s(y)$ for each state $s\in\mathcal{S}(M)$. As in Proposition 3.1, we can show that M/I is an Archimedean wPEMV-algebra, so that $\oplus$ is commutative. In what follows, without loss generality, we can assume that M is a wEMV-algebra with commutative $\oplus$, or we use M/I instead of M, if necessary.

Let m_1 and m_2 be strong measures on M. Since $m_i\le^+ m_1+m_2$, $i=1,2$, due Remark 6.1, there exists a measure $m=m_1\vee m_2$. We show that m is a strong measure on M. If m_1 or m_2 is the zero measure, then trivially m is a strong measure. We can assume that both measures are non-zero measures. Since m_1 and m_2 are strong measures, there are elements $a_1,a_2\in M$ such that $m_i(a_i)=\sup\{m_i(x)\colon x\in M\}$ for $i=1,2$. Define $m_0=m_1+m_2$, clearly, m_0 is a strong measure because it takes the greatest value in $a_0=a_1\vee a_2$. We define the kernel Ker also for strong measures in the same way as for states. Let $J=\mathrm{Ker}(m)$, $J_i=\mathrm{Ker}(m_i)$, $i=1,2$. As in Proposition 3.1(iv), we define $\widetilde m,\widetilde m_1,\widetilde m_2$ on M/J, M/J_1 and M/J_2, respectively, by $\widetilde m(x/J)=m(x)$ $(x\in M)$; similarly we define $\widetilde m_1$ and $\widetilde m_2$. Moreover, M/J, M/J_1 and M/J_2 are bounded wPEMV-algebras.

Since M/J is a bounded wEMV-algebra, we can assume that there is an element $a\in M$ with $a\ge a_1,a_2$ such that a/J is an idempotent element in M/J and $\widetilde m_i(a)=m_i(a)=m_i(a_i)=\widetilde m_i(a_i)$ for $i=1,2$. Moreover, a/J_i is an idempotent element in M/J_i for $i=1,2$ because $J\subseteq J_1\cap J_2$. Let $r=\sup\{m(x)\colon x\in M\}$. Given $\epsilon>0$, there is an element $x\in M$ such that $r<m(x)+\epsilon$. Due to Remark 6.1, for this x, there are elements $x_1,x_2\in M$ with $x=x_1+x_2$ and $m(x)<m_1(x_1)+m_2(x_2)+\epsilon$. Then

$$\begin{aligned} r &< m(x)+\epsilon<m_1(x_1)+m_2(x_2)+2\epsilon\\ &= \widetilde m_1(x_1/J_1)+\widetilde m_2(x_2/J_2)+2\epsilon\\ &= \widetilde m_1((x_1\wedge a)/J_1)+\widetilde m_2((x_2\wedge a)/J_2)+2\epsilon\\ &= m_1(x_1\wedge a)+m_2(x_2\wedge a)+2\epsilon\\ &\le m((x_1\wedge a)+(x_2\wedge a))+2\epsilon\\ &= \widetilde m(((x_1\wedge a)+(x_2\wedge a))/J)+2\epsilon\\ &= \widetilde m(((x_1+x_2)\wedge a)/J)+2\epsilon\\ &\le m(a)+2\epsilon, \end{aligned}$$

where we have used a fact that $(x_1/J\wedge a/J)+(x_2/J\wedge a/J)=x/J\wedge a/J$ for $a\in\mathcal{I}(M/J)$, see [8, Prop 8.5]. Then $r\le m(a)$, so that $r=m(a)$, and m is a strong measure.

Since $m_1,m_2\le^+ m$, we have $m-m_1,m-m_2$ are strong measures on M and $(m-m_1)\vee(m-m_2)$ is a strong measure by the previous paragraphs. Then $m-(m_1\wedge m_2)=(m-m_1)\vee(m-m_2)$ which yields $m_1\wedge m_2$ is also a strong measure.

Now let $m_1,m_2\in\mathcal{J}_s(M)$. Then $m_i=m_i^+-m_i^-$, so that $m_i\le^+ m_1^++m_1^-+m_2^++m_2^-$, giving $\mathcal{J}_s(M)$ is directed with the positive cone consisting of all strong measures. Therefore, $\mathcal{J}_s(M)$ is an ℓ-group which is also a Riesz space. $\square$

Now, we present some notions about simplices, see also [1, 9].

We recall that a *convex cone* in a real linear space V is any subset C of V such that (i) $0\in C$, (ii) if $x_1,x_2\in C$, then $\alpha_1x_1+\alpha_2x_2\in C$ for any $\alpha_1,\alpha_2\in\mathbb{R}^+$. A *strict cone* is any convex cone C such that $C\cap -C=\{0\}$, where $-C=\{-x:\ x\in C\}$. A *base* for a convex cone C is any convex subset K of C such that every non-zero element $y\in C$ may be uniquely expressed in the form $y=\alpha x$ for some $\alpha\in\mathbb{R}^+$ and some $x\in K$.

We recall that in view of [9, Prop 10.2], if K is a non-void convex subset of V, and if we set

$$C=\{\alpha x:\ \alpha\in\mathbb{R}^+,\ x\in K\},$$

then C is a convex cone in V, and K is a base for C iff there is a linear functional f on V such that $f(K)=1$ iff K is contained in a hyperplane in V which misses the origin.

Any strict cone C of V defines a partial order $\le_C$ on V via $x\le_C y$ iff $y-x\in C$. It is clear that $C=\{x\in V:\ 0\le_C x\}$. A *lattice cone* is any strict convex cone C in V such that C is a lattice under $\le_C$.

A *simplex* in a linear space V is any convex subset K of V that is affinely isomorphic to a base for a lattice cone in some real linear space. A simplex K in a locally convex Hausdorff space is said to be (i) *Choquet* if K is compact, and (ii) *Bauer* if K and ∂K are compact, where ∂K is the set of extreme points of K.

Theorem 6.3. *Let M be a wPEMV-algebra. Then the state space $\mathcal{S}(M)$ of M is either the empty set or a non-empty simplex. Moreover, $\mathcal{S}(M)$ is a Choquet simplex if and only if it is a Bauer simplex.*

Proof. Assume the state space is non-empty. Theorem 6.2 says the space $\mathcal{J}_s(M)$ of strong Jordan signed measures on M is an ℓ-group. The space of strong Jordan measures is its positive cone whose base is the set of states on M. Therefore, $\mathcal{S}(M)$ is a simplex.

In addition, let $\mathcal{S}(M)$ be a Choquet simplex. Then $\mathcal{S}(M)$ is compact. Since $\mathcal{SM}(M)$ is closed, see Proposition 3.2, it is compact, so that it is a Bauer simplex. The converse statement is trivial. □

We note that due to [6, Thm 9.1], it is possible to show that if M is an associated wPEMV-algebra, then M admits a top element iff $\mathcal{S}(M)$ is a Choquet simplex iff it is a Bauer simplex. Example 5.2 shows that this equivalence does not hold for all wPEMV-algebras.

Proposition 6.4. *Let M be a wPEMV-algebra whose state space is non-empty. If $I = \bigcap\{\mathrm{Rad}(s)\colon s \in \mathcal{S}(M)\}$, then I is a normal ideal such that M/I is a commutative Archimedean wPEMV-algebra and $\mathcal{S}(M/I)$ is a simplex that is affinely homeomorphic to the simplex $\mathcal{S}(M)$ in the weak topology of states.*

Proof. As in the proof of Theorem 6.2, we have that I is a normal ideal of M, so that M/I is an Archimedean wPEMV-algebra such that $x/I = y/I$ iff $s(x) = s(x \wedge y) = s(y)$ for each $s \in \mathcal{S}(M)$. In addition, M/I is commutative.

If we define, for each $s \in \mathcal{S}(M)$, the function $\hat{s}(x/I) := s(x)$ $(x \in I)$, then $\hat{s}$ is a state on the wPEMV-algebra M/I. Conversely, if μ is a state on M/I, then $s_\mu : x \mapsto \mu(x/I)$, $x \in M$, is a state on M. The mapping $\Phi : \mathcal{S}(M) \to \mathcal{S}(M/I)$ defined by $\Phi(s) := \hat{s}$, $s \in \mathcal{S}(M)$, is an affine isomorphism (an isomorphism preserving convex combinations) of the state spaces, so it injectively maps state-morphisms of M onto state-morphisms of M/I. Moreover, a net $(s_\alpha)_\alpha$ converges weakly to a state s iff $(\hat{s}_\alpha)_\alpha$ converges to $\hat{s}$. □

Example 6.5. *We note that if M is a wPEMV-algebra that is not associated, then it can happen, for each state s on M, there is an idempotent element $a \in \mathcal{I}(M)$ such that $s(a) = 1$. Indeed, let $M_1 = M_2 = [0,1]$ and $M_3 = \mathbb{Z}^+$. Then $M = M_1 \times M_2 \times M_3$ is a non-associated wPEMV-algebra with this property, because $\mathcal{SM}(M) = \{s_1, s_2\}$, where $s_1(u,v,z) = u$, $s_2(u,v,z) = v$, $(u,v,z) \in M$ and each state s on M is of the unique form $s = \lambda s_1 + (1-\lambda)s_2$. Then $s(1,1,0)$ is an idempotent element such that $s(1,1,0) = 1$.*

For the main result of the present section, we need the following notions from [9].

Let $\mathcal{B}(K)$ be the Borel σ-algebra of a Hausdorff topological space K generated by all open subsets of K. Every element of $\mathcal{B}(K)$ is said to be a *Borel set* and each σ-additive measure on it is said to be a *Borel measure.* We recall that a Borel measure μ on $\mathcal{B}(K)$ is called *regular* if

$$\inf\{\mu(O):\ Y \subseteq O,\ O \text{ open}\} = \mu(Y) = \sup\{\mu(C):\ C \subseteq Y,\ C \text{ compact}\}, \tag{5}$$

for any $Y \in \mathcal{B}(K)$. For example, let δ_x be the Dirac measure concentrated at the point $x \in K$, i.e., $\delta_x(Y) = 1$ iff $x \in Y$, otherwise $\delta_x(Y) = 0$, then every Dirac measure is a regular Borel probability measure whenever K is a Hausdorff locally compact space, as it can be easily seen.

The following theorem is a generalization of [4, Thm 7.2] and [6, Thm 9.2] which were proved for integral representations of states on EMV-algebras and pseudo EMV-algebras. The proof of the next theorem follows the basic steps from the proof of [4, Thm 7.2] improved for our states on wPEMV-algebras which are not necessarily associated wPEMV-algebras.

Theorem 6.6. [Integral Representation of States] *Let M be a wPEMV-algebra with a non-empty state-space. For every state s on M, there is a unique regular Borel probability measure μ_s on the Borel σ-algebra $\mathcal{B}(\mathcal{S}(M))$ with $\mu_s(\mathcal{SM}(M)) = 1$ such that*

$$s(x) = \int_{\mathcal{SM}(M)} \hat{x}(t)\, \mathrm{d}\mu_s(t), \quad x \in M, \tag{6}$$

where $\hat{x}$ $(x \in M)$ is a continuous affine mapping from $\mathcal{S}(M)$ into the interval $[0,1]$ *such that $\hat{x}(s) := s(x)$, $s \in \mathcal{S}(M)$.*

Proof. According to Proposition 6.4, if $I = \bigcap\{\mathrm{Rad}(s)\colon s \in \mathcal{S}(M)\}$, the state spaces of M and M/I are affinely homeomorphic simplices and the conditions of Theorem are satisfied also for the commutative wPEMV-algebra M/I. So we can assume without loss of generality that $M \cong M/I$.

(1) If M is with top element, then M is equivalent to an MV-algebra and states on M are in fact states and the integral representation holds for s due to [11, 12] or due to [4, Thm 7.2]. Thus in what follows, we assume that M is without top element. Then due to Theorem 5.7, $\mathcal{SM}(M)$ is locally compact but not necessarily compact.

(2) *Existence of a Borel probability measure* μ_s. Given $x \in M$, we define a mapping $\hat{x} : \mathcal{S}(M) \to [0,1]$ by $\hat{x}(s) := s(x)$, $s \in \mathcal{S}(M)$. Then $\hat{x}$ is a continuous and affine mapping (= it preserves convex combinations). Every $\hat{x}$ can be uniquely extended by Proposition 4.3 to a continuous affine mapping $\bar{\hat{x}}$ defined on the Bauer simplex $\mathcal{S}(N)$.

In a similar way, for each $y \in N$, we define $\hat{y} : \mathcal{S}(N) \to [0,1]$ such that $\hat{y}(s) = s(y)$, $s \in \mathcal{S}(N)$. If $y \in M$, then $\bar{\hat{y}}(\tilde{s}) = \hat{y}(s)$, $s \in \mathcal{SM}(M)$.

Since N is in fact an MV-algebra, by (1), for every state $s \in \mathcal{S}(N)$, there is a unique regular Borel probability measure μ_s on $\mathcal{B}(\mathcal{S}(N))$ with $\mu_s(\mathcal{SM}(N)) = 1$ such that

$$s(y) = \int_{\mathcal{SM}(N)} \hat{y}(u)\,\mathrm{d}\mu_s(u), \quad y \in N,$$

so that the probability measure μ_s is concentrated on $\mathcal{SM}(N)$.

Define a mapping $\phi : \mathcal{S}(M) \to \mathcal{S}(N)$ by $\phi(s) = \tilde{s}$, $s \in \mathcal{S}(M)$, where $\tilde{s}$ is given by (4.1). Then ϕ is injective, affine, continuous, and open.

Now let s be a state on M, $\tilde{s}$ be its unique extension to a state on N defined by (4.1), and let $x \in M$. There is a unique regular Borel measure $\mu_{\tilde{s}}$ on $\mathcal{B}(\mathcal{S}(N))$ with $\mu_{\tilde{s}}(\mathcal{SM}(N)) = 1$ such that, for all $x \in M$, we have

$$s(x) = \tilde{s}(x) = \int_{\mathcal{SM}(N)} \bar{\hat{x}}(u)\,\mathrm{d}\mu_{\tilde{s}}(u).$$

Since s is a state on M, there is $x_0 \in M$ such that $s(x_0) = 1$, that is

$$1 = s(x_0) = \tilde{s}(x_0) = \int_{\mathcal{SM}(N)} \overline{\widehat{x_0}}(u)\,\mathrm{d}\mu_{\tilde{s}}(u).$$

But $\overline{\widehat{x_0}}(u) \le 1$ for each $u \in \mathcal{SM}(N)$, so that, the set $\{u \in \mathcal{SM}(N) \colon \overline{\widehat{x_0}}(u) < 1\}$ is open and $\mu_{\tilde{s}}(\{u \in \mathcal{SM}(N) \colon \overline{\widehat{x_0}}(u) < 1\}) = 0$. Moreover, the singleton $\{s_\infty\}$ is a closed set and it is a subset of $\{u \in \mathcal{SM}(N) \colon \overline{\widehat{x_0}}(u) < 1\}$, therefore, $\mu_{\tilde{s}}(\{s_\infty\}) = 0$ for each state s on M.

Put $\mu_s(A) = \mu_{\tilde{s}}(\phi(A))$, $A \in \mathcal{B}(\mathcal{S}(M))$. Then μ_s is a σ-additive measure on $\mathcal{B}(\mathcal{S}(M))$ such that $\mu_s(\mathcal{SM}(M)) = \mu_{\tilde{s}}(\phi(\mathcal{SM}(M))) = \mu_{\tilde{s}}(\phi(\mathcal{SM}(M))) + \mu_{\tilde{s}}(\{s_\infty\}) = \mu_{\tilde{s}}(\mathcal{SM}(N)) = 1$, so μ_s is a probability measure, and from the above, we have

$$s(x) = \int_{\phi(\mathcal{SM}(M))} \bar{\hat{x}}(u)\,\mathrm{d}\mu_{\tilde{s}}(u) = \int_{\mathcal{SM}(M)} \bar{\hat{x}}(\phi(t))\,\mathrm{d}\mu_{\tilde{s}}(\phi(t)) = \int_{\mathcal{SM}(M)} \hat{x}(t)\,\mathrm{d}\mu_s(t).$$

(3) μ_s *is a regular measure on* $\mathcal{B}(\mathcal{S}(M))$. We claim that μ_s is a regular Borel probability measure on $\mathcal{B}(\mathcal{S}(M))$. Let $Y \in \mathcal{B}(\mathcal{S}(M))$. If we put $\mu_s^*(Y) := \inf\{\mu_s(O) \colon Y \subseteq O, O \text{ open in } \mathcal{S}(M)\}$, then

$$\begin{aligned}
\mu_s^*(Y) &= \inf\{\mu_s(O) \colon Y \subseteq O, O \text{ open in } \mathcal{S}(M)\}\\
&= \inf\{\mu_{\tilde{s}}(\phi(O)) \colon Y \subseteq O, O \text{ open in } \mathcal{S}(M)\}\\
&= \inf\{\mu_{\tilde{s}}(\phi(O)) \colon \phi(Y) \subseteq \phi(O), O \text{ open in } \mathcal{S}(M)\}\\
&\ge \inf\{\mu_{\tilde{s}}(O_N) \colon \phi(Y) \subseteq O_N, O_N \text{ open in } \mathcal{S}(N)\}\\
&= \mu_{\tilde{s}}(\phi(Y)) = \mu_s(Y).
\end{aligned}$$

Given O_N open in $\mathcal{S}(N)$ with $\phi(Y) \subseteq O_N$, we have $Y \subseteq \phi^{-1}(O_N)$ and $\phi(Y) \subseteq \phi(\phi^{-1}(O_N))$. So for O_N, there is O (= $\phi^{-1}(O_N)$) open in $\mathcal{S}(M)$ such that $\phi(Y) \subseteq \phi(O)$, so that

$$\begin{aligned}
\mu_s^*(Y) &\ge \inf\{\mu_{\tilde{s}}(O_N) \colon \phi(Y) \subseteq O_N, O_N \text{ open in } \mathcal{S}(N)\}\\
&\ge \inf\{\mu_{\tilde{s}}(\phi(O)) \colon \phi(Y) \subseteq \phi(O), O \text{ open in } \mathcal{S}(M)\} = \mu_s^*(Y)\\
&= \inf\{\mu_{\tilde{s}}(O_N) \colon \phi(Y) \subseteq O_N, O_N \text{ open in } \mathcal{S}(N)\}.
\end{aligned}$$

We note, if C is a compact set of $\mathcal{S}(M)$, then $\phi(C)$ is compact in $\mathcal{S}(N)$. On the other hand, if C_N is a compact set in $\mathcal{S}(N)$ such that $C_N \subseteq \phi(Y)$, then $C = \phi^{-1}(C_N)$ is a compact set in $\mathcal{S}(M)$ because every net $(s_\alpha)_\alpha$ in C has a convergent subnet $(s_{\alpha_\beta})_\beta$ converging to some state on M belonging to C, moreover, $C \subseteq Y$. Hence, if we set $\mu_*(Y) = \sup\{\mu_s(C) \colon C \subseteq Y, C \text{ compact in } \mathcal{S}(M)\}$, we have

$$
\begin{aligned}
\mu_*(Y) &= \sup\{\mu_s(C)\colon C \subseteq Y, C \text{ compact in } \mathcal{S}(M)\}\\
&= \sup\{\mu_{\tilde{s}}(\phi(C))\colon C \subseteq Y, C \text{ compact in } \mathcal{S}(M)\\
&= \sup\{\mu_{\tilde{s}}(\phi(C))\colon \phi(C) \subseteq \phi(Y), C \text{ compact in } \mathcal{S}(M)\}\\
&= \sup\{\mu_{\tilde{s}}(C_N)\colon C_N \subseteq \phi(Y), C_N \text{ compact in } \mathcal{S}(N)\}\\
&= \mu_{\tilde{s}}(\phi(Y)) = \mu_s(Y),
\end{aligned}
$$

which proves μ_s is a regular Borel probability measure on $\mathcal{B}(\mathcal{S}(M))$ with $\mu_s(\mathcal{SM}(M)) = 1$.

(4) *Uniqueness of* μ_s. Using ideas from [4, Thm 7.2] and the proof of (3), we can show: If ν is a regular Borel probability measure on $\mathcal{B}(\mathcal{S}(M))$ with $\nu(\mathcal{SM}(M)) = 1$, then $\nu^\phi(A) := \nu(\phi^{-1}(A))$, $A \in \mathcal{S}(N)$, is a regular Borel probability measure on $\mathcal{B}(\mathcal{S}(N))$. From that it is possible to establish the uniqueness of μ_s. □

We have to note that the uniqueness of μ_s in the latter theorem is guaranteed by the assumptions $\mu_s(\mathcal{SM}(M)) = 1$. Without this condition, we can have more regular probability measures μ for a state s satisfying equation $s(x) = \int_{\mathcal{S}(M)} \hat{x}(t)\,\mathrm{d}\mu(t)$, $x \in M$. For example, let δ_s be the Dirac measure concentrated in the point $s \in \mathcal{S}(M)$. If s is not a state-morphism, then $\delta_s(\mathcal{SM}(M)) = 0$ although for this measure, we have $s(x) = \int_{\mathcal{S}(M)} \hat{x}(t)\,\mathrm{d}\delta_s(t)$, $x \in M$. Therefore, if $s \in \mathcal{SM}(M)$, then the uniqueness of the Borel regular probability measure μ_s entails $\mu_s = \delta_s$.

On the other side, if μ is a regular Borel probability measure, $\mu(\mathcal{SM}(M)) = 1$, then the right-hand side of formula (6) defines always a pre-state on M. If $\mathcal{SM}(M)$ is compact, then the pre-state is a state. But if $\mathcal{SM}(M)$ is not compact, only locally compact, the pre-state is not necessarily a state. For example, if M has at most countably many state-morphisms, $s_1, s_2, \ldots$, then if s is a state on M, there is a unique sequence $(\lambda_n)_n$ of reals from the interval $[0,1]$, $\sum_n \lambda_n = 1$, such that $s = \sum_n \lambda_n s_n$. Indeed, if we take $\lambda_n = \mu_s(\{s_n\})$ for each n, then s is a σ-convex combination of s_n's. But not every σ-convex combinations of s_n's is a state on M. Indeed, take Example 4.1, then $s : M \to [0,1]$ is a state iff s is a finite convex combinations of s_n's, i.e. $\mathcal{S}(M) = \mathrm{Con}(\mathcal{SM}(M))$. Consequently, a σ-additive probability measure $\mu = \sum_n \lambda_n \delta_{s_n}$ with $\lambda_n \in (0,1)$ and $\sum_n \lambda_n = 1$ defines by (6) only a pre-state and not a state, and it does define a state iff μ is only a finite convex combination of δ_{s_n}'s.

Using ideas from the latter theorem and the proof of [4, Thm 7.2], we can establish the following result.

Theorem 6.7. *Let M be a proper wPEMV-algebra. There is a one-to-one correspondence between the set of regular Borel probability measures on $\mathcal{B}(\mathcal{SM}(M))$ and the set of regular Borel probability measures on $\mathcal{B}(\mathcal{SM}(N))$ vanishing at $\{s_\infty\}$, where N is its representing EMV-algebra with top element.*

Remark 6.8. *Let a wPEMV-algebra M satisfy the conditions of Theorem* 6. *Using the homeomorphism between $\mathcal{SM}(M)$ and* $\mathrm{MaxN}_0(M)$ *established in Theorem* 5.5*, the integral representation* (6) *of states on M can be reformulated in an equivalent way: For any state s on the wPEMV-algebra M, there is a unique regular probability measure ν_s on the Borel σ-algebra $\mathcal{B}(\mathrm{MaxN}_0(M))$ with $\nu_s(\mathrm{MaxN}_0(M)) = 1$ such that*

$$
s(x) = \int_{\mathrm{MaxN}_0(M)} x^*\,\mathrm{d}\nu_s, \quad x \in M,
$$

where $x^*(I) = x/I \in [0,1]$, $I \in \mathrm{MaxN}_0(M)$.

7 Conclusion

We have introduced states and state-morphisms on a wPEMV-algebra M. If M has a top element, then M is equivalent to a pseudo MV-algebra, so the properties of states are known. Therefore, the main essence of the paper is to study states when M has no top element. In such a case, the situation is more complicated and the results are generalizing ones known for pseudo EMV-algebras. In some cases the proofs are similar to original ones but in general, it was necessary to present a new machinery.

The paper is divided into two parts.

Part I: States are defined as finitely additive mappings with values in the real interval $[0,1]$ attaining the value 1. We have shown that even non-trivial commutative wPEMV-algebras can be stateless which in the case of MV-algebras or EMV-algebras is impossible. A characterization of state-morphisms as extremal states was established in Theorem 3.7 and they are in a one-to-one correspondence with maximal and normal ideals with a special property, see Theorem 3.6.

Part II: The weak topology of states is introduced and it is shown that every state is a weak limit of a net of convex combinations of state-morphisms, Theorem 5.1, but not every such a limit defines a state on M if M has no top element. We have established that if M has no top element, the set of state-morphisms on a wPEMV-algebra without top element is locally compact, Theorem 5.7, whose one-point compactification is the set of state-morphisms on the representing wPEMV-algebra, Theorem 5.7. Moreover, the set of state-morphisms is homeomorphic to a special set of maximal and normal ideals, see Theorem 5.5. Finally, we show that every state can be represented as a standard integral over a unique regular Borel σ-additive measure, Theorem 6.6. It generalizes an analogous situation for states on MV-algebras established in [11, 12].

The paper is gathering and generalizing important results for states on wPEMV-algebras. We underline that such algebras do not possess necessarily top element. Many results were possible to formulate for each wPEMV-algebra and all states or we have assumed that, for any state s, there is an idempotent element $a \in M$ such that $s(a) = 1$. The question, is this assumption necessary or is it superfluous?

The future research on states will be devoted to states on unital proper wPEMV-algebras with a fixed strong unit where states are additive positive real-valued mappings taking the value 1 in the fixed strong unit. These algebras allow also states of this form on conic wPEMV-algebras.

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