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Original paper

The modularity equation for Mayor's aggregation operators and uninorms

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Abstract

The focus of this paper is to investigate the modularity equation involving uninorms and Mayor's aggregation operators. Necessary and sufficient conditions are established for this equation. And, it finds that the modularity equation of a Mayor's aggregation operator over a uninorm is reduced to the modularity of a commutative semi-t-norm over a uninorm and the modularity equation of a uninorm over a Mayor's aggregation operator is reduced to the modularity equation of a uninorm over a commutative semi-t-conorm. Among them, the cases of a uninorm which is locally internal on the boundary are studied in [35]. In this paper, we consider whether the neutral element e of the uninorm is idempotent element of the Mayor's aggregation operator in modularity equation to get solutions in the corresponding cases.

Keywords: Modularity equation, Mayor's aggregation operators, uninorms.

1 Introduction

The theory of aggregation operators (e.g. [2, 3, 4, 17]) has been proved to be an essential tool in various fields, such as applied mathematics, computer sciences, fuzzy set theory and so on.

Mayor's aggregation operators are introduced by Mayor in [27] generalized t-norms and t-conorms by omitting associativity and changing boundary conditions. Moreover, for Mayor's aggregation operators, there have been various discussion, such as, the distributivity equation for Mayor's aggregation operators has been studied (e.g. [5, 32, 43]).

Uninorm is a special class of aggregation operators which is the extension of t-norm and t-conorm by changing the neutral element. They firstly use the term uninorm in [40] with the idea of allowing a certain kind of aggregation operators to combine maximum and minimum, depending on changing the neutral element. Moreover, uninorms are widely studied in theory (e.g. [6, 9, 12, 16, 37] and applications (e.g. [29, 39]). Especially there have been many rich results in the study of functional equations involving uninorms.

In addition, one of the important topics in theoretical research of aggregation operators is to characterize aggregation operator pairs satisfying functional equations [1]. The functional equations involving aggregation operations (e.g. [3, 9, 10, 11, 15, 19, 24, 32]) are an important link in the field of fuzzy sets and fuzzy logic. Modularity equation is one of the important functional equations involving aggregation operators, and there has been a lot of literature work on its research. The modularity equation between two operators is derived from modular lattice. It can also be regarded as a generalized associativity equation with a constraint, as well as a special case of the distributivity equation. The former plays an essential role in fuzzy set theory, and the latter is highly useful in fuzzy logic.

Up to now, the known researches concerning the modularity equation have reached rich conclusions. Below we will list some of them. In [26], Mas et al. solved the modularity equation for semi-t-operators and two kinds of special uninorms. In [31], the solutions to the modularity equation for nullnorms and uninorms continuous in $(0,1)^2$ are given. Moreover, Su et al. [35] discussed the modularity of uninorms with more general structure. But up to now, the modularity equation involving uninorms is still not completely characterized.

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For other operators, there has been a lot of research. In [41, 42], Zhan et al. investigated the modularity equation with semi-uninorms and semi-t-operators by considering two special classes of semi-uninoms. Fechner and Rak et al. [13, 33] considered the modularity equation involving some common classes of 2-uninorms. In addition, the modularity equations involving overlap and grouping functions have also been studied (see [38, 44, 45] for example).

The remainder of this paper is organized as follows. In Section 2, we will recall some results and structures related to basic fuzzy connectives used in this paper. In Sections 3-5, we will characterize all solutions of the modularity equations for Mayor's aggregation operators and uninorms. Section 6 will conclude this paper and give ideas for the further research.

2 Preliminaries

Firstly, we recall some necessary basic notions.

Definition 2.1. [28] *An aggregation operator is a function $A^n : [0,1]^n \to [0,1]$ that is increasing in each variable and fulfills the following boundary conditions*

$$A^n(0,\ldots,0) = 0 \quad \text{and} \quad A^n(1,\ldots,1) = 1.$$

Since the topic of this paper is the binary aggregation operators, we denote simply A^2 by A.

2.1 Mayor's aggregation operators

Definition 2.2. [27] *A binary operator $F : [0,1]^2 \to [0,1]$ is called a Mayor's aggregation operator (GM aggregation operator for short) if it is commutative aggregation operator and satisfies the boundary conditions $F(0,x) = F(0,1)x$ and $F(1,x) = (1 - F(0,1))x + F(0,1)$ for all $x \in [0,1]$.*

Let $\mathbb{GM}$ denote the family of all GM aggregation operators. The following properties of the GM aggregation operators are essential for the further investigation.

Theorem 2.3. [27] *Let $F : [0,1]^2 \to [0,1]$ be a GM aggregation operator. Then, the following results hold:*

(i) *F is associative if and only if F is a t-norm or a t-conorm;*

(ii) *$F = \min$ or $F = \max$ if and only if $F(0,1) = 0$ or $F(0,1) = 1$, and $F(x,x) = x$ for all $x \in [0,1]$;*

(iii) *F is idempotent if and only if $\min \leq F \leq \max$.*

2.2 Uninorms

Definition 2.4. [40] *A uninorm $U : [0,1]^2 \to [0,1]$ is commutative, associative, increasing in each variable and has a neutral element $e \in [0,1]$.*

Remark 2.5. [20]

(i) *If $e = 0$, then U is a t-conorm and if $e = 1$, then U is a t-norm.*

(ii) *The neutral element e corresponding to a uninorm U is unique.*

(iii) *For any uninorm U we have $U(0,1) \in \{0,1\}$. A uninorm U with $U(0,1) = 0$ is said to be conjunctive, otherwise it is disjunctive.*

(iv) *The structure of a uninorm U with the neutral element $e \in (0,1)$ is always the following. It is like a t-norm on the square $[0,e]^2$, like a t-conorm on the square $[e,1]^2$ and it takes values between minimum and maximum in the other cases $A(e)$.*

The most studied classes of uninorms are:

- Uninorms in $\mathcal{U}_{\min}$ (respectively $\mathcal{U}_{\max}$) [16].
- Representable uninorms $\mathcal{U}_{\text{rep}}$ with additive generators, which were firstly introduced in [16].

- Uninorms continuous in the open unit square $(0,1)^2$, i.e. $\mathcal{U}_{\cos}$, which were characterized in [18] and included the representable uninorms.
- Idempotent uninorms $\mathcal{U}_{\text{ide}}$, that is $U(x,x)=x$ for all $x\in[0,1]$. Their characterization was given in [25, 34].
- Locally internal uninorms $\mathcal{U}_{\text{int}}$, that is $U(x,y)\in\{x,y\}$ for all $(x,y)\in A(e)$. More detailed studies can be found in [6, 7, 8, 9, 12]. This class includes $\mathcal{U}_{\text{ide}}$, $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$.
- Uninorms with continuous underlying operators. This class has had some partial cases found in [12, 14, 21, 22, 30] and further characterization were found in [23, 36, 37] . And obviously, this class includes all the previous ones except for cases of $\mathcal{U}_{\min}$, $\mathcal{U}_{\max}$ and $\mathcal{U}_{\text{int}}$.

Definition 2.6. [35]

(i) *Let U be a conjunctive uninorm. We will say that U is locally internal on the boundary if it satisfies*

$$U(1,y)\in\{y,1\} \quad \textit{for all} \quad y\in[0,1]. \tag{1}$$

(ii) *Let U be a disjunctive uninorm. We will say that U is locally internal on the boundary if it satisfies*

$$U(0,y)\in\{y,0\} \quad \textit{for all} \quad y\in[0,1]. \tag{2}$$

Obviously, the class of U satisfying locally internal on the boundary includes $\mathcal{U}_{\text{ide}}$, $\mathcal{U}_{\min}$, $\mathcal{U}_{\max}$, $\mathcal{U}_{\cos}$ and $\mathcal{U}_{\text{rep}}$.

Theorem 2.7. **[18]** *Suppose U is a uninorm continuous in $(0,1)^2$ with neutral element $e\in(0,1)$. Then one of the following two cases is satisfied:*

(i) *There exist $u\in[0,e)$, $\lambda\in[0,u]$, two continuous t-norms T_1,T_2 and a representable uninorm R such that U can be represented as*

$$U(x,y)=\begin{cases}\lambda T_1(\frac{x}{\lambda},\frac{y}{\lambda}) & \textit{if } x,y\in[0,\lambda],\\ \lambda+(u-\lambda)T_2(\frac{x-\lambda}{u-\lambda},\frac{y-\lambda}{u-\lambda}) & \textit{if } x,y\in[\lambda,u],\\ u+(1-u)R(\frac{x-\lambda}{1-u},\frac{y-\lambda}{1-u}) & \textit{if } x,y\in(u,1),\\ 1 & \textit{if } \min(x,y)\in(\lambda,1] \ \textit{ and } \max(x,y)=1,\\ \lambda \textit{ or } 1 & \textit{if } (x,y)\in\{(\lambda,1),(1,\lambda)\},\\ \min(x,y) & \textit{otherwise.}\end{cases} \tag{3}$$

The above uninorm is denoted by $U_{\cos,\min}$ and the class of all such uninorms by $\mathcal{U}_{\cos,\min}$.

(ii) *There exist $\nu\in(e,1]$, $\omega\in[\nu,1]$, two continuous t-conorms S_1,S_2 and a representable uninorm R such that U can be represented as*

$$U(x,y)=\begin{cases}\nu+(\omega-\nu)S_1(\frac{x-\nu}{\omega-\nu},\frac{y-\nu}{\omega-\nu}) & \textit{if } x,y\in[\nu,\omega],\\ \omega+(1-\omega)S_2(\frac{x-\omega}{1-\omega},\frac{y-\omega}{1-\omega}) & \textit{if } x,y\in[\omega,1],\\ \nu R(\frac{x}{\nu},\frac{y}{\nu}) & \textit{if } x,y\in(0,\nu),\\ 0 & \textit{if } \max(x,y)\in[0,\omega) \ \textit{ and } \min(x,y)=0,\\ \omega \textit{ or } 0 & \textit{if } (x,y)\in\{(0,\omega),(\omega,0)\},\\ \max(x,y) & \textit{otherwise.}\end{cases} \tag{4}$$

The above uninorm is denoted by $U_{\cos,\max}$ and the class of all such uninorms by $\mathcal{U}_{\cos,\max}$.

Let $\mathcal{U}_{\cos}$ denote the family of all such uninorms in Theorem 2.7.

2.3 Modularity equation

Let us first recall the modularity equation of the binary operators.

Definition 2.8. [26] *Let $F,\ G:[0,1]^2\to[0,1]$ be any binary operator. We say that F is modular over G if*

$$z\le x\Rightarrow F(x,G(y,z))=G(F(x,y),z) \quad \textit{for all } x,y,z\in[0,1]. \tag{5}$$

Theorem 2.9. [26] *Let $F,\ G : [0,1]^2 \to [0,1]$ be any binary operators. The following statements holds:*

(i) *if F and G are t-norms, then they fulfill (5) if and only if $F = G$;*

(ii) *if F and G are t-conorms, then they fulfill (5) if and only if $F = G$;*

(iii) *if F is a t-norm, and G is a t-conorm, then they fulfill (5) if and only if $F = \min$ and $G = \max$;*

(iv) *if F is a t-conorm, and G is a t-norm, then Eq. (5) has no solutions.*

3 Modularity equation between Mayor's aggregation operators and t-norms/t-conorms

In this section, we discuss the modularity between t-norms/t-conorms and Mayor's aggregation operators.

Theorem 3.1. *Let F be a GM aggregation operator with $F(0,1) = k \in [0,1]$ and T be a t-norm. Then*

(i) *F is modular over T if and only if $F = T$.*

(ii) *T is modular over F if and only if $F = T$, or $T = \min$ and $F = \max$.*

Proof. **1**. Let F be modular over T, i.e. $F(x, T(y,z)) = T(F(x,y), z)$ for all $z \leq x$.

Applying $x = 1, y = 0$ and $z = 0$, we have that $k = F(1,0) = F(1, T(0,0)) = T(F(1,0), 0) = 0$. Furthermore, applying $y = 1$ and $z \leq x$, we have $F(x,z) = F(x, T(1,z)) = T(F(x,1), z) = T(x,z)$. Hence, according to the commutativity of F and T, it holds that $F = T$.

2. Let T be modular over F, i.e. $T(x, F(y,z)) = F(T(x,y), z)$ for all $z \leq x$.

Applying $y = 0$ and $x = z$, we have $T(x, kx) = T(x, F(0,x)) = F(T(x,0), x) = F(0,x) = kx$. Again taking $z = 0$ and $x = y$, we have $T(x, kx) = T(x, F(x,0)) = F(T(x,x), 0) = kT(x,x)$. That is $kx = kT(x,x)$ for any $x \in [0,1]$, which means either $k = 0$ or $T(x,x) = x$.

Considering $k = 0$, taking $y = 1$, for any $z \leq x \in [0,1]$, we have that $T(x,z) = T(x, F(1,z)) = F(T(x,1), z) = F(x,z)$. According to the commutativity of F and T, it holds that $F = T$.

Considering $k \neq 0$, it means that $T(x,x) = x$. That is $T = \min$. Furthermore, taking $x = k$, $y = 1$ and $z = 0$, we have that $k = T(k,k) = T(k, F(1,0)) = F(T(k,1), 0) = k^2$, which means $k = 1$. So, taking $y = 1$ and $z \leq x$, it holds that $\max(x,z) = x = T(x,1) = T(x, F(1,z)) = F(T(x,1), z) = F(x,z)$. Hence, from the commutativity of F, we can obtain that $G = \max$. To summarize, $F = \max$ and $T = \min$.

Conversely, the results are obvious, so we omit the proof. □

Theorem 3.2. *Let F be a GM aggregation operator with $F(0,1) = k \in [0,1]$ and S be a t-conorm. Then*

(i) *F is modular over S if and only if $F = S$, or $F = \min$ and $S = \max$.*

(ii) *S is modular over F if and only if $F = S$.*

Obviously, a t-norm/t-conorm is a special case of a uninorm with neutral element $e = 0/e = 1$. So in the following content, we will only consider the case of a proper uninorm.

4 Modularity equation of $F \in \mathbb{GM}$ over $G \in \mathcal{U}$

In this section, we will discuss the modularity equation of $F \in \mathbb{GM}$ over $G \in \mathcal{U}$. In particular, we will consider $G \in \mathcal{U}$ in some different cases.

First, we have the following lemma to support this section.

Lemma 4.1. *Let F be a GM aggregation operator with $F(0,1) = k \in [0,1]$, and G be a uninorm with neutral element $e \in (0,1)$. If F is modular over G, then the following statements hold:*

(i) *$k = 0$.*

(ii) *$G(x,y) = \max(x,y)$ for all $(x,y) \in [e,1]^2$.*

Proof. (i) Taking $y = 0, z = 0, 0 < x < e$ in Eq. (5), from the structure of the uninorm and $kx \le e$, we have that $kx = F(x,0) = F(x, G(0,0)) = G(F(x,0),0) = G(kx,0) = 0$. Then $k = 0$.

(ii) From $k = 0$, it means $F(0,x) = 0$ and $F(1,x) = x$. Applying $y = 1, e \le z \le x$ in Eq. (5), it follows from the structure of the uninorm that $x = F(x,1) = F(x, G(1,z)) = G(F(x,1),z) = G(x,z)$. Then because of the commutativity of G, it follows that $G(x,y) = \max(x,y)$ for all $(x,y) \in [e,1]^2$. □

Lemma 4.2. *Let F be a GM aggregation operator with $F(0,1) = k \in [0,1]$, and G be a uninorm with neutral element $e \in (0,1)$. If F is modular over G, then $G(1,0) = 0$, i.e. G is a conjunctive uninorm.*

Proof. For the uninorm G, we have $G(0,1) \in \{0,1\}$. And from Lemma 4.1, it follows that $k = 0$, i.e. $F(0,x) = 0$ and $F(1,x) = x$.

First, assume that $G(1,0) = 1$. Applying $x \in (0,e)$, $y = 1$ and $z = 0$ in Eq. (5), we have that $x = F(x,1) = F(x, G(1,0)) = G(F(x,1),0) = G(x,0) = 0$, which contradicts $x > 0$.

Hence $G(1,0) = 0$, i.e. G is a conjunctive uninorm. □

Remark 4.3. *If $F \in \mathbb{GM}$ is modular over $G \in \mathcal{U}$, then*

(i) *the GM aggregation operator F must be a commutative semi-t-norm;*

(ii) *for the uninorm $G \in \mathcal{U}$ with underlying operators T_G and S_G, we have $S_G = \max$;*

(iii) *the uninorm G must be a conjunctive uninorm.*

Lemma 4.4. *Let F be a GM aggregation operator with $F(0,1) = k \in [0,1]$, and G be a uninorm with neutral element $e \in (0,1)$. If F is modular over G, then $0 < G(x,1) < 1$ for all $x \in (0,e)$.*

Proof. For $x \in (0,e)$, it follows from the structure of uninorms that $x \le G(1,x) \le 1$. Then $G(1,x) > 0$ for all $x \in (0,e)$. Assume that there exists $x_0 < e$ such that $G(1,x_0) = 1$, i.e. $G(1,x) = 1$ for all $x \in [x_0,1]$. If F is modular over G, taking $x_0 \le z < x = e$ and $y = 1$, it has $e = F(e,1) = F(e, G(1,z)) = G(F(e,1),z) = G(e,z) = z$, which contradicts $z < e$. Thus $G(1,x) < 1$ for all $x \in (0,e)$. To sum up, $0 < G(x,1) < 1$ for all $x \in (0,e)$. □

Remark 4.5. *The modularity of a Mayor's aggregation operator over a uninorm is reduced to the modularity of a commutative semi-t-norm over a uninorm. Theorem 4 in* [35] *is perfectly suitable for the characterization of the modularity of a commutative semi-t-norm over a uninrom which is locally internal on the boundary.*

From Theorem 4 in [35] and Lemma 4.4, it's easy to find that $G \notin (\mathcal{U}_{\cos} \cup \mathcal{U}_{rep} \cup \mathcal{U}_{\max})$ when a Mayor's aggregation is modular over a uninorm G.

Since the GM aggregation operator F is a commutative semi-t-norm when F is modular over a uninorm, we have $F(e,e) \le e$. Now, let's talk about the case when $F(e,e) = e$.

Theorem 4.6. *Let $F \in \mathbb{GM}$ such that $F(0,1) = k \in [0,1]$ and G be a uninorm with neutral element $e \in (0,1)$. If $F(e,e) = e$, then F is modular over G if and only if*

$$F(x,y) = \begin{cases} eT(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ \min(x,y) & \text{otherwise}, \end{cases} \tag{6}$$

$$G(x,y) = \begin{cases} eT(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ \max(x,y) & \text{if } (x,y) \in [e,1]^2, \\ \min(x,y) & \text{otherwise}, \end{cases} \tag{7}$$

where $T : [0,1]^2 \to [0,1]$ is a t-norm.

Proof. (⇒) Let F be modular over G and $F(e,e) = e$. $F(x,e) = e$ for all $x \in [e,1]$.

Applying $y = e \le x$, $z \in [0,1]$ and $z \le x$, it has $F(x,z) = F(x, G(e,z)) = G(F(x,e),z) = z$. Then by the commutativity of F, it holds that $F = \min$ on $[0,1]^2 \setminus [0,e]^2$.

Again taking $z < e = x \le y$, it has $\min(e, G(y,z)) = F(e, G(y,z)) = G(F(e,y),z) = G(e,z) = z$. Then by $z < e$, it holds that $G(y,z) = z$. That is $G = \min$ on $[0,e) \times [e,1] \cup [e,1] \times [0,e)$.

Furthermore, applying $z \le x \le e$ and $y = 1$, it has $F(x,z) = F(x, G(1,z)) = G(F(x,1),z) = G(x,z)$ when $z < e$, and $F(x,e) = G(x,e)$. Then by the commutativity, it holds that $F = G$ on $[0,e]^2$.

Thus together with Lemma 4.1 (ii), there exists a t-norm T such that F has the form (6) and G has the form (7).

(⇐) The result is obvious by Theorem 4 in [35], so we omit it. □

For the case of $F(e,e) < e$, we do not discuss it in this paper because the solution is more complicated. But, We will continue to discuss the case of $F(e,e) < e$ in future work.

Remark 4.7. *By comparing Theorem 4.6 with Theorem 4 in* [35]*, we can easily find that the solution to the modularity equation is the same. the additional condition $F(e,e) = e$ is considered in Theorem 4.6, but Theorem 4 in* [35] *requires that the uninorm is locally internal on the boundary. In other words, for the modularity equation of a Mayor's aggregation operator over a uninorm, the equation satisfying $F(e,e) = e$ is equivalent to that requiring the uninorm is locally internal on the boundary.*

5 Modularity equation of $F \in \mathcal{U}$ over $G \in \mathbb{GM}$

Similarly, in this section, we will discuss the modularity equation of $F \in \mathcal{U}$ over $G \in \mathbb{GM}$ by considering some different cases.

Lemma 5.1. *Let F be a uninorm with neutral element $e \in (0,1)$, and G be a GM aggregation operator with $G(0,1) = k \in [0,1]$. If F is modular over G, then*

(i) *$F(x, ky) = kF(x,y)$ for all $x, y \in [0,1]$.*

(ii) *$F(1,k) = k$.*

(iii) *$k = 1$ or $F(x,0) = 0$ for all $x \in [0,1]$.*

Proof. (i) If F is modular over G, then taking $z = 0$ in Eq. (5), we have that $F(x, ky) = F(x, G(y,0)) = G(F(x,y),0) = kF(x,y)$ for all $x, y \in [0,1]$.

(ii) Considering $x = y = 1$ in $F(x, ky) = kF(x,y)$, we that $F(1,k) = k$.

(iii) Similarly, considering $y = 0$ in $F(x, ky) = kF(x,y)$, we have $F(x,0) = kF(x,0)$, which means $k = 1$ or $F(x,0) = 0$ for all $x \in [0,1]$. □

Lemma 5.2. *Let $F \in \mathcal{U}$ with neutral element $e \in (0,1)$, and $G \in \mathbb{GM}$ such that $G(0,1) = k \in [0,1]$. If F is modular over G, then $k = 1$ and $F(0,1) = 1$.*

Proof. When F is modular over G, we obtain from Lemma 5.1 (iii) that $k = 1$ or $F(x,0) = 0$.

Assume that $F(x,0) = 0$ for all $x \in [0,1]$. According to the modularity equation, we have the following discussion.

Applying $y = 0$ and $x = 1$ and $z = e$ in Eq. (5), we have $F(1, ke) = F(1, G(0,e)) = G(F(1,0),e) = G(0,e) = ke$. Furthermore, applying $y = e$ and $x = 1$ and $z = 0$ in Eq. (5), we have $F(1, ke) = F(1, G(e,0)) = G(F(1,e),0) = G(1,0) = k$. That is $ke = k$. And from the definition of GM aggregation operators, we have $G(1,e) = (1-k)e + k = e - ke + k = e$.

Moreover, considering $y = 1$, $z = e$ and $x > e$ in Eq. (5), we have $x = F(x,e) = F(x, G(1,e)) = G(F(x,1),e) = G(1,e) = e$, which contradicts $x > e$.

Hence, $F(x,0) = 0$ for all $x \in [0,1]$ is impossible. That is $F(0,1) > 0$. Note that $F(0,1) \in \{0,1\}$. Then $F(0,1) = 1$. From Lemma 5.1 (iii) and $F(0,1) > 0$, it has $k = 1$. □

Lemma 5.3. *Let $F \in \mathcal{U}$ with neutral element $e \in (0,1)$, and $G \in \mathbb{GM}$ such that $G(0,1) = k \in [0,1]$. If F is modular over G, then $F(x,y) = \min(x,y)$ for all $(x,y) \in [0,e]^2$.*

Proof. It's easy to get the result by applying $y = 0$ and $0 \le z \le x \le e$ in Eq. (5). □

Remark 5.4. *By the above lemma, we find that F is a disjunctive uninorm and G is a commutative semi-t-conorm, when $F \in \mathcal{U}$ is modular over $G \in \mathbb{GM}$.*

Then, obviously, the modularity of a uninorm over a Mayor's aggregation operator is reduced to the modularity of a uninorm over a commutative semi-t-conorm. Theorem 6 in [35] will completely apply in the characterization of the modularity of a uninrom which is locally internal on the boundary over a commutative semi-t-conorm.

Lemma 5.5. *Let $F \in \mathcal{U}$ with neutral element $e \in (0,1)$, and $G \in \mathbb{GM}$ such that $G(0,1) = k \in [0,1]$. If F is modular over G, then $0 < F(x,0) < 1$ for all $x \in (e,1)$.*

Proof. Assume that there exists $x_1 \in (e,1]$ such that $F(0,x_1) = 0$. If F is modular over G, taking $e = z < x = x_1$ and $y = 0$, it has $x_1 = F(x_1,e) = F(x_1, G(0,e)) = G(F(x_1,0),e) = G(0,e) = e$, which contradicts $e < x_1$. Thus $0 < F(x,0) < 1$ for all $x \in (e,1)$. □

Remark 5.6. *From Lemma 5.5 and Theorem 6 in* [35]*, it's easy to find that* $F \notin (\mathcal{U}_{\cos} \cup \mathcal{U}_{rep} \cup \mathcal{U}_{\min})$ *when* $F \in \mathcal{U}$ *is modular over* $G \in \mathbb{GM}$*.*

Since the GM aggregation operator G is a commutative semi-t-conorm when a uninorm is modular over G, we have $G(e,e) \geq e$. Next, we start with the case $G(e,e) = e$.

Theorem 5.7. *Let* F *be a uninorm with neutral element* $e \in (0,1)$*, and* $G \in \mathbb{GM}$ *such that* $G(0,1) = k \in [0,1]$*. If* $G(e,e) = e$*, then* F *is modular over* G *if and only if*

$$F(x,y) = \begin{cases} \min(x,y) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)S(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ \max(x,y) & \text{otherwise}, \end{cases} \tag{8}$$

$$G(x,y) = \begin{cases} e + (1-e)S(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ \max(x,y) & \text{otherwise}, \end{cases} \tag{9}$$

where $S : [0,1]^2 \to [0,1]$ *is a t-conorm.*

Proof. It is similar to that of Theorem 4.6, so we omit it. □

Similarly, for the case of $G(e,e) > e$, we do not discuss it in this paper because the solution is more complicated. But, We will continue to discuss the case of $G(e,e) > e$ in future work.

Remark 5.8. *By comparing Theorem 5.7 with Theorem 6 in* [35]*, we can easily find that the solution to the modularity equation is the same. the additional condition* $G(e,e) = e$ *is considered in Theorem 5.7, but Theorem 6 in [35] requires that the uninorm is locally internal on the boundary. In other words, for the modularity equation of a uninorm over a Mayor's aggregation operator, the equation with the assumption that e is idempotent element of the Mayor's aggregation operator has same solution to that requiring the uninorm is locally internal on the boundary.*

6 Conclusion and further work

In this paper, we have investigated the modularity between Mayor's aggregation operators and uninorms. Firstly, we have considered the modularity equation for a Mayor's aggregation operator and a t-norm/t-conorm in Section 3, and found that they are either the same t-norm, or the same t-conorm, or $F = \min$ and $G = \max$. Next, in Section 4 and 5, by adding some conditions, we considered that one of the unknown functions in the modularity equation is a Mayor's aggregation operator, and the other is a uninorm. In the future work, we will concentrate on the modularity equation with the neutral element e which is not the idempotent element of Mayor's aggregation operator.

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