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Original paper

# Are multidimensional RDM interval arithmetic and constrained interval arithmetic one and the same?

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**Abstract**

This article discusses the comments made by some scientists that multidimensional interval arithmetic (MIA) is the same as constraint interval arithmetic (CIA) and multidimensional fuzzy arithmetic (MFA) is the same as constraint fuzzy arithmetic (CFA). Both types of arithmetic are briefly presented and then the difference in their dimensions, calculation methods, differences in the obtained results and the way they are used in complex calculations are shown. The answer to the question posed is presented in the conclusions.

*Keywords:* Constraint interval arithmetic, constraint fuzzy arithmetic, multidimensional fuzzy arithmetic, RDM interval arithmetic, RDM fuzzy arithmetic.

## 1 Introduction

Interval arithmetic (IA) and fuzzy (FA) are two types of uncertainty arithmetic (UA) that are constantly evolving and increasingly important. The reason for this is that most real problems have uncertain data. Such problems usually cannot be reliably solved using classical arithmetic and math of crisp numbers. The importance of UA is also evidenced by the fact that scientists are constantly interested in it and that they develop new and new methods of calculations on uncertain numbers. The sheer number of types of IA currently in existence now exceeds 12, and yet new types of this arithmetic are constantly being developed [3]. The same is with FA. However, scientists are working not only on new and improved types of UA, but also on their practical applications in all fields. Very often these are problems of decision making, i.e. the selection of optimal solutions to a problem from the set of all possible solutions to it. Examples of such problems are the optimal allocation of resources solved with the use of fuzzy linear programming [6], finding the shortest path in networks with uncertain weights [7], optimal solution of transport tasks with uncertain transport costs [8], solving various linear programming tasks [6]. It should be noted that to solve problems with fuzzy data, interval arithmetic can always be used, because fuzzy numbers can be decomposed into a set of alpha-cuts which are intervals. Hence all kinds of IA can be used in fuzzy problems. Therefore, IA seems to be the most important arithmetic of all kinds of uncertainty arithmetic. The existence of many types of UA causes a certain scientific chaos, which was noticed and described by A. Sotoudeh Anvari in his article [41]. On the one hand, this chaos consists in different types of UA delivering (usually) different results for one and the same problem. On the other hand, 2 different types of UA may sometimes seem very similar or even identical to some scientists, when in fact they are not. This may be the case with constrained interval arithmetic and multidimensional RDM IA. This is evidenced by some of the opinions that appear. Therefore, it is necessary to explain the differences between these types of IA, which is also the goal and motivation of the authors of this article.

So, there is constraint interval arithmetic (CIA), the concept of which was developed by W.A. Lodwick [15]. Then, with his colleagues he applied it in fuzzy arithmetic in the form of constraint fuzzy arithmetic (CFA), [4, 16, 17, 18],

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which is based on $\alpha$-cuts. There are also multidimensional interval arithmetic (MIA) and multidimensional fuzzy arithmetic (MFA), the concept of which was first proposed by A. Piegat and developed later with his colleagues. MIA and MFA [14, 28, 29, 33, 34, 35, 36] aroused the interest of scientists from all over the world. This is evidenced by their applications in decision-making problems [2, 38, 39], in the arithmetic of Z-numbers [21], in the control of dynamic systems [21, 19, 25, 26], in problems of differentiability of fuzzy functions [20, 22, 27], in technical problems [1, 43], in solving uncertain linear and nonlinear equations [11, 12, 13], in development of multidimensional fuzzy arithmetic Type-2 [30, 31]. The number of publications on MIA and MFA and their applications is now approaching 90. It is alleged in [23] that the MIA is the same as the CIA and, accordingly, that the MFA is the same as the CFA. Hence the motivation of our article is to present the differences between both arithmetic types. In fact, there is a certain similarity between them. But there are also significant differences in their philosophy, dimensionality, and the type and amount of results delivered. Explaining these differences will prevent the formation of misconceptions about multidimensional MIA and MFA. The aim of our article is also to deepen understanding of the benefits of a multidimensional approach to interval and fuzzy arithmetic. All currently existing types of interval arithmetic are 1-dimensional and of fuzzy arithmetic are 2-dimensional. This low dimensionality is the cause of the chaos that exists today in uncertainty arithmetic, which was very aptly described in [41] by A. Sotoudeh-Anvari. This article makes us realize that it is time to move from low dimensional arithmetic to a more precise multidimensional one.

Section 2 will introduce the CIA and the CFA, Section 3 will introduce the MIA and the MFA. Section 4 will discuss the differences between these types of arithmetic and the problem of universality and unnatural behavior of interval and fuzzy models, and Section 5 will give examples of the difference in computational results of the compared types of arithmetic. Section 6 contains conclusions and then references. In the article, the abbreviations given here will be used to reduce its volume. To facilitate its study, it is recommended to print the list of these abbreviations and then use it reading the paper. List of abbreviations: CIA-constraint (constrained) interval arithmetic, CFA-constraint (constrained) fuzzy arithmetic, MIA-multidimensional interval arithmetic, MFA- multidimensional fuzzy arithmetic, RBIA-rule-based interval arithmetic, SIA-standard interval arithmetic, FN- fuzzy number, RDM-relative distance measure, FA- fuzzy arithmetic, IA-interval arithmetic, MF-membership function, UBM-phenomenon-unnatural behavior in modeling phenomenon, UEF-united extension form.

## 2 Constraint interval arithmetic and constraint fuzzy arithmetic based on it

The first interval arithmetic (IA) was the arithmetic of Warmus, Sunaga, and Moore (WSMIA) [24, 42, 44], also called rule-based interval arithmetic (RBIA) and standard interval arithmetic (SIA). This arithmetic gave simple but strict rules (1) for the implementation of interval calculations, where $a$ and $c$ are lower borders, and $b$ and $d$ are upper borders of intervals $[a, b]$ and $[c, d]$.

$$\begin{array}{ll} \text{1. Addition rule} & [a,b]+[c,d]=[a+c,b+d], \\ \text{2. Subtraction rule} & [a,b]-[c,d]=[a-d,b-c], \\ \text{3. Multiplication rule} & [a,b]\cdot[c,d]=[\min\{ac,ad,bc,bd\},\max\{ac,ad,bc,bd\}], \\ \text{4. Division rule} & [a,b]/[c,d]=[a,b]\cdot[1/d,1/c],\ 0\notin[c,d]. \end{array} \tag{1}$$

It should be noted that SIA calculations are performed only with the use of borders $a$, $b$, $c$, $d$ of intervals. Intermediate values are not included in the calculations. The advantage of SIA is the simplicity and intuitiveness of its use and the certainty that the true value of the operation result will always be included in the result interval given by this arithmetic. The disadvantage of SIA is overestimation, i.e. sometimes too large result intervals that did not give any practical benefits. Other disadvantages are the inability to solve some problems, sometimes providing incorrect results and the dependence of the result on the mathematical form of the formulas used. The main causes of SIA defects were the lack of additive inverse ($A - A = 0$) in this arithmetic, lack of multiplicative inverse ($A/A = 1$), and not satisfying the distributive law ($A(B + C) = AB + AC$). Dependencies sometimes existing between variables could not be included in the SIA calculations. This increased the overestimation of results. The elimination of all these drawbacks of SIA was the goal of the constraint interval arithmetic developed by W.A. Lodwick [15]. In the CIA "a real interval $[\underline{x}, \overline{x}]$ is the graph of the real single-valued function $X'(\lambda_x)$, where

$$X'(\lambda_x) = \lambda_x \underline{x} + (1 - \lambda_x)\overline{x},\ 0 \leq \lambda_x \leq 1. \tag{2}$$

It should be noted, however, that the interval model (2) is not an original idea of the CIA founders, because an identical model can be found in Zimmermann's book from 1985 [46], page 15, and in Klir's [10] book from 1995, page 10. This interval model was used there in the definition of convexity of a fuzzy set. Formula (2) is illustrated in Figure 1.

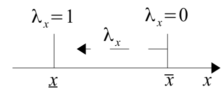


Figure 1: Illustration of the model (2) of a real interval $[\underline{x}, \overline{x}]$.

In the original CIA-interval model (2), the value $\lambda_x = 0$ correspond to the upper border $\overline{x}$ of interval. However, the equivalent interval model (3) given in [16] can also be used, where the interval $[\underline{x}, \overline{x}]$ is represented by the function $f(\gamma_x)$ defined by the formula (3).

$$f(\gamma_x) = \underline{x} + \omega_x \gamma_x,\ \omega_x = \overline{x} - \underline{x} \geq 0,\ 0 \leq \gamma_x \leq 1. \tag{3}$$

The interval thus determined is called constraint interval and is shown in Figure 2.

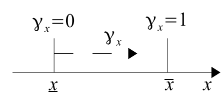


Figure 2: Illustration of an alternative way (3) of modeling the interval in the CIA.

The modeling ways (2) and (3) are equivalent because dependence $\lambda_x = 1 - \gamma_x,\ \lambda_x, \gamma_x \in [0, 1]$ occurs. When using version (2) of the interval model, the basic algebraic operations in the sense of the CIA should be performed [18] according to the formula (4).

$$\begin{aligned} Z = [\underline{z}, \overline{z}] &= X \circ Y = \{z | z = x \circ y, \forall x \in X'(\lambda_x), \forall y \in Y'(\lambda_y), 0 \leq \lambda_x \leq 1, 0 \leq \lambda_y \leq 1\} \\ &= \{z | z = (\lambda_x \underline{x} + (1-\lambda_x)\overline{x}) \circ (\lambda_y \underline{y} + (1-\lambda_y)\overline{y}), 0 \leq \lambda_x \leq 1, 0 \leq \lambda_y \leq 1\}, \\ &\text{where } \underline{z} = \min_{\lambda_x, \lambda_y}\{z\},\ \overline{z} = \max_{\lambda_x, \lambda_y}\{z\},\ \circ \in \{+, -, \cdot, /\}. \end{aligned} \tag{4}$$

The result of these arithmetic operations is always an interval.

According to W.A. Lodwick [15] "constraint interval arithmetic is an extension of interval arithmetic so that it possesses the same properties as interval arithmetic (see Moore 1966 [24]). In addition, constrained interval arithmetic has an additive, a multiplicative inverse and the distributive law holds". Since "use of CIA to evaluate arbitrary expressions requires the computing of the global optimum of a constrained optimization problem" W.A. Lodwick recommends the use of Lagrangians for this purpose. An important observation of the CIA author is [15] the observation that arithmetic operations should be performed differently on dependent and independent intervals: "if there are no dependencies present: that is, each interval in the arithmetic expression is independent, the usual rules (of SIA) for interval arithmetic hold". This means that the CIA should only be used when there are dependent intervals in the arithmetic expression. If such intervals do not occur, then there is no need to use the CIA, because then the results are identical as in the case of the computationally easy SIA. In [15] examples of the use of Lagrangians are given. In formula (4), it should be noted that the result of operations on $X$, $Y$ intervals is also an interval $Z = [\underline{z}, \overline{z}]$ , so the same 1-dimensional mathematical object as the input intervals $X$ and $Y$. This is consistent with the closure property [40]. Interval arithmetic, according to the principle of decomposing a fuzzy set into a set of interval $\alpha$-cuts, can be successfully used to create fuzzy arithmetic. This also applies to the CIA, which was used [18] to create constraint fuzzy arithmetic (CFA). The authors of [18] defined the CFA as follows: "Fuzzy arithmetic on fuzzy intervals using constraint interval arithmetic on each $\alpha$-cut is called constraint fuzzy arithmetic". Basic definitions of CFA are given in [18].

**Definition of a fuzzy interval**

A fuzzy interval $M$, defined by its membership function $\mu_M(\bullet)$, is a fuzzy subset of the real line such that, if $x, y, z \in \mathbb{R}$, $z \in [x, y]$, then

$$\mu_M(z) \geq \min\{\mu_M(x), \mu_M(y)\}. \tag{5}$$

**Definition of the $\alpha$-cut $M_\alpha$ of a fuzzy interval**

$$M_\alpha = \{x | \mu_M(x) \geq \alpha > 0\}. \tag{6}$$

The above definition [45] means that the $\alpha$-cut of a fuzzy interval is a closed interval. The way of performing algebraic operations by CFA is shown in [18] on the example of multiplication of two FIs $\tilde{A}$ and $\tilde{B}$. If FI $\tilde{A}$ is defined by MF $\mu_{\tilde{A}}(a)$ and FI $\tilde{B}$ by $\mu_{\tilde{B}}(b)$, then we calculate the inverse membership functions $\underline{a}(\alpha)$, $\overline{a}(\alpha)$, $\underline{b}(\alpha)$, $\overline{b}(\alpha)$ of both

FIs. We can define them in the form $\tilde{A} = [\underline{a}(\alpha), \overline{a}(\alpha)]$ and $\tilde{B} = [\underline{b}(\alpha), \overline{b}(\alpha)]$. Then we work out "united extension forms (UEFs) for both intervals, formula (7).

$$\lambda_{\tilde{A}}(\alpha) = \lambda_a \underline{a}(\alpha) + (1 - \lambda_a)\overline{a}(\alpha),\ \lambda_a \in [0,1]; \qquad , \lambda_{\tilde{B}}(\alpha) = \lambda_b \underline{b}(\alpha) + (1 - \lambda_b)\overline{b}(\alpha),\ \lambda_b \in [0,1]. \tag{7}$$

UEF $\lambda_{\tilde{C}}(\alpha)$ of the product $\tilde{C} = \tilde{A} \cdot \tilde{B}$ is defined by (8).

$$\lambda_{\tilde{C}}(\alpha) = \lambda_{\tilde{A}}(\alpha) \cdot \lambda_{\tilde{B}}(\alpha). \tag{8}$$

Finally we calculate the resulting fuzzy interval $\tilde{C} = [\underline{c}(\alpha), \overline{c}(\alpha)]$ from formula (9).

$$\tilde{C} = [\underline{c}(\alpha), \overline{c}(\alpha)] = [\min_{\gamma_a, \gamma_b \in [0,1]} \lambda_{\tilde{C}}(\alpha), \max_{\gamma_a, \gamma_b \in [0,1]} \lambda_{\tilde{C}}(\alpha)]. \tag{9}$$

The other CFA algebraic operations are performed in a similar way. It should be noted here that the result of the CFA computation is 2D fuzzy interval $\tilde{C} = [\underline{c}(\alpha), \overline{c}(\alpha)]$, hence the same mathematical form as the input intervals $\tilde{A}$ and $\tilde{B}$. The result of the CFA is not a multidimensional mathematical object. In no article on CIA and CFA arithmetic their authors mention multidimensionality of these arithmetic types, also in the last article [23]. It is similar in articles describing other types of interval and fuzzy arithmetic. This means that the authors of all types of low-dimensional arithmetic strictly adhere to the closure principle. Its text is as follows [40]: "If a set of numbers is such that a given operation on the numbers results in a number from that set, then the set is closed under that operation". Moreover, it seems that in uncertainty arithmetic, this principle does not have to apply. Also for SIA and CIA interval arithmetic. Examples of the implementation of computational operations according to the CIA and CFA rules with details can be found in [3, 15, 16, 17, 18].

## 3 Multidimensional interval and fuzzy arithmetic

Multidimensional interval arithmetic (MIA) and multidimensional fuzzy arithmetic (MFA) are types of arithmetic, the concept of which was developed by A. Piegat (AP) in 2010-11, after reaching the conclusion based on the research that all problems and irregularities occurring in the existing types of arithmetic cannot be solved in low 1D space (in the case of MIA) or 2D space (in the case of MFA) because their nature is multidimensional. It is difficult even for IA and FA specialists to understand this truth, as evidenced by the fact that almost all current IAs are 1-dimensional and FAs are 2-dimensional. In MFA, the concept of the horizontal membership function (HMF) was proposed [33, 29]. An extension of HMF to the fuzzy set of numbers is the concept of a horizontal fuzzy number used to solve fuzzy systems of linear equations [14]. To explain the multidimensionality issue, the problem of 2 tanks $A$ and $B$ will be presented.

**2 tanks problem**

In the quasi-cylindrical tank $A$ with deformed side walls there is an amount of water $a[m^3]$ which is not known exactly. Based on the measurements of the tank and the water level, it was found that $a \in [7.9]m^3$. Water from tank $A$ was poured into empty tank $B$, which is also a quasi-cylindrical tank with distorted side walls. Then, an unknown amount of water $x[m^3]$ was added to the tank $B$ from the tap. Based on the measurements of tank $B$ and the water level in it, it was found that the total amount of water in $B$ is $b \in [12.13]m^3$. Greater accuracy in determining the water in tank $B$ is due to the fact that the side walls of this tank are less distorted than the borders of tank $A$. The task is to determine the amount of water $x[m^3]$ that was added to the tank $B$ from the tap.

**Solution of the problem**

If the real, amount of water in tank $A$ is $a$, and in tank $B$ is $b$, and the exact amount of water added from the tap is marked as $x$, then the system of tanks can be interpreted as an addition system that realizes the equation (10).

$$a + x = b:\ a \in A = [7,9], b \in B = [12,13], x = ? \tag{10}$$

Let's try to solve the problem of tanks using SIA. A symbolic and specific description of this problem is given by (11), where $X = [\underline{x}, \overline{x}]$.

$$\begin{gathered} A + X = B, \\ [7,9] + [\underline{x}, \overline{x}] = [12,13]. \end{gathered} \tag{11}$$

Using the popular SIA arithmetic [24] we obtain the solution $X = [\underline{x}, \overline{x}] = [5,4]$ being an improper interval, which does not make any practical sense, because it means that the real, added amount of water $x[m^3]$ would have to satisfy simultaneously two contradictory dependencies $x \geq 5$ and $x \leq 4$, which is impossible. The problem of tanks presented here cannot be solved at all if we assume that its solution is an interval $[\underline{x}, \overline{x}]$. This problem is more complicated than it initially seems. So let's assume that the solution $X$ is not an interval but has some other character. The problem of tanks is then described by Equation (12).

$$\begin{gathered} A + X = B, \\ [7,9] + X = [12,13], X \neq [\underline{x}, \overline{x}]. \end{gathered} \tag{12}$$

Let us now ask ourselves: how many unknowns are there in the problem of tanks (12)? On the surface, there seems to be only one unknown $X$, as is the case with the crisp arithmetic equations, e.g. $7 + x = 12$. However, it is not so. In equation (12) we do not know not only the real amount of added water $x[m^3]$, but also the real amount of water $a[m^3]$ in $A$ and $b[m^3]$ in $B$. In equation (12) there are therefore not one but 3 unknowns $(a, x, b)$! Is it possible to solve one equation with 3 unknowns at all? It seems not! As will be shown, in order to do this we need to slightly revise our current understanding of the term "solution". In the case of the mathematics of uncertain numbers, the solution to the problem should be considered not the exact value of the result (which cannot be obtained, with some exceptions), but the knowledge about this value. This knowledge is approximated and, as it will be shown, may take various forms of varying quality. The best form of knowing the exact result of $x$ will be the form that will contain the most information about the result (maximum informativeness). Let us now return to the problem of tanks. Since there are 3 unknowns in this problem (real values of $a$, $b$, $x$), we can only define a set of possible states $(a, b, x)$ of the tank system that could have occurred. In this set, the variable $x$ depends on $a$ and $b$ $(x = b - a)$. So it cannot assume its values freely. Each of the possible states of the system must be consistent with the possessed knowledge, i.e. $a \in [7, 9]$, $b \in [12, 13]$, $x = b - a$. Examples of triples of system states $(a, b, x)$ are $(7.00, 12.00, 5.00)$, $(7.01, 12.00, 4.99)$, etc. The set of possible states (solutions) of the system can be marked as $S$ and expressed by the formula (13).

$$S = \{(a, b, x) | a \in A, b \in B, x = b - a \in X\}. \tag{13}$$

An extension of the notation form (13) used in MIA is the form using the variable RDM (relative distance measure), which denotes the possible relative distance of the real value of the variable from the left interval border. The concept of the RDM variable was introduced by A. Piegat (AP) in 2010 in [32] on their own, without knowing about the existence of the CIA, which AP only got to know around 2016 from [16]. In [32] AP introduced the concept of a fuzzy model based not on a (hyper-) rectangular, regular partition of the input space of the fuzzy model, but on an irregular one. An irregular partition makes it possible to significantly reduce the number of fuzzy model rules and to differentiate the influence zone of individual rules. In this concept, RDM variables made it possible to describe non-rectangular sectors of rules with the help of local quasi-Cartesian systems. In the problem of tanks, the RDM variables $[0, 1]$ enable to express the knowledge about the problem in a detailed way shown in (14).

$$\begin{aligned} &A : a \in A, B : b \in B, X : x = b - a \in X, \\ &S = \{(a, b, x) | a \in A, b \in B, x = b - a \in X\}, \\ &a = a(\gamma_a) = \underline{a} + \gamma_a(\overline{a} - \underline{a}) = 7 + 2\gamma_a, \gamma_a \in [0, 1]; \quad b = b(\gamma_b) = \underline{b} + \gamma_b(\overline{b} - \underline{b}) = 12 + \gamma_b, \gamma_b \in [0, 1], \\ &x = x(\gamma_a, \gamma_b) = b(\gamma_b) - a(\gamma_a) = \underline{b} - \underline{a} + \gamma_b(\overline{b} - \underline{b}) - \gamma_a(\overline{a} - \underline{a}) = 5 + \gamma_b - 2\gamma_a, \gamma_a, \gamma_b \in [0, 1]. \end{aligned} \tag{14}$$

The set $S$ is here the set of possible solutions (states of the tank system), while the variable $x(\gamma_a, \gamma_b)$ is the result variable $(x = b - a)$. The set $X$ is the set of possible values for this variable. This set should be distinguished from the set of possible solutions $S$ defined by the formula (15). The set $S$ is shown in Figure 3.

$$\begin{aligned} &S = \{(a, b, x) | a \in A, b \in B, x = b - a \in X\}, \\ &a = a(\gamma_a) = \underline{a} + \gamma_a(\overline{a} - \underline{a}) = 7 + 2\gamma_a, \gamma_a \in [0, 1]; \quad b = b(\gamma_b) = \underline{b} + \gamma_b(\overline{b} - \underline{b}) = 12 + \gamma_b, \gamma_b \in [0, 1], \\ &x = x(\gamma_a, \gamma_b) = b(\gamma_b) - a(\gamma_a), \gamma_a, \gamma_b \in [0, 1]. \end{aligned} \tag{15}$$

By inserting into the formula (15) selected numerical $\gamma_a$, $\gamma_b$ values from $[0, 1]$ we generate one possible state (possible solution) in the form of triple $(a, b, x)$ falling into the set $S$. This set contains an infinitely large number of possible triples $(a, b, x)$. However, in a real system of tanks only one triple (one state) could exist, which can be written as a condition for the set $S : card(S) = 1$.

**Indicators of the 3D $S$-set of possible states of the system**

The $S$-set of possible solutions (states) of the tank system exists in this case in 3D space and thus can be visualized (Figure 3). However, if the number of variables is higher and the knowledge of these variables is fuzzy, then the visualization is not possible. In any case, however, you can get some idea of the $S$-set thanks to the 2D indicators that can be seen and will be presented in the sequel. Let's call the variables $a$ and $b$ information inputs and the variable $x$ information output. The most frequently used indicator of the resulting $S$-set is the span of this set along the $x$ axis. It is equal to the span $SP_X$ of the set $X$ of possible values of the variable $x$. The spans of $A$- and $B$-sets are easy to

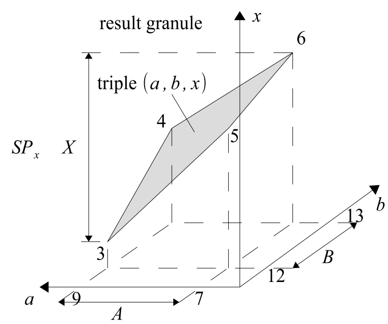


Figure 3: Illustration of 3D solution of tanks problem. $S$ - set of possible solutions (states) of the system, $X$ - set of possible values of the result variable $x$, $SP_X = [\underline{x}, \overline{x}]$ - span of the set $X$.

determine: $SP_A = [\underline{a}, \overline{a}] = [7, 9]$, $SP_B = [\underline{b}, \overline{b}] = [12, 13]$. The span of the set $X$ should be calculated from the formula (16).

$$SP_X = [\min_{\gamma_a, \gamma_b} x(\gamma_a, \gamma_b), \max_{\gamma_a, \gamma_b} x(\gamma_a, \gamma_b)], \gamma_a, \gamma_b \in [0, 1]. \tag{16}$$

Formula (17) is a specific form of the set (16).

$$SP_X = [\min_{\gamma_a, \gamma_b \in [0,1]} (5 + \gamma_b - 2\gamma_a), \max_{\gamma_a, \gamma_b \in [0,1]} (5 + \gamma_b - 2\gamma_a)] = [3, 6]. \tag{17}$$

It is easy to check that the left border $x$ of the span is obtained for the values $\gamma_a = 1$, $\gamma_b = 0$ and the right border $x = 6$ for the value $\gamma_a = 0$, $\gamma_b = 1$. The $SP_X$ of the set $X$ of the possible values of the variable $x$ in all currently existing types of interval arithmetic (except MIA) is taken as the result. Meanwhile, in the problem of tanks there is no solution in form of a real "proper interval" $[\underline{x}, \overline{x}]$ that would satisfy the interval equation of the system $[7, 9] + [\underline{x}, \overline{x}] = [12, 13]$. Such equation cannot be solved at all assuming that the result is an interval, because it is not. This assumption is generally incorrect. Similarly, the span $SP_X = [3, 6]$ being the interval calculated by the MIA is also not an algebraic solution to this equation because after inserting it we do not get the equating of the equation sides $[7, 9]+[3, 6] = [12, 13]$. It is easy to check that the result variable $x(\gamma_a, \gamma_b) = 5 + \gamma_b - 2\gamma_b$ satisfies the equation (18) of the tank system.

$$(7 + 2\gamma_a) + (5 + \gamma_b - 2\gamma_a) = (12 + \gamma_b), \gamma_a, \gamma_b \in [0, 1]. \tag{18}$$

If we use the CIA to solve the tanks problem, we get "CIA-result" $= [\underline{x}, \overline{x}]_{\text{CIA}} = [3, 6]$ the same as the span $SP_X$ calculated by the MIA. However, "CIA-result" $= [3, 6]$, as already shown, is not an algebraic result as it does not satisfy the interval equation of the system $[7, 9] + [3, 6]_{\text{CIA}} = [12, 13]$. The second simplified 2D-indicator of the set $S$ is the cardinality distribution of the result variable $x$. Figure 4 shows the projection of the result granule $S$ from the 3D space $A \times B \times X$ to the 2D space $A \times B$.

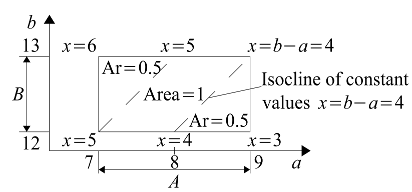


Figure 4: Illustration of how the cardinality measure $Mcard_x$ of the result variable $x$ is calculated with use of the area $Ar$ lying between isoclines of constant $x$-values.

Between the individual isoclines there is an infinite number of possible system states $(a, b, x)$. The measure $M$ of the number of solutions between $x = n$ and $x = n + 1$ is the area $Ar$ lying between them. The $Mcard_x$ -distribution of the tanks problem thus calculated is shown in Figure 5.

**Important!** The histogram $Mcard_x$ is not a probability density distribution (not normalized) because in the problem of tanks we do not have any knowledge about the pdf probability density distributions of the variable $a$ and $b$. The histogram $Mcard_x$ informs us about the number of possible results in individual sub-ranges of the variable $x$. However, if $Mcard_x$ has the highest value in the range $x \in [4, 5]$, we can intuitively expect that the apriori probability

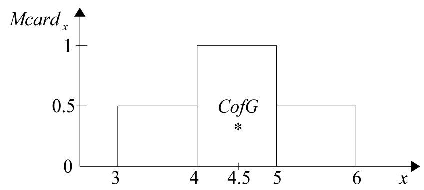


Figure 5: Histogram $Mcard_x$ of the cardinality measure of possible values of the result variable $x$ in the tanks' problem.

in this sub-range for the true value of $x$ is the highest (probability not confirmed experimentally). The $Mcard_x$ distribution provides us with qualitatively different information than the span $SP_X$. In addition to the span $SP_X$ and the distribution $Mcard_x$ a third simplified information about the result set $S$ can be specified in the form of the center of gravity $CofG_x$ , which can be calculated by a generally known method from the distribution $Mcard_x$ or approximately by the Monte Carlo method. $CofG_x$ value = 4.5.

**Operations of epistemic MIA interval arithmetic**

There are given intervals $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$ representing our knowledge about the exact values of variables $a$ and $b$ that are unknown to us. We denote the arithmetic operation with the sign $\circ \in \{+, -, \cdot, /\}$. The way of its implementation in terms of MIA is given in the formula (19).

$$\begin{array}{l} A : a \in A, B : b \in B, \\ a = a(\gamma_a) = \underline{a} + \gamma_a(\overline{a} - \underline{a}), \gamma_a \in [0,1]; \quad b = b(\gamma_b) = \underline{b} + \gamma_b(\overline{b} - \underline{b}), \gamma_b \in [0,1], \\ A \circ B = S = \{(a, b, x) | a \in A, b \in B, x = a \circ b \in X\}, \\ X : x(\gamma_a, \gamma_b) = a(\gamma_a) \circ b(\gamma_b), \circ \in \{+, -, \cdot, /\}. \end{array} \tag{19}$$

In the case of $\circ = /$, $0 \notin B$. The set $S$ is called the resulting set of possible states of an arithmetic system out of which only one state is true. The set $X$ is the set of possible values of the result variable $x$ of which only one value is true ($cardX = 1$). The simplified indicators of set $X$ are the span $SP_x$, the $Mcard_x$ histogram of the cardinality measure, and the center of gravity $GofG_x$. Important! If we want to use the result of the arithmetic operation in further calculations, we insert into them not the span $SP_x$ being the interval but the result variable $x(\gamma_a, \gamma_b) = a \circ b$, so as not to cause a calculation error.

**Epistemic MFA operations**

If the knowledge of the true values of the variables $a$ and $b$ is given in the form of vertical, classic MFs with left borders $\mu_A^L(a)$, $\mu_B^L(b)$ and right borders $\mu_A^R(a)$, $\mu_B^R(b)$, where $\mu_A^L(a)$, $\mu_B^L(b)$ are normal non-decreasing functions and $\mu_A^R(a)$, $\mu_B^R(b)$ are normal non-increasing functions, Figure 6, then on the basis of these functions we define their inverse, horizontal equivalents $a^L(\mu)$, $b^L(\mu)$, $a^R(\mu)$, $b^R(\mu)$ and using the RDM variables $\gamma_a, \gamma_b \in [0, 1]$ we develop horizontal MFs $a(\mu, \gamma_a)$ and $b(\mu, \gamma_b)$, modeling not only borders but also the entire interior of MFs of $A$ and $B$, Figure 6.

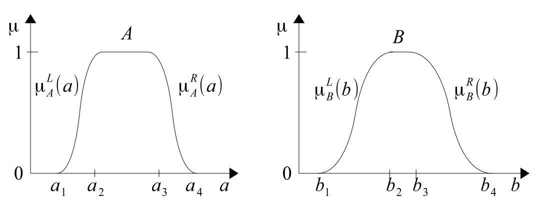


Figure 6: Denotation of parameters and borders ($L$-left, $R$-right) of vertical membership functions.

The MFs used in the MFA may be of any shape, as long as their left borders are not decreasing and their right borders are not increasing. In practice, the most commonly used are trapezoidal and triangular MFs, Figure 7.

The formulas needed to describe the trapezoidal, horizontal MF from Figure 7 are given by (20).

$$\begin{array}{l} a^L(\mu) = a_1 + \mu(a_2 - a_1), a^R(\mu) = a_4 - \mu(a_4 - a_3), \mu \in [0, 1], \\ a(\mu, \gamma_a) = a^L(\mu) + \gamma_a[a^R(\mu) - a^L(\mu)], \\ a(\mu, \gamma_a) = a_1 + \mu(a_2 - a_1) + \gamma_a[a_4 - a_1 - \mu(a_4 - a_3 + a_2 - a_1)], \mu, \gamma_a \in [0, 1]. \end{array} \tag{20}$$

The formulas needed to describe the triangular, horizontal MF from Figure 7 are given by (21).

$$\begin{array}{l} b^L(\mu) = b_1 + \mu(b_2 - b_1), b^R(\mu) = b_3 - \mu(b_3 - b_2), \mu \in [0, 1], \\ b(\mu, \gamma_b) = b^L(\mu) + \gamma_b[b^R(\mu) - b^L(\mu)], \\ b(\mu, \gamma_b) = b_1 + \mu(b_2 - b_1) + \gamma_b[b_3 - b_1 - \mu(b_3 - b_1)], \mu, \gamma_b \in [0, 1]. \end{array} \tag{21}$$

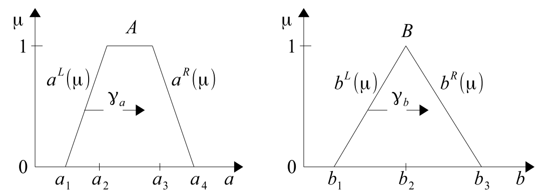


Figure 7: Trapezoidal and triangular MF and denotations used in horizontal MFs.

If $\circ$ denotes an arithmetic operation, $\circ \in \{+, -, \cdot, /\}$ and $A$ and $B$ denote fuzzy normal numbers, then the arithmetic operations of MFA are given by the formula (22).

$$\begin{aligned}&A : a \in A, B : b \in B,\\ &a = a(\mu, \gamma_a) = a^L(\mu) + \gamma_a[a^R(\mu) - a^L(\mu)], \mu, \gamma_a \in [0,1];\ \ b = b(\mu, \gamma_b) = b^L(\mu) + \gamma_b[b^R(\mu) - b^L(\mu)], \mu, \gamma_b \in [0,1],\\ &A \circ B = S\{a(\mu,\gamma_a), b(\mu,\gamma_b), a(\mu,\gamma_a) \circ b(\mu,\gamma_b) = x(\mu,\gamma_a,\gamma_b) | a \in A, b \in B, \mu, \gamma_a, \gamma_b \in [0,1], x \in X\},\\ &X : x(\mu,\gamma_a,\gamma_b) = a(\mu,\gamma_a) \circ b(\mu,\gamma_b), \mu, \gamma_a, \gamma_b \in [0,1].\end{aligned} \tag{22}$$

The set $S$ is the set of possible states of the system implementing the operation and consists of states $(a(\mu,\gamma_a)$, $b(\mu,\gamma_b)$, $x(\mu,\gamma_a,\gamma_b))$ of which only one state is true. The set $X$ is the set of possible values of the result variable $x$ of which only one value is true. The algebraic result of arithmetic operations in MFA is not the span function $SP_x$ as suggested by CFA and other types of fuzzy arithmetic, but the result variable $x(\mu,\gamma_a,\gamma_b)$.

# 4 Differences between constraint interval and fuzzy arithmetic and multidimensional interval and fuzzy arithmetic

**D1.** In the CIA, the result of arithmetic calculations is the interval $[x] = [\underline{x}, \overline{x}]$, which generally does not satisfy the equations to be solved. This in turn means that it is not an algebraic solution. This can be demonstrated by showing already 1 example confirming the above statement. Let us consider the interval equation (23) presented in the SIA version.

$$\begin{aligned}&[\underline{a}, \overline{a}][\underline{x}, \overline{x}] = [\underline{b}, \overline{b}],\\ &[1,2][\underline{x}, \overline{x}] = [5,6].\end{aligned} \tag{23}$$

In the CIA version, equation (23) has the form (24).

$$(1 + \gamma_a)(\underline{x} + \gamma_x(\overline{x} - \underline{x})) = 5 + \gamma_b, \gamma_a, \gamma_b, \gamma_x \in [0,1]. \tag{24}$$

The solution of the equation is given by the formula (25).

$$\begin{aligned}&\underline{x} + \gamma_x(\overline{x} - \underline{x}) = \tfrac{5+\gamma_b}{1+\gamma_a}, \gamma_a, \gamma_b, \gamma_x \in [0,1],\\ &\underline{x} = \min_{\gamma_x=0,\gamma_a,\gamma_b} \tfrac{5+\gamma_b}{1+\gamma_a} = 2.5;\ \ \overline{x} = \max_{\gamma_x=1,\gamma_a,\gamma_b} \tfrac{5+\gamma_b}{1+\gamma_a} = 6,\\ &[\underline{x}, \overline{x}]_{\text{SIA}} = [2.5, 6]_{\text{SIA}},\\ &[\underline{x}, \overline{x}]_{\text{CIA}} = 2.5 + 3.5\gamma_x, \gamma_x \in [0,1].\end{aligned} \tag{25}$$

After inserting the solution $[x]_{\text{CIA}}$ to the left side of the equation (24) we get (26).

$$\begin{aligned}&(1 + \gamma_a)(1.5 + 3.5\gamma_x) = L, \gamma_a, \gamma_x \in [0,1],\\ &\underline{L} = \min_{\gamma_a,\gamma_x} L = 2.5,\ \text{for } \gamma_a = 0, \gamma_x = 0;\ \ , \overline{L} = \max_{\gamma_a,\gamma_x} L = 12,\ \text{for } \gamma_a = 1, \gamma_x = 1,\\ &[L] = [\underline{L}, \overline{L}] = [2.5, 12].\end{aligned} \tag{26}$$

Since the right side $R$ of the equation (24) is the interval $[\underline{R}, \overline{R}] = [5,6]$ and the left side $[L] = [\underline{L}, \overline{L}] = [2.5, 12]$ it means that $[L] = [2.5, 12] \neq [R] = [5,6]$ Hence, the interval "solution" of equation (25) obtained with the use of the CIA is not an algebraic solution to this equation. When using MIA, a multi-dimensional solution is obtained as shown below. The problem notation in the SIA version gives the formula (27).

$$\begin{aligned}&AX = B, [\underline{a}, \overline{a}][\underline{x}, \overline{x}] = [\underline{b}, \overline{b}],\\ &[1,2][\underline{x}, \overline{x}] = [5,6].\end{aligned} \tag{27}$$

We transform the SIA version into the MIA version, formula (28).

$$\begin{aligned} &AX = B, A : a(\gamma_a) = 1 + \gamma_a \in A, B : b(\gamma_b) = 5 + \gamma_b \in B, X = B/A, \gamma_a, \gamma_b \in [0, 1],\\ &(1 + \gamma_a)x(\gamma_a, \gamma_b) = (5 + \gamma_b), \gamma_a, \gamma_b \in [0, 1]. \end{aligned} \tag{28}$$

The set of solutions to equation (28) is the set $S$ defined by (29).

$$\begin{aligned} &S = \{(a(\gamma_a), b(\gamma_b), x(\gamma_a, \gamma_b) = b(\gamma_b)/a(\gamma_a))|\gamma_a, \gamma_b \in [0, 1]\},\\ &S = \{((1 + \gamma_a), (5 + \gamma_b), (5 + \gamma_b)/(1 + \gamma_a))|\gamma_a, \gamma_b \in [0, 1]\}. \end{aligned} \tag{29}$$

The set $S$ consists of possible triples $(a, b, x)$ that could have occurred in the multiplication system $ax = b$, e.g. $\{(1, 5, 5), (2.5, 2.5), (1.5, 6.4), \ldots\}$. Only one triple of the entire $S$ set is true. The last value of $x$ in a triple is always dependent on the previous two values $a$ and $b$. It is easy to check that the value of the result variable $x(\gamma_a, \gamma_b)$ always, for any combination of $a(\gamma_a)$ and $b(\gamma_b)$, satisfies the equation (28). After inserting the result $x(\gamma_a, \gamma_b)$ into it, we always obtain the equality of the left and right sides of equation (28), as shown in (30).

$$\begin{aligned} &(1 + \gamma_a)x(\gamma_a, \gamma_b) = (5 + \gamma_b), \gamma_a, \gamma_b \in [0, 1],\\ &(1 + \gamma_a)\tfrac{5+\gamma_b}{1+\gamma_a} = (5 + \gamma_b), \gamma_a, \gamma_b \in [0, 1],\\ &(5 + \gamma_b) = (5 + \gamma_b),\ L = P. \end{aligned} \tag{30}$$

So the variable $x(\gamma_a, \gamma_b) = (5+\gamma_b)/(1+\gamma_a)$ is the real algebraic result variable of equation (28). If we want to use the solution of equation (28) in the next stages of calculations, then the multidimensional result variable $x(\gamma_a, \gamma_b)$ should be inserted into next formulas, not its span $SP_x$. This span, as simplified information about the solution $x(\gamma_a, \gamma_b)$, can of course be computed with formula (31). However, it should not be used in the further stages of calculations, because it will lead to errors in the final result.

$$SP_x = \left[\min_{\gamma_a,\gamma_b} \frac{5 + \gamma_b}{1 + \gamma_a}, \max_{\gamma_a,\gamma_b} \frac{5 + \gamma_b}{1 + \gamma_a}\right] = [2.5, 6]. \tag{31}$$

In addition to the span $SP_x$, you can also determine the $Mcard_x$ distribution and the position of the $CofG_x$ center of gravity according to the previously given formulas. CIA-result in the form of span (31) is not an algebraic solution to equation (28). Therefore, it does not satisfy the equation to be solved. All of the above considerations apply not only to the CIA and MIA but also to their fuzzy counterparts CFA and MFA.

**D2. The universality of the MIA and MFA results and the non-universality of the CIA and CFA results**

The problem of universality of the results of interval and fuzzy arithmetic was pointed out by L. Dymowa in [5], although she did not call it that. Also M. Mazandarani described this problem in [20] calling it Unnatural Behavior in Modeling (UBM-phenomenon). L. Dymowa in [5] introduced the concept of “interval extension of a function”, given below. “Let $f$ is a usual or interval function of real valued arguments $x_1, x_2, \ldots, x_n$. An interval extension of function $f$ is then the interval function $F$ of interval arguments $X_1, X_2, \ldots, X_n$, such that for real valued arguments $f(x_1, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)$”. If $f(x_1, x_2) = x_1 + x_2$ then the interval extension of this function is $F(x_1, x_2) = [\underline{x}_1, \overline{x}_1] + [\underline{x}_2, \overline{x}_2]$. In addition to the interval extension of the function $f(x_1, x_2, \ldots, x_n)$ we can also use the concept of fuzzy number extension of the real valued function, where $X_1, X_2, \ldots, X_n$ are fuzzy numbers. L. Dymowa in [5] also drew attention to the fact that SIA often calculates different final results depending on the mathematical form into which the original function will be transformed. She gave an interesting example, Example 3.3, which we will now introduce.

Consider the following 4 formal expressions of the same function $f(x)$:

$$f_1(x) = x(x + 1); \quad f_2(x) = xx + x; \quad f_3(x) = x^2 + x; \quad f_4(x) = (x + 1/2)^2 - 1/4. \tag{32}$$

Evaluations of their interval extensions for $[x] = [-1, 1]$ are given by (33).

$$\begin{aligned} &[F_1]([x]) = [x]([x] + 1) = [-2, 2],\\ &[F_2]([x]) = [x][x] + [x] = [-2, 2],\\ &[F_3]([x]) = [x]^2 + [x] = [-1, 2],\\ &[F_4]([x]) = ([x] + 1/2)^2 - 1/4 = [-1/2, 2]. \end{aligned} \tag{33}$$

If a given IA provides 3 different evaluations of the interval extension of one and the same function $f(x)$ depending on the mathematical form of the extensions, then such arithmetic is imperfect, it should be investigated, its errors detected and improved, because without conducting deeper research, it is not known which of the provided results is correct. It will also make it very difficult to solve problems in which, in order to find a final solution, it is necessary to transform the initial formulas into their new forms. Let us now consider the problem of the universality of the results delivered by the CIA and MIA arithmetic on the example of addition. We can interpret this operation as an operation performed by a (physical) addition system that adds two independent input quantities $x_1$ and $x_2$ and generates an output quantity $y = x_1 + x_2$. The operation of such a system can be described by 4 equivalent mathematical forms $f_1 - f_4$, formulas (34).

$$f_1 : y = x_1 + x_2; \quad f_2 : y - x_1 = x_2; \quad f_3 : y - x_2 = x_1; \quad f_4 : y - x_1 - x_2 = 0. \tag{34}$$

Each of the above forms is equivalent and if we know the real-valued values, e.g. $x_1 = 3$, $x_2 = 5$, then the value $y = 8$ calculated from the form $f_1$ will satisfy all other forms $f_2$, $f_3$, $f_4$. Likewise, all forms $f_1 - f_4$ yield the same result of adding $y = 8$. Suppose now that we do not know the exact values $x_1$, $x_2$, but only have the interval knowledge $x_1 \in [1, 3]$, $x_2 \in [5, 8]$. In the case of the CIA, the result of the computation is the interval $[\underline{y}, \overline{y}]$, which we denote $[\underline{y}, \overline{y}]_{\mathrm{CIA}}$. The crisp-valued form $f_1$ (34) corresponds to the extension interval (interval-valued form) (35).

$$F1 : [\underline{x}_1, \overline{x}_1]_{\mathrm{CIA}} + [\underline{x}_2, \overline{x}_2]_{\mathrm{CIA}} = [\underline{y}, \overline{y}]_{\mathrm{CIA}}. \tag{35}$$

For $[x_1] = [1, 3]$ and $[x_2] = [5, 8]$ the equation (35) takes the form (36).

$$F1 : [1, 3]_{\mathrm{CIA}} + [5, 8]_{\mathrm{CIA}} = [\underline{y}, \overline{y}]_{\mathrm{CIA}}. \tag{36}$$

After inserting the result $[\underline{y}, \overline{y}]_{\mathrm{CIA}} = [6, 11]$ into the equation to be solved, we obtain the "equation" (37).

$$F1 : [1, 3]_{\mathrm{CIA}} + [5, 8]_{\mathrm{CIA}} = [6, 11]_{\mathrm{CIA}}. \tag{37}$$

It's easy to see that the sides of this equation are equal. Now let's check what the situation will look like in the case of the next possible forms of equation (35). The crisp-valued form $f_2$, $y - x_1 = x_2$ (34) corresponds to the interval-valued form $F_2$ (38).

$$F2 : [\underline{y}, \overline{y}]_{\mathrm{CIA}} - [\underline{x}_1, \overline{x}_1]_{\mathrm{CIA}} = [\underline{x}_2, \overline{x}_2]_{\mathrm{CIA}}. \tag{38}$$

After inserting the interval result (36) to (38) we obtain the equation (39).

$$F2 : [6, 11]_{\mathrm{CIA}} - [1, 3]_{\mathrm{CIA}} = [5, 8]_{\mathrm{CIA}}. \tag{39}$$

It is easy to check that in (39) the sides are not equal. Crisp-valued form $f_3$ (34) $y - x_2 = x_1$ corresponds to the interval extension $F_3$ (40).

$$F3 : [\underline{y}, \overline{y}]_{\mathrm{CIA}} - [\underline{x}_2, \overline{x}_2]_{\mathrm{CIA}} = [\underline{x}_1, \overline{x}_1]_{\mathrm{CIA}}. \tag{40}$$

After inserting the interval result (36) to (40) we get $F_3$ (41).

$$F3 : [6, 11]_{\mathrm{CIA}} - [5, 8]_{\mathrm{CIA}} = [1, 3]_{\mathrm{CIA}}. \tag{41}$$

It is easy to check that the sides of the equation (41) are not equal. Crisp-form $f_4$ $y - x_1 - x_2 = 0$ corresponds to the interval extension $F_4$ (42).

$$F4 : [\underline{y}, \overline{y}]_{\mathrm{CIA}} - [\underline{x}_1, \overline{x}_1]_{\mathrm{CIA}} - [\underline{x}_2, \overline{x}_2]_{\mathrm{CIA}} = 0. \tag{42}$$

After inserting the numerical values into (42) we get (43).

$$F4 : [6, 11]_{\mathrm{CIA}} - [1, 3]_{\mathrm{CIA}} - [5, 8]_{\mathrm{CIA}} = 0. \tag{43}$$

Also in this case the sides of equation (43) are not equal. The exact forms of the CIA-intervals used in the above equations, in CIA terms, are given in (44).

$$[6, 11]_{\mathrm{CIA}} = 6 + 5\gamma_y, \gamma_y \in [0, 1]; \quad [1, 3]_{\mathrm{CIA}} = 1 + 2\gamma_{x1}, \gamma_{x1} \in [0, 1]; \quad [5, 8]_{\mathrm{CIA}} = 5 + 3\gamma_{x2}, \gamma_{x2} \in [0, 1]. \tag{44}$$

The presented example shows that the CIA result $[\underline{y}, \overline{y}]_{\mathrm{CIA}}$ is not a universal result, because it does not satisfy all possible forms of the interval addition (35). For computational practice, this means that if we insert this result into transformed formulas, we may obtain an incorrect final formula. And what about the multidimensional result provided by MIA? The MIA models of the interval knowledge about the true value of the variable $x_1$ and $x_2$ are given by the formula (45).

$$\begin{aligned}&X_1 + X_2 = Y,\\ &X_1 : x_1 = x_1(\gamma_1) = 1 + 2\gamma_1, \gamma_1 \in [0,1],\\ &X_2 : x_2 = x_2(\gamma_2) = 5 + 3\gamma_2, \gamma_2 \in [0,1],\\ &Y : y = y(\gamma_1,\gamma_2) = x_1(\gamma_1) + x_2(\gamma_2) = 6 + 2\gamma_1 + 3\gamma_2, \gamma_1,\gamma_2 \in [0,1].\end{aligned} \tag{45}$$

In the case of the $F_1$ form, the addition dependence is given by the formula (46).

$$\begin{aligned}&F_1 : y(\gamma_1,\gamma_2) = x_1(\gamma_1) + x_2(\gamma_2),\\ &(6 + 2\gamma_1 + 3\gamma_2) = (1 + 2\gamma_1) + (5 + 3\gamma_2), \gamma_1,\gamma_2 \in [0,1],\\ &L = R.\end{aligned} \tag{46}$$

It is easy to check that the right and left sides of the equation are equal. In the case of the $F_2$ form, the addition dependence is given by the formula (47).

$$\begin{aligned}&F_2 : y(\gamma_1,\gamma_2) - x_1(\gamma_1) = x_2(\gamma_2),\\ &(6 + 2\gamma_1 + 3\gamma_2) - (1 + 2\gamma_1) = (5 + 3\gamma_2), \gamma_1,\gamma_2 \in [0,1],\\ &L = R.\end{aligned} \tag{47}$$

Here, too, both sides of the equation are equal. In the case of the $F_3$-form, the addition dependence is given by the formula (48).

$$\begin{aligned}&F_3 : y(\gamma_1,\gamma_2) - x_2(\gamma_2) = x_1(\gamma_1),\\ &(6 + 2\gamma_1 + 3\gamma_2) - (5 + 3\gamma_2) = (1 + 2\gamma_1), \gamma_1,\gamma_2 \in [0,1],\\ &L = R.\end{aligned} \tag{48}$$

In this dependence both sides of the equation are the same. In the case of the $F_4$ form, the addition dependence is given by the formula (49).

$$\begin{aligned}&F_4 : y(\gamma_1,\gamma_2) - x_1(\gamma_1) - x_2(\gamma_2) = 0,\\ &(6 + 2\gamma_1 + 3\gamma_2) - (1 + 2\gamma_1) - (5 + 3\gamma_2) = 0, \gamma_1,\gamma_2 \in [0,1],\\ &L = R.\end{aligned} \tag{49}$$

Here, too, both sides of the equation are equal. Thus, it has been shown above that for MIA the multidimensional addition result $y(\gamma_1,\gamma_2)$ satisfies all forms $F_1 - F_4$ of the addition dependence for the specific case of $x_1 = 1 + 2\gamma_1$, $\gamma_1 \in [0,1]$ and $x_2 = 5 + 3\gamma_2$, $\gamma_2 \in [0,1]$. However, this can also be shown without difficulty for the general case $x_1 = \underline{x}_1 + \gamma_1(\overline{x}_1 - \underline{x}_1)$, $\gamma_1 \in [0,1]$, $x_2 = \underline{x}_2 + \gamma_2(\overline{x}_2 - \underline{x}_2)$, $\gamma_2 \in [0,1]$. The same is true for MFA arithmetic, where $x_1 = x_1(\mu,\gamma_1)$ and $x_2 = x_2(\mu,\gamma_2)$. The span of the result set $Y : y(\gamma_1,\gamma_2) \in Y$ can be determined from the formula (50).

$$SP_Y = [\min_{\gamma_1,\gamma_2\in[0,1]} y(\gamma_1,\gamma_2), \max_{\gamma_1,\gamma_2\in[0,1]} y(\gamma_1,\gamma_2)] = [\min_{\gamma_1,\gamma_2\in[0,1]}(6 + 2\gamma_1 + 3\gamma_2), \max_{\gamma_1,\gamma_2\in[0,1]}(6 + 2\gamma_1 + 3\gamma_2)] = [6, 11]. \tag{50}$$

The span value $[6, 11]$ is the same as that of the CIA. However, in MIA, span is not an algebraic result, it is just one of the 3 simplified pieces of information (span, distribution of cardinality measure, center of gravity) about a multidimensional algebraic result. The presented comparison of the CIA and MIA shows that MIA has more favorable mathematical properties than the CIA.

# 5 Example of different results obtained with CIA, CFA and MIA, MFA

**Example 5.1.** *The publication [18] gives an example of a problem solved by the creators of CFA using this arithmetic (Example 2). The content of the example (original denotations retained) is as follows: "Let $A = -2/-1/1/2$ and $B = -1/0/1$ be a trapezoidal and triangular fuzzy intervals respectively, and we wish to compute $C = A(A + B)$. For $0 \le \alpha \le 1$ the $\alpha$-cuts of $A$ are $[\underline{a}(\alpha), \overline{a}(\alpha)] = [\alpha - 2, -\alpha + 2]$ and that of $B$ are $[\underline{b}(\alpha), \overline{b}(\alpha)] = [\alpha - 1, -\alpha + 1]$. These intervals in the united extension form are, at each $\alpha$- cut:*

$$\begin{aligned}&\lambda_A(\alpha) = \{\lambda_a(\alpha - 2) + (1 - \lambda_a)(-\alpha + 2)|\lambda_a \in [0,1]\} = \{2(\alpha - 2)\lambda_a - \alpha + 2|\lambda_a \in [0,1]\},\\ &\lambda_B(\alpha) = \{\lambda_b(\alpha - 1) + (1 - \lambda_b)(-\alpha + 1)|\lambda_b \in [0,1]\} = \{2(\alpha - 1)\lambda_b - \alpha + 1|\lambda_b \in [0,1]\},\\ &\lambda_C(\alpha) = \{\lambda_A(\alpha)(\lambda_A(\alpha) + \lambda_B(\alpha))|\alpha \in [0,1]\} = \{\lambda_A^2(\alpha) + \lambda_A(\alpha)\lambda_B(\alpha)|\alpha \in [0,1]\}.\end{aligned} \tag{51}$$

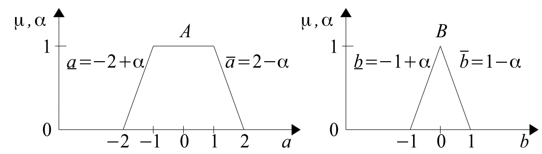


Figure 8: Membership functions of fuzzy intervals occurring in Example 2 in [18] of calculating $C = A(A + B)$ using CIA.

*Membership functions of fuzzy intervals $A$, $B$ are presented in Figure 8.*

*According to the calculations carried out by the authors of [18], the $C$-result obtained with the use of CFA is given by the formula (52). This result is 2-dimensional similarly as the input intervals $A$ and $C$.*

$$[\underline{c}(\alpha), \overline{c}(\alpha)] = [\min_{\lambda_a,\lambda_b\in[0,1]} \lambda_C(\alpha), \max_{\lambda_a,\lambda_b\in[0,1]} \lambda_C(\alpha)] = [-\alpha + 2, 2\alpha^2 - 7\alpha + 6], \alpha \in [0, 1]. \quad (52)$$

*The calculated result $C$ is shown in Figure 9.*

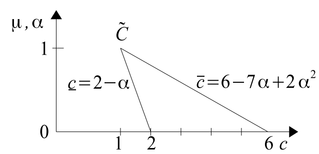


Figure 9: Calculation result $C = A(A + B)$ from Example 2 in [18] determined by the formula (52) and obtained with the use of CFA.

*The result of $C$ according to (52) is not a fuzzy interval because its left border (profile) $\underline{c} = 2 - \alpha$ is not a non-decreasing function with respect to the variable $c$ (it is a decreasing function) which is required by the definition of the fuzzy interval. The result $C$ is the so-called gradual interval [3] with respect to which such requirements are not met. Using the Instance Testing method, it is easy to check that this result is partially wrong. According to (52), for the cut level $\mu = \alpha = 0$, the result of $c < 2$ is not possible. Meanwhile, there is an infinitely large number of pairs $(a, b)$ for which the dependence $c = a(a + b) < 2$ is satisfied. An example of such a pair is $a = 0.5$ and $b = -1$, which give the result $c = -0.25$. The result (52), Figure 9 is not only numerically incorrect. It is also illogical, incompatible with common sense. It does not comply with the principle of inclusion of results presented in detail in [36]. Briefly, this principle can be formulated as follows: if the domain of computations $[\underline{a}(\alpha + \Delta\alpha), \overline{a}(\alpha + \Delta\alpha)] \times [\underline{b}(\alpha + \Delta\alpha), \overline{b}(\alpha + \Delta\alpha)]$ is contained in the larger domain $[\underline{a}(\alpha), \overline{a}(\alpha)] \times [\underline{b}(\alpha), \overline{b}(\alpha)]$ which it includes, then all the results of calculations of any function $y = f(a, b)$ for pairs $(a, b)$ contained in the first domain must also be included in the result set of the second domain and cannot extend beyond this domain. Hence, the result (52), Figure 9, in which the result set $c(\alpha = 0.5) = [1.5, 3.5]$ is not included in the result set $c(\alpha = 0)$ which is the interval $[2, 6]$, is not possible. And now Example 2 from [18] will be solved using MFA. The problem formulation is determined by the formulas (53).*

$$\begin{array}{l} A : a \in A, a = a(\gamma_a, \mu) = (-2 + \mu) + \gamma_a(4 - 2\mu), \mu, \gamma_a \in [0, 1], \\ B : b \in B, b = b(\gamma_b, \mu) = (-1 + \mu) + \gamma_b(2 - 2\mu), \mu, \gamma_b \in [0, 1], \\ A(A + B) = A^2 + AB = C, C : c \in C, \\ c = c(\gamma_a, \gamma_b, \mu) = a^2(\gamma_a, \mu) + a(\gamma_a, \mu)b(\gamma_b, \mu), \\ S : (a(\gamma_a, \mu), b(\gamma_b, \mu), c(\gamma_a, \gamma_b, \mu)) \in S. \end{array} \quad (53)$$

*The set $S$ is a multidimensional set of possible solutions to the problem. It allows you to generate all triples $(a, b, c)$ for each level of membership $\mu$ and the values of RDM variables $\gamma_a$, $\gamma_b$, The variable $c(\gamma_a, \gamma_b, \mu)$ is the result variable coupled with the specified (non-free) values of the variables $a$ and $b$. The set $C$, which is the set of all possible values of $c$ satisfying the problem conditions, has its span $SP_C$, which according to CFA is the only result of the calculations and according to MFA it is only a simplified 2D information about the 4D set $C$. Span function $SP_C$ can be calculated from formula (54).*

$$SP_C = [\min_{\gamma_1,\gamma_2\in[0,1]} c(\gamma_1, \gamma_2, \mu), \max_{\gamma_1,\gamma_2\in[0,1]} c(\gamma_1, \gamma_2, \mu)]. \quad (54)$$

*The value of $c(\gamma_a, \gamma_b, \mu)$ is given by the formula (53). After inserting the values of $a(\gamma_a, \mu)$, $b(\gamma_b, \mu)$ into the formula for $c$, we obtain a detailed form of the formula $c(\gamma_a, \gamma_b, \mu)$ given by (55).*

$$c(\gamma_a, \gamma_b, \mu) = [(-2 + \mu) + \gamma_a(4 - 2\mu)]^2 + [(-2 + \mu) + \gamma_a(4 - 2\mu)][(1 - \mu) + \gamma_b(2 - 2\mu)], \mu, \gamma_a, \gamma_b \in [0, 1]. \quad (55)$$

*Analysis of formula (54) shows that* $\min c(\gamma_a, \gamma_b, \mu)$ *is obtained for* $a(\gamma_a, \mu) = -b(\gamma_b, \mu)$ *and* $\max c(\gamma_a, \gamma_b, \mu)$ *for* $\gamma_a = 1$ *and* $\gamma_b = 1$*, formula (56).*

$$SP_C = [-0.25(1-\mu), 2\mu^2 - 7\mu + 6], \mu \in [0,1]. \tag{56}$$

*Figure 10 shows the span function* $SP_C$ *of result variable c.*

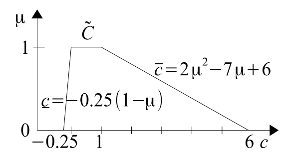


Figure 10: The span function $SP_C$ of the set of possible values of the result variable $c = a(a+b)$ in Example 2 from [18] obtained with the use of the MFA arithmetic.

*If we compare the result of the calculations* $C = A(A+B)$*, Figure 9 and the span* $SP_C$ *obtained with the MFA, Figure 10, it can be seen that the two spans are different. Span-CFA is not a fuzzy number but a pure granular number and does not include many possible values of the variable c. Span-MFA, Figure 10, is more comprehensive and is a fuzzy number consistent with the results inclusion principle [36]. Is it correct? This can be easily checked with the Instance Testing method. From the MFs of sets A and B, Figure 8, it can be seen that for the level* $\mu = 0$ *values* $a = -0.5$ *and* $b = 1$ *are possible, giving the value* $c = a(a+b) = -0.25$*. According to MFA, Figure 10, this value is possible. According to the CFA, Figure 9, this is an impossible value. Then, at the level of* $\mu = 0.5$*, Figure 8, the possible values are* $a = -0.25$*,* $b = 0.5$*, giving the result* $c = a(a+b) = -0.125$*. However, according to CFA, Figure 9 this result is not possible. According to the MFA, Figure 10 is possible. At the level* $\mu = 1$*, according to Figure 8, the values of* $a = 0$ *and* $b = 0$ *are possible, giving the result* $c = 0$*. However, according to CFA, Figure 9, this result is not possible. According to the MFA, Figure 10, it is possible. This means that the left border of the "result" calculated by the CFA is incorrect. However, the right border is correct and is in line with the right border of span of the* $SP_C$*, Figure 10, calculated by the MFA.*

**Example 5.2.** *Calculating function values by the CIA and MIA.*

*There are two functions* $F_1([a][x])$ *and* $F_2([a]x)$*. The function* $F_1([a][x])$ *is given by the formula (57).*

$$F_1([a][x]) = [a][x] = [-1,1][\underline{x}, \overline{x}]. \tag{57}$$

*The function* $F_2([a]x)$ *is a special case of the function* $F_1([a][x])$ *where* $\underline{x} = \overline{x} = x$ *(degenerate interval), occurs, formula (58).*

$$F_2([a]x) = [-1,1]x, \ \underline{x} = \overline{x} = x, \ x > 0. \tag{58}$$

***Calculating values of*** $F_1([a][x])$ ***and*** $F_2([a]x)$ ***using a 1-dimensional CIA***

*The interval* $[a] = [-1,1]$ *corresponds to the function* $a(\lambda_a)$*, formula (59).*

$$a(\lambda_a) = \lambda_a \underline{a} + (1-\lambda_a)\overline{a} = 1 - 2\lambda_a, \lambda_a \in [0,1]. \tag{59}$$

*The interval* $[x] = [\underline{x}, \overline{x}]$ *corresponds to the function x (), formula (60).*

$$x(\lambda_x) = \lambda_x \underline{x} + (1-\lambda_x)\overline{x}, \ \underline{x} < \overline{x}, \underline{x}, \overline{x} > 0. \tag{60}$$

*The operation of calculating the function* $F_1([a][x])$ *by the CIA is performed according to the formula (4).*

$$F_{1CIA} = [\underline{f}_1, \overline{f}_2] = \{f_1 | f_1 = a(\lambda_a)x(\lambda_x) = (1-2\lambda_a)[\lambda_x \underline{x} + (1-\lambda_x)\overline{x}], \lambda_a, \lambda_x \in [0,1]\}. \tag{61}$$

*where* $\underline{f}_1 = \min_{\lambda_a, \lambda_x}\{f_1\} = -\overline{x}$ *for* $\lambda_a = 1$*,* $\lambda_x = 0$*,* $\overline{f}_1 = \max_{\lambda_a, \lambda_x}\{f_1\} = \overline{x}$ *for* $\lambda_a = 0$*,* $\lambda_x = 0$*.*

*We get the result (62).*

$$F_{1CIA} = [-\overline{x}, \overline{x}]. \tag{62}$$

*In the case of the function* $F_2([a]x) = [-1,1]x$*, where* $\underline{x} = \overline{x} = x$*, the interval* $[a]$ *is represented by the function* $a(\lambda_a)$*, formula (63).*

$$a(\lambda_a) = \lambda_a \underline{a} + (1-\lambda_a)\overline{a} = 1 - 2\lambda_a, \lambda_a \in [0,1]. \tag{63}$$

*The value of $x$ is represented by (64).*

$$x(\lambda_x) = x = \underline{x} = \overline{x}. \tag{64}$$

*Hence the function $F_{2CIA}$ is given by the formula (65).*

$$F_{2CIA} = [\underline{f}_2, \overline{f}_2] = \{f_2 | f_2 = a(\lambda_a)x = (1 - 2\lambda_a)x, \lambda_a \in [0,1]\}. \tag{65}$$

*The lower and upper values of this function are obtained from (66).*

$$\underline{f}_2 = \min_{\lambda_a}\{f_2\} = -x \text{ for } \lambda_a = 1,\ \overline{f}_2 = \max_{\lambda_a}\{f_2\} = x \text{ for } \lambda_a = 0. \tag{66}$$

*Since $x = \underline{x} = \overline{x}$ then $\underline{f}_2 = -x$ and $\overline{f}_2 = x$. Hence the result of $F_{2CIA}$ is given by the formula (65). $F_{2CIA} = [-x, x]$. This means that, according to the CIA, the values of both functions are identical: $F_{1CIA} = F_{2CIA} = [-x, x]$. This result is rather inconsistent with common sense because in $F_1 = [-1,1][x]$ there is a normal interval $[\underline{x}, \overline{x}]$, $\underline{x} < \overline{x}$, while in $F_2 = [-1,1]x$ there is a degenerate interval $[\underline{x}, \overline{x}]$, $x = \underline{x} = \overline{x}$, being a point-value.*

*Figure 11 shows a visualization of the resulting set $F_{1CIA}$ and $F_{2CIA}$ of the possible values of the function.*

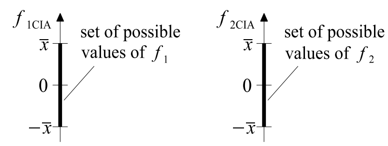


Figure 11: Visualization of the set of possible values of interval-valued functions $F_1([-1,1][\underline{x}, \overline{x}])$ and $F_2([-1,1], x)$ calculated according to the CIA rules.

*As Fig 11 shows, sets of possible values of the function $F_1([-1,1][\underline{x}, \overline{x}]$ and $F_2([-1,1], x)$ are identical, meaning that the CIA cannot distinguish between these functions because $F_{1CIA} = F_{2CIA} = [-x, x]$. This is especially important if we send these results to further stages of calculations (further stages of problem solving).*

***Calculation of the function values $F_1([a][x])$ and $F_2([a]x)$ by MIA***

*In the function $F_1([a][x]) = [-1,1][\underline{x}, \overline{x}])$ there are 2 unknown, uncertain, interval-valued-variables $a$ and $x$. In MIA terms, a mathematical model of the unknown value of a-variable is given by the formula (67).*

$$[a]_{MIA} : a(\alpha_a) = \underline{a} + \alpha_a(\overline{a} - \underline{a}) = -1 + 2\alpha_a, \alpha_a \in [0,1]. \tag{67}$$

*The mathematical model of the unknown value of the variable $x$ is given by the formula (68).*

$$[x]_{MIA} : x(\alpha_x) = \underline{x} + \alpha_x(\overline{x} - \underline{x}), \alpha_x \in [0,1]. \tag{68}$$

*The mathematical model of the unknown result value $f_1$ is given by the formula (69).*

$$F_1 = f_1 : f_1 = a(\alpha_a)x(\alpha_x) = (-1 + 2\alpha_a)[\underline{x} + \alpha_x(\overline{x} - \underline{x})],\ \alpha_a, \alpha_x \in [0,1]. \tag{69}$$

*The 3-dimensional set of possible function values is the set $(A, X, F_1)$ is given by (67), (68), (69) and presented in Figure 12 in the projection on the 2-D space $A \times X$. The resulting values of the function $f_1(a, x)$ are given in the marked points.*

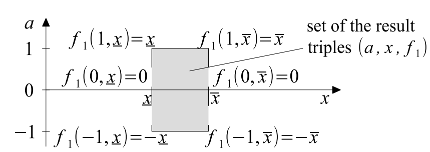


Figure 12: 2D-projection of the 3-dimensional set of solutions $(A, X, F_1)$ into the space of inputs $A \times X$. In selected points the value of the function $f_1(a, x)$ is given.

*For the interval-valued function $F_2 = [\underline{a}, \overline{a}]x = [-1.1]x$ the mathematical model of the resulting value of the function is given by (70).*

$$F_{2MIA} = \{f_2(a,x)\} = \{(a(\alpha_a)x)\} : \begin{array}{l} a(\alpha_a) = \underline{a} + \alpha_a(\overline{a} - \underline{a}) = -1 + 2\alpha_a, \alpha_a \in [0,1], \\ x = \underline{x} + \alpha_x(\overline{x} - \underline{x}), x = \underline{x} = \overline{x} : x = \overline{x}. \end{array} \tag{70}$$

*Since crisp value of $x$ can be interpreted as a degenerate interval $x(\alpha_x) = \underline{x} + \alpha_x(\overline{x} - \underline{x})$ where $x = \underline{x} = \overline{x}$ then the $F_{2MIA}$ function can be interpreted as $F_2([a], x)$, formula (71).*

$$F_{2MIA} = F_2([a]x) = \{(-1 + 2\alpha_a)\overline{x}\}, \alpha_a \in [0, 1], \overline{x} = \underline{x} = x. \tag{71}$$

*The set of possible values of the $F_{2MIA}$ function is shown in Figure 13.*

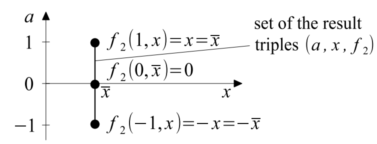


Figure 13: Projection of the multidimensional set of solutions $(A, x, F_2)$ on the input space $A \times x$. The value of the function $f_2(a, x)$ is given in selected points.

*A comparison of the sets of possible solutions of the functions $F_{1MIA}$ and $F_{2MIA}$ shown in Figure 12 and Figure 13 shows that the sets are different. This is consistent with common sense. The difference between these results means that the MIA distinguishes between the two functions considered. On the other hand, the reason for the lack of distinction between them by the CIA is the erroneous assumption that the result of calculations on the intervals is also the interval.*

**Example 5.3.** *Let us consider fuzzy linear system (72),* [9, 14, 23].

$$\begin{aligned} x_1 - x_2 &= (\alpha, 2 - \alpha), \\ x_1 + 3x_2 &= (4 + \alpha, 7 - 2\alpha), \end{aligned} \tag{72}$$

*where* $\alpha \in [0, 1]$.

*The solution to the fuzzy linear system (72) according to CFA are fuzzy numbers (73), Figure 14, see solution by CFA in [23] Example 2.*

$$\begin{aligned} [x_1(\alpha)] &= [\underline{x}_1(\alpha), \overline{x}_1(\alpha)] = [1 + \alpha, 13/4 - 5\alpha/4], \\ [x_2(\alpha)] &= [\underline{x}_2(\alpha), \overline{x}_2(\alpha)] = [1/2 + \alpha/2, 7/4 - 3/4\alpha], \end{aligned} \tag{73}$$

*where* $\alpha \in [0, 1]$.

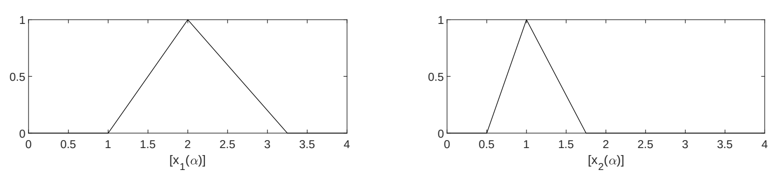


Figure 14: Solution $[x_1(\alpha)]$ and $[x_2(\alpha)]$ obtained by the CFA, see [23].

*The solution (called direct solution) obtained by MFA are multidimensional granule of information (74), Figure 15, see Example 3 in [14]. Whereas result (73) in MFA is the indicator of the direct solution (a secondary result).*

$$\begin{aligned} x_1(\alpha, \alpha_{x1}, \alpha_{x2}) &= 1 + \alpha + 1.5\alpha_{x1}(1 - \alpha) + 0.75\alpha_{x2}(1 - \alpha), \\ x_2(\alpha, \alpha_{x1}, \alpha_{x2}) &= 1 - 0.5\alpha_{x1}(1 - \alpha) + 0.75\alpha_{x2}(1 - \alpha). \end{aligned} \tag{74}$$

*where* $\alpha, \alpha_{x1}, \alpha_{x2} \in [0, 1]$.

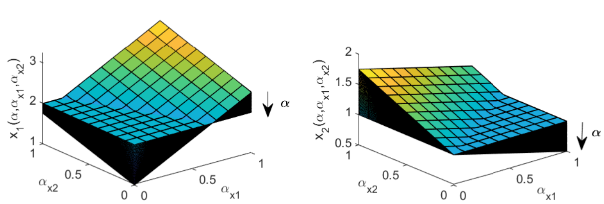


Figure 15: Solution $x_1(\alpha, \alpha_{x1}, \alpha_{x2})$ and $x_2(\alpha, \alpha_{x1}, \alpha_{x2})$ obtained by the MFA, see [14].

*It can be proven by inserting CFA solutions (73) (shown in Figure 14) into equations (72) that they do not give equality of the left and right sides of these equations. MFA solutions (74) are algebraic solutions of equations (72) and give equality of the left and right sides of these equations. CFA solutions are mathematical objects in 2D and MFA are objects in 3D.*

# 6 Conclusions

The article discusses whether MIA and CIA, MFA and CFA are identical types of interval and fuzzy arithmetic, and identifies the differences given in the sequel.

1. CIA is 1-dimensional arithmetic (in terms of the input variables and the result obtained).

2. CFA is 2-dimensional arithmetic (in the sense as above) while MFA is multidimensional.

3. Solutions to equations with unknowns provided by the CIA and CFA (in general) do not satisfy these equations because they are not universal algebraic solutions. On the other hand, the solutions calculated by MIA and MFA satisfy the solved equations and have the feature of universality, i.e. they do not depend on the mathematical form of equations.

4. The low-dimensional calculation results provided by the CIA and CFA (in general) cannot be inserted into the further steps of the calculation, as this may lead to an erroneous final result and the Unnatural Behavior in Modeling (UBM) phenomenon. The above remark does not apply to results provided by the MIA and the MFA.

5. According to the authors of the CIA and CFA, these types of arithmetic belong to gradual arithmetic [18, 3], which sometimes produces results that are 2D fuzzy numbers, and at other times pure 2D-gradual numbers that are not fuzzy numbers, which do not meet the principle of results' inclusion [36]. Some examples of CFA calculations given in [18] indicate that also CFA sometimes generates pure gradual numbers. However, this is not the case with MFA, the spans of which comply with the principle of results' inclusion.

6. There is only one type of outcome in the CIA and CFA: interval and fuzzy interval, respectively. In MIA and MFA, there is a multidimensional algebraic result and its 3 simplified indicators: span, distribution of cardinality measure, and center of gravity. These indicators can be referred to as simplified secondary results. The CIA and CFA generate only one of these 3 secondary results which is actually the span of the primary result. This span does not contain complete information about the multidimensional primary result.

7. The CIA applies the rule that if "each interval in the arithmetic expression is independent, the usual rules (of SIA) for interval arithmetic should be used". In the MIA, this rule does not apply as the SIA leads only to the interval results and MIA to multidimensional ones.

8. In general, the CIA may compute identical results for two different interval-valued functions. This means that the CIA may not be able to distinguish between different functions. The same is true for the CIA-based CFA. In the case of MIA and MFA, for two different functions we obtain two different multidimensional result sets. This means distinguishing between these functions.

9. The CIA and the CFA based on it have 5 mathematical properties (*commutativity of addition and multiplication,* associativity of add & mult, *additive and multiplicative neutral elements, *inverse elements for add & mult, *cancellation law only for addition but not for multiplication). MIA and MFA have 7 important mathematical properties (*commutativity law for add & mult, *associativity law for add & mult, *neutral elements for add & mult, *inverse elements for add & mult,* restoration law for add & mult, *providing algebraic solutions for interval and fuzzy equations,* independence of the results from the mathematical form of the problem description). The small number of CIA properties is due to its assumption that the result of operations on intervals is also an interval.

Summarizing the above component conclusions, it can be stated that the CIA and CFA, despite some similarity, are not the same types of arithmetic as MIA and MFA.

The current and further work of the authors and associates concerns the use of multidimensional interval arithmetic to improve and develop the existing computational methods of uncertainty mathematics. In 2022, an article [37] was prepared and published, showing how, in a new way, based on the concept of robustness on data uncertainty, one can solve the basic interval equation $[\underline{a}, \overline{a}]X = [\underline{b}, \overline{b}]$, also for cases which previously could not be solved. The method presented in [37] is the basis for a new method of solving interval linear systems of equations and interval nonlinear equations. Then it will be used to solve the fuzzy versions of the equations mentioned.

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