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Original paper

L-R representation of TA fuzzy arithmetic and its application to solving fuzzy equations

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Abstract

Fuzzy arithmetic with standard methods such as the extension principle and α-cut lead to restricted possibilities for solving fuzzy equations. The procedures to find a solution to a fuzzy equality with these methods require strong assumptions and high computation costs. Among several approaches dealing with this restrictions this paper focuses on the Transmission Average (TA) fuzzy arithmetic. The shape preservation of the TA arithmetic operations on L-R fuzzy numbers is proven. These properties together with some other algebraic properties investigated in the paper are applied to solve fuzzy polynomial equations as well as systems of linear fuzzy equations in general form. Several examples in the paper present the advantages of TA arithmetic in solving fuzzy equations. It is shown that the results in this paper support the fact that TA arithmetic is an easy to implement approach in fuzzy modeling.

Keywords: Fuzzy arithmetic, fuzzy equality, L-R fuzzy numbers, transmission average (TA).

1 Introduction

Fuzzy equations are important tools in modeling real-world problems which occur in a wide range of disciplines such as economy and engineering in [8, 9, 10]. For this reason solution methods for fuzzy equations are of interest for many researchers. However, the problem of solving fuzzy equations faces serious difficulties which stem from several different reasons. Maybe the most important reason is the non-existence of inverses and reciprocals of fuzzy numbers in standard fuzzy arithmetic using Zadeh's extension principle (EP). The lack of inverses and reciprocals of fuzzy numbers leads to the result that even in the linear case, a single variable fuzzy equation fails to have a solution most of the time. Another reason is the lack of shape preservation, for instance the product of two triangular fuzzy numbers is not a triangular fuzzy number anymore. Finally, another main issue is the growing width effect: the support of the solutions grow so rapidly that, in the end, it becomes difficult to interpret the solution. There are so many different attempts to solve the mentioned or other issues, that we will recall only some of them. One group of advances provide approximate solutions. Abbasbandy et al. introduced fuzzy versions of several numerical methods in [1, 2, 3] and Hanss in [12], proposed a discrete method for fuzzy arithmetic. Buckley and Qu in [8] suggested three different solutions to a fuzzy linear system. An approach that is reducing the growing width effect and is based on the interval extended zero is proposed in [19]. The Relative Distance Measure arithmetic that uses horizontal membership function representations linked to Piegat and Landowski in [17, 18] provided an alternative perspective for solving fuzzy equations. Another group of researches are about using different t-norms in the fuzzy arithmetic. T-norms, preserving shape under addition, are investigated in [14, 16]. Hong in [13] proved that for multiplication, the drastic (weak) t-norm is the only shape preserving t-norm.

A rather new approach to fuzzy arithmetic is the transmission average (TA) method. The algebraic structure of the transmission average fuzzy arithmetic was introduced in [5] followed by the papers [4, 6] where solutions to fuzzy equations are explored. The method seems to be useful in the sense that it not only provides a broader space of solvable fuzzy equations but also decreases the growing width effect of the operations.

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This paper focuses on the investigation of the TA arithmetic in the L-R representation. Shape preservation properties and some other algebraic properties which are helpful in solving fuzzy equations are proved. Advantages gained by these properties are explained by examples. The findings are applied to the solution of fuzzy polynomial equations of any order and explicit solutions are obtained. Moreover the solution method for systems of linear fuzzy equations is discussed in the light of the findings. In this way it is shown that many type of problems which have no solutions with EP fuzzy arithmetic are solvable with TA arithmetic with the help of shape preservation theorems and some other algebraic properties proven in this paper.

2 Preliminaries

In order to introduce arithmetic operations based on TA we begin with the definition of pseudo-trapezoidal fuzzy numbers:

Definition 2.1. [15] *Let $l_{\widetilde{A}}(x)$ and $r_{\widetilde{A}}(x)$ be non-increasing and non-decreasing functions respectively. A fuzzy number $\widetilde{A}$ with the membership function,*

$$\mu_{\widetilde{A}}(x) = \begin{cases} l_{\widetilde{A}}(x), & \underline{a} \le x \le a_1, \\ 1, & a_1 \le x \le a_2, \\ r_{\widetilde{A}}(x), & a_2 \le x \le \overline{a}, \\ 0, & otherwise. \end{cases},$$

is called a pseudo-trapezoidal fuzzy number and is denoted by $\widetilde{A} = (\underline{a}, a_1, a_2, \overline{a}, l_{\widetilde{A}}(x), r_{\widetilde{A}}(x))$.

Consider the two fuzzy numbers,

$$\widetilde{A} = (\underline{a}, a_1, a_2, \overline{a}, l_{\widetilde{A}}(x), r_{\widetilde{A}}(x)), \widetilde{B} = (\underline{b}, b_1, b_2, \overline{b}, l_{\widetilde{B}}(x), r_{\widetilde{B}}(x)),$$

with the following α-cut forms:

$$A = \cup_{\alpha\in(0,1]}\alpha.A_\alpha, A_\alpha = [\underline{A}_\alpha, \overline{A}_\alpha], B = \cup_{\alpha\in(0,1]}\alpha.B_\alpha, B_\alpha = [\underline{B}_\alpha, \overline{B}_\alpha].$$

Let $\phi = \frac{a_1+a_2}{2}$, $\psi = \frac{b_1+b_2}{2}$, then TA based arithmetic operations are defined as following in [5].

· Addition

$$A + B = \cup_{\alpha\in(0,1]}\alpha.(A + B)_\alpha, (A + B)_\alpha = [\underline{(A + B)}_\alpha, \overline{(A + B)}_\alpha],$$

where,

$$\underline{(A + B)}_\alpha = \frac{\phi + \psi}{2} + (\frac{\underline{A}_\alpha + \underline{B}_\alpha}{2}), \overline{(A + B)}_\alpha = \frac{\phi + \psi}{2} + (\frac{\overline{A}_\alpha + \overline{B}_\alpha}{2}).$$

· Subtraction

Firstly,

$$-B = \cup_{\alpha\in(0,1]}\alpha.(-B), (-B)_\alpha = [(\underline{-B})_\alpha, (\overline{-B})_\alpha],$$

where $(\underline{-B})_\alpha = -2\psi + \underline{B}_\alpha$ and $(\overline{-B})_\alpha = -2\psi + \overline{B}_\alpha$.

Finally, $A - B = A + (-B)$,

$$A - B = \cup_{\alpha\in(0,1]}\alpha.(A - B)_\alpha, (A - B)_\alpha = [\underline{(A - B)}_\alpha, \overline{(A - B)}_\alpha],$$

where,

$$\underline{(A - B)}_\alpha = \frac{\phi - 3\psi}{2} + (\frac{\underline{A}_\alpha + \underline{B}_\alpha}{2}), \overline{(A - B)}_\alpha = \frac{\phi - 3\psi}{2} + (\frac{\overline{A}_\alpha + \overline{B}_\alpha}{2}).$$

· Multiplication

$$A.B = \cup_{\alpha\in(0,1]}\alpha.(A.B)_\alpha,$$
$$(A.B)_\alpha = [\underline{(A.B)}_\alpha, \overline{(A.B)}_\alpha],$$

where,

$$[\underline{(A.B)}_\alpha, \overline{(A.B)}_\alpha] = \begin{cases} [(\frac{\psi}{2})\underline{A}_\alpha + (\frac{\phi}{2})\underline{B}_\alpha, (\frac{\psi}{2})\overline{A}_\alpha + (\frac{\phi}{2})\overline{B}_\alpha], & \phi \geq 0, \psi \geq 0 \\ [(\frac{\psi}{2})\overline{A}_\alpha + (\frac{\phi}{2})\underline{B}_\alpha, (\frac{\psi}{2})\underline{A}_\alpha + (\frac{\phi}{2})\overline{B}_\alpha], & \phi \geq 0, \psi \leq 0 \\ [(\frac{\psi}{2})\overline{A}_\alpha + (\frac{\phi}{2})\overline{B}_\alpha, (\frac{\psi}{2})\underline{A}_\alpha + (\frac{\phi}{2})\underline{B}_\alpha], & \phi \leq 0, \psi \leq 0 \\ [(\frac{\psi}{2})\underline{A}_\alpha + (\frac{\phi}{2})\overline{B}_\alpha, (\frac{\psi}{2})\overline{A}_\alpha + (\frac{\phi}{2})\underline{B}_\alpha], & \phi \leq 0, \psi \geq 0 \end{cases}.$$

· Division

Firstly,

$$B^{-1} = \cup_{\alpha \in (0,1]} \alpha.(B^{-1})_\alpha,$$
$$(B^{-1})_\alpha = [\underline{(B^{-1})}_\alpha, \overline{(B^{-1})}_\alpha],$$

where $\underline{(B^{-1})}_\alpha = \left(\frac{1}{\psi^2}\right)\underline{B^{-1}}_\alpha$, $\overline{(B^{-1})}_\alpha = \left(\frac{1}{\psi^2}\right)\overline{B^{-1}}_\alpha$.

Finally,

$$A.B^{-1} = \cup_{\alpha \in (0,1]} \alpha.(A.B^{-1})_\alpha,$$
$$(A.B^{-1})_\alpha = [\underline{(A.B^{-1})}_\alpha, \overline{(A.B^{-1})}_\alpha],$$

$$[\underline{(A.B^{-1})}_\alpha, \overline{(A.B^{-1})}_\alpha] = \begin{cases} [(\frac{1}{2\psi})\underline{A}_\alpha + (\frac{\phi}{2\psi^2})\underline{B}_\alpha, (\frac{1}{2\psi})\overline{A}_\alpha + (\frac{\phi}{2\psi^2})\overline{B}_\alpha], & \phi \geq 0, \psi > 0 \\ [(\frac{1}{2\psi})\overline{A}_\alpha + (\frac{\phi}{2\psi^2})\underline{B}_\alpha, (\frac{1}{2\psi})\underline{A}_\alpha + (\frac{\phi}{2\psi^2})\overline{B}_\alpha], & \phi \geq 0, \psi < 0 \\ [(\frac{1}{2\psi})\overline{A}_\alpha + (\frac{\phi}{2\psi^2})\overline{B}_\alpha, (\frac{1}{2\psi})\underline{A}_\alpha + (\frac{\phi}{2\psi^2})\underline{B}_\alpha], & \phi \leq 0, \psi < 0 \\ [(\frac{1}{2\psi})\underline{A}_\alpha + (\frac{\phi}{2\psi^2})\overline{B}_\alpha, (\frac{1}{2\psi})\overline{A}_\alpha + (\frac{\phi}{2\psi^2})\underline{B}_\alpha], & \phi \leq 0, \psi > 0 \end{cases}.$$

We continue by recalling $L-R$ Fuzzy Numbers as in [11].

Definition 2.2. *The functions L and R, defined on $\mathbb{R}_0^+$, are called shape functions if they possess the following properties:*

1) $L(x) \in [0,1]$, $\forall x$ *and* $R(x) \in [0,1]$, $\forall x$.
2) $L(0) = R(0) = 1$.
3) $L(x)$ *and* $R(x)$ *are decreasing in* $[0, \infty]$.
4) $L(1) = 0$ *if* $\min_x L(x) = 0$,
5) $\lim_{x\to\infty} L(x) = 0$ *if* $L(x) > 0$, $\forall x$,
6) $R(1) = 0$ *if* $\min_x R(x) = 0$,
7) $\lim_{x\to\infty} R(x) = 0$ *if* $R(x) > 0$, $\forall x$.

Definition 2.3. *Let L and R be shape functions. An $L-R$ fuzzy number $\tilde{A}$ has the following membership function*

$$\mu_{\tilde{A}}(x) = \begin{cases} \mu_l(x) = L[\frac{a-x}{\alpha}], & x < a, \\ \mu_r(x) = R[\frac{x-a}{\beta}], & x \geq a. \end{cases}$$

With the help of this definition a fuzzy number $\tilde{A}$ is characterized by its modal value a and the spreads α and β, corresponding to the left-hand and right-hand curves of the membership function, respectively. We will use the abbreviated notation $A = \langle a, \alpha, \beta\rangle_{L,R}$ for an $L-R$ fuzzy number $\tilde{A}$. In this setting, for example the triangular fuzzy number with support $[1,8]$ and modal value 3 will be denoted by $\langle 3, 2, 5\rangle_{1-x,1-x}$. The same fuzzy number is denoted by $(1,3,8)$ in the standard notation.

3 Properties of TA fuzzy arithmetic in L-R setting

This section mainly contains theorems about the shape preservation of TA arithmetic followed by some algebraic properties that will be useful in the solution process of fuzzy equations. Besides, comparisons of TA arithmetic with EP arithmetic are made and advantages of the TA arithmetic over EP arithmetic are discussed with some examples.

Theorem 3.1. *The TA addition is shape-preserving.*

Proof. Let $A = \langle a, \alpha_1, \beta_1 \rangle_{L,R}$ and $B = \langle b, \alpha_2, \beta_2 \rangle_{L,R}$ be two fuzzy numbers. We will assume that both L and R are continuous and strictly monotonic. This provides the existence of their inverses L^{-1} and R^{-1}. But it should be noted that even for non-continuous or non-monotonic functions the proof can be made with proper definitions of pseudo-inverses of L and R.

We have that;

$$\mu_A(x) = \begin{cases} L[\frac{a-x}{\alpha_1}] & x < a \\ R[\frac{x-a}{\beta_1}] & x \geq a, \end{cases}$$

$$\mu_B(x) = \begin{cases} L[\frac{b-x}{\alpha_2}] & x < b \\ R[\frac{x-b}{\beta_2}] & x \geq b \end{cases}.$$

For $\gamma \in [0,1]$ let $\mu_A(x) = \gamma$. This implies that for some x and y, $L[\frac{a-x}{\alpha_1}] = \gamma$ and $R[\frac{y-a}{\beta_1}] = \gamma$. We apply the inverse functions and obtain, $L^{-1}L[\frac{a-x}{\alpha_1}] = L^{-1}(\gamma)$ and $R^{-1}R[\frac{y-a}{\beta_1}] = R^{-1}(\gamma)$. Therefore we observe,

$$x = a - \alpha_1 L^{-1}(\gamma) \text{ and } y = a + \beta_1 R^{-1}(\gamma).$$

This means that the γ-cuts of A and B are

$$A_\gamma = [a - \alpha_1 L^{-1}(\gamma), a + \beta_1 R^{-1}(\gamma)],$$
$$B_\gamma = [b - \alpha_2 L^{-1}(\gamma), b + \beta_2 R^{-1}(\gamma)].$$

Now we apply TA addition;

$$\underline{(A+B)} = \frac{a+b}{2} + \frac{a - \alpha_1 L^{-1}(\gamma) + b - \alpha_2 L^{-1}(\gamma)}{2} = a + b - \frac{(\alpha_1 + \alpha_2) L^{-1}(\gamma)}{2}, \tag{1}$$

and

$$\overline{(A+B)} = \frac{a+b}{2} + \frac{a + \beta_1 R^{-1}(\gamma) + b + \beta_2 R^{-1}(\gamma)}{2} = a + b + \frac{(\beta_1 + \beta_2) R^{-1}(\gamma)}{2}. \tag{2}$$

Combining (1) and (2) we obtain:

$$(A+B) = \left\langle a+b, \frac{\alpha_1 + \alpha_2}{2}, \frac{\beta_1 + \beta_2}{2} \right\rangle_{L,R}.$$

□

Following theorems show the shape preservation properties of the TA subtraction, multiplication and division. Their proofs use similar reasoning to the proof of Theorem 3.1 and therefore are kept short.

Theorem 3.2. *The TA subtraction is shape preserving.*

Proof. Let $A = \langle a, \alpha_1, \beta_1 \rangle_{L,R}$ and $B = \langle b, \alpha_2, \beta_2 \rangle_{L,R}$ be two fuzzy numbers. Then,

$$A - B = \left\langle a-b, \frac{\alpha_1 + \alpha_2}{2}, \frac{\beta_1 + \beta_2}{2} \right\rangle_{L,R}.$$

For the γ-cuts of $A - B$ we observe the following:

$$(\underline{A-B})_\gamma = \frac{a-3b}{2} + \frac{a - \alpha_1 L^{-1}(\gamma) + b - \alpha_2 L^{-1}(\gamma)}{2} = a - b - \frac{(\alpha_1 + \alpha_2) L^{-1}(\gamma)}{2}, \tag{3}$$

and

$$\overline{(A-B)}_\gamma = \frac{a-3b}{2} + \frac{a + \beta_1 R^{-1}(\gamma) + b + \beta_2 R^{-1}(\gamma)}{2} = a - b + \frac{(\beta_1 + \beta_2) R^{-1}(\gamma)}{2}. \tag{4}$$

Combining (3) and (4) we obtain the desired result. □

Theorem 3.3. *Let $A = \langle a, \alpha_1, \beta_1 \rangle_{L,R}$ and $B = \langle b, \alpha_2, \beta_2 \rangle_{L,R}$ be two fuzzy numbers with $a, b \geq 0$.*
$A.B = \left\langle ab, \frac{b\alpha_1 + a\alpha_2}{2}, \frac{b\beta_1 + a\beta_2}{2} \right\rangle_{L,R}$.

Proof. We observe following for the γ-cuts of the product,

$$\underline{(A.B)}_\gamma = \frac{b}{2}(a - \alpha_1 L^{-1}(\gamma)) + \frac{a}{2}(b - \alpha_2 L^{-1}(\gamma)) = ab - \frac{b}{2}\alpha_1 L^{-1}(\gamma) - \frac{a}{2}\alpha_2 L^{-1}(\gamma) \tag{5}$$

$$= ab - \left(\frac{b\alpha_1 + a\alpha_2}{2} \right) L^{-1}(\gamma). \tag{6}$$

Similarly,

$$\overline{(A.B)}_\gamma = ab + \left(\frac{b\beta_1 + a\beta_2}{2} \right) R^{-1}(\gamma). \tag{7}$$

(1) and (2) implies,

$$A.B = \left\langle ab, \frac{b\alpha_1 + a\alpha_2}{2}, \frac{b\beta_1 + a\beta_2}{2} \right\rangle_{L,R}.$$

□

Other cases are as following. If $a, b < 0$,

$$A.B = \left\langle ab, \frac{-b\beta_1 - a\beta_2}{2}, \frac{-b\alpha_1 - a\alpha_2}{2} \right\rangle_{R,L}.$$

If $A = \langle a, \alpha_1, \beta_1 \rangle_{L,R}$ and $B = \langle b, \alpha_2, \beta_2 \rangle_{R,L}$ with $a \geq 0$ and $b < 0$ then,

$$A.B = \left\langle ab, \frac{a\alpha_2 - b\beta_1}{2}, \frac{a\beta_2 - b\alpha_1}{2} \right\rangle_{R,L}.$$

Finally if $A = \langle a, \alpha_1, \beta_1 \rangle_{L,R}$ and $B = \langle b, \alpha_2, \beta_2 \rangle_{R,L}$ with $a < 0$ and $b \geq 0$ then,

$$\left\langle ab, \frac{b\alpha_1 - a\beta_2}{2}, \frac{b\beta_1 - a\alpha_2}{2} \right\rangle_{L,R}.$$

Remark 3.4. *The multiplication by TA arithmetic returns the same result with the tangent approximation* [11] *to the EP multiplication but with half the spreads.*

Theorem 3.5. *Let $A = \langle a, \alpha_1, \beta_1 \rangle_{L,R}$ and $B = \langle b, \alpha_2, \beta_2 \rangle_{L,R}$ be two fuzzy numbers. For $(a, b \geq 0)$,*

$$A.B^{-1} = \left\langle \frac{a}{b}, \frac{b\alpha_1 + a\alpha_2}{2b^2}, \frac{b\beta_1 + a\beta_2}{2b^2} \right\rangle_{L,R}.$$

Proof. First we obtain the lower and upper bounds of the γ-cuts of the division,

$$\underline{(A.B^{-1})}_\gamma = \frac{1}{2b}\left(a - \alpha_1 L^{-1}(\gamma)\right) + \frac{a}{2b^2}(b - \alpha_2 L^{-1}(\gamma)) = \frac{a}{b} - \left(\frac{\alpha_1}{2b} L^{-1}(\gamma) + \frac{a\alpha_2}{2b^2} L^{-1}(\gamma)\right)$$

$$= \frac{a}{b} - \left(\frac{b\alpha_1 + a\alpha_2}{2b^2} \right) L^{-1}(\gamma). \tag{8}$$

Similarly we get,

$$\overline{(A.B^{-1})}_\gamma = \frac{a}{b} + \left(\frac{b\beta_1 + a\beta_2}{2b^2} \right) R^{-1}(\gamma).$$

These two bounds combined imply that,

$$A.B^{-1} = \left\langle \frac{a}{b}, \frac{b\alpha_1 + a\alpha_2}{2b^2}, \frac{b\beta_1 + a\beta_2}{2b^2} \right\rangle_{L,R}.$$

□

Below is the result for all cases. If $a, b < 0$,

$$A.B^{-1} = \left\langle \frac{a}{b}, \frac{-b\beta_1 - a\beta_2}{2b^2}, \frac{-b\alpha_1 - a\alpha_2}{2b^2} \right\rangle_{R,L}.$$

If $A = \langle a, \alpha_1, \beta_1 \rangle_{L,R}$ and $B = \langle b, \alpha_2, \beta_2 \rangle_{R,L}$ with $a \geq 0$ and $b < 0$ then,

$$A.B^{-1} = \left\langle \frac{a}{b}, \frac{a\alpha_2 - b\beta_1}{2b^2}, \frac{a\beta_2 - b\alpha_1}{2b^2} \right\rangle_{R,L}.$$

Finally if $A = \langle a, \alpha_1, \beta_1 \rangle_{L,R}$ and $B = \langle b, \alpha_2, \beta_2 \rangle_{R,L}$ with $a < 0$ and $b > 0$ then,

$$A.B^{-1} = \left\langle \frac{a}{b}, \frac{b\alpha_1 - a\beta_2}{2b^2}, \frac{b\beta_1 - a\alpha_2}{2b^2} \right\rangle_{L,R}.$$

Remark 3.6. *The division matches this time the double tangent approximation* [11] *of EP division with half spreads.*

In the following we provide a comparison of TA arithmetic with standard EP arithmetic. Let us consider the sum, difference, product and division of the triangular fuzzy numbers $A = \langle 6, 2, 2 \rangle_{1-x,1-x}$ and $B = \langle 2, 1, 1 \rangle_{1-x,1-x}$ with TA arithmetic and EP arithmetic. The membership functions of A and B are graphed in Figure 1. Figure 2 includes both the TA and EP sum of A and B. For addition we observe that both operations preserve the linearity but there is a much grater growth in width (also called dependency effect) for EP arithmetic. The fact that TA arithmetic results always with less growing width compared to EP arithmetic is proven in [5]. The difference $A - B$ is compared in Figure 3. Again the linearity is preserved for both TA and EP-difference. When it comes to the product or division this will be no longer the case for EP as seen in Figure 4 and Figure 5.

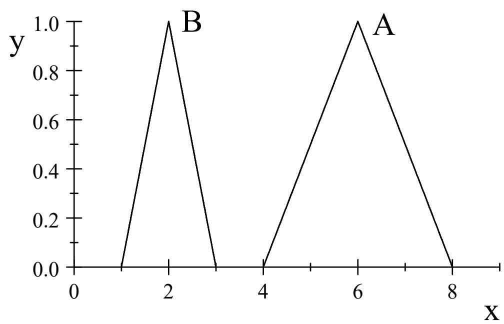

Figure 1: Triangular fuzzy numbers $A = \langle 6, 2, 2 \rangle_{1-x,1-x}$ and $B = \langle 2, 1, 1 \rangle_{1-x,1-x}$.

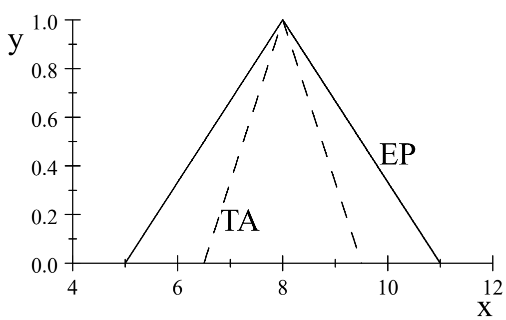

Figure 2: $A + B$ with both TA and EP fuzzy arithmetic.

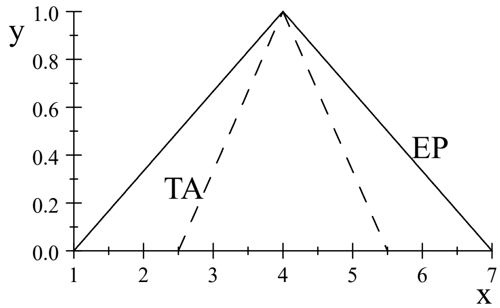

Figure 3: $A - B$ with both TA and EP fuzzy arithmetic.

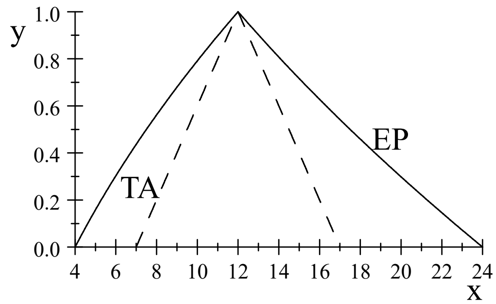

Figure 4: $A \cdot B$ with both TA and EP fuzzy arithmetic.

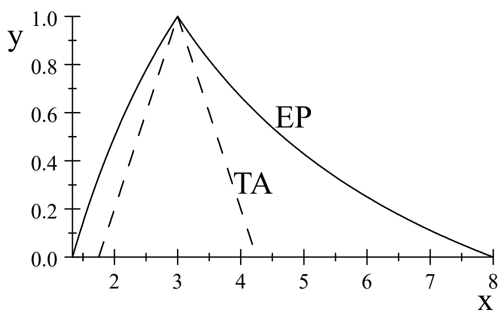

Figure 5: A/B with both TA and EP fuzzy arithmetic.

Applications of fuzzy arithmetic mostly include triangular fuzzy numbers (fuzzy numbers where the shape functions are linear, i.e. $L = R = 1 - x$). Therefore it is reasonable to expect to be able to solve equations in which triangular fuzzy numbers are involved. We give two simple but effective examples showing the advantages of TA arithmetic in case of solving such equations. First consider the fuzzy equation $(-3, 2, 3) + X = (1, 4, 5)$. Using standard EP arithmetic this equation has no solution ($X = (4, 2, 2)$ is not a fuzzy number). Thanks to the controlled growing width effect of TA arithmetic this equation has the solution $X = (1, 2, 3)$. Next consider the equation $(1, 2, 3)X = (5, 8, 11)$. Once again there exists no solution with EP arithmetic since the right hand-side of such an equation can remain a triangular fuzzy number if and only if X is a crisp real number. Since it is shape preserving this is not an issue for TA arithmetic and it can be verified that $X = (3, 4, 5)$ is the solution.

Following propositions prove some useful algebraic properties of TA arithmetic. Let $A = \langle a, \alpha_1, \beta_1 \rangle_{L,R}$, $B = \langle b, \alpha_2, \beta_2 \rangle_{L,R}$ and $C = \langle c, \alpha_3, \beta_3 \rangle_{L,R}$ be fuzzy numbers with shape functions L and R. We will omit the L, R sub-scripts whenever it causes no confusion. We begin with the commutativity of addition and multiplication. The results are straightforward.

Proposition 3.7. *TA addition is commutative.*

Proof. By the definition of TA addition we observe,

$$\begin{aligned} A+B &= \langle a,\alpha_1,\beta_1\rangle + \langle b,\alpha_2,\beta_2\rangle = \langle a+b, \frac{\alpha_1+\alpha_2}{2}, \frac{\beta_1+\beta_2}{2}\rangle = \langle b+a, \frac{\alpha_2+\alpha_1}{2}, \frac{\beta_2+\beta_1}{2}\rangle \\ &= \langle b,\alpha_2,\beta_2\rangle + \langle a,\alpha_1,\beta_1\rangle = B+A. \end{aligned}$$

□

Proposition 3.8. *TA multiplication is commutative.*

Proof. For $a, b > 0$ we have by definition,

$$\begin{aligned} A\cdot B &= \langle a,\alpha_1,\beta_1\rangle \cdot \langle b,\alpha_2,\beta_2\rangle = \langle ab, \frac{a\alpha_2+b\alpha_1}{2}, \frac{a\beta_2+b\beta_1}{2}\rangle = \langle ba, \frac{b\alpha_1+a\alpha_2}{2}, \frac{b\beta_1+a\beta_2}{2}\rangle \\ &= \langle b,\alpha_2,\beta_2\rangle \cdot \langle a,\alpha_1,\beta_1\rangle = B\cdot A. \end{aligned}$$

□

Although none of the TA arithmetic operators are associative in general, the property holds under certain conditions as shown below.

Proposition 3.9. *If $\alpha_1 = \alpha_3$ and $\beta_1 = \beta_3$ then $(A+B)+C = A+(B+C)$.*

Proof. By definition we may write,

$$\begin{aligned} (A+B)+C &= (\langle a,\alpha_1,\beta_1\rangle + \langle b,\alpha_2,\beta_2\rangle) + \langle c,\alpha_3,\beta_3\rangle \\ &= \left\langle a+b, \frac{\alpha_1+\alpha_2}{2}, \frac{\beta_1+\beta_2}{2}\right\rangle + \langle c,\alpha_3,\beta_3\rangle \\ &= \left\langle a+b+c, \frac{\alpha_1+\alpha_2+2\alpha_3}{4}, \frac{\beta_1+\beta_2+2\beta_3}{4}\right\rangle \\ &= \left\langle a+b+c, \frac{\alpha_2+\alpha_3+2\alpha_1}{4}, \frac{\beta_2+\beta_3+2\beta_1}{4}\right\rangle \text{ (by assumption)} \\ &= \langle a,\alpha_1,\beta_1\rangle + \left\langle b+c, \frac{\alpha_2+\alpha_3}{2}, \frac{\beta_2+\beta_3}{2}\right\rangle \\ &= \langle a,\alpha_1,\beta_1\rangle + (\langle b,\alpha_2,\beta_2\rangle + \langle c,\alpha_3,\beta_3\rangle) \\ &= A+(B+C). \end{aligned}$$

□

Proposition 3.10. *If $c\alpha_1 = a\alpha_3$ and $c\beta_1 = a\beta_3$ then $(A\cdot B)\cdot C = A\cdot(B\cdot C)$.*

Proof.

$$\begin{aligned} (A\cdot B)\cdot C &= (\langle a,\alpha_1,\beta_1\rangle \cdot \langle b,\alpha_2,\beta_2\rangle)\cdot\langle c,\alpha_3,\beta_3\rangle \\ &= \left(\langle ab, \frac{a\alpha_2+b\alpha_1}{2}, \frac{a\beta_2+b\beta_1}{2}\rangle\right)\cdot\langle c,\alpha_3,\beta_3\rangle \\ &= \left\langle abc, \frac{ca\alpha_2+cb\alpha_1+2ab\alpha_3}{4}, \frac{ca\beta_2+cb\beta_1+2ab\beta_3}{4}\right\rangle \\ &= \left\langle abc, \frac{ac\alpha_2+ab\alpha_3+2bc\alpha_1}{4}, \frac{ac\beta_2+ab\beta_3+2bc\beta_1}{4}\right\rangle \text{ (by assumption)} \\ &= \langle a,\alpha_1,\beta_1\rangle\cdot\left(\langle bc, \frac{b\alpha_3+c\alpha_2}{2}, \frac{b\beta_3+c\beta_2}{2}\rangle\right) \\ &= (\langle a,\alpha_1,\beta_1\rangle\cdot(\langle b,\alpha_2,\beta_2\rangle)\cdot\langle c,\alpha_3,\beta_3\rangle) \\ &= A\cdot(B\cdot C). \end{aligned}$$

□

Lack of complete Associativity is maybe the main weakness of TA arithmetic. In applications on has to decide for in which order the computations should be made. During the text we will adopt the preference of order from the left to the right.

Distributivity is a property that fails to hold in many cases in EP arithmetic. For TA arithmetic we need a specific condition as well.

Proposition 3.11. *If $b+c=0$ then $A\cdot(B+C)=A\cdot B+A\cdot C$*

Proof.

$$\begin{aligned}
A\cdot(B+C) &= \langle a,\alpha_1,\beta_1\rangle\cdot(\langle b,\alpha_2,\beta_2\rangle+\langle c,\alpha_3,\beta_3\rangle)\\
&= \langle a,\alpha_1,\beta_1\rangle\cdot\left(\langle b+c,\frac{\alpha_2+\alpha_3}{2},\frac{\beta_2+\beta_3}{2}\rangle\right)\\
&= \left\langle a\,(b+c)\,,\frac{a\alpha_2+a\alpha_3+2b\alpha_1+2c\alpha_1}{4},\frac{a\beta_2+a\beta_3+2b\beta_1+2c\beta_1}{4}\right\rangle\\
&= \left\langle ab+ac,\frac{a\alpha_2+b\alpha_1+c\alpha_1+a\alpha_3}{4},\frac{a\beta_2+b\beta_1+c\beta_1+a\beta_3}{4}\right\rangle \text{ (by assumption)}\\
&= \left\langle ab,\frac{a\alpha_2+b\alpha_1}{2},\frac{a\beta_2+b\beta_1}{2}\right\rangle+\left\langle ac,\frac{c\alpha_1+a\alpha_3}{2},\frac{c\beta_1+a\beta_3}{2}\right\rangle\\
&= \langle a,\alpha_1,\beta_1\rangle\cdot\langle b,\alpha_2,\beta_2\rangle+\langle a,\alpha_1,\beta_1\rangle\cdot\langle c,\alpha_3,\beta_3\rangle\\
&= A\cdot B+A\cdot C.
\end{aligned}$$

□

Neutral and inverse elements of TA arithmetic are discussed in [5]. Here we will just provide examples. Consider the fuzzy number $A=\langle 2,1,1\rangle_{L,R}$, for $0_A=\langle 0,1,1\rangle_{L,R}$ we have $A-A=0_A$ and $A+0_A=A$. Similarly for $1_A=\left\langle 1,\frac{1}{2},\frac{1}{2}\right\rangle_{L,R}$ we have $A\cdot 1_A=A$ and for $A^{-1}=\left\langle\frac{1}{2},\frac{1}{4},\frac{1}{4}\right\rangle_{L,R}$, $A\cdot A^{-1}=1_A$. We finalise this section with some further algebraic properties that are useful in solving fuzzy equations.

Theorem 3.12. *Let $n\in R$ and $A=\langle a,\alpha,\beta\rangle_{L,R}$ be a fuzzy number. If $n>0$,*

$$nA=\left\langle na,\frac{n\alpha}{2},\frac{n\beta}{2}\right\rangle_{L,R},$$

if $n<0$,

$$nA=\left\langle na,\frac{-n\beta}{2},\frac{-n\alpha}{2}\right\rangle_{L,R}.$$

Proof. We may write $n=\langle n,0,0\rangle_{L,R}$ and $A=\langle a,\alpha,\beta\rangle_{L,R}$ and apply multiplication with the help of Theorem3.3,for $n>0$ we have,

$$nA=\left\langle na,\frac{a\cdot 0+n\alpha}{2},\frac{a\cdot 0+n\beta}{2}\right\rangle_{L,R} \text{ and so,}$$

$$nA=\left\langle na,\frac{n\alpha}{2},\frac{n\beta}{2}\right\rangle_{L,R}.$$

For negative n the proof is similar. □

Theorem 3.13. *Let n be a positive natural number and $A=\langle a,\alpha,\beta\rangle_{L,R}$ be a fuzzy number. Then we have,*

$$\sum_{i=1}^{n}A=\langle na,\alpha,\beta\rangle_{L,R}.$$

Proof. We will prove the theorem by induction on n. The equality holds for $n=1$:

$$\sum_{i=1}^{1}A=\langle a,\alpha,\beta\rangle_{L,R}=A$$

Suppose that the equality holds for $n = k$,

$$\sum_{i=1}^{k} A = \langle ka, \alpha, \beta \rangle_{L,R}.$$

We will show that it also holds for $n = k + 1$. Since,

$$\sum_{i=1}^{k+1} A = \langle (k+1)a, \alpha, \beta \rangle_{L,R} = \sum_{i=1}^{k} A + \sum_{i=1}^{1} A = \langle (k+1)a, \alpha, \beta \rangle_{L,R},$$

we get the desired result. □

When multiplying with scalars one should be careful. For instance with the help of the last property it is easy to observe $nA = \sum_{i=1}^{n} A$ if and only if $n = 2$ and $2(2A)$ is not equal to $4A$.

Theorem 3.14. *Let n be a positive natural number and $A = \langle a, \alpha, \beta \rangle_{L,R}$ be a positive fuzzy number. Then,*

$$A^n = \left\langle a^n, a^{n-1}\alpha, a^{n-1}\beta \right\rangle_{L,R}.$$

Proof. We will prove the theorem by induction. The theorem is obviously true for $n = 1$. Suppose that for $n = k$,

$$A^k = \left\langle a^k, a^{k-1}\alpha, a^{k-1}\beta \right\rangle_{L,R}.$$

Let us show that it is also true for $n = k + 1$:

$$\begin{aligned} A^{k+1} &= A^k.A = \left\langle a^k, a^{k-1}\alpha, a^{k-1}\beta \right\rangle_{L,R} \cdot \langle a, \alpha, \beta \rangle_{L,R} \\ &= \left\langle a^{k+1}, \frac{a^k\alpha + a\left(a^{k-1}\right)\alpha}{2}, \frac{a^k\beta + a\left(a^{k-1}\right)\beta}{2} \right\rangle_{L,R} \\ &= \left\langle a^{k+1}, a^k\alpha, a^k\beta \right\rangle. \end{aligned}$$

□

Theorem 3.15. *Let n be a positive natural number and $A = \langle a, \alpha, \beta \rangle_{L,R}$ be a negative fuzzy number. Then*

$$A^n = \begin{matrix} \left\langle a^n, a^{n-1}\alpha, a^{n-1}\beta \right\rangle_{L,R}, & \text{if } n \text{ is odd} \\ \left\langle a^n, -a^{n-1}\beta, -a^{n-1}\alpha \right\rangle_{R,L}, & \text{if } n \text{ is even.} \end{matrix}$$

The proof is similar to the proof of Theorem3.14 and is omitted.

4 Application to solutions of polynomial equations

In this section we apply the obtained results to solve polynomial fuzzy equations and systems of linear fuzzy equations. Findings about shape preservation will help to recognize the type of the solution whereas the algebraic findings such as Theorem 5, Theorem 6, Theorem 7 and Theorem 8 will be useful in the computational part of the solutions. We begin with polynomial fuzzy equations.

Let $P_n(X) = A_nX^n + A_{n-1}X^{n-1} + ... + A_0 = B$ be an equality where the coefficients $A_0, A_1, ..., A_n$ and B are fuzzy numbers of the same $L - R$ type and positive (positiveness is not necessary for B). The shape preservation suggests that the solution X -if it exists- can be assumed to be of $L - R$ type as well. In the following we discuss the procedure to solve this equation.

1. step

The equation is put into the standard form,

$$A_nX^n + A_{n-1}X^{n-1} + ... + A_0 = \langle b, \alpha_s, \beta_s \rangle_{L,R}.$$

Note that α_s and β_s are half the left and right spreads of B respectively. We denote, $A_i = \langle a_i, \alpha_i, \beta_i \rangle_{L,R}$, for all $i \in \{0, 1, 2, ..., n\}$. And assume that the solution is $X = \langle x, \alpha, \beta \rangle_{L,R}$. Now we have,

$$\sum_{i=0}^{n} a_i x^i = b.$$

This equality then is solved for x analytically if possible. A suitable numerical technique has to be applied in case of non-existence of an analytical solution.

2. step For a non-negative solution x $(x \geq 0)$:

For any $i \in \{1, 2, ..., n\}$ by Theorem3.14 we have,

$$X^i = \left\langle x^i, \alpha x^{i-1}, \beta x^{i-1} \right\rangle_{L,R},$$

and by Theorem3.3,

$$A_i X^i = \left\langle a_i x^i, \frac{\alpha_i x^i + \alpha a_i x^{i-1}}{2}, \frac{\beta_i x^i + \beta a_i x^{i-1}}{2} \right\rangle_{L,R}.$$

Using Theorem3.1 it is not difficult to obtain that,

$$\sum_{i=0}^{n} A_i X^i = \left\langle \sum_{i=0}^{n} a_i x^i, \frac{2^n\alpha_0 + x^n\alpha_n + \alpha x^4 a_5 + \sum_{i=1}^{n-1}(2^{n-1-i}(\alpha x^{i-1}a_i + x^i\alpha_i))}{2^{n+1}}, \frac{2^n\beta_0 + x^n\beta_n + \beta x^4 a_5 + \sum_{i=1}^{n-1}(2^{n-1-i}(\beta x^{i-1}a_i + x^i\beta_i))}{2^{n+1}} \right\rangle.$$

Using the solution for x in the first step we set,

$$2^n\alpha_0 + x^n\alpha_n + \alpha x^4 a_5 + \sum_{i=1}^{n-1}(2^{n-1-i}(\alpha x^{i-1}a_i + x^i\alpha_i)) = 2^{n+1}\alpha_s,$$

and

$$2^n\beta_0 + x^n\beta_n + \beta x^4 a_5 + \sum_{i=1}^{n-1}(2^{n-1-i}(\beta x^{i-1}a_i + x^i\beta_i)) = 2^{n+1}\beta_s.$$

Solving for α, β yields,

$$\alpha = \frac{2^{n+1}\alpha_s - 2^{n-1}\alpha_0 - x^n\alpha_n - \sum_{i=0}^{n-1} 2^{n-1-i}x^i\alpha_i}{x^{n-1}a_n + \sum_{i=1}^{n-1} 2^{n-1-i}x^{i-1}a_i}, \tag{9}$$

$$\beta = \frac{2^{n+1}\beta_s - 2^{n-1}\beta_0 - x^n\beta_n - \sum_{i=0}^{n-1} 2^{n-1-i}x^i\beta_i}{x^{n-1}a_n + \sum_{i=1}^{n-1} 2^{n-1-i}x^{i-1}a_i}. \tag{10}$$

The validity of the solution depends now on the conditions, $\alpha \geq 0$ and $\beta \geq 0$. Note that the fuzzy computations are made in the order from the left to the right.

3. step For negative solutions $(x < 0)$:

The case of negative roots requires the further assumption that all the coefficients are semi-symmetric i.e. $L = R$ in order to obtain an explicit formula for the solution. Here its explained why: For any odd $i \in \{1, 2, ..., n\}$,

$$X^i = \left\langle x^i, \alpha x^{i-1}, \beta x^{i-1} \right\rangle_{L,R}, \tag{11}$$

and if i is even,

$$X^i = \left\langle x^i, -\beta x^{i-1}, -\alpha x^{i-1} \right\rangle_{R,L}. \tag{12}$$

The fact that the shape functions R and L swap in case of even i is the reason behind the assumption $R = L$. Multiplying with the coefficients for odd i we have,

$$A_i X^i = \left\langle a_i x^i, \frac{-\beta_i x^i + \alpha a_i x^{i-1}}{2}, \frac{-\alpha_i x^i + \beta a_i x^{i-1}}{2} \right\rangle_{L,L},$$

and for even i,

$$A_i X^i = \left\langle a_i x^i, \frac{\alpha_i x^i - \beta a_i x^{i-1}}{2}, \frac{\beta_i x^i - \alpha a_i x^{i-1}}{2} \right\rangle_{L,L}.$$

The resulting left and right spreads of the sum $\sum_{i=0}^{n} A_i X^i$ depending on whether n is odd or even is set equal to α_s and β_s respectively and a system of two linear equations with the variables α and β are obtained. For even n :

$$\alpha_s = \alpha \sum_{i=1}^{\frac{n}{2}} 2^{-2i-1} a_{2i-1} x^{2i-2} - \beta 2^{-n-2} a_n x^{n-1} - \beta \sum_{i=1}^{\frac{n}{2}} 2^{-2i-2} a_{2i} x^{2i-1} + \frac{1}{2}\alpha_0 + \frac{1}{2^{n+2}} \alpha_n x^n + \sum_{i=1}^{\frac{n}{2}} 2^{-2i-2} \alpha_{2i} x^{2i} - \sum_{i=1}^{\frac{n}{2}} 2^{-2i-1} \beta_{2i-1} x^{2i-1}, \tag{13}$$

$$\beta_s = \beta \sum_{i=1}^{\frac{n}{2}} 2^{-2i-1} a_{2i-1} x^{2i-2} - \alpha 2^{-n-2} a_n x^{n-1} - \alpha \sum_{i=1}^{\frac{n}{2}} 2^{-2i-2} a_{2i} x^{2i-1} + \frac{1}{2}\beta_0 + \frac{1}{2^{n+2}} \beta_n x^n + \sum_{i=1}^{\frac{n}{2}} 2^{-2i-2} \beta_{2i} x^{2i} - \sum_{i=1}^{\frac{n}{2}} 2^{-2i-1} \alpha_{2i-1} x^{2i-1}. \tag{14}$$

The system may look complicated but as we will show with numerical examples the computation is straightforward. For odd n the system is as follows:

$$\alpha_s = 2^{-n-1} x^{n-1} \alpha a_n + \alpha \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} 2^{-2i-1} a_{2i-1} x^{2i-2} - \beta \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} 2^{-2i-2} a_{2i} x^{2i-1} + \frac{1}{2}\alpha_0 + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} 2^{-2i-2} \alpha_{2i} x^{2i} - 2^{-n-1} x^n \beta_n - \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} 2^{-2i-1} \beta_{2i-1} x^{2i-1}, \tag{15}$$

$$\beta_s = 2^{-n-1} x^{n-1} \beta a_n + \beta \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} 2^{-2i-1} a_{2i-1} x^{2i-2} - \alpha \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} 2^{-2i-2} a_{2i} x^{2i-1} + \frac{1}{2}\beta_0 + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} 2^{-2i-2} \beta_{2i} x^{2i} - 2^{-n-1} x^n \alpha_n - \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} 2^{-2i-1} \alpha_{2i-1} x^{2i-1}. \tag{16}$$

Here $\lfloor \cdot \rfloor$ denotes the Greatest Integer Function. Using the solutions of the corresponding 2×2 linear system, $X = \langle x, \alpha, \beta \rangle_{L,L}$ is a root of the polynomial fuzzy equality as long as $\alpha \geq 0$ and $\beta \geq 0$.

Example 4.1. *Consider the fuzzy equality,* $\langle 1, 0.5, 2 \rangle_{1-x,1-x} X^2 + \langle 1, 1, 1 \rangle_{1-x,1-x} X + \langle -6, 1, 1 \rangle_{1-x,1-x} = \langle 0, \frac{37}{16}, \frac{15}{4} \rangle_{1-x,1-x}$.

Solution 4.2. *It should be noted that no solution exists for this equality with EP arithmetic since the result is a triangular fuzzy number. This is not an issue for TA arithmetic. To solve the equation we follow steps 1,2 and 3 with the case* $n = 2$.

1. step: we solve the equality $x^2 + x - 6 = 0$ *for* x *and take the positive solution* $x = 2$.

2. step: using $x = 2$ *and* $n = 2$ *in the equations (9) and (10) we compute*

$$\alpha = \frac{2^{2+1}\alpha_s - 2^{2-1}\alpha_0 - x^2\alpha_2 - \sum_{i=0}^{2-1} 2^{2-1-i} x^i \alpha_i}{x^{2-1} a_2 + \sum_{i=1}^{2-1} 2^{2-1-i} x^{i-1} a_i} = -\frac{1}{x a_2 + a_1}\left(\alpha_2 x^2 + \alpha_1 x + 4\alpha_0 - 8\alpha_s\right) = \frac{7}{2},$$

and

$$\beta = \frac{2^{2+1}\beta_s - 2^{2-1}\beta_0 - x^2\beta_2 - \sum_{i=0}^{2-1} 2^{2-1-i} x^i \beta_i}{x^{2-1} a_2 + \sum_{i=1}^{2-1} 2^{n-1-i} x^{i-1} a_i} = -\frac{1}{x a_2 + 2^{n-2} a_1}\left(\beta_2 x^2 + \beta_1 x + 4\beta_0 - 8\beta_s\right) = \frac{16}{3}.$$

Since the existence conditions hold, $X = \left\langle 2, \frac{7}{2}, \frac{16}{3} \right\rangle_{1-x,1-x}$ *is a solution to the fuzzy equation. We verify this result by putting* $X = \left\langle 2, \frac{7}{2}, \frac{16}{3} \right\rangle_{1-x,1-x}$ *into the equation:*

$X = \left\langle 2, \frac{7}{2}, \frac{16}{3} \right\rangle$, *so by Theorem 3.14* $X^2 = \left\langle 4, 7, \frac{32}{3} \right\rangle$. *By Theorem 3.3* $A_2X^2 = \langle 1, 0.5, 2 \rangle \cdot \left\langle 4, 7, \frac{32}{3} \right\rangle = \left\langle 4, \frac{9}{2}, \frac{56}{6} \right\rangle$ *and* $A_1X = \langle 1, 1, 1 \rangle \cdot \left\langle 2, \frac{7}{2}, \frac{16}{3} \right\rangle = \left\langle 2, \frac{11}{4}, \frac{22}{6} \right\rangle$. *So by Theorem 3.1,* $A_2X^2 + A_1X = \left\langle 6, \frac{29}{8}, \frac{39}{6} \right\rangle$ *and* $A_2X^2 + A_1X + A_0 = \left\langle 6, \frac{29}{8}, \frac{39}{6} \right\rangle + \langle -6, 1, 1 \rangle = \left\langle 0, \frac{37}{16}, \frac{15}{4} \right\rangle$ *where this is equal to given B.*

3. step: for the negative solution $x = -3$ *using the equations (13) and (14) with* $n = 2$ *we obtain,*

$\frac{1}{2}\alpha_0 - \frac{1}{8}x\beta_1 + \frac{1}{8}\alpha a_1 + \frac{1}{8}x^2\alpha_2 - \frac{1}{8}x\beta a_2 = \alpha_s$ *and*

$\frac{1}{2}\beta_0 - \frac{1}{8}x\alpha_1 + \frac{1}{8}\beta a_1 + \frac{1}{8}x^2\beta_2 - \frac{1}{8}x\alpha a_2 = \beta_s$. *This yields to the system*

$$\frac{1}{8}\alpha + \frac{3}{8}\beta + \frac{23}{16} = \frac{37}{16},$$
$$\frac{3}{8}\alpha + \frac{1}{8}\beta + \frac{25}{8} = \frac{15}{4}.$$

the solution of this system is: $[\alpha = 1, \beta = 2]$ *.Since the existence conditions hold again,* $X = \langle -3, 1, 2 \rangle_{1-x,1-x}$ *is the second solution to the fuzzy equation. We will verify the solution:*

$X = \langle -3, 1, 2 \rangle$, *so by Theorem 3.15* $X^2 = \langle 9, 6, 3 \rangle$,

By Theorem 3.3 $A_2X^2 = \langle 1, 0.5, 2 \rangle \cdot \langle 9, 6, 3 \rangle = \left\langle 9, \frac{21}{4}, \frac{21}{2} \right\rangle$ *and*

$A_1X = \langle 1, 1, 1 \rangle \langle -3, 1, 2 \rangle = \left\langle -3, 2, \frac{5}{2} \right\rangle$,

$A_2X^2 + A_1X = \left\langle 9, \frac{21}{4}, \frac{21}{2} \right\rangle + \left\langle -3, 2, \frac{5}{2} \right\rangle = \left\langle 6, \frac{29}{8}, \frac{26}{4} \right\rangle$ *and finally,*

$A_2X^2 + A_1X + A_0 = \left\langle 6, \frac{29}{8}, \frac{26}{4} \right\rangle + \langle -6, 1, 1 \rangle = \left\langle 0, \frac{37}{16}, \frac{30}{4} \right\rangle$ *where this is equal to given B.*

In the foregoing example n was even and we could verify equations (9)-(14). In our next example $n = 3$ and so we will also verify equations (15) and (16).

Example 4.3. *Consider the equation* $A_3X^3 + A_2X^2 + A_1X + A_0 = \langle 1, 4, 3 \rangle$ *for any shapes* L, R *with* $L = R$. *Let* $A_3 = \langle 1, 1, 1 \rangle$, $A_2 = \langle 4, 2, 1 \rangle$, $A_1 = \langle 1, 1, 2 \rangle$, $A_0 = \langle -5, 2, 1 \rangle$, $B = \langle 1, 4, 3 \rangle$.

Solution 4.4. *1. step: we solve the equation* $x^3 + 4x^2 + x - 5 = 1$ *and get the roots* $x = -2$, $x = -3$ *and* $x = 1$.

2. step: for the positive root $x = 1$ *we employ equations (9) and (10) with* $n = 3$:

$$\alpha = \frac{2^{3+1}\alpha_s - 2^{3-1}\alpha_0 - x^3\alpha_3 - \sum_{i=0}^{3-1} 2^{3-1-i}x^i\alpha_i}{x^{3-1}a_3 + \sum_{i=1}^{3-1} 2^{3-1-i}x^{i-1}a_i} = -\frac{\alpha_3x^3 + \alpha_2x^2 + 2\alpha_1x + 4\alpha_0 - 16\alpha_s + 4\alpha_0}{a_3x^2 + a_2x + 2a_1},$$

$$\beta = \frac{2^{3+1}\beta_s - 2^{3-1}\beta_0 - x^3\beta_3 - \sum_{i=0}^{3-1} 2^{3-1-i}x^i\beta_i}{x^{3-1}a_3 + \sum_{i=1}^{3-1} 2^{3-1-i}x^{i-1}a_i} = -\frac{\beta_3x^3 + \beta_2x^2 + 2\beta_1x + 4\beta_0 - 16\beta_s + 4\beta_0}{a_3x^2 + a_2x + 2a_1}.$$

We plug in the given data and get the solutions, $\alpha = \frac{43}{7}$, *and* $\beta = \frac{34}{7}$. *So* $X = \left\langle 1, \frac{43}{7}, \frac{34}{7} \right\rangle$ *is the first solution of the fuzzy equality.*

3. step: We begin with the first negative root, $x = -2$. *Since n is odd we will use equations (15) and (16). With* $n = 3$ *we get,*

$\frac{1}{2}\alpha_0 + \frac{1}{8}\alpha a_1 - \frac{1}{8}x\beta_1 + \frac{1}{16}x^2\alpha_2 - \frac{1}{16}x^3\beta_3 - \frac{1}{16}x\beta a_2 + \frac{1}{16}x^2\alpha a_3 = \alpha_s$,

$\frac{1}{2}\beta_0 + \frac{1}{8}\beta a_1 - \frac{1}{8}x\alpha_1 + \frac{1}{16}x^2\beta_2 - \frac{1}{16}x^3\alpha_3 - \frac{1}{16}x\alpha a_2 + \frac{1}{16}x^2\beta a_3 = \beta_s$

With the given data this yields to the system,

$$\frac{3}{8}\alpha + \frac{1}{2}\beta + \frac{5}{2} = 4,$$
$$\frac{1}{2}\alpha + \frac{3}{8}\beta + \frac{3}{2} = 3,$$

where the solution is $\left[\alpha = \frac{12}{7}, \beta = \frac{12}{7}\right]$. *So we obtain a second solution to the given fuzzy equality as* $X = \left\langle -2, \frac{12}{7}, \frac{12}{7} \right\rangle$. *This time we will verify this result since it was the first case involving equations (15) and (16):*

$X = \left\langle -2, \frac{12}{7}, \frac{12}{7} \right\rangle$,

$X^2 = \left\langle 4, \frac{24}{7}, \frac{24}{7} \right\rangle$,

$X^3 = \left\langle -8, \frac{48}{7}, \frac{48}{7} \right\rangle$,
$A_3X^3 = \langle 1,1,1 \rangle \cdot \left\langle -8, \frac{48}{7}, \frac{48}{7} \right\rangle = \left\langle -8, \frac{104}{14}, \frac{104}{14} \right\rangle$,
$A_2X^2 = \langle 4,2,1 \rangle \cdot \left\langle 4, \frac{24}{7}, \frac{24}{7} \right\rangle = \left\langle 16, \frac{152}{14}, \frac{124}{14} \right\rangle$,
$A_1X = \langle 1,1,2 \rangle \cdot \left\langle -2, \frac{12}{7}, \frac{12}{7} \right\rangle = \left\langle -2, \frac{40}{14}, \frac{26}{14} \right\rangle$,
$A_0 = \langle -5,2,1 \rangle$,
$A_3X^3 + A_2X^2 = \left\langle -8, \frac{104}{14}, \frac{104}{14} \right\rangle + \left\langle 16, \frac{152}{14}, \frac{124}{14} \right\rangle = \left\langle 8, \frac{256}{28}, \frac{228}{28} \right\rangle$,
$A_3X^3 + A_2X^2 + A_1X = \left\langle 8, \frac{256}{28}, \frac{228}{28} \right\rangle + \left\langle -2, \frac{40}{14}, \frac{26}{14} \right\rangle = \left\langle 6, \frac{336}{56}, \frac{280}{56} \right\rangle$,
$A_3X_1^3 + A_2X_1^2 + A_1X_1 + A_0 = \left\langle 6, \frac{336}{56}, \frac{280}{56} \right\rangle + \langle -5,2,1 \rangle = \langle 1,4,3 \rangle$, *where this equal to B and verifies the result.*
Finally for the root $x=-3$ using equations (15) and (16) similar to the case $x=-2$ we obtain the system,

$$\frac{11}{16}\alpha + \frac{3}{4}\beta + \frac{73}{16} = 4,$$
$$\frac{3}{4}\alpha + \frac{11}{16}\beta + \frac{25}{8} = 3,$$

and the solution is $\left[\alpha = \frac{75}{23}, \beta = -\frac{86}{23}\right]$. Since β is negative this solution is not valid.

Systems of linear fuzzy equations will be the next application. TA arithmetic provides a good solution possibility for this type of systems. In general, a $n \times n$ fuzzy system can be equivalently transformed into three $n \times n$ crisp systems. In order to enlighten this fact we will discuss the general solution of a 2×2 fuzzy system and give a numerical example.

Let us consider the linear system $\begin{array}{c} aX + bY = V \\ cX + dY = W \end{array}$ where a, b, c, d are real coefficients, V, W are fuzzy numbers in $L-R$ setting and X, Y are fuzzy unknowns. Employing shape preservance of TA arithmetic the unknowns X and Y can be assumed to be fuzzy numbers with same shape functions L and R. In the sequel -for sake of simplicity- a fuzzy number A of type $\langle a, \alpha, \beta \rangle_{L,R}$ will be denoted by (a_1, a_2, a_3) where $a_1 = a - \alpha$, $a_2 = a$ and $a_3 = a + \beta$. As long as L and R are fixed shape functions their notation will be dropped. So the given system can be shown as $\begin{array}{c} a\,(x_1, x_2, x_3) + b(y_1, y_2, y_3) = (v_1, v_2, v_3) \\ c\,(x_1, x_2, x_3) + d(y_1, y_2, y_3) = (w_1, w_2, w_3) \end{array}$. We apply TA arithmetic rules to the first equality in the system and observe:

$$\begin{aligned}(v_1, v_2, v_3) &= a\,(x_1, x_2, x_3) + b(y_1, y_2, y_3) \\ &= \left(\frac{a(x_1 + x_2)}{2}, ax_2, \frac{a(x_3 + x_2)}{2} \right) + \left(\frac{b(y_1 + y_2)}{2}, by_2, \frac{b(y_3 + y_2)}{2} \right) \\ &= \left(\frac{ax_1 + by_1 + 3(ax_2 + by_2)}{4}, ax_2 + by_2, \frac{ax_3 + by_3 + 3(ax_2 + by_2)}{4} \right).\end{aligned}$$

Since TA arithmetic is preserving shapes we can transform the last fuzzy equation above into the following three crisp equalities.

$$ax_1 + by_1 + 3(ax_2 + by_2) = 4v_1, \tag{17}$$
$$ax_2 + by_2 = v_2, \tag{18}$$
$$ax_3 + by_3 + 3(ax_2 + by_2) = 4v_3. \tag{19}$$

Next we compute the second equation of the system similarly,

$$\begin{aligned}(w_1, w_2, w_3) &= c\,(x_1, x_2, x_3) + d(y_1, y_2, y_3)\,(w_1, w_2, w_3) \\ &= \left(\frac{c(x_1 + x_2)}{2}, cx_2, \frac{c(x_3 + x_2)}{2} \right) + \left(\frac{d(y_1 + y_2)}{2}, dy_2, \frac{d(y_3 + y_2)}{2} \right) \\ &= \left(\frac{cx_1 + dy_1 + 3(cx_2 + dy_2)}{4}, cx_2 + dy_2, \frac{cx_3 + dy_3 + 3(cx_2 + dy_2)}{4} \right).\end{aligned}$$

The equation above yields following three crisp equalities.

$$cx_1 + dy_1 + 3(cx_2 + dy_2) = 4w_1, \tag{20}$$
$$cx_2 + dy_2 = w_2, \tag{21}$$
$$cx_3 + dy_3 + 3(cx_2 + dy_2) = 4w_3. \tag{22}$$

After some simple algebra these 6 crisp equations become the following system:

$$\begin{aligned} ax_1 + by_1 &= 4v_1 - 3v_2, \\ cx_1 + dy_1 &= 4w_1 - 3w_2, \\ ax_2 + by_2 &= v_2, \\ cx_2 + dy_2 &= w_2, \\ ax_3 + by_3 &= 4v_3 - 3v_2, \\ cx_3 + dy_3 &= 4w_3 - 3w_2. \end{aligned}$$

In this way we observe that we actually ended up with three $n \times n$ crisp linear systems. Their solutions follows.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, now provided that $\det(A) \neq 0$ the solution for the first pair of equations, namely, $\begin{array}{l} ax_1 + by_1 = 4v_1 - 3v_2 \\ cx_1 + dy_1 = 4w_1 - 3w_2 \end{array}$ is $x_1 = -\frac{1}{\det(A)}(4bw_1 - 3bw_2 - 4dv_1 + 3dv_2)$ and $y_1 = \frac{1}{\det(A)}(4aw_1 - 3aw_2 - 4cv_1 + 3cv_2)$.

Similarly we obtain the solutions of the second and third pairs of equalities as, $x_2 = -\frac{bw_2 - dv_2}{\det(A)}$, $y_2 = \frac{aw_2 - cv_2}{\det(A)}$, $x_3 = \frac{1}{\det(A)}(3bw_2 - 4bw_3 - 3dv_2 + 4dv_3)$ and $y_3 = -\frac{1}{\det(A)}(3aw_2 - 4aw_3 - 3cv_2 + 4cv_3)$.

It should be noted that the existence of the solution of the original fuzzy system requires the additional conditions $x_1 \leq x_2 \leq x_3$ and $y_1 \leq y_2 \leq y_3$. If desired, these conditions can be checked before entirely computing the solution using following equivalent conditions:

$$\left.\begin{array}{l} b(w_2 - w_1) \leq d(v_2 - v_1) \\ b(w_3 - w_2) \leq d(v_3 - v_2) \\ c(v_2 - v_1) \leq a(w_2 - w_1) \\ c(v_3 - v_2) \leq a(w_3 - w_2) \end{array}\right\},$$ for $\det(A) > 0$. If $\det(A) < 0$ all inequalities are reversed.

We complete this section with a numerical example.

Example 4.5. *Consider the fuzzy linear system* $\begin{array}{l} 2X + 4Y = \langle 14, 1.5, 2 \rangle_{L,R} \\ 3X + 5Y = \langle 19, 2, 2.75 \rangle_{L,R} \end{array}$ *for any given shape functions L and R.*

Solution 4.6. *Since* $14 - 1.5 = 12.5$, $14 + 2 = 16$, $19 - 2 = 17$ *and* $19 + 2.75 = 21.75$ *the system can be re-written in form of,* $\begin{array}{l} 2(x_1, x_2, x_3) + 4(y_1, y_2, y_3) = (12.5, 14, 16) \\ 3(x_1, x_2, x_3) + 5(y_1, y_2, y_3) = (17, 19, 21.75) \end{array}$. *We observe that* $\det(A) = -2$. *We employ TA arithmetic rules and obtain following three* 2×2 *crisp linear systems:*

i) $\begin{array}{l} 2x_1 + 4y_1 = 8 \\ 3x_1 + 5y_1 = 11 \end{array}$, *solution is:* $[x_1 = 2, y_1 = 1]$,

ii) $\begin{array}{l} 2x_2 + 4y_2 = 14 \\ 3x_2 + 5y_2 = 19 \end{array}$, *solution is:* $[x_2 = 3, y_2 = 2]$,

iii) $\begin{array}{l} 2x_3 + 4y_3 = 22 \\ 3x_3 + 5y_3 = 30 \end{array}$, *solution is:* $[x_3 = 5, y_3 = 3]$.

Since $x_1 \leq x_2 \leq x_3$ *and* $y_1 \leq y_2 \leq y_3$ *we get to the solutions* $(x_1, x_2, x_3) = (2, 3, 5)$ *and* $(y_1, y_2, y_3) = (1, 2, 3)$. *So, transforming back into the* $L - R$ *setting we have* $X = \langle 3, 1, 2 \rangle_{L,R}$ *and* $Y = \langle 2, 1, 1 \rangle_{L,R}$.

5 Conclusion

As it is pointed out in details in [20], traditional approaches to fuzzy arithmetic such as EP and α-cut are computationally complex and expensive. Moreover they introduce a serious amount of overestimation due to the growing width effect. Lack of shape preservation of multiplication and division even in the linear case makes the task of solving fuzzy equations too difficult or impossible in most cases. The TA fuzzy arithmetic promises certain advantages in all these departments. The growing width effect is significantly reduced and the space of solvable fuzzy equations is widened with this approach. The shape preservation properties proven in this paper in the most general form together with some proven algebraic properties provide an easy to understand and implement fuzzy computational environment for fuzzy modeling with TA arithmetic.

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