International Journal Of Industrial Electronics Control and Optimization
Print ISSN: 3517-2645
Online ISSN: 3568-2645
Homepage: https://ieco.usb.ac.ir/

Design of Optimal Sliding Mode Control Based on Linear Matrix Inequality for Fractional Time-Varying Delay Systems

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Article Info

Article type:
Research Article

Article history:
Received: 20 April 2022
Received in revised form: 18 Sep 2022
Accepted: 19 Sep 2022
Published online: 25 Nov 2022

Keywords:
Delay systems,
Integral sliding mode,
Linear quadratic regulator,
Fractional order systems,
Optimal sliding mode.

ABSTRACT

This paper considers an optimal sliding mode control based on the cost control guaranteed approach using the linear quadratic regulator method to stabilize delay fractional under involved disturbance. We propose an approach to an open research problem in the design of an LMI-based sliding mode controller in which there are some constraints such as optimizing system performance. The sliding mode technique is well-known as an effective tool for calculating the transient response of the system and achieving robust system performance. LQR classic techniques are less effective for studying an optimal fractional system in the presence of disturbance due to nonlinearity, so we use the optimal sliding mode approach control law designed for the nominal system and, then, combined it with a fractional sliding mode controller. By using the Razumikhin theorem for the stability of fractional order systems with delay and linear matrix inequality, conditions on asymptotically stabilization were obtained. The presented controller stabilizes the nominal system and guarantees an adequate level of system performance. The sliding mode controller presented in the article, in addition to eliminating the effect of disturbance in the system, is independent of the delay A numerical example was provided to illustrate the effectiveness of the main results.

I. Introduction

The linear quadratic regulator (LQR) method is a major problem in optimal control [1,2,3]. From an engineering point of view, it is desirable to design control systems that are only asymptotically stable, but can also ensure an appropriate level of system performance. As the cornerstone of modern control theory, the optimal control technique has a major limitation because it usually requires an accurate mathematical model of the system. Therefore, a precise definition of the system is required to determine the optimal controller. If the system is subject to any uncertainty due to parameter changes or external disturbances during the process, including input and output interference, sensor noise and actuator noise, the optimal controller will most likely not work properly [4]. However, all the physical systems are affected by uncertainty and deviate from the desired value [5]. Such uncertain conditions may even lead to system instability. Therefore, an optimal controller must be robust to ensure attenuated performance in the event of uncertainty. In the field of robust control, sliding mode control has become widely popular as a successful control strategy for nonlinear systems [6,7,8 ,9,10,11]. In the early 1970, sliding mode control gained considerable attention, due to the simplicity and inherent strength of uncertainties [12,13]. Sliding mode controllers consist of a discontinuous control input that directs the controlled system to a specified sliding surface. When the system is on the sliding surface, it is protected from uncertainty. In [14], a linear fractional system with delayed input was studied. The sliding surface used was a fractional sliding surface. In [15], the LQR problem was studied for a class of nonlinear and non-fractional systems in the presence of perturbation. The fractional sliding mode was

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DOI: 10.22111/IECO.2022.42132.1423

employed to eliminate the perturbation effect. Lyapunov theorem was adopted for the stability of the system. In this study, the delay effect was not considered. Given that LQR plays a key role in the design of optimal control [16,17] and is widely used in engine control, chemical process control , aircraft flight control, etc., research on the design of sliding mode controllers using the LQR strategy for nonlinear systems is still an important research topic [18,19]. Usually, the most conservative scenario is designed for the worst-case scenario where system uncertainties and external disturbances predominate. In such cases, stability is the main concern that needs serious attention. However, during nominal system operation, stability is not the only concern of controller design, criteria such as minimizing control input gain and faster convergence to the point of equilibrium also merit attention in the nominal system process stage. On the other hand, near the point of equilibrium, when uncertainties become the main factor, the sliding mode controller is ideally suited as the controller. To ensure this dual purpose, the optimal control strategy is integrated with the sliding mode controller to achieve optimal sliding mode control (OSMC) [8]. Research on optimal sliding mode control is ongoing. In the optimal sliding mode method, optimal control is designed for a nominal system such that it is stable has optimal performance and is then combined with a sliding mode controller. For this, we find an upper bound for the LQR cost function that ensures its performance. The plan of the switching surface can guarantee asymptotic stability and desired performance. In recent years, different results have been derived for nonlinear fractional-order systems [4,11,20,21]. The combination of optimal control with SMC-based methods preserves the main benefits of this technique. Tanks to its, simplicity and inherent robustness towards the matched uncertainty, it also provides greater precision and optimal performance. The tool that has been considered in recent research is linear matrix inequality (LMI). Using an LMI and based on several variables called decision variables, the negative definite of a certain matrix can be determined [22]. In [19], an optimal sliding mode controller was presented for non-fractional and uncertain systems by using an integral sliding surface. In [5], an optimal second-order sliding mode control method was utilized to stabilize non- fractional and nonlinear systems affected by uncertainties. In [23], a second-order sliding mode control was proposed for finite-time bounded and nonlinear fractional-order systems without delay. Given the existing work, slight attention has been paid to the problem of sliding mode design for nonlinear and fractional systems. The design of the LMI-based sliding mode controller, in which there are some constraints such as optimizing system performance, is still an open research problem. In his paper, we solve this problem for the asymptotic stability of a class of delayed nonlinear systems in the presence of perturbations. We design a delay-independent sliding mode controller to ensure system stability and optimization. This paper uses a cost control guaranteed approach which provides a high limit on the system performance index and ensures that the system performance reduction due to uncertainties is less than this limit. Based on this idea, many significant results are presented for nonlinear systems with an integer order. Fractional order systems have special features compared to non-fractional order systems, so analyzing the stability of fractional order systems is more complex and difficult than integer order systems. In particular, it is difficult to solve the control problem using the cost control guaranteed method for fractional order systems [23,24]. Section 2 is devoted to the necessary preliminaries of LMI and fractional calculus including Razumikhin stability for fractional order systems with delay. The problem statement is given in Section 3. In Section 4, we present an optimal state feedback control law for nominal system. In Segment 5, we plan a delay independent and optimal SMC technique, and a condition based on a linear matrix inequality and prove the stability of the system. In Section 6, the numerical example is simulated to demonstrate the reasonableness of our theory. Finally, we conclude the paper in Section 6.

II. FUNDAMENTAL CONCEPTS

Notations ${}_{t_0}^{C}D_t^{\alpha} z(t)$ and ${}_{t_0}I_t^{\alpha} z(t)$ denote fractional operators, the Caputo derivative, and the Riemann-Liouville integral of function z(t), respectively.

Lemma 1. [25]. For differentiable function $z(t)$, the following relation between ${}_{0}^{C}D_t^{\alpha} z(t)$ and ${}_{0}I_t^{\alpha} z(t)$ holds:

$$ {}_{0}I_t^{\alpha}\left({}_{0}^{C}D_t^{\alpha} z(t)\right) = z(t) - z(0), \qquad 0 < \alpha < 1. $$

Lemma 2. [26]. Let $z(t) \in \mathbb{R}^n$ be a vector of differentiable function, the following inequality holds:

$$ \frac{1}{2} D^{\alpha}\left(z^T(t) P z(t)\right) \le z^T(t) \Theta D^{\alpha} z(t), \qquad \forall \alpha \in (0,1). $$

where $\Theta \in \mathbb{R}^{n\times n}$ and $\Theta = \Theta^{\mathrm{T}} > 0$.

Lemma 3.. [27]. For constant matrices $\Theta_1, \Theta_2, \Theta_3$ where Θ_1 and Θ_2 are symmetric, and $\Theta_2 > 0$, then $\Theta_1 + {\Theta_3}^T {\Theta_2}^{-1} \Theta_3 < 0$, if and only if :

$$ \begin{bmatrix} \Theta_1 & {\Theta_3}^T \\ \Theta_3 & -\Theta_2 \end{bmatrix} < 0. $$

Lemma 4. [6]. For any matrix Θ_1 and Θ_2 with compatible dimensions, and for $\delta > 0$, and vectors $z, x \in \mathbb{R}^n$ we have

$$ 2 z^T \Theta_1 \Theta_2 x \le \delta\ z^T {\Theta_1}^T \Theta_1 z + \delta^{-1} x^T {\Theta_2}^T \Theta_2 x. $$

Theorem 1. [28]. For the delay fractional system as follows:

${}_{to}^{C}D_{t}^{\alpha}z(t) = f(t, z_\theta).$ (1)

where $z(t) \in \mathbb{R}^n$, $z_\theta(t) = z(t+\theta), -h \le \theta \le 0$, suppose that $v_1, v_2, v_3: \mathbb{R}^+ \to \mathbb{R}^+$ are scalar, continuous and non-decreasing functions, and $v_1(0) = v_2(0) = v_3(0) = 0$, $v_2(.)$ is strictly increasing, if there exists a continuous function $V: \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ such that:

(i) $v_1(\|z\|) \le V(t,z) \le v_2(\|z\|)$,

(ii) $D_{t}^{\alpha}V(t, z(t)) \le -v_3(\|z\|)$, $0 < \alpha \le 1$, provided

$V(t+\theta, z(t+\theta)) \le V(t, z(t))$, $\theta \in [-\tau, 0]$

for $q > 1$, $-h \le \theta \le 0$, and $t \ge 0$, then system (1) is uniformly stable.

III. PROBLEM STATEMENT

Consider the fractional system with state time-delay described by:

$$ {}_{t_0}^{C}D_t^{\alpha}z(t) = A_0 z(t) + A_d\, z(t-h) + Bu(t) + \omega(t, z(t)), \quad t \ge t_0. \quad (2)$$

$$z(t) = \Psi(t), \qquad t_0 - h \le t \le t_0.$$

where $0 < \alpha < 1$, and $z(t) \in \mathbb{R}^n$, and $u(t) \in \mathbb{R}^m$, are the state and control vectors, respectively, and A_0, A_d, B are constant matrices with appropriate dimensions. Moreover, the vector ω: $\mathbb{R}^n \to \mathbb{R}^n$, represents the system nonlinearity and any model uncertainties in the system including external disturbances, which is Lipschitz:

$$\exists \lambda > 0, \qquad \|\omega(z(t)\| \le \lambda \|z(t)\|, \forall z(t) \in \mathbb{R}^n. \quad (3)$$

$\Psi(t) \in C([t_0 - h, t_0], \mathbb{R}^n)$ is the continuous initial state function; and h > 0 is the constant time-delay of the system. In this paper, the control law is designed based on the sliding surface technique to stabilize the time-delay system (2) in the face of external disturbance. The task is to design an asymptotically stabilizing control law that also minimizes the following cost functional:

$$J = {}_0I_t^{\alpha}\left(z^T(t)Qz(t) + u^T(t)Ru(t)\right)dt, \quad (4)$$

where $Q \in \mathbb{R}^{n\times n}$, and $R \in \mathbb{R}^1$ are positive definite weighing matrices, respectively and integral I is the Riemann-Liouville fractional integral.

Assumption 1. The cost functional is minimized such that the following relation holds for the LQR cost function [25]:

$$J = {}_0I_t^{\alpha}(z^T(t)Qz(t) + u^T(t)Ru(t))dt < \tilde{J}, \tilde{J} \in \mathbb{R} \quad (5)$$

IV. OPTIMAL CONTROLLER DESIGN

In the following, we design two control laws. First, an optimal control law u_1 is designed for the nominal system such that the nominal system be stable and Eq. (5) holds for LQR cost function. Then, u_1 combines with a switching control law u_2 to reduce external disturbances. Neglecting the nonlinearity term, the nominal system with the delay of (2) becomes

$$ {}_{t_0}^{C}D_t^{\alpha}z(t) = A_0 z(t) + A_d\, z(t-h) + Bu_1(t), \quad (6)$$

The cost functional that should be minimized should hold to Eq. (5). In this section, we find an optimal state control $u_1 = Kz(t)$ introduced using the stability criteria in theorem (1) for system (6).

Theorem 2. The state feedback control law $u_1 = kz(t)$, stabilizes the nominal system defined in (6) by minimizing the performance index (5) if there is a symmetric positive definite matrix P, a matrix Y and two positive constants δ1 and δ2, such that the following LMIs holds:

$$\begin{bmatrix} M & I \\ I & -\delta_1 I \end{bmatrix} < 0,$$

$$\delta_1 \|\mathrm{A_d}\|^2 P < \delta_2 I, \quad (8)$$

where

$M = A_0P + PA_0^T + BY + Y^TB^T + PQP + Y^TRY + \delta_2 P$, and $K = YP^{-1}$.

Proof. We can consider the following Lyapunov function:

$$V(t) = z^T(t)P^{-1}z(t).$$

It can be easily observed that $V(t) = z^T(t)P^{-1}z(t)$ is bounded by $v_1(t) = \lambda_{min}(P^{-1})\|z\|^2$ and $v_2(t) = \lambda_{max}(P^{-1})\|z\|^2$. Therefore, the first condition of theorem (1) is satisfied. Now, according to theorem (1), we consider the second condition of the Razumikhin stability theorem. To this end we consider the closed-loop system as follows:

$$ {}_{t_0}^{C}D_t^{\alpha}z(t) = A_0 z(t) + A_d\, z(t-h) + Bu(t) = (A_0 + Bk)z(t) + A_d\, z(t-h). \quad (9)$$

According to lemma (2), we have

$$D^{\alpha}V(t) \le 2z^TP^{-1}D^{\alpha}z(t) \le 2z^TP^{-1}[(A_0 + BK)z(t) + (A_d)\, z(t-h) \le 2z^TP^{-1}(A_0 + BK)z(t)2z^TP^{-1}\ (A_d)\, z(t-h).$$

Considering the above relation as a sum of two terms, we have for the first term:

$$2z^TP^{-1}(A_0 + BK)z = z^T[P^{-1}A_0 + A_0^TP^{-1} + P^{-1}BK + K^TB^TP^{-1}]z. \quad (10)$$

and for the second term, according to lemma (4):

$$2z^TP^{-1}\ A_d\, z(t-\tau) \leq \ \delta_1 z^T(t-h)A_\tau^T A_\tau\, z(t-h) + \delta_1^{-1}z^TP^{-1}P^{-1}z. \quad (11)$$

By combining these two relations and considering Eq. (3), it is straightforward to see that:

$$D^\alpha V(t) \leq z^T[P^{-1}A_0 + A_0^TP^{-1} + P^{-1}BK + K^TB^TP^{-1} + \delta_1^{-1}P^{-1}P^{-1} + Q + K^TRK]z + \delta_2 z^T(t-h)P^{-1}z(t-h) - z^T[Q+K^TRK]z.$$

Now, if the following assumption holds

$$V\big(t+s, z(t+s)\big) = z^T(t+s)P^{-1}z(t+s) < q^* z^TP^{-1}z; \qquad s\in[-h,0],$$

where $q^* = 1+\delta, \delta > 0$, then by substituting $s=-h$, and from (4.3), $\delta_1\|\mathrm{A_d}\|^2P < \delta_2 I$, we have

$$D^\alpha V(t) \leq z^T[P^{-1}A_0 + A_0^TP^{-1} + P^{-1}BK + K^TB^TP^{-1} + \delta_1^{-1}P^{-1}P^{-1} + Q + K^TRK]z + q^*\delta_2 z^TP^{-1}z - z^T[Q+K^TRK]z.$$

Because $q^* = \delta + 1$ by $\delta \to 0$, and also P and Q are positive definite matrices, so

$$D^\alpha V(t) \leq z^TM_2z - z^T[Q+K^TRK]z \leq z^TM_2z.$$

where

$$M_2 = \quad P^{-1}A_0 + A_0^TP^{-1} + P^{-1}BK + K^TB^TP^{-1} + \delta_1^{-1}P^{-1}P^{-1} + Q + K^TRK + \ \delta_2P^{-1}.$$

Now, pre and post multiplying both sides of M_2 by P and let $K = YP^{-1}$ we have:

$$\mathrm{M_1} = A_0P + PA_0^T + BY + Y^TB^T + \delta_1^{-1}I + PQP + Y^TRY + \delta_2P.$$

we have $M_2 \ < 0$ if $\mathrm{M_1} \ < 0$. From the Schur complement lemma, $\mathrm{M_1} < 0$ is equal to

$$\begin{bmatrix} M & I \\ I & -\delta_1 I \end{bmatrix} < 0,$$

where $M = A_0P + PA_0^T + BY + Y^TB^T + PQP + Y^TRY + \delta_2P$.

Thus, according to theorem (1), the closed- loop system (9) is stable. Finally, we have

$V(t) = z^T(x)P^{-1}z(t) > 0$, also:

$$\lambda_{min}(P^{-1})\|z(0)\|^2 \leq V(t) \leq \lambda_{max}(P^{-1})\|z(0)\|^2$$

Moreover, $D^\alpha V(t) < 0$, Q and R are positive definite matrices, so by using lemma (1):

$$D^\alpha V(t) + z^T[Q+K^TRK]z \leq 0$$

$${}_0I_t^\alpha z^T[Q+K^TRK]z < - \ {}_0I_t^\alpha\big(D^\alpha V(t)\big) = V(0) - V(t) \quad < \lambda_{max}(P^{-1})\|z(0)\|^2.$$

Thus, we have $\tilde{J} = \lambda_{max}(P^{-1})\|z(0)\|^2$ which completes the proof.

V. SLIDING MODE CONTROLLER DESIGN

Here, finding the parameters of a sliding mode controller guarantee that the directions of the SMC system guided to the a presupposed sliding surface in a constrained moment and the closed-loop system is asymptotically stable subject to the influences of external disturbances. To do this, we plan an integral-type sliding surface for the system. From theorem (2), the optimal controller is equal to $u_1 = Kz(t)$, so the control law for system (2) can be defined as $u = u_1 + u_2$, where u_2 is a switching control law. Here, u is the optimal sliding mode control law. Consequently, we have:

$${}_{t_0}^{C}D_t^\alpha z(t) = A_0z(t) + A_d\ z(t-h) + Bu(t) + \omega\big(t, z(t)\big). \quad (12)$$

To combine the optimal controller with a sliding mode controller, an integral sliding surface $\Lambda(t)$ is designed as

$$\Lambda(t) = CI^{1-\alpha}z(t) - \int_{t_0}^{t} C(A_0 + BK)z(\xi)d\xi \quad (13)$$

where $\Lambda(t) \in \mathbb{R}^m$ is the sliding surface, $\mathrm{K} \in \mathbb{R}^{m\times n}$ of the gain matrix and $\mathrm{G} \in \mathbb{R}^{m\times n}$ is also a constant matrix that must be designed in such a way that $\mathrm{GB} \in \mathbb{R}^{m\times m}$ is non-singular. Integral I is the Riemann-Liouville fractional integral. Matrix K is designed using theorem (2). Now, we proceed to design a second-order SMC such that the reachability of the specified sliding surface $\Lambda(t)$ and $\dot{\Lambda}(t) = 0$. is ensured. Consider system (12) involving time delay, the control law u ensures the convergence of the state variables to the sliding surface inside a limited time T^*. The following theorem will prove the reachability law.

Theorem 3. Consider the time-delay system (12) with the sliding surface in (13). The following control law guarantees that all trajectories of the sliding surface reach it in the finite time:

$$u(t) = u_1(t) + u_2(t) = Kz(t) - (CB)^{-1}\rho\big(z(t)\big)sign\big(\Lambda(t)\big), \quad (14)$$

where $sign(.)$ is the sign function, and the switching gain $\rho(z(t))$ is given by:

$$\rho(z(t)) = \beta + (\|GA_d\| + \|G\|\lambda) sup\|z(t)\|, \quad (15)$$

with $\beta > 0$.

Proof. After derivating from (12), we have

$$\dot{\Lambda}(t) = GD^{\alpha}z(t) - G[(A_0 + BK)z(t)].$$

Using (11), one has

$$\dot{\Lambda}(t) = G(A_0 z(t) + (A_d)z(t-h) + Bu(t) + \omega(z(t)) - (G[(A_0 + BK)z(t)] = GA_d z(t-h) + GBu(t) + G\omega(z(t)) - CBKz(t).$$

from (13) and the above relation one can see that

$$\dot{\Lambda}(t) = GA_d z(t-h) - \rho sign(\Lambda(t)) + G\omega(z(t)). \quad (16)$$

The following Lyapunov function is considered:

$$V(t) = 0.5\Lambda(t)\Lambda^T(t). \quad (17)$$

According to Eq. (14), the time derivative of Eq. (16) is obtained as:

$$\dot{V}(t) = \Lambda(t)\dot{\Lambda}(t) = \Lambda(t)(GA_d z(t-h) - \rho\ sign(\Lambda(t)) + G\omega(t, z(t)) \le \|GA_d z(t-\tau)\|\|\Lambda(t)\| - \rho\|\Lambda(t)\| + \|G\omega\|\|\Lambda(t)\| \le (\|GA_d z(t-h)\| + \|G\omega\|)\|\Lambda(t)\| - \rho\|\Lambda(t)\|. \quad (18)$$

We also have:

$$\|z(t-\tau)\| < sup\|z(t)\|, \quad (19)$$

Substituting Eq. (19) in Eq. (18), yields

$$\dot{V}(t) \le (\|GA_d z(t-\tau)\| + \|G\omega\|)\|\Lambda(t)\| - \rho\|\Lambda(t)\| \le (\|GA_d\| + \|G\|\lambda) sup\|z(t)\|\|\Lambda(t)\| - \rho\|\Lambda(t)\| = (\|GA_\tau\| + \|G\|\lambda) sup\|z(t)\| - \rho)\|\Lambda(t)\| = -(-\|GA_d\| - \|G\|\lambda) sup\|z(t)\| + \rho)\|\Lambda(t)\|.$$

Thus, the control law in Eq. (14) satisfies the following condition in the presence of external disturbance, where $\rho = \beta + (\|GA_d\| + \|G\|\lambda) sup\|z(t)\|$ and $\beta > 0$:

$$\dot{V}(t) = -\beta\|\Lambda(t)\| \le 0. \quad (20)$$

Therefore, the reaching law is established. Using (20), it can be shown that the trajectories of the sliding surface reach it in a finite time T^*, where:

$$t_0 < T^* \le \beta^{-1}\sqrt{2V(t_0)} + t_0.$$

In fact, from (15), we have $\dot{V}(t) \le -\beta\sqrt{V(t)}$. By integrating from this inequality, the time occurrence of the reaching phase is calculated in this way:

$$\frac{\dot{V}(t)}{\sqrt{2V(t)}} \le -\beta \rightarrow \int_{t_0}^{t} \frac{\dot{V}(\xi)}{\sqrt{2V(\xi)}} d\xi \le \int_{t_0}^{t} -\beta\ d\xi,$$

By integrating, we have:

$$\int_{t_0}^{t} \frac{\dot{V}(\xi)}{\sqrt{2V(\xi)}} d\xi = \sqrt{2V(t)} - \sqrt{2V(t_0)} \le -\beta(t - t_0)$$

Now, when $\Lambda(t) = 0$, we have $V(t) = 0$, then

$$-\beta^{-1}\sqrt{2V(t_0)} - t_0 \le -T^* \rightarrow T^* \le \beta^{-1}\sqrt{2V(t_0)} + t_0.$$

This completes the proof.

By substituting u(t) from (14) in (2), the equation of the closed loop system can be derived as follows

$${}_{t_0}^{C}D_t^{\alpha} z(t) = A_0 z(t) + A_d z(t-h) + \omega(t, z(t)) - B(CB)^{-1}\rho(z(t)) sign(\Lambda(t).$$

According to theorem (3), state variables reach the sliding surface in finite time, so on the sliding surface for $t > T^*$, we have Λ(t)=0. As a result, the closed loop system in the sliding mode is as follows:

$$D^{\alpha}z(t) = (A_0 + BK)z(t) + A_d z(t-h) + w(t, z(t)). \quad (21)$$

Here, we present a lemma to find the necessary conditions for the parameters of theorem (2) so that the choice of matrices K and P can ensure the asymptotic stability of the closed-loop system on the sliding surface. In the following lemma, we show that for the stability of the closed-loop system (21), in addition to (8), a linear matrix inequality must also hold.

Lemma 5. In system (2), matrix $K = YP^{-1}$, closed loop systems (18) is asymptotically stable if there is a positive number δ_3, such that the following LMI holds:

$$\begin{bmatrix} N_1 & \lambda P & I \\ \lambda P & -\delta_3 I & 0 \\ I & 0 & -\delta_1 I \end{bmatrix} < 0, \quad (22)$$

where $N_1 = A_0 P + PA_0^T + BY + Y^T B^T + \delta_2 P + \delta_3 I$.

Proof. The following Lyapunov function is considered, where P is the same as P in theorem (2):

$$V(t) = z^T(t)P^{-1}z(t).$$

To satisfy the first condition of theorem (1), we have $\upsilon_1(t) = \lambda_{min}(P^{-1})\|z\|^2$ and $\upsilon_2(t) = \lambda_{max}(P^{-1})\|z\|^2$.

Considering Eq. (23) and lemma (2), the time derivative of Eq. (22) is obtained as:

$$D^{\alpha}V(t) \le 2z^T P^{-1} D^{\alpha} z(t)$$

$$\leq 2z^T P^{-1}\big[(A_0+BK)z(t) + (A_d)\, z(t-h) + \omega\big(t,z(t)\big)\big] + 2z^T P^{-1}\omega\big(t,z(t)\big). \quad (24)$$

By considering the above relation as a sum of three terms, and Eq. (3) and lemma (4), we have for the last term:

$$2z^T P^{-1}\omega \leq \delta_3 z^T P^{-1}P^{-1}z + \delta_3^{-1}\lambda^2 z^T z. \quad (25)$$

By substituting (24) in (23), and using (10) and (11), we have:

$$D^\alpha V(t) \leq z^T\Big[P^{-1}A_0 + A_0^T P^{-1} + P^{-1}BK + K^T B^T P^{-1} + \delta_1^{-1}\ P^{-1}P^{-1} + \delta_3 P^{-1}P^{-1} + \delta_3^{-1}\lambda^2 I\Big]z + \delta_1 z^T(t-h)A_d^T A_d\, z(t-h).$$

As for LMI (8) and for $q^* = \delta + 1 > 1$, the assumption $V\big(t+s,z(t+s)\big) < q^* V(t,z(t))$ for $s \in [-h,0]$ implies:

$$D^\alpha V(t) \leq z^T[A_0 P + P A_0^T + BY + Y^T B^T + \delta_1^{-1} I + \delta_2 P + \delta_3 I + \delta_3^{-1}\lambda^2 PP]z.$$

Therefore, according to the Schur complement lemma $D^\alpha V(t) < 0$, if

$$\begin{bmatrix} N_1 & \lambda P & I \\ \lambda P & -\delta_3 I & 0 \\ I & 0 & -\delta_1 I \end{bmatrix} < 0,$$

where, $N_1 = A_0 P + P A_0^T + BY + Y^T B^T + \delta_2 P + \delta_3 I$. Thus, whenever $V\big(t+s,z(t+s)\big) < q^* V(t,z(t))$ is satisfied, condition $D^\alpha V(t) < 0$ is also satisfied, by using the stability criteria in theorem (1), the closed-loop system (21) is stable.

Finally, the following procedure is suggested for the design of the optimal sliding mode controller:

(I) Solve **LMI (7)** and Find matrix **K**

(II) Use **K** to design sliding mode surface $\boldsymbol{\Lambda(t)}$

(III) Compute switching control $\mathbf{u_{sw}}$

(IV) Compute equivalent control $\mathbf{u_{eq}}$

(V) Compute $\mathbf{u} = \mathbf{u_{sw}} + \mathbf{u_{eq}}$

(VI) Check the satisfaction of the **LMI (22).** If this **LMI** is satisfied, **u** is **optimal sliding mode controller** for delayed fractional order system with disturbances,

VI. SIMULATION RESULTS

In this section, we provide a numerical example to demonstrate the effectiveness of our methods. The numerical simulation is created utilizing Ninteger and Yalmip toolbox in MATLAB software.

Example. Consider the nonlinear time-delay system (2) with $\alpha = 0.7$ and the following system characteristics:

$$\boldsymbol{D^\alpha z} = \begin{bmatrix} -9 & 5 & -8 \\ -4 & -3 & 5 \\ -4 & 9 & 0 \end{bmatrix} z(t) + \begin{bmatrix} 0.4 & -1.2 & 1.9 \\ 0.2 & -0.9 & 0.1 \\ -0.1 & -0.5 & 2.7 \end{bmatrix} z(t-h) + \begin{pmatrix} 4 \\ -1 \\ 12 \end{pmatrix} u(t) + \begin{pmatrix} 0.8\sin(z_1^2(t)) \\ 0.8\sin(z_2^2(t)) \\ 0.8\sin(z_3^2(t)) \end{pmatrix}.$$

Subject to:

$$J = {}_0 I_t^\alpha\big(z^T(t)Qz(t) + u^T(t)Ru(t)\big)dt.$$

where $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.9 \end{bmatrix}$, $R = 1$. Function $\omega(t,z(t))$ is Lipschitz, and $\lambda = 0.8$. This system is unstable for u = 0 as shown in Fig. 1. In this example, we choose h = 1, 3 and 5. By applying theorem (2) and lemma (5), the LMIs conditions hold if:

$$P = \begin{bmatrix} 9.3016 & -0.3530 & 6.2433 \\ -0.3530 & 0.8247 & -1.3964 \\ 6.2433 & -1.3964 & 7.3454 \end{bmatrix},$$

$$Y = [-4, 1, -12].$$

and $\delta_1 = 4.4250$, $\delta_2 = 17.7142$, and $\delta_3 = 0.4482$.

Then the controller gain matrix can be obtained as $K = YP^{-1} = [2.9474, -6.6859, -5.4098]$. Now, by using theorem (2), we have $\tilde{J} = \lambda_{max}(P^{-1})\|z(0)\|^2 = 14.7427\|z(0)\|^2$. Matrix G is chosen $G = [-0.1, 0.1, -0.1]$. By (12), the sliding surface function can be computed by movable parameters, $\beta = 1$. According to the conditions of theorem (3) we have GB is non-singular and $\beta > 0$. By applying the control law in theorem (3), we find that the condition of the theorem is satisfied. Figs. 2-4 display the trajectories of the states of the closed-loop system with optimal SMC controller law (13). Fig. 5 presents the simulation results of the feedback control law and Fig. 6 depicts the simulation results of the sliding function. Therefore, we show that the system is stable. These figures demonstrate the efficiency of our method for selecting controller gain matrix K to guarantee robustness against perturbations and minimize the LQR cost function. This indicates that the sliding mode controller, independent of the delay, stabilizes the system. To prevent the control signals from chattering, we replace $sign\big(\Lambda(t)\big)$ with $\Lambda(t) = \frac{\Lambda(t)}{\|\Lambda(t)\|+0.04}$.

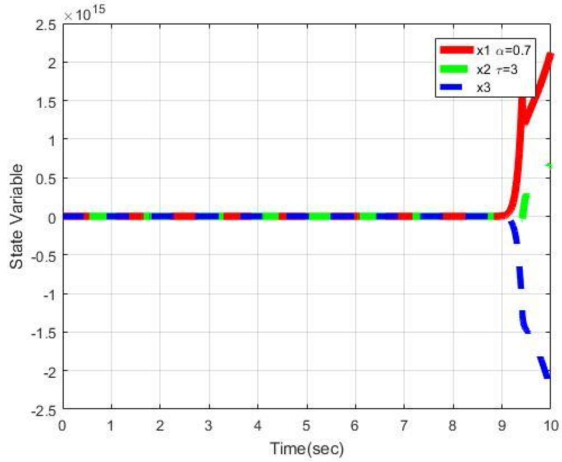

Fig. 1: Trajectories of the open-loop system with h = 1.

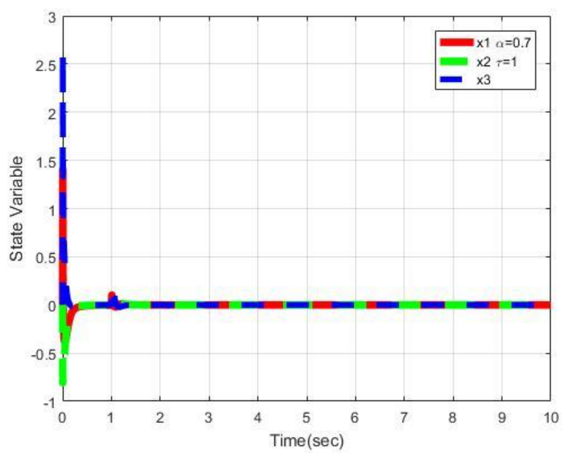

Fig. 2: Trajectories of the closed-loop system with h = 1.

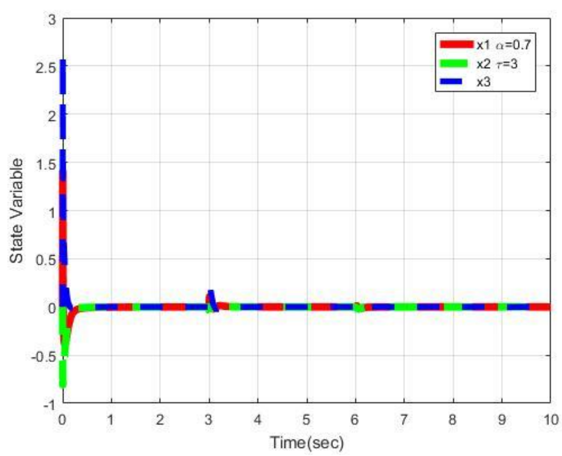

Fig. 3: Trajectories of the closed-loop system of the system with h = 3.

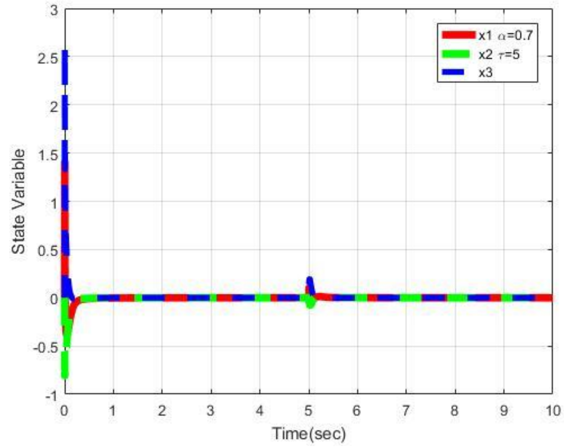

Fig. 4: Trajectories of the closed-loop system with h = 5.

Fig. 5: Feedback control law h = 3.

Fig. 6: Sliding Manifold with h = 5.

VII. Conclusions

In this paper, we studied the problem of asymptotically stabilization of a class of uncertain nonlinear fractional-order time-varying delayed systems subject to nonlinear disturbance on the basis of an LMI- based SMC with LQR cost function. We proposed new sufficient conditions expressed using LMI based on the Razumikhin approach. The performance of the main results was demonstrated using an example showing that the results of combining LQR with SMC could guarantee better efficiency of the optimal control system. We will apply the results obtained in this paper to the problem of robust FTB of fractional nonlinear time-varying delay systems using LQR-based SMC in our future work.

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