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Original paper

Generalization of rough fuzzy sets based on a fuzzy ideal

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Abstract

Since Pawlak defined the notion of rough sets in 1982, many authors made wide research studying rough sets in the ordinary case and the fuzzy case. This paper introduced a new style of rough fuzzy sets based on a fuzzy ideal ℓ on a universal finite set X. New lower and new upper fuzzy sets are introduced, and consequently, fuzzy interior and fuzzy closure operators of a rough fuzzy set are discussed. These definitions, if ℓ is restricted to $\ell^\circ = \{\overline{0}\}$, imply the fuzzification of previous definitions given in the ordinary case, and moreover in the crisp case, we get exactly these previous definitions. The new style gives us a better accuracy value of roughness than the previous styles. Rough fuzzy connectedness is introduced as a sample of applications on the recent style of roughness.

Keywords: Fuzzy approximation space, rough set, rough fuzzy set, rough fuzzy connectedness.

1 Introduction

The concept of rough sets was introduced by Pawlak [14] in 1982 based on there are some objects in a vague area called the boundary region that can not be determined by a set or its complement. Rough sets depends on a relation R defined on the universal finite set X, and the pair (X, R) is called an approximation space. Firstly, rough sets was given by some equivalence relation. Many authors studied rough sets based on more generalized relations on X, for example see [1, 2, 9, 19]. There are lower set, upper set and consequently a boundary region that became an essential role in artificial intelligence, granular computing and decision analysis. The generated topology τ on an approximation space (X, R) that represent the topological properties of rough sets were studied by many authors (ex. [17, 21]). Many fuzzification studies were given to generalize rough sets as in the literature [4, 5, 12, 13, 15, 18]. The effect of defining a fuzzy ideal in [16] on the fuzzy topological spaces and the fuzzy approximation spaces were studied in [3, 7, 10].

Based on the paper in [8], if we combined the definitions given in [11] and the definitions given in [9] that used an ideal on X, then we get a more general form of roughness and a better accuracy value of the rough set. Thus, assigning an ideal in defining the lower and upper sets in some approximation space is a generalization of roughness.

Considering a fuzzy ideal ℓ, as in [16], on a fuzzy approximation space (X, R), we will define rough fuzzy sets in a new pattern. Then, we show that under restrictions ($\ell = \{\overline{0}\}$), the new definitions give the fuzzification of the definitions of [1, 2, 9, 11, 14, 19], and in the crisp case, it will be exactly the same definitions of [1, 2, 9, 11, 14, 19] in as ordinary case. Defining a more generalized accuracy value is given in this paper. As a characterization of the definition of rough fuzzy sets and as a generalization of connectedness in fuzzy topological spaces given in [6], we introduce the concept of rough fuzzy connectedness.

Through the paper, let X be a finite set of objects and I the closed unit interval $[0, 1]$. I^X denotes all fuzzy subsets of X, and $\lambda^c(x) = 1 - \lambda(x)$, $\forall x \in X, \forall \lambda \in I^X$. A constant fuzzy set $\overline{t}$ for all $t \in I$ is defined by $\overline{t}(x) = t$, $\forall x \in X$. Infimum and supremum of a fuzzy set $\lambda \in I^X$ are given as: $\inf \lambda = \bigwedge_{x \in X} \lambda(x)$ and $\sup \lambda = \bigvee_{x \in X} \lambda(x)$. If $f : X \to Y$ is a mapping, $\mu \in I^X, \nu \in I^Y$, then $(f(\mu))(y) = \bigvee_{x \in f^{-1}(y)} \mu(x)\ \forall y \in Y$ and $f^{-1}(\nu) = (\nu \circ f)$.

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Recall that the fuzzy difference between two fuzzy sets was defined in [7] as:

$$(\lambda \bar{\wedge} \mu) = \begin{cases} \overline{0} & \text{if } \lambda \leq \mu, \\ \lambda \wedge \mu^c & \text{otherwise} \end{cases} \tag{1}$$

A subset $\ell \subset I^X$ is called a fuzzy ideal [16] on X if it satisfy the following conditions:

(1) $\overline{0} \in \ell$,

(2) If $\nu \leq \mu$ and $\mu \in \ell$, then $\nu \in \ell$ for all $\mu, \nu \in I^X$,

(3) If $\mu \in \ell$ and $\nu \in \ell$, then $(\mu \vee \nu) \in \ell$ for all $\mu, \nu \in I^X$.

Usually, we consider the proper fuzzy ideal ℓ ($\overline{1} \notin \ell$). Denote a fuzzy ideal ℓ° for the fuzzy ideal including only $\overline{0}$.

2 Lower, upper and boundary region fuzzy sets

In the ordinary case, we have the following:

(1) (Pawlak [14]) Let R be an equivalence relation on X, $[x]_R$ be the equivalence class containing x. For any subset A of X, the lower approximation $\underline{R}(A)$ and the upper approximation $\bar{R}(A)$ are defined as:

$$\underline{R}(A) = \{x \in X : [x]_R \subseteq A\}, \quad \bar{R}(A) = \{x \in X : [x]_R \cap A \neq \emptyset\}.$$

(2) (Yao [19]) Let R be a binary relation on X. For any subset A of X, the lower approximation $\underline{R}(A)$ and the upper approximation $\bar{R}(A)$ are defined as:

$$\underline{R}(A) = \{x \in X : xR \subseteq A\}, \quad \bar{R}(A) = \{x \in X : xR \cap A \neq \emptyset\},$$

where xR is called the after set of x defined as: $xR = \{y \in X : xRy\}$. Moreover, Rx is called the before set of x defined as: $Rx = \{y \in X : yRx\}$.

(3) (Allam [2]) Let R be a reflexive binary relation on X, $< p > R$ is the intersection of all the after sets xR containing p. Then, for any subset A of X, the lower approximation $\underline{R}(A)$ and the upper approximation $\bar{R}(A)$ are defined as:

$$\underline{R}(A) = \{x \in X : < x > R \subseteq A\}, \quad \bar{R}(A) = \{x \in X : < x > R \cap A \neq \emptyset\},$$

where $< p > R$ is defined as:

$$< p > R = \begin{cases} \bigcap_{p \in xR} xR & \text{if } \exists x : \ p \in xR, \\ \emptyset & \text{otherwise} \end{cases}$$

Moreover,

$$R < p > = \begin{cases} \bigcap_{p \in Rx} Rx & \text{if } \exists x : p \in Rx, \\ \emptyset & \text{otherwise} \end{cases}$$

(4) (Kandil [9]) Let R be a reflexive relation on X and I be an ideal on X. For any subset A of X, the lower approximation $\underline{R}(A)$ and the upper approximation $\bar{R}(A)$ are defined as:

$$\underline{R}(A) = \{x \in A : < x > R \cap A^c \in I\},$$

$$\bar{R}(A) = A \cup \{x \in X : < x > R \cap A \notin I\}.$$

(5) (Kozae [11]) Let R be a binary relation on X. For any subset A of X, the lower approximation $\underline{R}(A)$ and the upper approximation $\bar{R}(A)$ are defined as:

$$\underline{R}(A) = \{x \in X : R < x > R \subseteq A\},$$

$$\bar{R}(A) = \{x \in X : R < x > R \cap A \neq \emptyset\},$$

where $R < p > R$ is defined as: $R < p > R \ = < p > R \cap R < p >$.

The previous definitions are given for the roughness of an approximation space (X, R) in the ordinary case.

Here, we will define rough fuzzy sets in a fuzzy approximation space (X, R) in a generalized form.

Definition 2.1. *Let X be a finite set, R a fuzzy relation on X. Then, for any $x \in X$, define the fuzzy sets $xR, Rx \in I^X$ as follow* (see [20])*:*

$$xR(y) = R(x, y) \quad \text{and} \quad Rx(y) = R(y, x) \ \forall y \in X. \tag{2}$$

Define for any $a \in X$, the fuzzy sets $< a > R,\ R < a >\in I^X$ as follow:

$$< a > R = \bigwedge_{x \in X,\, R(x,a)>0} xR \quad \text{and} \quad R < a >= \bigwedge_{x \in X,\, R(a,x)>0} Rx. \tag{3}$$

For any $a \in X$, define $R < a > R : X \to I$ as follows:

$$R < a > R =< a > R \ \wedge \ R < a > . \tag{4}$$

Definition 2.2. *For every $x \in X$, define $\lambda_*, \lambda^* \in I^X$ of a fuzzy set $\lambda \in I^X$ by:*

$$\lambda_*(x) \ = \ \begin{cases} (\bigvee\limits_{z \in X} R < z > R\,(x))^c & \text{if } R < x > R \ \wedge \lambda^c \notin \ell \ \ \text{and} \ \ R < x > R \ \wedge \lambda \notin \ell \\ 1 & \text{if } R < x > R \ \wedge \lambda^c \in \ell \\ 0 & \text{if } R < x > R \ \wedge \lambda^c \notin \ell \ \ \text{and} \ \ R < x > R \ \wedge \lambda \in \ell \end{cases} \tag{5}$$

$$\lambda^*(x) \ = \ \begin{cases} \bigvee\limits_{z \in X} R < z > R\,(x) & \text{if } R < x > R \ \wedge \lambda \notin \ell \ \ \text{and} \ \ R < x > R \ \wedge \lambda^c \notin \ell \\ 0 & \text{if } R < x > R \ \wedge \lambda \in \ell \\ 1 & \text{if } R < x > R \ \wedge \lambda \notin \ell \ \ \text{and} \ \ R < x > R \ \wedge \lambda^c \in \ell \end{cases} \tag{6}$$

The roughness of a fuzzy set $\lambda \in I^X$ is defined by:

$$\lambda_R = \lambda \wedge \lambda_* \quad \text{and} \quad \lambda^R = \lambda \vee \lambda^*. \tag{7}$$

λ_R is the lower fuzzy set of λ and λ^R is the upper fuzzy set of λ. The boundary fuzzy region of λ is λ^B given by: $\lambda^B = \lambda^R \bar{\wedge} \lambda_R$. The pair (X, R) will be called rough fuzzy approximation space.

Definition 2.3. *For every rough fuzzy set $\lambda \in I^X$, define the accuracy fuzzy set, $\alpha(\lambda) \in I^X$, for all $x \in X$, by the following*

$$\alpha(\lambda)(x) \ = \ \begin{cases} 0 & \text{if } \lambda^R = \bar{1} \ \ \text{and} \ \ \lambda_R = \bar{0}, \\ (\lambda^R(x) - \lambda(x))^c \ \wedge (\lambda(x) - \lambda_R(x))^c & \text{if } \lambda^R \not\leq \lambda_R, \\ 1 & \text{otherwise,} \end{cases} \tag{8}$$

and moreover the accuracy value of the rough fuzzy set λ is given by $\text{Inf}(\alpha(\lambda))$.

Whenever λ^R be so that $\lambda^R \leq \lambda_R$, we get that $\lambda = \lambda_R = \lambda^R$ and then, $\lambda^B = \bar{0}$ and $\text{Inf}(\alpha(\lambda)) = 1$. If $\lambda_R = \bar{0}$ and $\lambda^R = \bar{1}$, then $\lambda^B = \bar{1}$ and $\text{Inf}(\alpha(\lambda)) = 0$. Otherwise, $\lambda^B = \lambda^R \wedge (\lambda_R)^c$ and $0 < \text{Inf}(\alpha(\lambda)) < 1$. That is, the largest boundary fuzzy set is associated with the lowest accuracy value and the converse is true. If $\text{Inf}(\alpha(\lambda)) = 1$, then λ is crisp with respect to R ($\lambda_R = \lambda^R$ and λ is precise with respect to R). If $\text{Inf}(\alpha(\lambda)) = 0$, then λ is totally rough with respect to R. Moreover, if $0 < \text{Inf}(\alpha(\lambda)) < 1$, then λ is rough with respect to R.

Lemma 2.4. *Let R be a fuzzy relation on X, ℓ be a fuzzy ideal on X and $\lambda, \mu \in I^X$. Then, the following properties hold:*

(1) $\lambda^* = ((\lambda^c)_*)^c$,

(2) $\overline{0}^* = \overline{0}$ and $\overline{1}_* = \overline{1}$,

(3) $\lambda \leq \mu$ implies $\lambda_* \leq \mu_*$ and $\lambda^* \leq \mu^*$,

(4) $(\lambda \wedge \mu)^* \leq \lambda^* \wedge \mu^*$,

(5) $(\lambda \vee \mu)_* \geq \lambda_* \vee \mu_*$,

(6) $(\lambda \wedge \mu)_* \leq \lambda_* \wedge \mu_*$,

(7) $(\lambda \vee \mu)^* \geq \lambda^* \vee \mu^*$,

(8) *If $\lambda \in \ell$, then $\lambda^* = \overline{0}$.* Moreover , *if $\lambda^c \in \ell$, then $\lambda_* = \overline{1}$.*

Proof. For (1): It is clear that $(\lambda^*)^c = (\lambda^c)_*$, $(\lambda_*)^c = (\lambda^c)^*$ and thus $\lambda^* = ((\lambda^c)_*)^c$ and $\lambda_* = ((\lambda^c)^*)^c$.

For (2): we have $(\overline{0})^*(x) = 0 \ \forall x \in X$ from Equation 6, and $(\overline{1})_*(x) = 1 \ \forall x \in X$ from Equation 5, and then $\overline{0}^* = \overline{0}$, $\overline{1}_* = \overline{1}$.

For (3): It is proved from the definition of the fuzzy ideal and Equations 5, 6.

For (4): we get it directly using the result in (3).

For (5): we get it directly using the result in (3).

For (6): we get that $\lambda_* \wedge \mu_* \geq (\lambda \wedge \mu)_*$ directly.

For (7): we get that $(\lambda \vee \mu)^* \geq \lambda^* \vee \mu^*$ directly.

For (8): Since $\lambda \in \ell$ implies that $R < x > R \wedge \lambda \in \ell$, and thus $\lambda^*(x) = 0 \ \forall x \in X$. Hence, $\lambda^* = \overline{0}$. Similarly, $\lambda^c \in \ell$ implies that $R < x > R \wedge \lambda^c \in \ell$, and thus $\lambda_*(x) = 1 \ \forall x \in X$. Hence, $\lambda_* = \overline{1}$. □

Note that: if we have the trivial fuzzy ideal $\ell = I^X$, then $\lambda_* = \overline{1}$ and $\lambda^* = \overline{0}$, and hence $\lambda_R = \lambda^R = \lambda$, and therefore any fuzzy set has accuracy value $\text{Inf}(\alpha(\lambda)) = 1$.

Remark 2.5. *Let R be a fuzzy relation on X, ℓ be a fuzzy ideal on X and $\lambda, \mu \in I^X$. Then, the following results hold in general.*

(1) $\lambda \nleq \lambda_* \nleq \lambda^* \nleq \lambda$ and $\lambda \nleq \lambda^* \nleq \lambda_* \nleq \lambda$,

(2) $(\lambda \wedge \mu)^* \ngeq \lambda^* \wedge \mu^*$ *and* $(\lambda \wedge \mu)_* \ngeq \lambda_* \wedge \mu_*$,

(3) $(\lambda \vee \mu)_* \nleq \lambda_* \vee \mu_*$ *and* $(\lambda \vee \mu)^* \nleq \lambda^* \vee \mu^*$,

(4) $\lambda^* = \overline{0} \nRightarrow \lambda \in \ell$.

(5) $\lambda_* = \overline{1} \nRightarrow \lambda^c \in \ell$.

The following example proves the results in Remark 2.5.

Example 2.6. *Let R be a fuzzy relation on a set $X = \{a, b, c, d\}$ as shown down.*

R	a	b	c	d
a	0	0.2	1	0.5
b	0.6	0	0.8	0.5
c	1	0.5	0.6	0.6
d	0.9	0.6	1	1

$aR = \{0, 0.2, 1, 0.5\}$, $bR = \{0.6, 0, 0.8, 0.5\}$, $cR = \{1, 0.5, 0.6, 0.6\}$, $dR = \{0.9, 0.6, 1, 1\}$ *and* $Ra = \{0, 0.6, 1, 0.9\}$, $Rb = \{0.2, 0, 0.5, 0.6\}$, $Rc = \{1, 0.8, 0.6, 1\}$, $Rd = \{0.5, 0.5, 0.6, 1\}$. *Then,* $< a > R = \{0.6, 0, 0.6, 0.5\}$, $< b > R = \{0, 0.2, 0.6, 0.5\}$, $< c > R = \{0, 0, 0.6, 0.5\}$, $< d > R = \{0, 0, 0.6, 0.5\}$ *and* $R < a >= \{0.2, 0, 0.5, 0.6\}$, $R < b >= \{0, 0.5, 0.6, 0.9\}$, $R < c >= \{0, 0, 0.5, 0.6\}$, $R < d >= \{0, 0, 0.5, 0.6\}$ *and then,* $R < a > R = \{0.2, 0, 0.5, 0.5\}$, $R < b > R = \{0, 0.2, 0.6, 0.5\}$, $R < c > R = \{0, 0, 0.5, 0.5\}$, $R < d > R = \{0, 0, 0.5, 0.5\}$.

(1) *Consider a fuzzy ideal* ℓ *on* X *so that* $\nu \in \ell \Leftrightarrow \nu \leq \overline{0.4}$*, then we compute* λ_*, λ^* *for a fuzzy set* $\lambda = \{0.1, 0.8, 0.4, 0.6\}$ *as follow:*
$\lambda_* = \{0.8, 0.8, 0.4, 0.5\}$, $\lambda^* = \{0.2, 0.2, 0.6, 0.5\}$. *Hence,* $\lambda \not\leq \lambda_* \not\leq \lambda^* \not\leq \lambda$ and $\lambda \not\leq \lambda^* \not\leq \lambda_* \not\leq \lambda$*, and thus* (1) *holds.*

(2) *Let* $\mu = \{0.6, 0.2, 0.9, 0.2\}$*, we get that* $\mu^* = \{0.2, 0.2, 0.6, 0.5\}$, $\mu_* = \{0.8, 0.8, 0.4, 0.5\}$*, and then* $\mu^* = \lambda^* = \{0.2, 0.2, 0.6, 0.5\}$, $\mu_* = \lambda_* = \{0.8, 0.8, 0.4, 0.5\}$.

For $\lambda \wedge \mu = \{0.1, 0.2, 0.4, 0.2\}$*, we get that* $(\lambda \wedge \mu)^* = \overline{0}$*. Hence,* $\lambda^* \wedge \mu^* = \{0.2, 0.2, 0.6, 0.5\} \not\leq \overline{0} = (\lambda \wedge \mu)^*$*. Also, we get that* $(\lambda \wedge \mu)_* = \overline{0} \not\geq \{0.8, 0.8, 0.4, 0.5\} = \lambda_* \wedge \mu_*$*. Thus,* (2) *is proved.*

(3) *Again, for* $\lambda \vee \mu = \{0.6, 0.8, 0.9, 0.6\}$*, we get that* $(\lambda \vee \mu)_* = \overline{1}$*. Hence,* $(\lambda \vee \mu)_* = \overline{1} \not\leq \lambda_* \vee \mu_* = \{0.8, 0.8, 0.4, 0.5\}$*. Also, we get that*

$$(\lambda \vee \mu)^* = \overline{1} \not\leq \{0.2, 0.2, 0.6, 0.5\} = \lambda^* \vee \mu^*.$$

Thus, (3) *is proved.*

(4) *Now define a fuzzy ideal* ℓ *on* X *so that* $\nu \in \ell \Leftrightarrow \nu \leq \overline{0.5}$*, and consider a fuzzy set* $\lambda = \{0.1, 0.8, 0.4, 0.6\}$*. We compute* λ_*, λ^* *as follow:*

$$\lambda^* = \{0, 0, 0, 0\} = \overline{0}, \ \lambda_* = \{1, 0, 1, 1\}.$$

That is, $\lambda^* = \overline{0}$ *while* $\lambda \notin \ell$*. That is* (4) *is proved.*

(5) *Moreover, if* ℓ *is defined on* X *so that* $\nu \in \ell \Leftrightarrow \nu \leq \overline{0.6}$*, then* $\lambda_* = \overline{1}$ *while* $\lambda^c \notin \ell$*, and thus* (5) *is proved.*

Lemma 2.7. *The lower and upper fuzzy sets of fuzzy sets satisfy the following properties:*

(1) $\lambda_R \leq \lambda \leq \lambda^R$,

(2) $\overline{0}_R = \overline{0}^R = \overline{0}$ and $\overline{1}_R = \overline{1}^R = \overline{1}$,

(3) $(\lambda \vee \mu)_R \geq \lambda_R \vee \mu_R \ \ \forall \lambda, \mu \in I^X$,

(4) $(\lambda \wedge \mu)^R \leq \lambda^R \wedge \mu^R \ \ \forall \lambda, \mu \in I^X$,

(5) $\lambda \leq \mu$ implies that $\lambda_R \leq \mu_R$ and $\lambda^R \leq \mu^R \ \ \forall \lambda, \mu \in I^X$,

(6) $(\lambda \vee \mu)^R \geq \lambda^R \vee \mu^R \ \ \forall \lambda, \mu \in I^X$,

(7) $(\lambda \wedge \mu)_R \leq \lambda_R \wedge \mu_R \ \ \forall \lambda, \mu \in I^X$,

(8) $(\lambda^R)^c = (\lambda^c)_R$ and $(\lambda_R)^c = (\lambda^c)^R$

(9) $(\lambda_R)^R \geq \lambda_R \geq (\lambda_R)_R$

(10) $(\lambda^R)_R \leq \lambda^R \leq (\lambda^R)^R$

Proof. From Equation 7 and Lemma 2.4, we get easily the proof of all these results. □

Remark 2.8. *As in the usual case, whenever* R *is a reflexive fuzzy relation on* X*, then we have* $\lambda_* \leq \lambda \leq \lambda^* \ \forall \lambda \in I^X$*. In this case, the equality hold in both of (6) and (7) in Lemma 2.4, and thus the equality hold in both of (6) and (7) in Lemma 2.7.*

Moreover, as in the usual case, if R *is a reflexive and transitive fuzzy relation, then* $(\lambda_R)_R = \lambda_R$ *and* $(\lambda^R)^R = \lambda^R$*.*

If R *is considered to be a reflexive fuzzy relation on* X*, then a fuzzy pretopology* τ_R *on the rough fuzzy approximation space* (X, R) *is generated by the following:*

$$\tau_R = \{\nu \in I^X : \nu = \nu_R\} \quad or \quad \tau_R = \{\nu \in I^X : \nu^c = (\nu^c)^R\}. \tag{9}$$

That is, the condition $(\lambda_R)_R = \lambda_R \ \forall \lambda \in I^X$ *is not satisfied.*

If R *is considered to be a reflexive and transitive fuzzy relation on* X*, then a fuzzy topology* τ_R *is generated on the rough fuzzy approximation space by Equation 9 as well. That is, the condition* $(\lambda_R)_R = \lambda_R, \ \forall \lambda \in I^X$ *is satisfied.*

In the following, it will be defined a weaker definition than Definition 2.2.

Definition 2.9. *For every $x \in X$, define $\lambda_{**}, \lambda^{**} \in I^X$ of a fuzzy set $\lambda \in I^X$ by:*

$$\lambda_{**}(x) = \begin{cases} (\bigvee_{z\in X} <z> R\ (x))^c & \text{if } <x> R\ \wedge \lambda^c \notin \ell \text{ and } <x> R\ \wedge \lambda \notin \ell, \\ 1 & \text{if } <x> R\ \wedge \lambda^c \in \ell \\ 0 & \text{if } <x> R\ \wedge \lambda^c \notin \ell \text{ and } <x> R\ \wedge \lambda \in \ell \end{cases} \tag{10}$$

$$\lambda^{**}(x) = \begin{cases} \bigvee_{z\in X} <z> R\ (x) & \text{if } <x> R\ \wedge \lambda \notin \ell \text{ and } <x> R\ \wedge \lambda^c \notin \ell, \\ 0 & \text{if } <x> R\ \wedge \lambda \in \ell \\ 1 & \text{if } <x> R\ \wedge \lambda \notin \ell \text{ and } <x> R\ \wedge \lambda^c \in \ell \end{cases} \tag{11}$$

The roughness of a fuzzy set $\lambda \in I^X$ is defined by:

$$\underline{\lambda} = \lambda \wedge \lambda_{**} \quad \text{and} \quad \bar{\lambda} = \lambda \vee \lambda^{**}. \tag{12}$$

where $\underline{\lambda}$ is the lower fuzzy set of λ and $\bar{\lambda}$ is the upper fuzzy set of λ.

The boundary fuzzy region of λ is $B(\lambda)$ given by: $B(\lambda) = \bar{\lambda} \bar{\wedge} \underline{\lambda}$.

All the results given in the section are satisfied exactly, only the main difference is coming from Equation 4. That is, $R <x> R \ \leq \ <x> R \ \forall x \in X$. Hence,
Definition 2.2 gives us a boundary region fewer than that of Definition 2.9.

Remark 2.10. *It is clear by definitions that:*

(1) $\lambda_{**} \ \leq \ \lambda_*$, and then $\underline{\lambda} \ \leq \ \lambda_R$,

(2) $\lambda^* \ \leq \ \lambda^{**}$, and then $\lambda^R \ \leq \ \bar{\lambda}$.

Example 2.11. *From Example 2.6 in case of $\nu \in \ell \ \Leftrightarrow \ \nu \leq \overline{0.4}$, we get that*

$$\lambda^* = \{0.2, 0.2, 0.6, 0.5\}, \ \lambda_* = \{0.8, 0.8, 0.4, 0.5\},$$

and then $\lambda^R = \{0.2, 0.8, 0.6, 0.6\}$, $\lambda_R = \{0.1, 0.8, 0.4, 0.5\}$. That is, $\lambda^B = \{0.2, 0.2, 0.6, 0.5\}$, and moreover $\alpha(\lambda) = \{0.9, 1, 0.8, 0.9\}$, and thus $\mathrm{Inf}(\alpha(\lambda)) = 0.8$.

If we used Definition 2.9, we get that $\lambda^{**} = \{0.6, 0.2, 0.6, 0.5\}$, $\lambda_{**} = \{0.4, 0.8, 0.4, 0.5\}$, and then $\bar{\lambda} = \{0.6, 0.8, 0.6, 0.6\}$, $\underline{\lambda} = \{0.1, 0.8, 0.4, 0.5\}$. That is, $B(\lambda) = \{0.6, 0.2, 0.6, 0.5\}$. Hence, the boundary region of Definition 2.2 is better than the boundary region of Definition 2.9.

In case of $\nu \in \ell \ \Leftrightarrow \ \nu \leq \overline{0.5}$, we get that $\lambda_* = \{1, 0, 1, 1\}$, $\lambda^* = \overline{0}$, and then $\lambda^R = \{0.1, 0.8, 0.4, 0.6\}$, $\lambda_R = \{0.1, 0, 0.4, 0.6\}$. That is, $\lambda^B = \{0.1, 0.8, 0.4, 0.4\}$. Moreover, $\alpha(\lambda) = \{1, 0.2, 1, 1\}$ and $\mathrm{Inf}(\alpha(\lambda)) = 0.2$.

Note that: if we discussed the crisp case or the classical case, then we have R as a classical binary relation on X that has at least $R(x, y) = 1$ for some $x, y \in X$ and an ideal $\ell = \{\overline{0}\}$. Hence, the first branch in Equation 5 goes to zero, and the first branch in Equation 6 goes to one. Thus, Definitions 2.2 and Definition 2.9 will be as follow:

$$\lambda_*(x) = \begin{cases} 1 & \text{if } R <x> R\ \wedge \lambda^c = \overline{0} \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda^*(x) = \begin{cases} 1 & \text{if } R <x> R\ \wedge \lambda \neq \overline{0}, \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda_{**}(x) = \begin{cases} 1 & \text{if } <x> R\ \wedge \lambda^c = \overline{0} \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda^{**}(x) = \begin{cases} 1 & \text{if } <x> R\ \wedge \lambda \neq \overline{0}, \\ 0 & \text{otherwise.} \end{cases}$$

That means Definition 2.2 will be the same meaning of rough sets in the ordinary case using the intersection of after and before sets of a classical binary relation R on X. Also, Definition 2.9 will be the same meaning in the ordinary case only using the after sets of the classical binary relation R on X.

Remark 2.12. *From Equations 5, 6, we get that:*

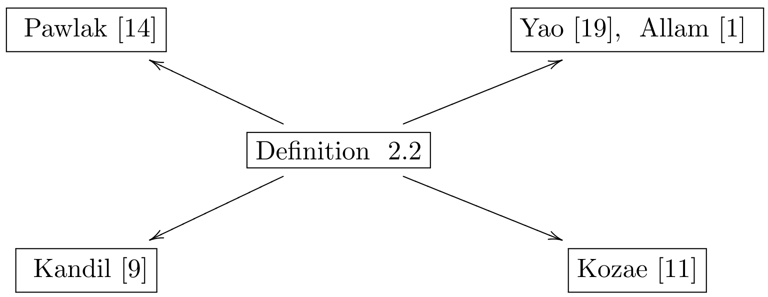
(1) *if we have an equivalence fuzzy relation R and $\ell = \ell^\circ$ on X, then our definition will be the fuzzification of the main definition given by Pawlak in* [14].

(2) *if we have a symmetric fuzzy relation R on X and the fuzzy ideal $\ell = \ell^\circ$ on X, then our definition will be the fuzzification of the definition given by Yao* [19] *and also the definition of Allam* [1].

(3) *if we have a reflexive and symmetric fuzzy relation R on X, then our definition will be the fuzzification of the definition given by Kandil* [9].

(4) *If we replace the fuzzy ideal ℓ on X by the fuzzy ideal $\ell = \ell^\circ$, then our definition will be the fuzzification of the definition given by Kozae* [11].

In the crisp case, we get exactly the definitions in the ordinary case as given in [1, 9, 11, 14, 19], *respectively.*



Example 2.13. *In Example 2.11, we computed λ_R, λ^R for a fuzzy set $\lambda = \{0.1, 0.8, 0.4, 0.6\}$ as follow:*

$$\lambda_R = \{0.1, 0.8, 0.4, 0.5\},\ \lambda^R = \{0.2, 0.8, 0.6, 0.6\},$$

and thus $\lambda^B = \{0.2, 0.2, 0.6, 0.5\}$.

If we restricte the fuzzy ideal ℓ to ℓ°, then the roughness computed for λ can not be restricted to a fuzzification of any of the previous definitions in [1, 9, 14] *because R here is not reflexive, not symmetric and not transitive. But, the roughness computed for λ is exactly the fuzzification of the definitions of* [11, 19].

Now, let R be a reflexive, symmetric not transitive fuzzy relation on X in the following.

R	a	b	c	d
a	1	0	0.1	0
b	0	1	0.1	0
c	0.1	0.1	1	0
d	0	0	0	1

Then, this example gives a fuzzification of the definition of [9]. *If $\ell = \ell^\circ$, then the roughness for λ will be the fuzzification of these computed with the definitions of [1, 11, 19] but not given in the sense of* [14].

If we suggested R as follows:

R	a	b	c	d
a	1	0.5	0.1	0
b	0.5	1	0.1	0
c	0.1	0.1	1	0
d	0	0	0	1

Then, R is an equivalence relation, and the roughness of λ will be the fuzzification of [9], *and moreover if $\ell = \ell^\circ$, then it will be the fuzzification of all definitions* [1, 11, 14, 19]. *Hence, according to the fuzzy relation and the fuzzy ideal, the computation of roughness of a fuzzy set is changed.*

For a fuzzy set λ in a rough fuzzy approximation space (X, R) where R is a reflexive fuzzy relation, we can define a Čech fuzzy interior operator $\text{int}_R\lambda \in I^X$ and a Čech fuzzy closure operator $\text{cl}_R\lambda \in I^X$ (that means $\text{int}_R(\text{int}_R\lambda) \neq \text{int}_R\lambda$) as follow:

$$\text{int}_R\lambda \;=\; \lambda_R, \quad \text{cl}_R\lambda \;=\; \lambda^R. \tag{13}$$

Also, from Equation 12, we can define for any $\lambda \in I^X$ a Čech fuzzy interior operator $\text{I}_R\lambda \in I^X$ and a Čech fuzzy closure operator $\text{C}_R\lambda \in I^X$ as follow:

$$\text{I}_R\lambda \;=\; \underline{\lambda}, \quad \text{C}_R\lambda \;=\; \bar{\lambda}. \tag{14}$$

It is clear from Remark 2.10 that: the Čech fuzzy interior operator int_R and the Čech fuzzy closure operator cl_R have the following properties related to the Čech fuzzy interior operator I_R and the Čech fuzzy closure operator C_R:

$$\text{int}_R\lambda \;\geq\; \text{I}_R\lambda \quad \text{and} \quad \text{cl}_R\lambda \leq \text{C}_R\lambda \quad \forall\lambda \in I^X. \tag{15}$$

Note that: the fuzzy operators cl_R , C_R of $\lambda = \{0.1, 0.8, 0.4, 0.6\}$ in Example 2.6 (where R was not reflexive) are not even Čech fuzzy closure operators while both are computed as

$$\text{cl}_R\lambda = \lambda^R = \{0.2, 0.8, 0.6, 0.6\} \leq \text{C}_R\lambda = \bar{\lambda} = \{0.6, 0.8, 0.6, 0.6\}.$$

Considering R a reflexive and transitive fuzzy relation, we have a fuzzy interior operator and a fuzzy closure operator on (X, R) generating a fuzzy topology τ_R as in Equation 9. In this case, the usual properties of fuzzy interior and fuzzy closure operators are satisfied as follow:

Lemma 2.14. *The following conditions are satisfied.*

(1) $\text{int}_R\overline{0} = \overline{0}$, $\text{int}_R\overline{1} = \overline{1}$,

(2) $\text{int}_R(\nu) \leq \nu \quad \forall\nu \in I^X$,

(3) $\nu \leq \eta \implies \text{int}_R(\nu) \leq \text{int}_R(\eta) \quad \forall\nu, \eta \in I^X$,

(4) $\text{int}_R(\nu \vee \eta) \geq \text{int}_R(\nu) \vee \text{int}_R(\eta), \quad \text{int}_R(\nu \wedge \eta) = \text{int}_R(\nu) \wedge \text{int}_R(\eta) \quad \forall\nu, \eta \in I^X$,

(5) $\text{int}_R(\text{int}_R(\nu)) = \text{int}_R(\nu) \ \forall\nu \in I^X$.

Proof. (1) and (2) are clear.
From (5) in Lemma 2.7, we get (3).
According to Remark 2.8, and from (3) and (7) in Lemma 2.7, we get (4).
Also, by Remark 2.8, and from (9) in Lemma 2.7, we get (5). □

Thus, $\text{int}_R\lambda$ is the fuzzy interior of λ in the rough fuzzy approximation space (X, R) generating a fuzzy topology defined by:

$$\varpi_R \;=\; \{\nu \in I^X : \;\; \nu \;=\; \text{int}_R(\nu)\}.$$

Note that $\text{I}_R\lambda$ of any $\lambda \in I^X$ in the rough fuzzy approximation space (X, R) with R as a reflexive and transitive fuzzy relation is also generating a fuzzy topology defined by:

$$\omega_R \;=\; \{\nu \in I^X : \;\; \nu \;=\; \text{I}_R(\nu)\},$$

and thus this is coarser than that one generated by $\text{int}_R\lambda$. That is, $\omega_R \leq \varpi_R$.

Note that:

$$\text{cl}_R(\nu^R) = \text{cl}_R(\nu) \ \forall\nu \in I^X, \quad \text{int}_R(\nu_R) = \text{int}_R(\nu) \ \forall\nu \in I^X,$$
$$\text{int}_R(\nu^c) \;=\; (\text{cl}_R(\nu))^c \quad \text{and} \quad \text{cl}_R(\nu^c) \;=\; (\text{int}_R(\nu))^c \ \forall\nu \in I^X.$$

Similarly, where R is a reflexive and transitive fuzzy relation on X we have the following:

Lemma 2.15. *The fuzzy closure operator satisfy the following conditions:*

(1) $\text{cl}_R\overline{0} = \overline{0}$, $\text{cl}_R\overline{1} = \overline{1}$,

(2) $\text{cl}_R(\nu) \geq \nu \ \forall\nu \in I^X$,

(3) $\nu \leq \eta \implies \mathrm{cl}_R(\nu) \leq \mathrm{cl}_R(\eta)\ \forall \nu, \eta \in I^X$,

(4) $\mathrm{cl}_R(\nu \wedge \eta) \leq \mathrm{cl}_R(\nu) \wedge \mathrm{cl}_R(\eta), \quad \mathrm{cl}_R(\nu \vee \eta) = \mathrm{cl}_R(\nu) \vee \mathrm{cl}_R(\eta) \quad \forall \nu, \eta \in I^X$,

(5) $\mathrm{cl}_R(\mathrm{cl}_R(\nu)) = \mathrm{cl}_R(\nu)\ \forall \nu \in I^X$.

Proof. Similar to Lemma 2.14. □

Hence, from $\mathrm{cl}_R(\nu^c) = (\mathrm{int}_R(\nu))^c$, cl_R is a fuzzy closure operator generating the same fuzzy topology given above as follows:

$$\varpi_R = \{\nu \in I^X : \ \nu^c = \mathrm{cl}_R(\nu^c)\}.$$

As an application of these generalized rough fuzzy sets, we discuss rough fuzzy connected spaces using fuzzy closure operators where R is reflexive and transitive fuzzy relation.

3 Connectedness in rough fuzzy approximation spaces

Definition 3.1. *Let (X, R) be a rough fuzzy approximation space and ℓ a fuzzy ideal on X. Then,*

(1) *the fuzzy sets $\mu, \nu \in I^X$ are called rough fuzzy approximation separated if*

$$\mathrm{cl}_R(\mu) \wedge \nu = \mu \wedge \mathrm{cl}_R(\nu) = \overline{0}.$$

(2) *A fuzzy set $\eta \in I^X$ is called rough fuzzy approximation disconnected (RF -disconnected) set if there exist rough fuzzy approximation separated sets $\mu, \nu \in I^X$, such that $\mu \vee \nu = \eta$. A fuzzy set η is called rough fuzzy approximation connected (RF -connected) if it is not rough fuzzy approximation disconnected (RF-disconnected). In other words, if there are no rough fuzzy approximation separated sets μ, ν except $\mu = \overline{0}$, or $\nu = \overline{0}$.*

(3) *(X, R) is called rough fuzzy approximation disconnected (RF-disconnected) space if there exist rough fuzzy approximation separated sets $\mu, \nu \in I^X$, such that $\mu \vee \nu = \overline{1}$. A fuzzy approximation space (X, R) is called rough fuzzy approximation connected (RF -connected) if it is not rough fuzzy approximation disconnected (RF-disconnected).*

Remark 3.2. *Any two rough fuzzy approximation separated sets μ, ν in I^X with respect to the fuzzy closure operator C_R defined by Equations 12, 14 are also rough fuzzy approximation separated sets as well from Equation 15. That is, rough fuzzy approximation disconnectedness with respect to the fuzzy closure operator C_R implies rough fuzzy approximation disconnectedness and thus, rough fuzzy approximation connectedness implies rough fuzzy approximation connectedness with respect to the fuzzy closure operator C_R.*

Example 3.3. *Let $X = \{a, b, c, d\}$, R a reflexive and transitive fuzzy relation defined by*

R	a	b	c	d
a	1	0	0.1	0
b	0	1	0.1	0
c	0	0	1	0
d	0	0	0.1	1

$<a>R = \{1, 0, 0.1, 0\}$, $<b>R = \{0, 1, 0.1, 0\}$, $<c>R = \{0, 0, 0.1, 0\}$, $<d>R = \{0, 0, 0.1, 1\}$ *and*
$R<a> = \{0.1, 0, 0, 0\}$, $R<b> = \{0, 0.1, 0, 0\}$, $R<c> = \{0.1, 0.1, 1, 0.1\}$, $R<d> = \{0, 0, 0, 0.1\}$. *Then,*
$R<a>R = \{0.1, 0, 0, 0\}$, $R<b>R = \{0, 0.1, 0, 0\}$, $R<c>R = \{0, 0, 0.1, 0\}$, $R<d>R = \{0, 0, 0, 0.1\}$.

Define a fuzzy ideal ℓ on X so that $\nu \in \ell \iff \nu \leq \overline{0.3}$, then for any $\lambda \in I^X$ we have $\lambda^* = \overline{0}$. That is, $\lambda^R = \mathrm{cl}_R\lambda = \lambda$ for any $\lambda \in I^X$. Hence, we can find $\lambda = \{0, 0, 0.2, 0.2\}$, $\mu = \{0.3, 0.3, 0, 0\}$ so that the fuzzy set $(\lambda \vee \mu) = \{0.3, 0.3, 0.2, 0.2\}$ is a rough fuzzy set for which $\mathrm{cl}_R\lambda \wedge \mu = \lambda \wedge \mathrm{cl}_R\mu = \lambda \wedge \mu = \overline{0}$. Thus, $\{0.3, 0.3, 0.2, 0.2\}$ is a rough fuzzy disconnected set.

The choice of R and ℓ played the main role of being $\lambda^* = \overline{0}$. That is, $\lambda^R = \mathrm{cl}_R\lambda = \lambda$ for any $\lambda \in I^X$, and so we could find a pair of rough fuzzy separated sets as shown above, and thus we found a fuzzy set which is a rough fuzzy disconnected.

If R is taken without these restrictions in the above choice, and ℓ is coarser enough, then we can not find any pair of rough fuzzy separated sets, and thus the whole space will be rough fuzzy connected space.

Remark 3.4. *Let (X,R) be a rough fuzzy approximation space. Then, (X,R) is a rough fuzzy connected which implies that it is a rough fuzzy connected with respect to the fuzzy closure operator C_R as defined in Equations 12, 14.*

Proposition 3.5. *Let (X,R) be a rough fuzzy approximation space and ℓ a fuzzy ideal on X. Then, the following are equivalent.*

(1) *(X,R) is rough fuzzy approximation connected.*

(2) *$\mu\wedge\nu=\overline{0}$, $\mathrm{int}_R(\mu)=\mu$, $\mathrm{int}_R(\nu)=\nu$ and $\mu\vee\nu=\overline{1}$ imply $\mu=\overline{0}$ or $\nu=\overline{0}$.*

(3) *$\mu\wedge\nu=\overline{0}$, $\mathrm{cl}_R(\mu)=\mu$, $\mathrm{cl}_R(\nu)=\nu$ and $\mu\vee\nu=\overline{1}$ imply $\mu=\overline{0}$ or $\nu=\overline{0}$.*

Proof. (1) ⇒ (2): Let $\mu,\nu\in I^X$ with $\mathrm{int}_R(\mu)=\mu$, $\mathrm{int}_R(\nu)=\nu$ such that $\mu\wedge\nu=\overline{0}$ and $\mu\vee\nu=\overline{1}$. Then, $\mathrm{cl}_R(\mu)=\mathrm{cl}_R(\nu^c)=\nu^c=\mu$, $\mathrm{cl}_R(\nu)=\mathrm{cl}_R(\mu^c)=\mu^c=\nu$.

Hence, $\mathrm{cl}_R(\mu)\wedge\nu=\mu\wedge\mathrm{cl}_R(\nu)=\mu\wedge\nu=\overline{0}$. That is, μ,ν are rough fuzzy approximation separated sets with $\mu\vee\nu=\overline{1}$. But (X,R) is rough fuzzy approximation connected which implies that $\mu=\overline{0}$ or $\nu=\overline{0}$.

(2) ⇒ (3): (3) ⇒ (1): Clear. □

Proposition 3.6. *Let (X,R) be a rough fuzzy approximation space and ℓ a fuzzy ideal on X. Then, for $\mu\in I^X$, the following are equivalent.*

(1) *μ is a rough fuzzy approximation connected set.*

(2) *If ν,ρ are rough fuzzy approximation separated sets with $\mu\le(\nu\vee\rho)$, then $\mu\wedge\nu=\overline{0}$ or $\mu\wedge\rho=\overline{0}$.*

(3) *If ν,ρ are rough fuzzy approximation separated sets with $\mu\le(\nu\vee\rho)$, then $\mu\le\nu$ or $\mu\le\rho$.*

Proof. (1) ⇒ (2): Let ν,ρ be rough fuzzy approximation separated sets with $\mu\le(\nu\vee\rho)$. That is, $\mathrm{cl}_R(\nu)\wedge\rho=\mathrm{cl}_R(\rho)\wedge\nu=\overline{0}$ with $\mu\le(\nu\vee\rho)$. Since

$$(\mathrm{cl}_R(\mu\wedge\nu)\wedge(\mu\wedge\rho) = \mathrm{cl}_R(\mu)\wedge\mathrm{cl}_R(\nu)\wedge(\mu\wedge\rho) = \mathrm{cl}_R(\mu)\wedge\mu\ \wedge\ \mathrm{cl}_R(\nu)\wedge\rho=\mu\wedge\overline{0}=\overline{0},$$

$$\mathrm{cl}_R(\mu\wedge\rho)\wedge(\mu\wedge\nu) = \mathrm{cl}_R(\mu)\wedge\mathrm{cl}_R(\rho)\wedge(\mu\wedge\nu) = \mathrm{cl}_R(\mu)\wedge\mu\ \wedge\ \mathrm{cl}_R(\rho)\wedge\nu=\mu\wedge\overline{0}=\overline{0}.$$

Then, $(\mu\wedge\nu)$ and $(\mu\wedge\rho)$ are rough fuzzy approximation separated sets with

$$\mu=(\mu\wedge\nu)\vee(\mu\wedge\rho).$$

But μ is rough fuzzy approximation connected, that is

$$\mu\wedge\nu=\overline{0}\text{ or }\mu\wedge\rho=\overline{0}.$$

(2) ⇒ (3): If $\mu\wedge\nu=\overline{0}$, $\mu\le(\nu\vee\rho)$, that is $\mu=\mu\wedge(\nu\vee\rho)=(\mu\wedge\nu)\vee(\mu\wedge\rho)=\mu\wedge\rho$, and thus $\mu\le\rho$. Also, if $\mu\wedge\rho=\overline{0}$, then $\mu\le\nu$.

(3) ⇒ (1): Let ν,ρ be rough fuzzy approximation separated sets with $\mu=\nu\vee\rho$. Then, $\mu\le\nu$ or $\mu\le\rho$.

If $\mu\le\nu$, then

$$\rho=(\nu\vee\rho)\wedge\rho=\mu\wedge\rho\le\nu\wedge\rho\ \le\ \mathrm{cl}_R(\nu)\wedge\rho=\overline{0}.$$

Also, if $\mu\le\rho$, then

$$\nu=(\nu\vee\rho)\wedge\nu=\mu\wedge\nu\le\rho\wedge\nu\ \le\ \mathrm{cl}_R(\rho)\wedge\nu=\overline{0}.$$

Hence, μ is a rough fuzzy approximation connected set. □

Define a rough fuzzy approximation continuous mapping $f:(X,R)\to(Y,R^*)$ by

$$\mathrm{cl}_R(f^{-1}(\nu))\le f^{-1}(\mathrm{cl}_{R^*}\nu)\quad\text{or}\quad \mathrm{int}_R(f^{-1}(\nu))\ge f^{-1}(\mathrm{int}_{R^*}\nu)\quad\forall\nu\in I^Y.$$

Theorem 3.7. *Let $(X,R),(Y,R^*)$ be rough fuzzy approximation spaces, ℓ,ℓ^* be fuzzy ideals on X,Y respectively and $f:(X,R)\to(Y,R^*)$ be a rough fuzzy approximation continuous mapping. Then, $f(\eta)\in I^Y$ is a rough fuzzy approximation connected set if η is a rough fuzzy approximation connected set in X.*

Proof. Let $\nu, \rho \in I^Y$ be rough fuzzy approximation separated sets with $f(\eta) = \nu \vee \rho$. That is,

$$\mathrm{cl}_{R^*}(\nu) \wedge \rho = \mathrm{cl}_{R^*}(\rho) \wedge \nu = \overline{0}.$$

Then, $\eta \leq (f^{-1}(\nu) \vee f^{-1}(\rho))$, and that f is rough fuzzy approximation continuous, we get that

$$\mathrm{cl}_R(f^{-1}(\nu)) \wedge f^{-1}(\rho) \leq f^{-1}(\mathrm{cl}_{R^*}(\nu)) \wedge f^{-1}(\rho) = f^{-1}(\mathrm{cl}_{R^*}(\nu) \wedge \rho) = f^{-1}(\overline{0}) = \overline{0},$$

and in a similar way, we have

$$\mathrm{cl}_R(f^{-1}(\rho)) \wedge f^{-1}(\nu) \leq f^{-1}(\mathrm{cl}_{R^*}(\rho)) \wedge f^{-1}(\nu) = f^{-1}(\mathrm{cl}_{R^*}(\rho) \wedge \nu) = f^{-1}(\overline{0}) = \overline{0}.$$

Hence, $f^{-1}(\nu)$ and $f^{-1}(\rho)$ are rough fuzzy approximation separated sets in X with $\eta \leq (f^{-1}(\nu) \vee f^{-1}(\rho))$. But from (3) in Proposition 3.6, we get that $\eta \leq f^{-1}(\nu)$ or $\eta \leq f^{-1}(\rho)$, which means that $f(\eta) \leq \nu$ or $f(\eta) \leq \rho$. Thus, from that η is rough fuzzy approximation connected set in X, and again from (3) in Proposition 3.6, we get that $f(\eta)$ is rough fuzzy approximation connected in Y. □

Definition 3.8. *Let (X, R) be a rough fuzzy approximation space and $\lambda \in I^X$. Then, λ is called rough fuzzy component (RF-component) if λ is a maximal RF-connected set in X, that is, if $\mu \geq \lambda$ and μ is RF-connected set, then $\lambda = \mu$.*

Proposition 3.9. *Let $\lambda \neq \overline{0}$ be RF-connected set in (X, R) and $\lambda \leq \mu \leq \mathrm{cl}_R(\lambda)$. Then, μ is RF-connected as well.*

Proof. Let ν, ρ be RF-separated sets in I^X such that $\mu = \nu \vee \rho$. That is,

$$\mathrm{cl}_R(\nu) \wedge \rho = \mathrm{cl}_R(\rho) \wedge \nu = \overline{0}.$$

Since $\lambda \leq \mu$ implies that $\lambda \leq (\nu \vee \rho)$ and λ is RF-connected, then from (3) in Proposition 3.6, we have $\lambda \leq \nu$ or $\lambda \leq \rho$.
From $\mu \leq \mathrm{cl}_R(\lambda)$ we get that:
If $\lambda \leq \nu$, then

$$\rho = (\nu \vee \rho) \wedge \rho = \mu \wedge \rho \leq \mathrm{cl}_R(\lambda) \wedge \rho \leq \mathrm{cl}_R(\nu) \wedge \rho = \overline{0}.$$

If $\lambda \leq \rho$, then

$$\nu = (\nu \vee \rho) \wedge \nu = \mu \wedge \nu \leq \mathrm{cl}_R(\lambda) \wedge \nu \leq \mathrm{cl}_R(\rho) \wedge \nu = \overline{0}.$$

Hence, μ is RF-connected. □

4 Conclusions

In this paper, it is given a generalization of rough fuzzy sets based on a fuzzy ideal ℓ defined on a fuzzy approximation space (X, R). Fuzzy interior and fuzzy closure operators are deduced in the general sense. In a proposed future work, a new generalization of rough fuzzy sets based on a fuzzy ideal ℓ will be introduced on a fuzzy approximation space (X, R) but in sense of Šostak. Thus, the proposed generalization will depend on the degrees $r \in [0, 1]$ and a new form of fuzzy relation R beside the fuzzy ideal ℓ where both are defined on the fuzzy approximation space.

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