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Original paper

Simplex algorithm for hesitant fuzzy linear programming problem with hesitant cost coefficient

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Abstract

In the real world, in most cases, such as industry, management, and even in daily life, we encounter optimization and decision-making problems that require the opinions of experts and masters on the problem to be able to make the best decision. In these cases, it is necessary to use an optimization problem with hesitant fuzzy parameters. There are few studies on hesitant fuzzy linear programming (HFLP) problems. Therefore, in this paper, we consider such problems. Especially, we study HFLP problems with hesitant cost coefficients. For this purpose, we propose the simplex method to solve the introduced optimization problems and draw a flowchart of the proposed method. Finally, by solving two illustrative examples with hesitant fuzzy information, we examine the applicability of the proposed method.

Keywords: Hesitant fuzzy linear programming, hesitant fuzzy number, ranking function, hesitant fuzzy simplex algorithm.

1 Introduction

These days, in most professions, such as industry, management, and even in everyday life, we deal with optimization and decision-making problems that do not have crisp parameters. We also need the opinions of various experts to solve them. In these cases, there is a wide range of tools that are able to deal with different types of uncertainties in many problems, such as type-2 fuzzy sets, interval-valued fuzzy sets, intuitionistic fuzzy sets, and hesitant fuzzy sets (HFSs). In these cases, it is very necessary and meaningful to study the optimization problems, especially linear programming (LP) problems with fuzzy parameters. For example, Tanaka, Okuda, and Asai [28] presented the concept of fuzzy mathematical programming at the general level. Zimmermann [42] was the first to formulate fuzzy linear programming (FLP) problems and later applied them to fuzzy environments [43], which then evolved qualitatively and continuously [22]. Ranking fuzzy numbers and their common methods were then reviewed by Wang and Kerre [32]. Gasimov and Yenilmez [9] discussed the solution to the FLP problems via linear ranking functions. Nehi, Maleki, and Mashinchi [23] proposed the lexicographic ranking function of the fuzzy numbers to solve FLP problems using fuzzy numbers. Buckley and Feuring [5] and Maleki [19] considered a kind of FLP problems and separately proposed an approach to solving them. Van Hop [31] solved LP problems under a fuzziness and randomness environment. Lotfi et al. [18] considered a fully fuzzy linear programming (FFLP) with triangular fuzzy numbers. Wu [33] studied optimality conditions for LP problems with fuzzy coefficients. Ezzati, Khorram, and Enayati [7] introduced a new algorithm to solve FFLP problems using the multiobjective linear programming problem. Baykasoglu and Subulan [3] suggested a different method based on the constricted fuzzy technical theory to solve FFLP.Another method to solve FLP problems is to use the simplex approach. Maleki, Tata, and Mashinchi [20] proved some theorems of LP problems for FLP problems and then proposed

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a primal fuzzy simplex algorithm to obtain the optimal solution to such problems. After that, Ebrahimnejad [6] proved that if an FLP problem with fuzzy cost coefficients has an optimal solution, then it also has a basic optimal solution.

HFSs, as the extension of the theory of fuzzy sets, have attracted the attention of many researchers in a short time because hesitant situations are very common in different real problems. In some cases, experts may disagree due to incomplete information about the values of the parameters. In these cases, we do not have only one membership degree for unknown parameters, but each parameter has several membership degrees. Torra and Narukawa [29, 30] presented HFSs that allow membership degrees to have a set of possible values. An HFS is proposed to describe the situations in which decision-makers are hesitant among several values to assess an indicator, alternative, variable, coefficient, and so on. A deep revision of the specialized recent literature shows the quick growth and applicability of HFSs, which have extended from different points of view. For example, in decision-making problems, the following studies can be mentioned [1, 8, 10, 11, 12, 13, 14, 15, 16, 17, 21, 34, 39, 41]. In the same way, in optimization problems, the following works can be mentioned [4, 24, 25, 26, 38, 40]. The first formulation of hesitant fuzzy linear programming (HFLP) problem was proposed by Ranjbar and Effati [24]. They suggested two new approaches to solve symmetric and right-hand-side HFLP problems. As new research, in this paper, we have limited ourselves to describing HFLP with hesitant cost coefficient using the simplex method. In the beginning, we introduce a suitable ranking for hesitant fuzzy numbers (HFNs) to compare them. Then we prove the optimality of the answers. Finally, we introduce two examples to describe and compare the proposed approach.

The rest of the article is organized as follows: In Section 2, we review some of the required concepts of the article. In Section 3, we state and prove theorems related to the optimality of the HFLP model. In Section 4, we propose an algorithm for the new approach of HFLP with HFNs. In Section 5, we solve two examples to show the applicability of the proposed technique. In Section 6, discussion and comparative analyses of the new approach are given. Finally, in Section 7, we state some of the conclusions obtained.

2 Preliminaries

In the present section, the information needed for the next topics will be stated.

Definition 2.1. [35] *Let S be a set of objects denoted by s. Then an HFS $\tilde{\tilde{H}}$ in S is a set of ordered pairs as follows:*

$$\tilde{\tilde{H}} = \{\big(s, h_{\tilde{\tilde{H}}}(s)\big) \ \mid \ s \in S\},$$

which is the values of membership degree $h_{\tilde{\tilde{H}}}(s)$ in $[0,1]$.

Definition 2.2. [2] *Let S be an infinite set of objects indicated by s. Then there is a uniformly hesitant fuzzy set (UHFS) $\tilde{\tilde{U}}$ in S, if there exists a number p such that $l(s) \leq p$ for every $s \in S$.*

Let S be a set and let $\tilde{\tilde{\mathrm{U}}} = \{\big(s, h_{\tilde{\tilde{\mathrm{U}}}}(s)\big) \mid s \in S\}$ be a UHFS with $Char(h_{\tilde{\tilde{\mathrm{U}}}}) = p$. Then we decompose $h_{\tilde{\tilde{U}}}(s) = \{h^{\sigma(1)}_{\tilde{\tilde{U}}}(s), \ldots, h^{\sigma(l(s))}_{\tilde{\tilde{U}}}(s)\}$, where $h^{\sigma(1)}_{\tilde{\tilde{U}}}(s) < \cdots < h^{\sigma(l(s))}_{\tilde{\tilde{U}}}(s)$ and $l(s) \leq p$ for all $s \in S$ (see [2]).

Definition 2.3. [26] *Let S be infinite as a special type of HFSs and let $\tilde{\tilde{N}} = \{\tilde{N}^j\}_{j=1}^p$ be a UHFS with $Char(h_{\tilde{\tilde{N}}}) = p$. If*

1. *$\tilde{N}^j \in F$, for all $j = 1, \ldots, p$. (F is a space of fuzzy numbers)*
2. *$\bigcap_{j=1}^p \tilde{N}_1^j \neq \emptyset$, ($\tilde{N}_1^j$ is a 1-cut for the jth element of the UHFS $\tilde{\tilde{N}}$)*

then we call $\tilde{\tilde{N}}$ an HFN.

Remark 2.4. *In this paper, for calculations of principal operators, we use the symbol $\underset{\approx}{*}$ for HFNs and the symbol $\underset{\sim}{*}$ for fuzzy numbers. We consider all trapezoidal hesitant fuzzy numbers (THFNs) as (a, b, c, d) with $a \leq b \leq c \leq d$. Note that if $b = c$, then we will have a triangular HFN.*
Also, we use the symbol ($\lesssim$ & $\gtrsim$) to indicate fuzzy inequalities and the symbol ($\underset{\approx}{<}$ & $\underset{\approx}{>}$) to represent a hesitant fuzzy inequalities.

Remark 2.5. *Let $X = \mathbb{R}$, and take the HFN $\tilde{\tilde{N}}$ as follows:*

$$\tilde{\tilde{N}} = \{\tilde{N}^{\sigma(j)}\}_{j=1}^{p} = \{\tilde{N}^1, \tilde{N}^2, \ldots, \tilde{N}^p\},$$

where $\tilde{N}^j = (a_j, b_j, c_j, d_j)$ for $j = 1, \ldots, p$ are trapezoidal fuzzy numbers. Then

1. $\tilde{N}^j \gtrsim 0$ *if* $a_j > 0,\ j = 1, \ldots, p,$
2. $\tilde{N}^j \lesssim 0$ *if* $d_j < 0,\ j = 1, \ldots, p.$

Based on, we define

1. $\tilde{\tilde{N}} \gtrapprox 0$ *if* $\tilde{N}^j \gtrsim 0$, *for all* $j = 1, \ldots, p,$
2. $\tilde{\tilde{N}} \lessapprox 0$ *if* $\tilde{N}^j \lesssim 0$, *for all* $j = 1, \ldots, p.$

Definition 2.6. [27] *Let $\tilde{\tilde{N}} = \{\tilde{N}_1^{\sigma(j)}\}_{j=1}^{p}$ and $\tilde{\tilde{V}} = \{\tilde{V}_1^{\sigma(j)}\}_{j=1}^{p}$ be HFNs . Then $\tilde{\tilde{N}} \underset{\approx}{*} \tilde{\tilde{V}}$ on R is defined as follows:*

$$\tilde{\tilde{N}} \underset{\approx}{*} \tilde{\tilde{V}}(z) = \bigcup_{j=1,\ldots,p} \{\sup_{z=x*y} \{\min\{\mu_{\tilde{N}^{\sigma(j)}}(x), \mu_{\tilde{V}^{\sigma(j)}}(y)\}\}\}, \tag{1}$$

for all $z \in R$. For availability, we can denote (1) *as follows:*

$$\tilde{\tilde{N}} \underset{\approx}{*} \tilde{\tilde{V}}(z) = \{\tilde{N}^{\sigma(j)} \underset{\sim}{*} \tilde{V}^{\sigma(j)}\}_{j=1}^{p}.$$

In the special case when numbers are THFNs, we have $\tilde{\tilde{\mathrm{N}}} = \{\tilde{N}^1, \tilde{N}^2, \ldots, \tilde{N}^r\}$ and $\tilde{\tilde{\mathrm{V}}} = \{\tilde{V}^1, \tilde{V}^2, \ldots, \tilde{V}^s\}$ in which r and s are the number of experts. For simplicity, we assume that $r = s = p$. Then

$$\tilde{\tilde{N}} \underset{\approx}{\oplus} \tilde{\tilde{V}} = \{\tilde{N}^1 \underset{\sim}{\oplus} \tilde{V}^1, \tilde{N}^2 \underset{\sim}{\oplus} \tilde{V}^2, \ldots, \tilde{N}^p \underset{\sim}{\oplus} \tilde{V}^p\}, \tag{2}$$

$$\tilde{\tilde{N}} \underset{\approx}{\ominus} \tilde{\tilde{V}} = \{\tilde{N}^1 \underset{\sim}{\ominus} \tilde{V}^1, \tilde{N}^2 \underset{\sim}{\ominus} \tilde{V}^2, \ldots, \tilde{N}^p \underset{\sim}{\ominus} \tilde{V}^p\}, \tag{3}$$

$$\tilde{\tilde{N}} \underset{\approx}{\otimes} \tilde{\tilde{V}} = \{\tilde{N}^1 \underset{\sim}{\otimes} \tilde{V}^1, \tilde{N}^2 \underset{\sim}{\otimes} \tilde{V}^2, \ldots, \tilde{N}^p \underset{\sim}{\otimes} \tilde{V}^p\}, \tag{4}$$

$$\tilde{\tilde{N}} \underset{\approx}{\oslash} \tilde{\tilde{V}} = \{\tilde{N}^1 \underset{\sim}{\oslash} \tilde{V}^1, \tilde{N}^2 \underset{\sim}{\oslash} \tilde{V}^2, \ldots, \tilde{N}^p \underset{\sim}{\oslash} \tilde{V}^p\}, \tag{5}$$

$$k\tilde{\tilde{N}} = \{k\tilde{N}^1, k\tilde{N}^2, \ldots, k\tilde{N}^p\}. \tag{6}$$

The basic computational operators for trapezoidal fuzzy numbers $\tilde{N}^j = (a_j, b_j, c_j, d_j)$ and $\tilde{V}^j = (e_j, f_j, g_j, h_j)$ for all $j = 1, \ldots, p$ are as follows:

$$-\tilde{N}^j = (-d_j, -c_j, -b_j, -a_j), \tag{7}$$

$$(\tilde{N}^j)^{-1} = \frac{1}{\tilde{N}^j} = (\frac{1}{d_j}, \frac{1}{c_j}, \frac{1}{b_j}, \frac{1}{a_j}), \tag{8}$$

$$\tilde{N}^j \underset{\sim}{\oplus} \tilde{V}^j = (a_j + e_j, b_j + f_j, c_j + g_j, d_j + h_j), \tag{9}$$

$$\tilde{N}^j \underset{\sim}{\ominus} \tilde{V}^j = (a_j - h_j, b_j - g_j, c_j - f_j, d_j - e_j), \tag{10}$$

$$k\tilde{N}^j = (ka_j, kb_j, kc_j, kd_j), \qquad k > 0, \tag{11}$$

$$k\tilde{N}^j = (kd_j, kc_j, kb_j, ka_j), \qquad k < 0, \tag{12}$$

$$\tilde{N}^j \underset{\sim}{\otimes} \tilde{V}^j = \begin{cases} (a_je_j, b_jf_j, c_jg_j, d_jh_j), & \tilde{N}^j \gtrsim 0, \tilde{V}^j \gtrsim 0, \\ (d_je_j, c_jf_j, b_jg_j, a_jh_j), & \tilde{N}^j \gtrsim 0, \tilde{V}^j \lesssim 0, \\ (a_jh_j, b_jg_j, c_jf_j, d_je_j), & \tilde{N}^j \lesssim 0, \tilde{V}^j \gtrsim 0, \\ (d_jh_j, c_jg_j, b_jf_j, a_je_j), & \tilde{N}^j \lesssim 0, \tilde{V}^j \lesssim 0, \end{cases} \tag{13}$$

$$\tilde{N}^j \oslash \tilde{V}^j = \begin{cases} (\frac{a_j}{h_j}, \frac{b_j}{g_j}, \frac{c_j}{f_j}, \frac{d_j}{e_j}), & \tilde{N}^j \gtrsim 0, \tilde{V}^j \gtrsim 0, \\ (\frac{d_j}{h_j}, \frac{c_j}{g_j}, \frac{b_j}{f_j}, \frac{a_j}{e_j}), & \tilde{N}^j \gtrsim 0, \tilde{V}^j \lesssim 0, \\ (\frac{a_j}{e_j}, \frac{b_j}{f_j}, \frac{c_j}{g_j}, \frac{d_j}{h_j}), & \tilde{N}^j \lesssim 0, \tilde{V}^j \gtrsim 0, \\ (\frac{d_j}{e_j}, \frac{c_j}{f_j}, \frac{b_j}{g_j}, \frac{a_j}{h_j}), & \tilde{N}^j \lesssim 0, \tilde{V}^j \lesssim 0, \end{cases} \tag{14}$$

Remark 2.7. *Sometimes, two HFNs have unequal cardinals. Then the shorter one should be extended so that both reach the same length when we operate arithmetic operations on them. In order to extend the shorter one, adding the same value multiple times to it is the best way. Actually, the shorter one can be extended by adding any value in it; see* [36]. *In this study, adding the maximum value in such conditions is proposed.*

Definition 2.8. *Let $\tilde{\tilde{N}} = (\tilde{N}^1, \tilde{N}^2, \ldots, \tilde{N}^p)$ be an HFN such that $\tilde{N}^j = (a_j, b_j, c_j, d_j)$, $j = 1, 2, \ldots, p$, are trapezoidal fuzzy numbers. We introduce a ranking HFN $\tilde{\tilde{N}}$ as follows:*

$$\Re(\tilde{\tilde{N}}) = \frac{1}{4p} \sum_{j=1}^{p} (a_j + b_j + c_j + d_j). \tag{15}$$

In the above definition, if $p = 1$, then the defined ranking will be the same as the Yager [37] ranking for trapezoidal fuzzy numbers.

Definition 2.9. *Let $\tilde{\tilde{N}}^1$ and $\tilde{\tilde{N}}^2$ be THFNs with n fuzzy number members that ordered by a ranking function $\Re$. Then we say $\tilde{\tilde{N}}^1 \lessapprox \tilde{\tilde{N}}^2$ if and only if $\Re(\tilde{\tilde{N}}^1) \leq \Re(\tilde{\tilde{N}}^2)$.*

In the next section, we want to solve LP problems whose objective function coefficients are HFNs.

3 Solving HFLP problem with hesitant cost coefficient

Consider an LP problem whose cost coefficients are THFNs. The general matrix form of HFLP problems is as follows:

$$\begin{aligned} \max \quad & \tilde{\tilde{Z}} = \tilde{\tilde{c}} x \\ \text{s.t.} \quad & Ax \leq b, \\ & x \geq 0, \end{aligned} \tag{16}$$

where $\tilde{\tilde{c}}$ is a hesitant fuzzy element, $b = (b_1, b_2, \ldots, b_m)^T \in \mathbb{R}^m$, $A = (a_{ij})_{m\times n} \in \mathbb{R}^{m\times n}$, and $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$. In the system of constraints (16), $rank(A) = m$ and it can be partitioned as $A = [B, N]$, where $B_{m\times m}$ is a basic matrix with $rank(B) = m$ and N is a nonbasic matrix.
In the following, we intend to generalize the simplex algorithm to HFLP problems with hesitant cost coefficients. For this purpose, we first introduce some required symbols. Then we state and prove theorems related to the optimality, optimal solution, and infinite solution for the HFLP model. Finally, we generalize the simplex algorithm to solve such problems.

Definition 3.1. *The solution $x = (x_B, x_N)^T = (B^{-1}b, 0)^T$ is called a basic solution to $Ax = b$. If $x_B \geq 0$, then this basic solution is primal feasible, and the hesitant fuzzy objective function value is obtained as $\tilde{\tilde{Z}} = \tilde{\tilde{c}}_B x_B$, where $\tilde{\tilde{c}}_B = (\tilde{\tilde{c}}_{B_1}, \tilde{\tilde{c}}_{B_2}, \ldots, \tilde{\tilde{c}}_{B_m})^T$.*
Suppose that the set of indices of nonbasic variables is denoted by J_N. For each nonbasic variable x_j, $j \in J_N$, the hesitant fuzzy variable $\tilde{\tilde{z}}_j$ is defined as $\tilde{\tilde{z}}_j = \tilde{\tilde{c}}_B B^{-1} a_j = \tilde{\tilde{c}}_B y_j$, where y_j is the solution to $By_j = a_j$.

In the following, some theorems are stated to investigate the optimal solution to the HFLP problem (16).

Theorem 3.2. *Let X^* be a basic feasible solution with basis B and objective value $\tilde{\tilde{Z}}$. For some nonbasic variable x_k while $y_k \nleq 0$, let $\tilde{\tilde{z}}_k \lessapprox \tilde{\tilde{c}}_k$. Then there is a new feasible solution with objective value $\tilde{\tilde{Z}}_{new} = \tilde{\tilde{Z}} \ominus (\tilde{\tilde{z}}_k \ominus \tilde{\tilde{c}}_k) x_k$, such that $\tilde{\tilde{Z}} \lessapprox \tilde{\tilde{Z}}_{new}$.*

Proof. Let $x = (x_B, x_N)^T = (B^{-1}b, 0)^T$ be the basic feasible solution to the matrix B. The basic variables and the hesitant fuzzy objective function in terms of nonbasic variables x_N read as follows:

$$x_B = B^{-1}b - B^{-1}Nx_N, \tag{17}$$

$$\tilde{\tilde{Z}} = \tilde{\tilde{c}}_B x_B \underset{\approx}{\oplus} \tilde{\tilde{c}}_N x_N = \tilde{\tilde{c}}_B(B^{-1}b - B^{-1}Nx_N) \underset{\approx}{\oplus} \tilde{\tilde{c}}_N x_N = \tilde{\tilde{c}}_B B^{-1}b \underset{\approx}{\ominus} (\tilde{\tilde{c}}_B B^{-1}N \underset{\approx}{\ominus} \tilde{\tilde{c}}_N)x_N.$$

Using the new solution

$$\begin{aligned} &x_{Br} = x_k = \theta = \frac{\bar{b}_r}{y_{rk}} = \min_{1\le i\le m}\{\frac{\bar{b}_i}{y_{ik}} | y_{ik} > 0\}, \\ &x_{Bi} = \bar{b}_i - y_{ik}\frac{\bar{b}_r}{y_{rk}}, \quad i = 1, 2, \dots, m, i \neq r, \\ &x_j = 0, \qquad j \in J_N, j \neq k, \end{aligned} \tag{18}$$

the value of the new hesitant fuzzy objective function is obtained as follows:

$$\tilde{\tilde{Z}}_{\text{new}} = \tilde{\tilde{c}}_B x_B \underset{\approx}{\oplus} \tilde{\tilde{c}}_N x_N = \textstyle\sum_{i=1, i\neq r}^{m} \left(\tilde{\tilde{c}}_{B_i}(\bar{b}_i - y_{ik}\frac{\bar{b}_r}{y_{rk}})\right) \underset{\approx}{\oplus} \tilde{\tilde{c}}_k \frac{\bar{b}_r}{y_{rk}} = \textstyle\sum_{i=1, i\neq r}^{m} \left(\tilde{\tilde{c}}_{B_i}(\bar{b}_i - y_{ik}\theta)\right) \underset{\approx}{\oplus} \tilde{\tilde{c}}_k \theta. \tag{19}$$

Since $\tilde{\tilde{c}}_{B_i}$ are THFNs, we have

$$\begin{aligned} \tilde{\tilde{Z}}_{\text{new}} &= \textstyle\sum_{i=1, i\neq r}^{m} \left\{\tilde{c}^1_{B_i}, \tilde{c}^2_{B_i}, \tilde{c}^3_{B_i}, \dots, \tilde{c}^p_{B_i}\right\}(\bar{b}_i - y_{ik}\theta) \underset{\approx}{\oplus} \left\{\tilde{c}^1_k, \tilde{c}^2_k, \tilde{c}^3_k, \dots, \tilde{c}^p_k\right\}\theta \\ &= \textstyle\sum_{i=1, i\neq r}^{m} \left\{(c^1_{1B_i}, c^1_{2B_i}, c^1_{3B_i}, c^1_{4B_i}), \dots, (c^p_{1B_i}, c^p_{2B_i}, c^p_{3B_i}, c^p_{4B_i})\right\}(\bar{b}_i - y_{ik}\theta) \\ &\quad \underset{\approx}{\oplus} \left\{(c^1_{1k}, c^1_{2k}, c^1_{3k}, c^4_{4k}), \dots, (c^p_{1k}, c^p_{2k}, c^p_{3k}, c^p_{4k})\right\}\theta \\ &= \textstyle\sum_{i=1, i\neq r}^{m} \left\{(c^1_{1B_i}, c^1_{2B_i}, c^1_{3B_i}, c^1_{4B_i})(\bar{b}_i - y_{ik}\theta), \dots, (c^p_{1B_i}, c^p_{2B_i}, c^p_{3B_i}, c^p_{4B_i})(\bar{b}_i - y_{ik}\theta)\right\} \\ &\quad \underset{\approx}{\oplus} \left\{(c^1_{1k}, c^1_{2k}, c^1_{3k}, c^4_{4k})\theta, \dots, (c^p_{1k}, c^p_{2k}, c^p_{3k}, c^p_{4k})\theta\right\}. \end{aligned} \tag{20}$$

By simplifying (20), we get

$$\begin{aligned} \tilde{\tilde{Z}}_{new} = \; &\Big\{(z^1_1 - \theta z'^1_{1k} + \theta c^1_{1k}, z^1_2 - \theta z'^1_{2k} + \theta c^1_{2k}, z^1_3 - \theta z'^1_{3k} + \theta c^1_{3k}, z^1_4 - \theta z'^1_{4k} + \theta c^1_{4k}), \\ &\dots, (z^p_1 - \theta z'^p_{1k} + \theta c^p_{1k}, z^p_2 - \theta z'^p_{2k} + \theta c^p_{2k}, z^p_3 - \theta z'^p_{3k} + \theta c^p_{3k}, z^p_4 - \theta z'^p_{4k} + \theta c^p_{4k})\Big\}, \end{aligned} \tag{21}$$

where for $t = 1, \dots, p$

$$z^t_j = \textstyle\sum_{i=1}^{m} c^t_{jB_i}\bar{b}_i, \qquad j = 1, 2, 3, 4, \tag{22}$$

and

$$z'^t_{jk} = \textstyle\sum_{i=1}^{m} c^t_{jB_i} y_{ik}, \qquad j = 1, 2, 3, 4. \tag{23}$$

In (21), it is clear that if $\theta = 0$, then

$$\tilde{\tilde{Z}}_{new} = \left\{(z^1_1, z^1_2, z^1_3, z^1_4), (z^2_1, z^2_2, z^2_3, z^2_4), \dots, (z^p_1, z^p_2, z^p_3, z^p_4)\right\} = \left\{\tilde{z}^1, \tilde{z}^2, \dots, \tilde{z}^p\right\} = \tilde{\tilde{Z}}.$$

On the other side, in a situation where $\theta > 0$, we must constitute the proof of $\Re(\tilde{\tilde{Z}}_{new}) \geq \Re(\tilde{\tilde{Z}})$, where $\Re$ is the ranking function in Definition 2.8.

For all nonbasic variable $x_j, j \in J_N$, we write the hesitant fuzzy variable $\tilde{\tilde{z}}_j$ as

$$\tilde{\tilde{z}}_j = \tilde{\tilde{c}}^T_B B^{-1}a_j = \tilde{\tilde{c}}^T_B y_j = \sum_{i=1}^{m}\{\tilde{c}^1_{B_i}, \tilde{c}^2_{B_i}, \dots, \tilde{c}^p_{B_i}\}y_{ij}.$$

Since $\tilde{c}^j_{B_i}$ for $j = 1, 2, \ldots, p$ are trapezoidal fuzzy numbers, we have

$$\begin{aligned}\tilde{\tilde{z}}_j &= \sum_{i=1}^{m} \Big\{(c^1_{1B_i}, c^1_{2B_i}, c^1_{3B_i}, c^1_{4B_i}), (c^2_{1B_i}, c^2_{2B_i}, c^2_{3B_i}, c^2_{4B_i}), \ldots, (c^p_{1B_i}, c^p_{2B_i}, c^p_{3B_i}, c^p_{4B_i})\Big\} y_{ij} \\ &= \sum_{i, y_{ij} \geq 0} \Big\{(c^1_{1B_i}, c^1_{2B_i}, c^1_{3B_i}, c^1_{4B_i}) y_{ij}, (c^2_{1B_i}, c^2_{2B_i}, c^2_{3B_i}, c^2_{4B_i}) y_{ij}, \ldots, (c^p_{1B_i}, c^p_{2B_i}, c^p_{3B_i}, c^p_{4B_i}) y_{ij}\Big\} \\ &\underset{\approx}{\oplus} \sum_{i, y_{ij} < 0} \Big\{(c^1_{4B_i}, c^1_{3B_i}, c^1_{2B_i}, c^1_{1B_i}) y_{ij}, (c^2_{4B_i}, c^2_{3B_i}, c^2_{2B_i}, c^2_{1B_i}) y_{ij}, \ldots, (c^p_{4B_i}, c^p_{3B_i}, c^p_{2B_i}, c^p_{1B_i}) y_{ij}\Big\}.\end{aligned}$$

By simplifying it, we get

$$\tilde{\tilde{z}}_j = \Big\{(z^1_{1j}, z^1_{2j}, z^1_{3j}, z^1_{4j}), (z^2_{1j}, z^2_{2j}, z^2_{3j}, z^2_{4j}), \ldots, (z^p_{1j}, z^p_{2j}, z^p_{3j}, z^p_{4j})\Big\} = \Big\{\tilde{z}^1_j, \tilde{z}^2_j, \tilde{z}^3_j, \ldots, \tilde{z}^p_j\Big\},$$

where for $t = 1, \ldots, p$

$$\begin{aligned} z^t_{1j} &= \sum_{\{i: y_{ij} \geq 0\}} c^t_{1B_i} y_{ij} + \sum_{\{i: y_{ij} < 0\}} c^t_{4B_i} y_{ij}, \\ z^t_{2j} &= \sum_{\{i: y_{ij} \geq 0\}} c^t_{2B_i} y_{ij} + \sum_{\{i: y_{ij} < 0\}} c^t_{3B_i} y_{ij}, \\ z^t_{3j} &= \sum_{\{i: y_{ij} \geq 0\}} c^t_{3B_i} y_{ij} + \sum_{\{i: y_{ij} < 0\}} c^t_{2B_i} y_{ij}, \\ z^t_{4j} &= \sum_{\{i: y_{ij} \geq 0\}} c^t_{4B_i} y_{ij} + \sum_{\{i: y_{ij} < 0\}} c^t_{1B_i} y_{ij}. \end{aligned} \tag{24}$$

Now, regarding Eqs. (23) and (24), for all $t = 1, \ldots, p$, we have

$$\begin{aligned} z^t_{1k} + z^t_{4k} &= \sum_{i: y_{ij} \geq 0} c^t_{1B_i} y_{ik} + \sum_{i: y_{ij} < 0} c^t_{4B_i} y_{ik} + \sum_{i: y_{ij} \geq 0} c^t_{4B_i} y_{ik} + \sum_{i: y_{ij} < 0} c^t_{1B_i} y_{ik} \\ &= \Big(\sum_{i: y_{ij} \geq 0} c^t_{1B_i} y_{ik} + \sum_{i: y_{ij} < 0} c^t_{1B_i} y_{ik}\Big) + \Big(\sum_{i: y_{ij} < 0} c^t_{4B_i} y_{ik} + \sum_{i: y_{ij} \geq 0} c^t_{4B_i} y_{ik}\Big) \\ &= z'^t_{1k} + z'^t_{4k}. \end{aligned} \tag{25}$$

Similarly, examining Eq. (25), we have

$$z^t_{1k} + z^t_{2k} + z^t_{3k} + z^t_{4k} = z'^t_{1k} + z'^t_{2k} + z'^t_{3k} + z'^t_{4k} \qquad \text{for all } t = 1, \ldots, p. \tag{26}$$

From the other side, since $\tilde{\tilde{z}}_k \lessapprox \tilde{\tilde{c}}_k$ and thus $\Re(\tilde{\tilde{z}}_k) \leq \Re(\tilde{\tilde{c}}_k)$, and by considering Eq. (26), we get

$$\frac{1}{4p} \sum_{t=1}^{p} \Big[(c^t_{1k} - z'^t_{1k}) + (c^t_{2k} - z'^t_{2k}) + (c^t_{3k} - z'^t_{3k}) + (c^t_{4k} - z'^t_{4k})\Big] > 0. \tag{27}$$

Now, according to Eq. (21), we have

$$\Re(\tilde{\tilde{Z}}_{new}) = \Re\Bigg(\tilde{z}^1 \underset{\sim}{\ominus} \theta \tilde{z}'^{\,1}_k \underset{\sim}{\oplus} \theta \tilde{c}_k{}^1, \tilde{z}^2 \underset{\sim}{\ominus} \theta \tilde{z}'^{\,2}_k \underset{\sim}{\oplus} \theta \tilde{c}_k{}^2, \ldots, \tilde{z}^p \underset{\sim}{\ominus} \theta \tilde{z}'^{\,p}_k \underset{\sim}{\oplus} \theta \tilde{c}_k{}^p\Bigg).$$

By using Definition 2.8, we have

$$\Re(\tilde{\tilde{Z}}_{new}) = \frac{1}{4p} \sum_{t=1}^{p} \Big[({z_1}^t - \theta {z'_{1k}}^t + \theta {c_{1k}}^t) + ({z_2}^t - \theta {z'_{2k}}^t + \theta {c_{2k}}^t) + (z^t_3 - \theta z'^t_{3k} + \theta c^t_{3k}) + (z^t_4 - \theta z'^t_{4k} + \theta c^t_{4k})\Big].$$

Simplifying implies that

$$\begin{aligned} \Re(\tilde{\tilde{Z}}_{new}) &= \frac{1}{4p} \sum_{t=1}^{p} \Big[(z^t_1 + z^t_2 + z^t_3 + z^t_4) + \theta\Big((c^t_{1k} - z'^t_{1k}) + (c^t_{2k} - z'^t_{2k}) + (c^t_{3k} - z'^t_{3k}) + (c^t_{4k} - z'^t_{4k})\Big)\Big] \\ &= \Re(\tilde{\tilde{z}}) + \frac{\theta}{4p} \sum_{t=1}^{p} \Big[(c^t_{1k} - z'^t_{1k}) + (c^t_{2k} - z'^t_{2k}) + (c^t_{3k} - z'^t_{3k}) + (c^t_{4k} - z'^t_{4k})\Big]. \end{aligned} \tag{28}$$

Therefore, regarding Eq. (27), it can be concluded that $\Re(\tilde{\tilde{Z}}_{new}) \geq \Re(\tilde{\tilde{Z}})$, that is, $\tilde{\tilde{Z}}_{new} \gtrapprox \tilde{\tilde{Z}}$. □

The next theorem states when the optimal solution to the HFLP problem is unbounded.

Theorem 3.3. *Let X^* be a basic feasible solution with basis B and objective value $\tilde{\tilde{z}}$. Then the optimal solution to the HFLP problem (16) is unbounded, if for some nonbasic variables x_k we have $\tilde{\tilde{z}}_k \lessapprox \tilde{\tilde{c}}_k$ while $y_k \leq 0$.*

Proof. In Eq. (28), $\Re(\tilde{\tilde{Z}}_{new}) = \Re(\tilde{\tilde{Z}}) + \theta\Re(\tilde{\tilde{c}}_k - \tilde{\tilde{z}}'_k)$. Since $\tilde{\tilde{z}}_k \lessapprox \tilde{\tilde{c}}_k$, thus $\Re(\tilde{\tilde{c}}_k - \tilde{\tilde{z}}_k) > 0$. This proves that in the situation $\theta > 0, \Re(\tilde{\tilde{Z}}_{new})$ can be as large as desired, and therefore, the optimal solution to HFLP problem (16) is unbounded. □

Theorem 3.4. *The basic solution $x = (x_B, x_N)^T = (B^{-1}b, 0)^T$ is an optimal solution to the HFLP problem (16) if $\tilde{\tilde{z}}_j \gtrapprox \tilde{\tilde{c}}_j$ for all $j \in J_N$.*

Proof. Let $x = (x_B, x_N)^T = (B^{-1}b, 0)^T$ be a feasible basic solution and let $\tilde{\tilde{Z}}_0 = \tilde{\tilde{c}}_B B^{-1}b = \tilde{\tilde{c}}_B \bar{b}$ be the hesitant fuzzy objective function value. In Eq. (10), we obtain $\tilde{\tilde{Z}} = \tilde{\tilde{Z}}_0 \underset{\approx}{\ominus} \sum_{j\in J_N}(\tilde{\tilde{z}}_j \underset{\approx}{\ominus} \tilde{\tilde{c}}_j)x_j$. Because $\tilde{\tilde{z}}_j \gtrapprox \tilde{\tilde{c}}_j$, that is, $\Re(\tilde{\tilde{z}}_j) \geq \Re(\tilde{\tilde{c}}_j)$ for all $j \in J_N$, we conclude $\Re(\tilde{\tilde{Z}}) \leq \Re(\tilde{\tilde{Z}}_0)$ or equally $\tilde{\tilde{Z}} \lessapprox \tilde{\tilde{Z}}_0$. □

4 Algorithm of the new approach for HFLP with HFNs

In this section, we introduce a simplex algorithm for the HFLP problem with HFNs in the cost coefficient.
Algorithm : Hesitant fuzzy simplex algorithm (The problem is maximization)
First, consider an initial basic feasible solution with basic B.

1. Compute the system $Bx_B = b$. Suppose $x_B = B^{-1}b$ and $x_N = 0$. The objective value is $\tilde{\tilde{z}} = \tilde{\tilde{c}}_B^T x_B$.
2. Calculate $\tilde{\tilde{w}}^T B = \tilde{\tilde{c}}_B^T$ and let $\tilde{\tilde{w}} = \tilde{\tilde{c}}_B^T B^{-1}$.
3. Calculate $\tilde{\tilde{z}}_j = \tilde{\tilde{c}}_B^T B^{-1} a_j = \tilde{\tilde{w}}^T a_j$, for all $j \in J_N$ and $\Re(\tilde{\tilde{z}}_j - \tilde{\tilde{c}}_j)$, for all $j \in J_N$.
   Let $\Re(\tilde{\tilde{z}}_k - \tilde{\tilde{c}}_k) = \min_{j\in J_N}\{\Re(\tilde{\tilde{z}}_j - \tilde{\tilde{c}}_j\}$. If $\Re(\tilde{\tilde{z}}_k - \tilde{\tilde{c}}_k) \geq 0$, then stop; the current solution is optimal.
4. Solve the system $By_k = a_k$ and let $y_k = B^{-1}a_k$. If $y_k \leq 0$, then stop. The problem is unbounded.
   If $y_k \nleq 0$, then x_k enters the basic and x_{Br} exits the basic given that
$$\frac{\bar{b}_r}{y_{rk}} = \min_{1\leq i\leq m}\{\frac{\bar{b}_i}{y_{ik}}|y_{rk} > 0\}.$$
5. Update the basic B, where a_k replaces a_{Br} update the index set J_N and go to step 3.

Figure 1 is a flowchart of the simplex algorithm for solving HFLP problems.

Example 4.1. *A pottery manufacturer can make four different types of dining room service sets: JJP English, Currier, Primrose, and Bluetail. Furthermore, each set uses clay, enamel, dry room time, ans kiln time, and results in a profit, as shown in Table 1 (Here, Ibs is the abbreviation for pounds).*

Table 1: Primal data of Example 4.1

Resources	x_1	x_2	x_3	x_4	$Total$
Clay(Ibs)	10	15	10	20	130
Enamel(Ibs)	1	2	2	1	13
Dry room(hours)	3	1	6	3	45
Kiln(hours)	2	4	2	3	23
Profit	$(49, 51, 51, 52)$ $(48, 51, 51, 53)$	$(101, 102, 102, 103)$ $(100, 101, 102, 105)$	$(65, 66, 66, 68)$ $(64, 65, 66, 67)$	$(87, 89, 90, 91)$ $(88, 89, 89, 90)$	0

The decision variables x_1, x_2, x_3, x_4 are the number of sets of type English, Currier, Primrose, and Bluetail, respectively.

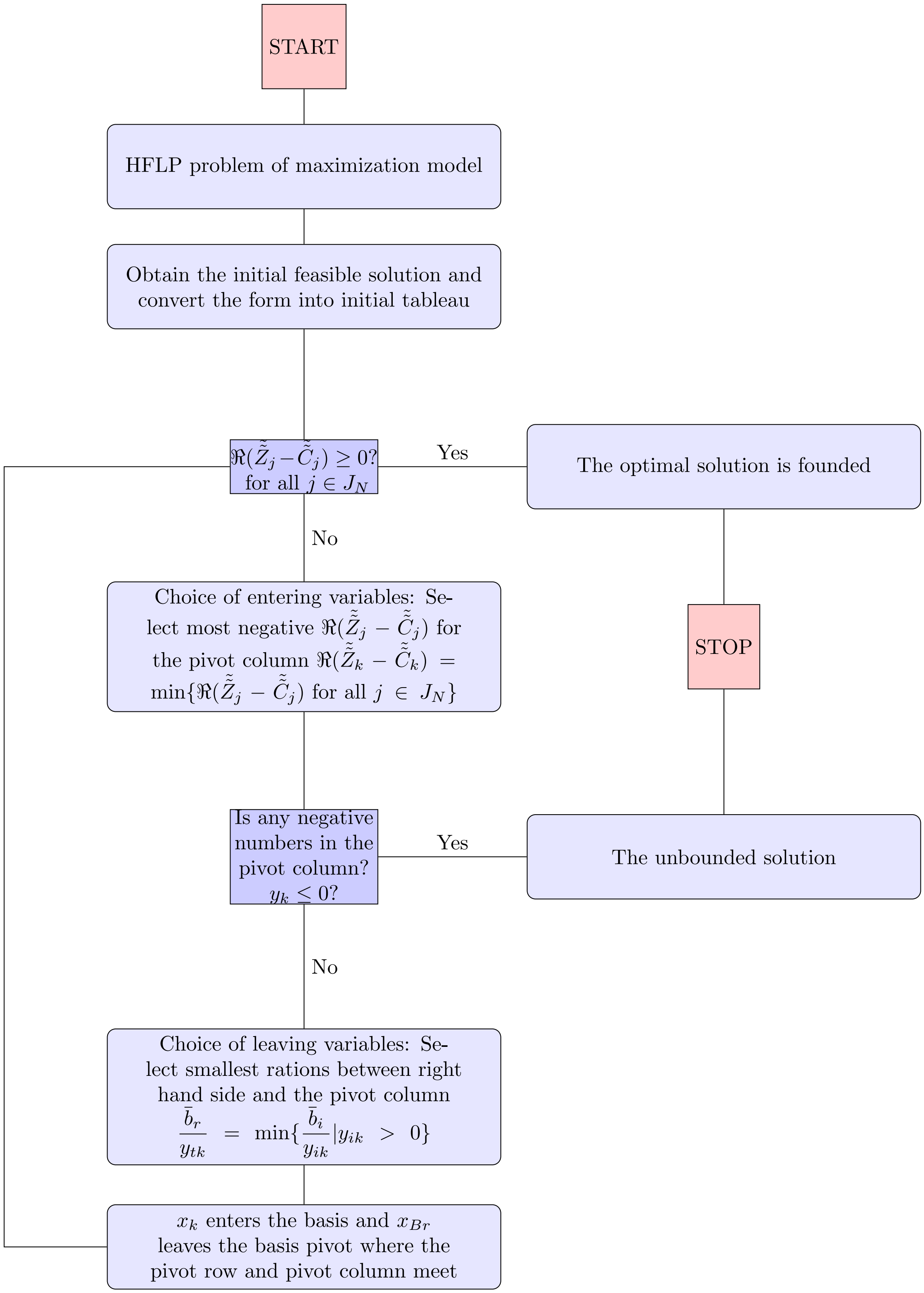

Figure 1: Flowchart of simplex method for HFLP problem

We assume, for the purposes of this problem, that the number of sets of each type can be fractional. The formulation of the profit maximization problem is given below:

$$\begin{array}{ll} maximize & \tilde{\tilde{51}}x_1 + \tilde{\tilde{102}}x_2 + \tilde{\tilde{66}}x_3 + \tilde{\tilde{89}}x_4 \\ s.t. & 10x_1 + 15x_2 + 10x_3 + 20x_4 \leq 130, \\ & x_1 + 2x_2 + 2x_3 + x_4 \leq 13, \\ & 3x_1 + x_2 + 6x_3 + 3x_4 \leq 45, \\ & 2x_1 + 4x_2 + 2x_3 + 3x_4 \leq 23, \\ & x_1, x_2, x_3, x_4 \geq 0. \end{array} \tag{29}$$

Iteration 1:
The primal simplex table is shown in Table 2.

Table 2: Iteration 1 of hesitant fuzzy simplex algorithm for Example 4.1

Basic \ Nonbasic variables	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
$\tilde{\tilde{z}}_j - \tilde{\tilde{c}}_j$	{(-52,-51,-51,-49) (-53,-51,-51,-48)}	{(-103,-102,-102,-101) (-105,-102,-101,-100)}	{(-68,-66,-66,-65) (-67,-66,-65,-64)}	{(-91,-90,-89,-87) (-90,-89,-89,-88)}	$\tilde{\tilde{0}}$	$\tilde{\tilde{0}}$	$\tilde{\tilde{0}}$	$\tilde{\tilde{0}}$	$\tilde{\tilde{0}}$
$\Re(\tilde{\tilde{z}}_j - \tilde{\tilde{c}}_j)$	-50.75	-102	-65.875	-89.125	0	0	0	0	0
x_5	10	15	10	20	1	0	0	0	130
x_6	1	2	2	1	0	1	0	0	13
x_7	3	1	6	3	0	0	1	0	45
x_8	2	$\boxed{4}$	2	3	0	0	0	1	23

Iteration 2:
Compute $\min \left\{ \Re(\tilde{\tilde{z}}_j - \tilde{\tilde{c}}_j), j = 1, 2, 3, 4 \right\} = \Re(\tilde{\tilde{z}}_2 - \tilde{\tilde{c}}_2)$. Thus x_2 enters the basis.
The following test determines the basic leaving variable x_{Br}:

$$\min \left\{ \frac{\bar{b}_1}{y_{12}}, \frac{\bar{b}_2}{y_{22}}, \frac{\bar{b}_3}{y_{32}}, \frac{\bar{b}_4}{y_{42}} \right\} = \min \left\{ \frac{130}{15}, \frac{13}{2}, \frac{45}{1}, \frac{23}{4} \right\} = \frac{23}{4} = 5.75.$$

Hence, $x_{N2} = x_2$ enters the basic and $x_{B4} = x_8$ leaves the basic. Table 3 is the updated simplex table.

Iteration 3:
After performing the calculations, in this step, the nonbasic variable x_3 is defined as the input variable, and the variable x_6 is defined as the output variable. Table 4 is the updated simplex table.

Iteration 4:
After performing the calculations, in this step, the nonbasic variable x_4 is defined as the input variable, and the variable x_5 is defined as the output variable. Table 5 is the updated simplex table.

Since $\Re(\tilde{\tilde{z}}_j - \tilde{\tilde{c}}_j) \geq 0$ for all $j = 1, 2, 3, 4, 5, 6, 7, 8$, Table 5 is optimal. The optimal solution to HFLP is $x^* = (x_1, x_2, x_3, x_4) = (0, 1, \frac{7}{2}, 4)$ and $\tilde{\tilde{z}}^* = \left\{ (658, 685.5, 693.75, 711.75), (661.25, 683.25, 693, 726) \right\}$.

Table 3: Iteration 2 of hesitant fuzzy simplex algorithm for Example 4.1

Basic \ Nonbasic variables	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
$\tilde{\tilde{z}}_j - \tilde{\tilde{c}}_j$	{(-1.5,0,0,2.5) (-3,-0.5,0,4.5)}	{(-2,0,0,2) (-5,-1,1,5)}	{(-17.5,-15,-15,-13.5) (-17,-15.5,-14,-11.5)}	{(-15.25,-13.5,-12.5,-9.75) (-15,-13.25,-12.5,-9.25)}	$\tilde{\tilde{0}}$	$\tilde{\tilde{0}}$	$\tilde{\tilde{0}}$	{(25.25,25.5,25.5,25.75) (25,25.25,25.5,26.25)}	{(580.75,586.5,586.5,592.25) (575,580.75,586.5,603.75)}
$\Re(\tilde{\tilde{z}}_j - \tilde{\tilde{c}}_j)$	0.25	0	-14.875	-12.625	0	0	0	25.5	586.5
x_5	$\frac{5}{2}$	0	$\frac{5}{2}$	$\frac{35}{4}$	1	0	0	$-\frac{15}{4}$	$\frac{175}{4}$
x_6	0	0	$\boxed{1}$	$-\frac{1}{2}$	0	1	0	$-\frac{1}{2}$	$\frac{3}{2}$
x_7	$\frac{5}{2}$	0	$\frac{11}{2}$	$\frac{9}{4}$	0	0	1	$-\frac{1}{4}$	$\frac{157}{4}$
x_2	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{3}{4}$	0	0	0	$\frac{1}{4}$	$\frac{23}{4}$

Table 4: Iteration 3 of hesitant fuzzy simplex algorithm for Example 4.1

Basic \ Nonbasic variables	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
$\tilde{\tilde{z}}_j - \tilde{\tilde{c}}_j$	{(-1.5,0,0,2.5) (-3,-0.5,0,4.5)}	{(-2,0,0,2) (-5,-1,1,5)}	{(-6,-1,0.5,3.5) (-3.5,-0.5,1,6)}	{(-24,-21,-20,-16.5) (-23.5,-21,-19.5,-15)}	$\tilde{\tilde{0}}$	{(11.5,14,15.5,17) (13.5,15,15,17.5)}	$\tilde{\tilde{0}}$	{(16.5,18,18,19) (16.5,17.5,18.5,20.5)}	{(598,607.5,609.75,617.75) (595.25,603.25,609,630)}
$\Re(\tilde{\tilde{z}}_j - \tilde{\tilde{c}}_j)$	0.25	0	0	-20.0625	0	14.875	0	18.0625	608.8125
x_5	$\frac{5}{2}$	0	0	$\boxed{10}$	1	$-\frac{5}{2}$	0	$-\frac{5}{2}$	40
x_3	0	0	1	$-\frac{1}{2}$	0	1	0	$-\frac{1}{2}$	$\frac{3}{2}$
x_7	$\frac{5}{2}$	0	0	5	0	$-\frac{11}{2}$	1	$\frac{5}{2}$	31
x_2	$\frac{1}{2}$	1	0	1	0	$-\frac{1}{2}$	0	0	5

Table 5: Iteration 4 of hesitant fuzzy simplex algorithm for Example 4.1

Basic \ Nonbasic variables	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
$\tilde{\tilde{z}}_j - \tilde{\tilde{c}}_j$	{(2.25,4.875,5.25,8.375)(1.125,4.5,5.25,1.5)}	{(-2,0,0,2)(-5,-1,1,5)}	{(-6,-1,0.5,3.5)(-3.5,-0.5,1,6)}	{(-9,-1.5,1,7)(-7,-1,1.5,9)}	{(1.5,1.95,2.1,2.35)(1.65,2.0,2.1,2.4)}	{(5.8,8.75,10.5,12.875)(7.625,9.75,10.125,13.75)}	$\tilde{\tilde{0}}$	{(10.5,12.75,13,14.875)(10.625,12.25,13.625,16.75)}	{(658,685.5,693.75,711.75)(661.25,683.25,693,726)}
$\Re(\tilde{\tilde{z}}_j - \tilde{\tilde{c}}_j)$	4.140625	0	0	0	2.00625	9.896875	0	13.046875	689.0625
x_4	$\frac{1}{4}$	0	0	1	$\frac{1}{10}$	$-\frac{1}{4}$	0	$-\frac{1}{4}$	4
x_3	$\frac{1}{8}$	0	1	0	$\frac{1}{20}$	$\frac{7}{8}$	0	$-\frac{5}{8}$	$\frac{7}{2}$
x_7	$\frac{5}{4}$	0	0	0	$-\frac{1}{2}$	$-\frac{17}{4}$	1	$\frac{15}{4}$	11
x_2	$\frac{1}{4}$	1	0	0	$-\frac{1}{10}$	$-\frac{1}{2}$	0	$\frac{1}{4}$	1

Example 4.2. *An HFLP problem with hesitant fuzzy cost coefficients is as follows:*

$$\begin{array}{ll} \max & \tilde{\tilde{z}} = \tilde{\tilde{12}}x_1 + \tilde{\tilde{8}}x_2 \\ \text{s.t.} & \\ & 2x_1 + 3x_2 \leq 100, \\ & 4x_1 + 2x_2 \leq 80, \\ & x_1, x_2 \geq 0, \end{array} \tag{30}$$

where

$$\tilde{\tilde{c}}_1 = \tilde{\tilde{12}} = \{(10,12,12,13),(11,12,12,15)\},$$
$$\tilde{\tilde{c}}_2 = \tilde{\tilde{8}} = \{(7,8,8,9),(5,8,8,12),(6,8,9,11)\}.$$

Iteration 1:
The primal simplex table is shown in Table 6.

Iteration 2:
Compute $\min\left\{\Re(\tilde{\tilde{z}}_1 - \tilde{\tilde{c}}_1), \Re(\tilde{\tilde{z}}_2 - \tilde{\tilde{c}}_2)\right\} = \Re(\tilde{\tilde{z}}_1 - \tilde{\tilde{c}}_1) = -12.125$. Thus x_1 enters the basic.
The following test determines the basic leaving variable x_{Br}:

$$\min\left\{\frac{\bar{b_1}}{y_{12}}, \frac{\bar{b_2}}{y_{22}}\right\} = \min\left\{\frac{100}{2}, \frac{80}{4}\right\} = \frac{80}{4} = 20.$$

Hence, $x_{N1} = x_1$ enters the basic and $x_{B2} = x_4$ leaves the basis. Table 7 shows the updated simplex table.

Iteration 3:
After performing the calculations, in this step, the nonbasic variable x_2 is defined as the input variable, and the variable x_3 is defined as the output variable. Table 8 shows the updated simplex table.
Since $\Re(\tilde{\tilde{z}}_j - \tilde{\tilde{c}}_j) \geq 0$ for all $j = 1,2,3,4$, Table 8 is optimal. The optimal solution to HFLP is $x^* = (5,30)$ and $\tilde{\tilde{z}}^* = \left\{(185,300,300,365),(145,300,300,495),(205,300,330,480)\right\}$.

Table 6: Iteration 1 of hesitant fuzzy simplex algorithm for Example 4.2

Basic \ Nonbasic variables	x_1	x_2	x_3	x_4	RHS
$\tilde{\tilde{z}}_j - \tilde{\tilde{c}}_j$	$\{(-15,-12,-12,-11)$ $(-13,-12,-12,-10)\}$	$\{(-11,-9,-8,-6)$ $(-12,-8,-8,-5)$ $(-9,-8,-8,-7)\}$	$\tilde{\tilde{0}}$	$\tilde{\tilde{0}}$	$\tilde{\tilde{0}}$
$\Re(\tilde{\tilde{z}}_j - \tilde{\tilde{c}}_j)$	-12.125	-8.25	0	0	0
x_3	2	3	1	0	100
x_4	$\boxed{4}$	2	0	1	80

Table 7: Iteration 2 of hesitant fuzzy simplex algorithm for Example 4.2

Basic \ Nonbasic variables	x_1	x_2	x_3	x_4	RHS
$\tilde{\tilde{z}}_j - \tilde{\tilde{c}}_j$	$\{(-5,0,0,2)$ $(-2,0,0,5)\}$	$\{(-6,-3,-2,0.5)$ $(-6.5,-2,-2,2.5)$ $(-3.5,-2,-2,0.5)\}$	$\tilde{\tilde{0}}$	$\{(2.5,3,3,3.25)$ $(2.75,3,3,3.75)\}$	$\{(200,240,240,260)$ $(220,240,240,300)\}$
$\Re(\tilde{\tilde{z}}_j - \tilde{\tilde{c}}_j)$	0	-2.125	0	3.03125	242.5
x_3	0	$\boxed{2}$	1	$-\frac{1}{2}$	60
x_1	1	$\frac{1}{2}$	0	$\frac{1}{4}$	20

Table 8: Iteration 3 of hesitant fuzzy simplex algorithm for Example 4.2

Basic \ Nonbasic variables	x_1	x_2	x_3	x_4	RHS
$\tilde{\tilde{z}}_j - \tilde{\tilde{c}}_j$	$\{(-5,0,0,2)$ $(-2,0,0,5)\}$	$\{(-6.5,-1,0,4)$ $(-9,0,0,9)$ $(-4,0,1,6.5)\}$	$\{(-0.25,1,1,1.75)$ $(-1.25,1,1,3.25)$ $(-0.25,1,1.5,3)\}$	$\{(1,2.25,2.5,3.375)$ $(1.125,2.5,2.5,4.375)$ $(1.875,2.5,2.5,3.875)\}$	$\{(185,300,300,365)$ $(145,300,300,495)$ $(205,300,330,480)\}$
$\Re(\tilde{\tilde{z}}_j - \tilde{\tilde{c}}_j)$	0	0	1.0625	2.53125	308.75
x_2	0	1	$\frac{1}{2}$	$-\frac{1}{4}$	30
x_1	1	0	$-\frac{1}{4}$	$\frac{3}{8}$	5

5 Discussion and comparative analyses

In this section, we are going to answer the question why this method is used and mention some of its advantages.

In this paper, the simplex method is implemented in a special case of HFLP problems for the model (16). This model can be converted to an LP problem as follows:

$$(LP) \qquad \begin{aligned} \max\ & Z = \Re(\tilde{\tilde{c}})x \\ s.t.\ & Ax \leq b \\ & x \geq 0, \end{aligned} \tag{31}$$

in which $\Re(\tilde{\tilde{c}})$ is the ranking function defined in Definition 2.8. Then the model (31) can be solved by the simplex method in the LP problems. Also, the HFLP problem in the model (16) can be converted into an FLP problem as follows:

$$(FLP) \qquad \begin{aligned} \max\ & Z = \tilde{c}x \\ s.t.\ & Ax \leq b \\ & x \geq 0, \end{aligned} \tag{32}$$

in which $\tilde{c} = (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n)^T$ are n-tuple of the fuzzy numbers that are calculated from the average of the boundaries and the cores of HFNs in $\tilde{\tilde{c}} = (\tilde{\tilde{c}}_1, \tilde{\tilde{c}}_2, \ldots, \tilde{\tilde{c}}_n)^T$. Then the model (32) can be solved by the simplex method in the FLP problems [19].

Now, instead of solving the HFLP problem, we can solve LP or FLP problems, which are equivalent, in the sense that the feasible solution sets to the LP, FLP, and HFLP problems are the same. These two methods are the easiest ways to solve the HFLP problem. However, these methods cause us to lose a lot of information about the problem in the process of problem-solving. Indeed using the proposed algorithm in this paper and using arithmetic operations for the HFNs, we can preserve the information of the problem until the end of the solution. This causes us to have more information on the optimal values of the objective function.

To compare and evaluate the results, we solve Example 4.1 for the case that the coefficients of the objective function are converted to the crisp numbers $\Big($i.e., $c_1 = 50.75$, $c_2 = 102$, $c_3 = 65.875$, $c_4 = 89.125\Big)$ and for the case where the objective function coefficients are converted to fuzzy numbers $\Big($i.e., $\tilde{c}_1 = (48.5, 51, 51, 52.5)$, $\tilde{c}_2 = (100.5, 101.5, 102, 104)$, $\tilde{c}_3 = (64.5, 65.5, 66, 67.5)$, $\tilde{c}_4 = (87.5, 89, 89.5, 90.5)\Big)$. Results are placed in Table 9.

Table 9: Comparing results LP, FLP, and $HFLP$ for Example 4.1

Optimal solutions \ Simplex methods	$x^* = (x_1^*, x_2^*, x_3^*, x_4^*)$	z^*	$\Re(z^*)$
HFLP to LP (model (33))	$(0, 1, \frac{7}{2}, 4)$	689.06	689.06
HFLP to FLP (model (34))	$(0, 1, \frac{7}{2}, 4)$	$\tilde{z}^* = (659.625, 682.625, 692.375, 718.325)$	688.2375
HFLP (model (16))	$(0, 1, \frac{7}{2}, 4)$	$\tilde{\tilde{z}}^* = \{(658, 685.5, 693.75, 711.75)$ $(661.25, 683.25, 693, 726)\}$	689.0625

Also, we solve Example 4.2 for the case that the coefficients of the objective function are converted to the crisp numbers $\Big($i.e., $c_1 = 12.125$, $c_2 = 8.25\Big)$ using the ranking (15), and for the case where the objective function coefficients are fuzzy numbers $\Big($i.e., $\tilde{c}_1 = (10.5, 12, 12, 14)$, $\tilde{c}_2 = (6, 8, 8.3, 10.6)\Big)$. Results are placed in Table 10.

The results obtained in Tables 9 and 10 show that the solutions and values obtained from all three approaches are close to each other, but it can be said that the results obtained in the proposed method are more comprehensive. It should be noted that we do not claim that the values obtained in the proposed algorithm are quantitatively better because it depends on the opinion of experts. Indeed it can be said that these answers are closer to reality due to the fact that they preserve the information of the problem in the process of solving the problem. Therefore, experts have more freedom in decision-making.

Table 10: Comparing results LP, FLP, and $HFLP$ for Example 4.2

Optimal solutions \ Simplex methods	$x^* = (x_1^*, x_2^*)$	z^*	$\Re(z^*)$
HFLP to LP (model (33))	$(5, 30)$	308.125	308.125
HFLP to FLP (model (34))	$(5, 30)$	$\tilde{z}^* = (180, 300, 309, 445.9)$	308.725
HFLP (model (16))	$(5, 30)$	$\tilde{\tilde{z}}^* = \{(185, 300, 300, 365)$ $(145, 300, 300, 495)$ $(205, 300, 330, 480)\}$	308.75

Note that in a special case, if the opinions of the experts are the same for the cost coefficients, then the HFLP problem becomes an FLP problem. For example, if the opinions of the experts are the same in Example 4.2, that is, the cost coefficients are in the form of $\tilde{\tilde{c}}_1 = \Big\{(10, 12, 12, 13), (10, 12, 12, 13)\Big\}$, $\tilde{\tilde{c}}_2 = \Big\{(7, 8, 8, 9), (7, 8, 8, 9), (7, 8, 8, 9)\Big\}$, then it will be the same as FLP. Results are placed in Table 11.

Table 11: Comparing results FLP and $HFLP$ for the special case of Example 4.2

Optimal solutions \ Simplex methods	$x^* = (x_1^*, x_2^*)$	z^*	$\Re(z^*)$
HFLP to FLP (model (34))	$(5, 30)$	$\tilde{z}^* = (215, 300, 300, 380)$	298.75
HFLP (model (16))	$(5, 30)$	$\tilde{\tilde{z}}^* = \{(215, 300, 300, 380)$ $(215, 300, 300, 380)$ $(215, 300, 300, 380)\}$	298.75

This approach can be considered for other types of HFLP problems. One of the most advantages of it can be the determination of hesitant fuzzy solutions. However, the related challenges can be more, which we will address in future studies.

6 Conclusion

In this paper, we solved the HFLP problem with hesitant cost coefficients. The method of converting an HFLP problem to an LP or FLP problem is one of the easiest ways to solve HFLP problems, but this method causes us to lose a lot of information about the problem. We thus decided to solve the HFLP problems by the simplex method because the simplex method causes us not to lose problem information, and outputs of the objective function are also in a hesitant space. Two revealing examples were provided to show the feasibility of the method. In future works, we will intend to solve other states of HFLP problems, such as HFLP problems with right-hand-side hesitant and full hesitant fuzzy problems by the proposed simplex scheme.

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