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Original paper

A new approach to solve intuitionistic fuzzy bi-matrix games involving multiple opinions

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Abstract

In the competitive business world, the whole thing is in a state of flux. It is not possible to know the exact outcomes of the strategies adopted by a company. Companies are always unsure of the customer's responses regarding their strategies, and the judgment of the decision-makers is correct to some extent but not always exact. To avoid erroneous estimations, the companies generally preferred the opinion of more than one expert. It is highly understood that no two experts will describe the similar payoffs for a mix of strategies used. Therefore, the payoff matrices, given by a group of experts, provide more information to select the best strategies for the companies. This paper presents an approach to solving bimatrix game problems with multiple experts in an intuitionistic fuzzy environment. Further, the applicability and superiority of the proposed method have been shown with the help of a real-life numerical example.

Keywords: Game theory, bi-matrix game, intuitionistic fuzzy numbers, intuitionistic fuzzy bi-matrix game.

1 Introduction

Game theory is one of the most effective tools to analyze conflict of interest among multiple players. It was initially originated by Neumann and Morgenstern [43] and published in the book entitled "*The Theory of Games and Economic Behavior*." The theory of games has played a significant role in the fields such as economics, management, and political sciences [15, 22, 28]. In recent years, many researchers have conducted in-depth studies on matrix games under different information environments [29, 30, 36, 37, 48]. In game theory, every person wants to make the best choice from a finite set of different strategies to get more profit. Two or more players take the decision, and it always affects the output of each other. But in reality, the players are not completely antagonistic. Thus, it is useful to study a bimatrix game, a two-person non-zero-sum non-cooperative game. In the bimatrix game theory, each player endeavors to attain as large as possible payoffs by choosing pure or mixed strategies. Bi-matrix game theory has been used in different fields, such as voting, artificial intelligence, and management.

In real-life situations, it is not easy to access crisp payoffs due to the uncertainty in the environment. Fuzzy set theory, proposed by Zadeh [52], provides an efficient tool to describe the uncertain information of payoffs in bi-matrix games. The concept of decision-making in the fuzzy environment was explained by Bellman and Zadeh [6]. Later, fuzzy methodologies have been effectively applied in various real-world competitive decision situations. Maeda [21] proposed two equilibrium concepts in any bi-matrix game with fuzzy payoffs. Vidyottama et al. [42] used a ranking function to solve bi-matrix fuzzy game problems with fuzzy goals and fuzzy payoffs. Bector and Chandra [5] introduced the duality concepts of fuzzy linear programming to solve bi-matrix games with fuzzy payoffs. Roy [27] formulated two different solution procedures for solving a two-person multi-criteria bi-matrix game. In addition, Han et al. [14] defined different concepts of minimax equilibrium strategies to solve the fuzzy bi-matrix game with fuzzy payoffs. Kudryavzsev

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et al. [16] used ordering operators to define Nash equilibrium and compare fuzzy numbers in bi-matrix games. Liu and Xing [20] introduced the method of interval value function to solve bi-matrix games based on fuzzy payoffs. Nash [24] explained that any bi-matrix game problem always has at least one equilibrium point in mixed strategies.

The fuzzy set theory uses the degree of membership to describe the uncertainty and vagueness in a complex situation. But in reality, both membership and non-membership degrees play an important role in describing the uncertainty and vagueness. Atanassov [3] introduced the theory of intuitionistic fuzzy set (IFS) to model uncertain concepts more precisely. Since its appearance, the IFS theory has been applied successfully in a variety of problems [35, 38, 39, 40, 41, 45, 51]. Recently, much attention has been paid by researchers to solve bimatrix games with intuitionistic fuzzy payoffs. Yang and Wang [50] solved the multi-objective bimatrix game problems with intuitionistic fuzzy payoffs. Seikh et al. [32] presented the concept of triangular intuitionistic fuzzy numbers and their applications. Nayak and Pal [25] used linear and nonlinear membership functions to solve the bimatrix game of intuitionistic fuzzy goals. Li and Yang [19] developed the auxiliary parametric bilinear programming model to solve the bi-matrix game in which payoffs are in the form of trapezoidal intuitionistic fuzzy numbers. Bhurjee [7] used an interval quadratic programming problem to find the equilibrium of the bimatrix game with payoffs as the closed interval.

Gao [11] proposed existence theorems and three concepts to find uncertain equilibrium strategies for uncertain bimatrix games. Yang et al. [49] created a difference-index-based ranking algorithm for solving bimatrix games using triangular intuitionistic fuzzy numbers as payoffs. To get the Pareto-Nash equilibrium strategies, Cunlin and Qiang [8] utilized the concept of crisp bi-matrix games. The Nash equilibrium strategy of two-person zero-sum games with trapezoidal fuzzy payoffs was investigated by Dutta and Gupta [9]. To solve IFPBiG, Fan et al. [10] focused on non-linear programming algorithms to solve bimatrix games with intuitionistic fuzzy payoffs. Seikh et al. [31] developed a solution procedure to solve bimatrix game problems with triangular intuitionistic fuzzy number payoffs. Li and Nan [18] considered two nonlinear bi-objective programming models to obtain the solution of matrix games with payoffs denoted by Atanassov's IFSs. An et al. [2] proposed a weighted mean-area ranking method to solve bimatrix games with payoffs intuitionistic fuzzy numbers. Nan et al. [23] developed a non-linear programming model to find a solution for bimatrix games in which payoffs are regarded as triangular intuitionistic fuzzy numbers. An and Li [1] discussed constrained bimatrix games and linear programming models to solve such game problems.

Aggregation operators play a very significant role in the field of optimization to fuse the information data. Several aggregation operators have been proposed to aggregate a collection of intuitionistic fuzzy numbers and apply them to solve many complex decision-making problems [12, 13, 34, 44, 46, 47]. Recently, Singla et al. [33] developed a novel method for solving matrix game problems with intuitionistic fuzzy payoffs in which the payoff values are based on the opinion of different experts. Based on the literature review, it has been observed that various methods were proposed to solve fuzzy bimatrix games in which only a single expert evaluated the payoff values. But in real-world competitive situations, a firm or company always considers the opinion of multiple experts for equilibrium points and optimum strategies. The existing methods can not solve the bimatrix game problems under an intuitionistic fuzzy information environment involving a group of experts. These limitations of existing methods motivate us to develop a novel approach to solving fuzzy bimatrix games with intuitionistic fuzzy payoffs, considering the opinion of a group of experts. Accordingly, the contributions of this investigation are as below:

- An aggregation operator-based technique is formulated to fuse the opinion of a group of experts to obtain the collective payoff matrices corresponding to both the players in the fuzzy bimatrix game with intuitionistic fuzzy payoffs.
- A solution procedure is designed to solve the fuzzy bimatrix game problem with the collective payoff matrices.
- The application of the developed approach is demonstrated with the help of a numerical example.

The rest of the paper is organized as follows. In Section 2, some basic definitions and concepts related to intuitionistic fuzzy sets, aggregation operators, and bimatrix games are introduced. Section 3 formulates the concept of a fuzzy bi-matrix game with intuitionistic fuzzy payoffs. In Section 4, the existing method of Li [17] on bi-matrix games with intuitionistic fuzzy payoffs is presented, and discuss some of its imitations. In Section 5, a new method is presented to find the solution to the fuzzy bimatrix game with intuitionistic fuzzy payoffs considering the opinion of a group of experts. The applicability of the proposed method is explained with the help of an example in Section 6. The conclusion of the study is given in Section 7.

2 Preliminaries

Definition 2.1. [3] *Let X be a nonempty set of the universe. An intuitionistic fuzzy set $\tilde{A}$ in X is a set $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)\rangle : x \in X\}$, where $\mu_{\tilde{A}}(x)$ is termed as the grade of membership of x in $\tilde{A}$ and $\nu_{\tilde{A}}(x)$ is termed as the grade of non-membership of x in $\tilde{A}$. Here $\mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) : X \to [0,1]$ are two functions from X to a space $[0,1]$ which is called membership space and satisfying $0 \leq \mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \leq 1 \,\forall x \in X$. The degree of hesitation is given by*

$\pi_{\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x) - \nu_{\tilde{A}}(x)$. *For a given* $x \in X$ *the pair* $\langle \mu(x), \nu(x) \rangle$ *is called an intuitionistic fuzzy number (IFN)* [13, 46] *and simply denoted by* $\alpha = \langle \mu, \nu \rangle$ *where* $\mu \in [0,1], \nu \in [0,1]$ *and* $\mu + \nu \leq 1$.

In the literature, several aggregation operators have been defined to aggregate a finite collection of intuitionistic fuzzy numbers by various researchers. Here, we present the definition of the intuitionistic fuzzy Einstein interactive weighted averaging (IFEIWA) operator, which will be used to develop our work further.

Definition 2.2. [13] *Let* $\alpha_z = \langle \mu_z, \nu_z \rangle \, (z = 1, 2, \ldots, r)$ *be a finite collection of different intuitionistic fuzzy numbers and* $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_r)$ *be the corresponding weight vector satisfying* $\varpi \in [0,1]$ *and* $\sum_{z=1}^{r} \varpi_z = 1$, *then IFEIWA operator is defined as*

$$IFIEWA(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left\langle \frac{\prod_{z=1}^{r}(1+\mu_z)^{\varpi_z} - \prod_{z=1}^{r}(1-\mu_z)^{\varpi_z}}{\prod_{z=1}^{r}(1+\mu_z)^{\varpi_z} + \prod_{z=1}^{r}(1-\mu_z)^{\varpi_z}}, \frac{2\left(\prod_{z=1}^{r}(1-\mu_z)^{\varpi_z} - \prod_{z=1}^{r}(1-\mu_z-\nu_z)^{\varpi_z}\right)}{\prod_{z=1}^{r}(1+\mu_z)^{\varpi_z} + \prod_{z=1}^{r}(1-\mu_z)^{\varpi_z}} \right\rangle. \quad (1)$$

Bimatrix Game

A bimatrix game can be defined as an extension of the matrix game. It is a two-person non-zero-sum finite game, and there is at least one situation where the sum of the payoffs of the two players, A_1 and A_2 is not equal to zero. Bi-matrix game can be reduced to a matrix game. Consider a bimatrix game defined by (P, Q) of $m \times n$ matrices. Let $M = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ $N = \{\delta_1, \delta_2, \ldots, \delta_n\}$ be the sets of pure strategies for players A_1 and A_2, respectively. If player A_1 chooses any pure strategy $\gamma_i \in M$ $(i = 1, 2, \ldots, m)$ and player A_2 chooses any pure strategy $\delta_j \in N$ $(j = 1, 2, \ldots, n)$, then the payoffs of player A_1 at (γ_i, δ_j) can be expressed as a_{ij} and the payoffs of player A_2 at (γ_i, δ_j) by b_{ij}. Thus the payoff matrices corresponding to player A_1 and player A_2 can be represented as follows:

$$P = \begin{array}{c} \\ \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_m \end{array} \begin{array}{c} \begin{matrix} \delta_1 & \delta_2 & \cdots & \delta_n \end{matrix} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} \quad \text{and} \quad Q = \begin{array}{c} \\ \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_m \end{array} \begin{array}{c} \begin{matrix} \delta_1 & \delta_2 & \cdots & \delta_n \end{matrix} \\ \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} \end{array}.$$

In the concept of mixed strategies, let $p_i \, (i = 1, 2, 3, ..., m)$ and $q_j \, (j = 1, 2, 3, ..., n)$ be the probabilities of players A_1 and A_2 choosing the pure strategy $\gamma_i \in M$ and $\delta_j \in N$ to employ their strategies.

Further, assume that $R = \{p | \sum_{i=1}^{m} p_i = 1; p_i \geq 0\}$ and $S = \left\{q | \sum_{j=1}^{n} q_j = 1; q_j \geq \right\}$ denote the mixed strategies sets of players A_1 and A_2, respectively.

In any bimatrix game, the expected payoffs of players A_1 and A_2 with mixed strategies can be calculated by the following expressions:

$$E_P(p,q) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i \, a_{ij} \, q_j \text{ and } E_Q(p,q) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i \, b_{ij} \, q_j \text{ respectively.}$$

Definition 2.3. [24, 26] *A pair of strategies* $(p^*, q^*) \in R \times S$ *is a Nash equilibrium point of a bi-matrix game if* (p^*, q^*) *satisfies the conditions for any pair of strategies* $(p, q) \in R \times S$ *defined as under:*

(i) $p^T P q^* \leq {p^*}^T P q^*$ *for any strategy* $p \in R$,

(ii) ${p^*}^T Q q \leq {p^*}^T Q q^*$ for any strategy $q \in S$.

Here $(p^*, q^*, \psi_1^*, \psi_2^*)$ is called the Nash equilibrium solution of the bi-matrix game, where p^* and q^* are called Nash equilibrium strategies and $\psi_1^* = {p^*}^T P q^*$, $\psi_2^* = {p^*}^T Q q^*$ are called Nash equilibrium values of players A_1 and A_2, respectively.

Theorem 2.4. *Any bi-matrix game* (P, Q) *has at least one Nash equilibrium solution.*

Theorem 2.5. *Any bi-matrix game* (P, Q) *has a Nash equilibrium solution* $(p^*, q^*, \psi_1^*, \psi_2^*)$ *if and only if it is an optimal solution of the following bilinear programming model given as:*

$$\max \left\{ p^T (P + Q) q - \psi_1 - \psi_2 \right\},$$

$$\text{s.t} \quad \begin{cases} Pq \leq \psi_1 e^m \\ Q^T p \leq \psi_2 e^n \\ p^T e^m = 1 \text{ and } q^T e^n = 1 \\ p \geq 0, q \geq 0. \end{cases}$$

Here, $e^m = (1,1,...,1)^T \in R^m, e^n = (1,1,...,1)^T \in R^n$.
If $(p^*, q^*, \psi_1^*, \psi_2^*)$ is an optimal solution to the bilinear programming model, then $\psi_1^* = {p^*}^T P q^*, {\psi_2}^* = {p^*}^T Q q^*$.

3 Fuzzy bimatrix game with payoffs represented by the intuitionistic fuzzy numbers

A fuzzy bimatrix game is a two-person non-zero-sum finite game with payoffs denoted by intuitionistic fuzzy numbers. In a bimatrix game, each player wants to maximize their own payoffs. Let $M = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$, $N = \{\delta_1, \delta_2, \ldots, \delta_n\}$ be the set of pure strategies for players A_1 and A_2, respectively. If player A_1 and player A_2 choose any pure strategy $\gamma_i \in M$ $(i = 1,2,\ldots,m)$ and $\delta_j \in N$ $(j = 1,2,\ldots,n)$ respectively then players A_1 and A_2 gain their payoffs in the form of intuitionistic fuzzy numbers $\langle \mu_{ij}, \nu_{ij} \rangle$ at (γ_i, δ_j). Thus, the payoffs of players A_1 and A_2 in all pure strategy situations (γ_i, δ_j) can be concisely expressed in the form of an intuitionistic fuzzy index matrix (IFIM) [4] as follows:

$$\hat{P} = \left(M, N, \left\langle \mu_{ij}^{A_1}, \nu_{ij}^{A_1} \right\rangle\right) = \begin{array}{c} \\ \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_m \end{array} \begin{array}{c} \begin{array}{cccc} \delta_1 & \delta_2 & \cdots & \delta_n \end{array} \\ \begin{pmatrix} \left\langle \mu_{11}^{A_1}, \nu_{11}^{A_1} \right\rangle & \left\langle \mu_{12}^{A_1}, \nu_{12}^{A_1} \right\rangle & \cdots & \left\langle \mu_{1n}^{A_1}, \nu_{1n}^{A_1} \right\rangle \\ \left\langle \mu_{21}^{A_1}, \nu_{21}^{A_1} \right\rangle & \left\langle \mu_{22}^{A_1}, \nu_{22}^{A_1} \right\rangle & \cdots & \left\langle \mu_{2n}^{A_1}, \nu_{2n}^{A_1} \right\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \left\langle \mu_{m1}^{A_1}, \nu_{m1}^{A_1} \right\rangle & \left\langle \mu_{m2}^{A_1}, \nu_{m2}^{A_1} \right\rangle & \cdots & \left\langle \mu_{mn}^{A_1}, \nu_{mn}^{A_1} \right\rangle \end{pmatrix} \end{array},$$

and

$$\hat{Q} = \left(M, N, \left\langle \mu_{ij}^{A_2}, \nu_{ij}^{A_2} \right\rangle\right) = \begin{array}{c} \\ \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_m \end{array} \begin{array}{c} \begin{array}{cccc} \delta_1 & \delta_2 & \cdots & \delta_n \end{array} \\ \begin{pmatrix} \left\langle \mu_{11}^{A_2}, \nu_{11}^{A_2} \right\rangle & \left\langle \mu_{12}^{A_2}, \nu_{12}^{A_2} \right\rangle & \cdots & \left\langle \mu_{1n}^{A_2}, \nu_{1n}^{A_2} \right\rangle \\ \left\langle \mu_{21}^{A_2}, \nu_{21}^{A_2} \right\rangle & \left\langle \mu_{22}^{A_2}, \nu_{22}^{A_2} \right\rangle & \cdots & \left\langle \mu_{2n}^{A_2}, \nu_{2n}^{A_2} \right\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \left\langle \mu_{m1}^{A_2}, \nu_{m1}^{A_2} \right\rangle & \left\langle \mu_{m2}^{A_2}, \nu_{m2}^{A_2} \right\rangle & \cdots & \left\langle \mu_{mn}^{A_2}, \nu_{mn}^{A_2} \right\rangle \end{pmatrix} \end{array}.$$

In the concept of mixed strategies, let $p_i; (i = 1,2,3,...,m)$ and $q_j; (j = 1,2,3,...,n)$be the probabilities of players A_1 and A_2 for choosing the pure strategy $\gamma_i \in M$ and $\delta_j \in N$ to employ their strategies.

Let $R = \{p \mid \sum_{i=1}^m p_i = 1;\ p_i \geq 0\}$ and $S = \left\{q \mid \sum_{j=1}^n q_j = 1;\ q_j \geq 0\right\}$ be the sets of mixed strategies of players A_1 and A_2, respectively. In any fuzzy matrix game, if player A_1 chooses any mixed strategy p and player A_2 chooses any mixed strategy q, then the expected payoffs of player A_1 and player A_2 can be calculated based on IFWA operator [46] as follows :

$$E_{A_1}(p,q) = \begin{pmatrix} p_1 & p_2 & \ldots & p_m \end{pmatrix} \begin{pmatrix} \left\langle \mu_{11}^{A_1}, \nu_{11}^{A_1} \right\rangle & \left\langle \mu_{12}^{A_1}, \nu_{12}^{A_1} \right\rangle & \cdots & \left\langle \mu_{1n}^{A_1}, \nu_{1n}^{A_1} \right\rangle \\ \left\langle \mu_{21}^{A_1}, \nu_{21}^{A_1} \right\rangle & \left\langle \mu_{22}^{A_1}, \nu_{22}^{A_1} \right\rangle & \cdots & \left\langle \mu_{2n}^{A_1}, \nu_{2n}^{A_1} \right\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \left\langle \mu_{m1}^{A_1}, \nu_{m1}^{A_1} \right\rangle & \left\langle \mu_{m2}^{A_1}, \nu_{m2}^{A_1} \right\rangle & \cdots & \left\langle \mu_{mn}^{A_1}, \nu_{mn}^{A_1} \right\rangle \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$$

$$= \left\langle 1 - \prod_{j=1}^n \prod_{i=1}^m \left(1 - \mu_{ij}^{A_1}\right)^{p_i q_j}, \prod_{j=1}^n \prod_{i=1}^m \left(\nu_{ij}^{A_1}\right)^{p_i q_j} \right\rangle,$$

and

$$E_{A_2}(p,q) = \begin{pmatrix} p_1 & p_2 & \ldots & p_m \end{pmatrix} \begin{pmatrix} \left\langle \mu_{11}^{A_2}, \nu_{11}^{A_2} \right\rangle & \left\langle \mu_{12}^{A_2}, \nu_{12}^{A_2} \right\rangle & \cdots & \left\langle \mu_{1n}^{A_2}, \nu_{1n}^{A_2} \right\rangle \\ \left\langle \mu_{21}^{A_2}, \nu_{21}^{A_2} \right\rangle & \left\langle \mu_{22}^{A_2}, \nu_{22}^{A_2} \right\rangle & \cdots & \left\langle \mu_{2n}^{A_2}, \nu_{2n}^{A_2} \right\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \left\langle \mu_{m1}^{A_2}, \nu_{m1}^{A_2} \right\rangle & \left\langle \mu_{m2}^{A_2}, \nu_{m2}^{A_2} \right\rangle & \cdots & \left\langle \mu_{mn}^{A_2}, \nu_{mn}^{A_2} \right\rangle \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$$

$$= \left\langle 1 - \prod_{j=1}^n \prod_{i=1}^m \left(1 - \mu_{ij}^{A_2}\right)^{p_i q_j}, \prod_{j=1}^n \prod_{i=1}^m \left(\nu_{ij}^{A_2}\right)^{p_i q_j} \right\rangle.$$

4 Li's method for fuzzy bimatrix games with intuitionistic fuzzy payoffs

This section presents the method developed by Li [17] to solve fuzzy bimatrix game problems with payoffs represented by intuitionistic fuzzy numbers. This approach assumes that the payoffs are given by only one expert or the aggregated data of the fuzzy matrices. Li [17] suggested the following steps to find the solutions to non-linear programming models of fuzzy bi-matrix games under an intuitionistic fuzzy environment.

Step 1: The payoff matrices of the fuzzy bimatrix game under an intuitionistic fuzzy environment corresponding to the players A_1 and A_2 are given as follows:

$$\hat{P} = \left(M, N, \left\langle \mu_{ij}^{A_1}, \nu_{ij}^{A_1} \right\rangle\right) = \begin{array}{c} \\ \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_m \end{array} \begin{array}{c} \begin{array}{cccc} \delta_1 & \delta_2 & \cdots & \delta_n \end{array} \\ \begin{pmatrix} \left\langle \mu_{11}^{A_1}, \nu_{11}^{A_1} \right\rangle & \left\langle \mu_{12}^{A_1}, \nu_{12}^{A_1} \right\rangle & \cdots & \left\langle \mu_{1n}^{A_1}, \nu_{1n}^{A_1} \right\rangle \\ \left\langle \mu_{21}^{A_1}, \nu_{21}^{A_1} \right\rangle & \left\langle \mu_{22}^{A_1}, \nu_{22}^{A_1} \right\rangle & \cdots & \left\langle \mu_{2n}^{A_1}, \nu_{2n}^{A_1} \right\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \left\langle \mu_{m1}^{A_1}, \nu_{m1}^{A_1} \right\rangle & \left\langle \mu_{m2}^{A_1}, \nu_{m2}^{A_1} \right\rangle & \cdots & \left\langle \mu_{mn}^{A_1}, \nu_{mn}^{A_1} \right\rangle \end{pmatrix} \end{array},$$

and

$$\hat{Q} = \left(M, N, \left\langle \mu_{ij}^{A_2}, \nu_{ij}^{A_2} \right\rangle\right) = \begin{array}{c} \\ \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_m \end{array} \begin{array}{c} \begin{array}{cccc} \delta_1 & \delta_2 & \cdots & \delta_n \end{array} \\ \begin{pmatrix} \left\langle \mu_{11}^{A_2}, \nu_{11}^{A_2} \right\rangle & \left\langle \mu_{12}^{A_2}, \nu_{12}^{A_2} \right\rangle & \cdots & \left\langle \mu_{1n}^{A_2}, \nu_{1n}^{A_2} \right\rangle \\ \left\langle \mu_{21}^{A_2}, \nu_{21}^{A_2} \right\rangle & \left\langle \mu_{22}^{A_2}, \nu_{22}^{A_2} \right\rangle & \cdots & \left\langle \mu_{2n}^{A_2}, \nu_{2n}^{A_2} \right\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \left\langle \mu_{m1}^{A_2}, \nu_{m1}^{A_2} \right\rangle & \left\langle \mu_{m2}^{A_2}, \nu_{m2}^{A_2} \right\rangle & \cdots & \left\langle \mu_{mn}^{A_2}, \nu_{mn}^{A_2} \right\rangle \end{pmatrix} \end{array}.$$

Step 2: We can define the bilinear mathematical model of the fuzzy bimatrix game $\left(\hat{P}, \hat{Q}\right)$ as follows:

$$\max\left\{\sum_{j=1}^{n}\sum_{i=1}^{m} p_i[\mu_{ij}^{A_1} + \lambda_1(1-\mu_{ij}^{A_1}-\nu_{ij}^{A_1}) + \mu_{ij}^{A_2} + \lambda_2(1-\mu_{ij}^{A_2}-\nu_{ij}^{A_2})]q_j - \kappa_1(\lambda_1) - \kappa_2(\lambda_2)\right\}$$

$$s.t \begin{cases} \sum_{j=1}^{n}[\mu_{ij}^{A_1} + \lambda_1(1-\mu_{ij}^{A_1}-\nu_{ij}^{A_1})]q_j \le \kappa_1(\lambda_1) \\ \sum_{i=1}^{m}[\mu_{ij}^{A_2} + \lambda_2(1-\mu_{ij}^{A_2}-\nu_{ij}^{A_2})]p_i \le \kappa_2(\lambda_2) \\ \sum_{i=1}^{m} p_i = 1 \text{ and } \sum_{j=1}^{n} q_j = 1 \\ \kappa_1(\lambda_1) \ge 0,\ \kappa_2(\lambda_2) \ge 0 \text{ and } p_i \ge 0,\ q_j \ge 0,\ (i = 1,2,...,m,\ j = 1,2,...,n) \end{cases}$$

where $\kappa_1(\lambda_1)$ and $\kappa_2(\lambda_2)$ are decision variables.

Step 3: Solve the above-defined bilinear programming model using MATLAB or LINGO software and obtain the Nash equilibrium values and strategies of both the players by choosing different values of the parameters$\lambda_1, \lambda_2 \in [0,1]$.

Some limitations of Li's method

Estimating the future payoffs with reasonable accuracy is a great challenge in bimatrix games where the losses of one player do not match exactly with the gains of another player. Such situations generally exist in businesses with low penetration and possess massive scope for future growth. In such a business, the strategies adopted by one player also influence the overall industrial growth besides gaining market share from rivals. Such a wider impact of strategies adopted by competitors in the case of bimatrix games invites bigger chances of error in estimations. To hedge against such risks that emerge from erroneous estimations, it is quite common in the industry to seek the opinions of more than one expert while making estimations about expected payoffs corresponding to different combinations of strategies of the two players. Then, the opinions of different experts are assigned appropriate weights depending on their relative importance, knowledge, experience, and expertise. After assigning weights to the solicited opinions, the aggregation obtains the single set of intuitionistic fuzzy payoffs in such bimatrix games. The final bimatrix game so obtained is solved by applying the game principles. However, the existing method of Li [17] cannot be applied in such types of fuzzy bimatrix situations where the participating players prefer to obtain opinions of more than one expert rather than believing a single opinion. These types of situations are quite common in the real world. It is reasonable to say that the existing method of Li [17] fails to apply in the majority of real-world competitive situations. So, to resolve this issue, next, we present a modified version of Li's approach to solving a large class of complex real-world problems.

5 Method to solve fuzzy bimatrix games based on the judgment of different experts

In competitive situations, the company gives its judgment only to some extent due to the uncertainty of the customer responses to their strategies. So, companies never depend upon the judgment of one expert. They always seek the opinions of more than one expert. In such situations, aggregation operators provide an essential tool to aggregate the different opinions of a set of experts. The steps of the proposed method are as under:

Step 1: Let A_1 and A_2 be two players. Let $M = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ and $N = \{\delta_1, \delta_2, \ldots, \delta_n\}$ be the set of pure strategies of players A_1 and A_2, respectively. Let $\{E_1, E_2, \ldots, E_t\}$ be the set of experts who gave their opinions about the strategies of the players. Thus the payoffs of player A_1, given by different experts with m x n strategies, can be expressed in the form of intuitionistic fuzzy index matrices as follows:

$$\hat{P}^{(E_1)} = \left(M, N, \left\langle {}^{(E_1)}\mu_{ij}^{A_1}, {}^{(E_1)}\nu_{ij}^{A_1} \right\rangle\right) = \begin{array}{c} \\ \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_m \end{array} \begin{array}{cccc} \delta_1 & \delta_2 & \cdots & \delta_n \\ \left\langle {}^{(E_1)}\mu_{11}^{A_1}, {}^{(E_1)}\nu_{11}^{A_1} \right\rangle & \left\langle {}^{(E_1)}\mu_{12}^{A_1}, {}^{(E_1)}\nu_{12}^{A_1} \right\rangle & \cdots & \left\langle {}^{(E_1)}\mu_{1n}^{A_1}, {}^{(E_1)}\nu_{1n}^{A_1} \right\rangle \\ \left\langle {}^{(E_1)}\mu_{21}^{A_1}, {}^{(E_1)}\nu_{21}^{A_1} \right\rangle & \left\langle {}^{(E_1)}\mu_{22}^{A_1}, {}^{(E_1)}\nu_{22}^{A_1} \right\rangle & \cdots & \left\langle {}^{(E_1)}\mu_{2n}^{A_1}, {}^{(E_1)}\nu_{2n}^{A_1} \right\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \left\langle {}^{(E_1)}\mu_{m1}^{A_1}, {}^{(E_1)}\nu_{m1}^{A_1} \right\rangle & \left\langle {}^{(E_1)}\mu_{m2}^{A_1}, {}^{(E_1)}\nu_{m2}^{A_1} \right\rangle & \cdots & \left\langle {}^{(E_1)}\mu_{mn}^{A_1}, {}^{(E_1)}\nu_{mn}^{A_1} \right\rangle \end{array},$$

$$\hat{P}^{(E_2)} = \left(M, N, \left\langle {}^{(E_2)}\mu_{ij}^{A_1}, {}^{(E_2)}\nu_{ij}^{A_1} \right\rangle\right) = \begin{array}{c} \\ \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_m \end{array} \begin{array}{cccc} \delta_1 & \delta_2 & \cdots & \delta_n \\ \left\langle {}^{(E_2)}\mu_{11}^{A_1}, {}^{(E_2)}\nu_{11}^{A_1} \right\rangle & \left\langle {}^{(E_2)}\mu_{12}^{A_1}, {}^{(E_2)}\nu_{12}^{A_1} \right\rangle & \cdots & \left\langle {}^{(E_2)}\mu_{1n}^{A_1}, {}^{(E_2)}\nu_{1n}^{A_1} \right\rangle \\ \left\langle {}^{(E_2)}\mu_{21}^{A_1}, {}^{(E_2)}\nu_{21}^{A_1} \right\rangle & \left\langle {}^{(E_2)}\mu_{22}^{A_1}, {}^{(E_1)}\nu_{22}^{A_1} \right\rangle & \cdots & \left\langle {}^{(E_2)}\mu_{2n}^{A_1}, {}^{(E_2)}\nu_{2n}^{A_1} \right\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \left\langle {}^{(E_2)}\mu_{m1}^{A_1}, {}^{(E_2)}\nu_{m1}^{A_1} \right\rangle & \left\langle {}^{(E_2)}\mu_{m2}^{A_1}, {}^{(E_2)}\nu_{m2}^{A_1} \right\rangle & \cdots & \left\langle {}^{(E_2)}\mu_{mn}^{A_1}, {}^{(E_2)}\nu_{mn}^{A_1} \right\rangle \end{array},$$

$$\vdots$$

$$\hat{P}^{(E_t)} = \left(M, N, \left\langle {}^{(E_t)}\mu_{ij}^{A_1}, {}^{(E_t)}\nu_{ij}^{A_1} \right\rangle\right) = \begin{array}{c} \\ \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_m \end{array} \begin{array}{cccc} \delta_1 & \delta_2 & \cdots & \delta_n \\ \left\langle {}^{(E_t)}\mu_{11}^{A_1}, {}^{(E_t)}\nu_{11}^{A_1} \right\rangle & \left\langle {}^{(E_t)}\mu_{12}^{A_1}, {}^{(E_t)}\nu_{12}^{A_1} \right\rangle & \cdots & \left\langle {}^{(E_t)}\mu_{1n}^{A_1}, {}^{(E_t)}\nu_{1n}^{A_1} \right\rangle \\ \left\langle {}^{(E_t)}\mu_{21}^{A_1}, {}^{(E_t)}\nu_{21}^{A_1} \right\rangle & \left\langle {}^{(E_t)}\mu_{22}^{A_1}, {}^{(E_t)}\nu_{22}^{A_1} \right\rangle & \cdots & \left\langle {}^{(E_t)}\mu_{2n}^{A_1}, {}^{(E_t)}\nu_{2n}^{A_1} \right\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \left\langle {}^{(E_t)}\mu_{m1}^{A_1}, {}^{(E_t)}\nu_{m1}^{A_1} \right\rangle & \left\langle {}^{(E_t)}\mu_{m2}^{A_1}, {}^{(E_t)}\nu_{m2}^{A_1} \right\rangle & \cdots & \left\langle {}^{(E_t)}\mu_{mn}^{A_1}, {}^{(E_t)}\nu_{mn}^{A_1} \right\rangle \end{array}.$$

Similarly, the payoffs of player A_2, given by different experts with m x n strategies, can be expressed in the form of intuitionistic fuzzy index matrices as follows:

$$\hat{Q}^{(E_1)} = \left(M, N, \left\langle {}^{(E_1)}\mu_{ij}^{A_2}, {}^{(E_1)}\nu_{ij}^{A_2} \right\rangle\right) = \begin{array}{c} \\ \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_m \end{array} \begin{array}{cccc} \delta_1 & \delta_2 & \cdots & \delta_n \\ \left\langle {}^{(E_1)}\mu_{11}^{A_2}, {}^{(E_1)}\nu_{11}^{A_2} \right\rangle & \left\langle {}^{(E_1)}\mu_{12}^{A_2}, {}^{(E_1)}\nu_{12}^{A_2} \right\rangle & \cdots & \left\langle {}^{(E_1)}\mu_{1n}^{A_2}, {}^{(E_1)}\nu_{1n}^{A_2} \right\rangle \\ \left\langle {}^{(E_1)}\mu_{21}^{A_2}, {}^{(E_1)}\nu_{21}^{A_2} \right\rangle & \left\langle {}^{(E_1)}\mu_{22}^{A_1}, {}^{(E_1)}\nu_{22}^{A_1} \right\rangle & \cdots & \left\langle {}^{(E_1)}\mu_{2n}^{A_2}, {}^{(E_1)}\nu_{2n}^{A_2} \right\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \left\langle {}^{(E_1)}\mu_{m1}^{A_2}, {}^{(E_1)}\nu_{m1}^{A_2} \right\rangle & \left\langle {}^{(E_1)}\mu_{m2}^{A_2}, {}^{(E_1)}\nu_{m2}^{A_2} \right\rangle & \cdots & \left\langle {}^{(E_1)}\mu_{mn}^{A_2}, {}^{(E_1)}\nu_{mn}^{A_2} \right\rangle \end{array},$$

$$\hat{Q}^{(E_2)} = \left(M, N, \left\langle {}^{(E_2)}\mu_{ij}^{A_2}, {}^{(E_2)}\nu_{ij}^{A_2} \right\rangle\right) = \begin{array}{c} \\ \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_m \end{array} \begin{array}{cccc} \delta_1 & \delta_2 & \cdots & \delta_n \\ \left\langle {}^{(E_2)}\mu_{11}^{A_2}, {}^{(E_2)}\nu_{11}^{A_2} \right\rangle & \left\langle {}^{(E_2)}\mu_{12}^{A_2}, {}^{(E_2)}\nu_{12}^{A_2} \right\rangle & \cdots & \left\langle {}^{(E_2)}\mu_{1n}^{A_2}, {}^{(E_2)}\nu_{1n}^{A_2} \right\rangle \\ \left\langle {}^{(E_2)}\mu_{21}^{A_2}, {}^{(E_2)}\nu_{21}^{A_2} \right\rangle & \left\langle {}^{(E_2)}\mu_{22}^{A_1}, {}^{(E_2)}\nu_{22}^{A_1} \right\rangle & \cdots & \left\langle {}^{(E_2)}\mu_{2n}^{A_2}, {}^{(E_2)}\nu_{2n}^{A_2} \right\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \left\langle {}^{(E_2)}\mu_{m1}^{A_2}, {}^{(E_2)}\nu_{m1}^{A_2} \right\rangle & \left\langle {}^{(E_2)}\mu_{m2}^{A_2}, {}^{(E_2)}\nu_{m2}^{A_2} \right\rangle & \cdots & \left\langle {}^{(E_2)}\mu_{mn}^{A_2}, {}^{(E_2)}\nu_{mn}^{A_2} \right\rangle \end{array},$$

$$\vdots$$

$$\hat{Q}^{(E_t)}=\left(M,N,\left\langle{}^{(E_t)}\mu_{ij}^{A_2},{}^{(E_t)}\nu_{ij}^{A_2}\right\rangle\right)=\begin{matrix} \\ \gamma_1\\ \gamma_2\\ \vdots\\ \gamma_m\end{matrix}\begin{matrix}\begin{matrix}\delta_1 & \delta_2 & \cdots & \delta_n\end{matrix}\\ \begin{pmatrix}\left\langle{}^{(E_t)}\mu_{11}^{A_2},{}^{(E_t)}\nu_{11}^{A_2}\right\rangle & \left\langle{}^{(E_t)}\mu_{12}^{A_2},{}^{(E_t)}\nu_{12}^{A_2}\right\rangle & \cdots & \left\langle{}^{(E_t)}\mu_{1n}^{A_2},{}^{(E_t)}\nu_{1n}^{A_2}\right\rangle\\ \left\langle{}^{(E_t)}\mu_{21}^{A_2},{}^{(E_t)}\nu_{21}^{A_2}\right\rangle & \left\langle{}^{(E_t)}\mu_{22}^{A_1},{}^{(E_t)}\nu_{22}^{A_1}\right\rangle & \cdots & \left\langle{}^{(E_t)}\mu_{2n}^{A_2},{}^{(E_t)}\nu_{2n}^{A_2}\right\rangle\\ \vdots & \vdots & \vdots & \vdots\\ \left\langle{}^{(E_t)}\mu_{m1}^{A_2},{}^{(E_t)}\nu_{m1}^{A_2}\right\rangle & \left\langle{}^{(E_t)}\mu_{m2}^{A_2},{}^{(E_t)}\nu_{m2}^{A_2}\right\rangle & \cdots & \left\langle{}^{(E_t)}\mu_{mn}^{A_2},{}^{(E_t)}\nu_{mn}^{A_2}\right\rangle\end{pmatrix}\end{matrix}.$$

Step 2: Aggregate the above payoff matrices provided by different experts with the help of using the IFEIWA operator given in Definition 3.

After applying the IFEIWA operator on the payoff matrices $\hat{P}^{(E_1)},\hat{P}^{(E_2)},\ldots,\hat{P}^{(E_t)}$of player A_1 with the experts' weight vector $w_k\in[0,1]$ and $\sum_{k=1}^{t}w_k=1$, then we get the aggregated intuitionistic fuzzy payoff matrix $\bar{\hat{P}}$ for player A_1 given by

$$\bar{\hat{P}}=\left(M,N,\left\langle\breve{\mu}_{ij}^{A_1},\breve{\nu}_{ij}^{A_1}\right\rangle\right)=\begin{matrix} \\ \gamma_1\\ \gamma_2\\ \vdots\\ \gamma_m\end{matrix}\begin{matrix}\begin{matrix}\delta_1 & \delta_2 & \cdots & \delta_n\end{matrix}\\ \begin{pmatrix}\left\langle\breve{\mu}_{11}^{A_1},\breve{\nu}_{11}^{A_1}\right\rangle & \left\langle\breve{\mu}_{12}^{A_1},\breve{\nu}_{12}^{A_1}\right\rangle & \cdots & \left\langle\breve{\mu}_{1n}^{A_1},\breve{\nu}_{1n}^{A_1}\right\rangle\\ \left\langle\breve{\mu}_{21}^{A_1},\breve{\nu}_{21}^{A_1}\right\rangle & \left\langle\breve{\mu}_{22}^{A_1},\breve{\nu}_{22}^{A_1}\right\rangle & \cdots & \left\langle\breve{\mu}_{2n}^{A_1},\breve{\nu}_{2n}^{A_1}\right\rangle\\ \vdots & \vdots & \vdots & \vdots\\ \left\langle\breve{\mu}_{m1}^{A_1},\breve{\nu}_{m1}^{A_1}\right\rangle & \left\langle\breve{\mu}_{m2}^{A_1},\breve{\nu}_{m2}^{A_1}\right\rangle & \cdots & \left\langle\breve{\mu}_{mn}^{A_1},\breve{\nu}_{mn}^{A_1}\right\rangle\end{pmatrix}\end{matrix},$$

where

$$\left\langle\breve{\mu}_{ij}^{A_1},\breve{\nu}_{ij}^{A_1}\right\rangle=\left\langle\frac{\prod_{k=1}^{t}\left(1+{}^{(E_t)}\mu_{ij}^{A_1}\right)^{w_k}-\prod_{k=1}^{t}\left(1-{}^{(E_t)}\mu_{ij}^{A_1}\right)^{w_k}}{\prod_{k=1}^{t}\left(1+{}^{(E_t)}\mu_{ij}^{A_1}\right)^{w_k}+\prod_{k=1}^{t}\left(1-{}^{(E_t)}\mu_{ij}^{A_1}\right)^{w_k}},\frac{2\left(\prod_{k=1}^{t}\left(1-{}^{(E_t)}\mu_{ij}^{A_1}\right)^{w_k}-\prod_{k=1}^{t}\left(1-{}^{(E_t)}\mu_{ij}^{A_1}-{}^{(E_t)}\nu_{ij}^{A_1}\right)^{w_k}\right)}{\prod_{k=1}^{t}\left(1+{}^{(E_t)}\mu_{ij}^{A_1}\right)^{w_k}+\prod_{k=1}^{t}\left(1-{}^{(E_t)}\mu_{ij}^{A_1}\right)^{w_k}}\right\rangle. \tag{2}$$

Similarly, the aggregated intuitionistic fuzzy payoff matrice $\bar{\hat{Q}}$ for player A_2 is represented as

$$\bar{\hat{Q}}=\left(M,N,\left\langle\widehat{\mu}_{ij}^{A_2},\widehat{\nu}_{ij}^{A_2}\right\rangle\right)=\begin{matrix} \\ \gamma_1\\ \gamma_2\\ \vdots\\ \gamma_m\end{matrix}\begin{matrix}\begin{matrix}\delta_1 & \delta_2 & \cdots & \delta_n\end{matrix}\\ \begin{pmatrix}\left\langle\widehat{\mu}_{11}^{A_2},\widehat{\nu}_{11}^{A_2}\right\rangle & \left\langle\widehat{\mu}_{12}^{A_2},\widehat{\nu}_{12}^{A_2}\right\rangle & \cdots & \left\langle\widehat{\mu}_{1n}^{A_2},\widehat{\nu}_{1n}^{A_2}\right\rangle\\ \left\langle\widehat{\mu}_{21}^{A_2},\widehat{\nu}_{21}^{A_2}\right\rangle & \left\langle\widehat{\mu}_{22}^{A_1},\widehat{\nu}_{22}^{A_1}\right\rangle & \cdots & \left\langle\widehat{\mu}_{2n}^{A_2},\widehat{\nu}_{2n}^{A_2}\right\rangle\\ \vdots & \vdots & \vdots & \vdots\\ \left\langle\widehat{\mu}_{m1}^{A_2},\widehat{\nu}_{m1}^{A_2}\right\rangle & \left\langle\widehat{\mu}_{m2}^{A_2},\widehat{\nu}_{m2}^{A_2}\right\rangle & \cdots & \left\langle\widehat{\mu}_{mn}^{A_2},\widehat{\nu}_{mn}^{A_2}\right\rangle\end{pmatrix}\end{matrix},$$

where

$$\left\langle\widehat{\mu}_{ij}^{A_1},\widehat{\nu}_{ij}^{A_1}\right\rangle=\left\langle\frac{\prod_{k=1}^{t}\left(1+{}^{(E_t)}\mu_{ij}^{A_2}\right)^{w_k}-\prod_{k=1}^{t}\left(1-{}^{(E_t)}\mu_{ij}^{A_2}\right)^{w_k}}{\prod_{k=1}^{t}\left(1+{}^{(E_t)}\mu_{ij}^{A_2}\right)^{w_k}+\prod_{k=1}^{t}\left(1-{}^{(E_t)}\mu_{ij}^{A_2}\right)^{w_k}},\frac{2\left(\prod_{k=1}^{t}\left(1-{}^{(E_t)}\mu_{ij}^{A_2}\right)^{w_k}-\prod_{k=1}^{t}\left(1-{}^{(E_t)}\mu_{ij}^{A_2}-{}^{(E_t)}\nu_{ij}^{A_2}\right)^{w_k}\right)}{\prod_{k=1}^{t}\left(1+{}^{(E_t)}\mu_{ij}^{A_2}\right)^{w_k}+\prod_{k=1}^{t}\left(1-{}^{(E_t)}\mu_{ij}^{A_2}\right)^{w_k}}\right\rangle. \tag{3}$$

Step 3: Next, we define the bilinear mathematical model of the fuzzy bimatrix game $\left(\bar{\hat{P}},\bar{\hat{Q}}\right)$ with aggregated payoff matrices $\bar{\hat{P}}$ and $\bar{\hat{Q}}$ as follows:

$$\max\left\{\sum_{j=1}^{n}\sum_{i=1}^{m}p_i[\widehat{\mu}_{ij}^{A_1}+\lambda_1(1-\widehat{\mu}_{ij}^{A_1}-\widehat{\nu}_{ij}^{A_1})+\breve{\mu}_{ij}^{A_1}+\lambda_2(1-\breve{\mu}_{ij}^{A_1}-\breve{\nu}_{ij}^{A_1})]q_j-\xi_1(\lambda_1)-\xi_2(\lambda_2)\right\}$$

$$s.t\begin{cases}\sum_{j=1}^{n}[\widehat{\mu}_{ij}^{A_1}+\lambda_1(1-\widehat{\mu}_{ij}^{A_1}-\widehat{\nu}_{ij}^{A_1})]q_j\leq\xi_1(\lambda_1)\\ \sum_{i=1}^{m}[\breve{\mu}_{ij}^{A_1}+\lambda_2(1-\breve{\mu}_{ij}^{A_1}-\breve{\nu}_{ij}^{A_1})]p_i\leq\xi_2(\lambda_2)\\ \sum_{i=1}^{m}p_i=1\text{ and }\sum_{j=1}^{n}q_j=1\\ \xi_1(\lambda_1)\geq0,\ \xi_2(\lambda_2)\geq0\ and\ p_i\geq0,\ q_j\geq0,\ (i=1,2,...,m,\ j=1,2,...,n),\end{cases} \tag{4}$$

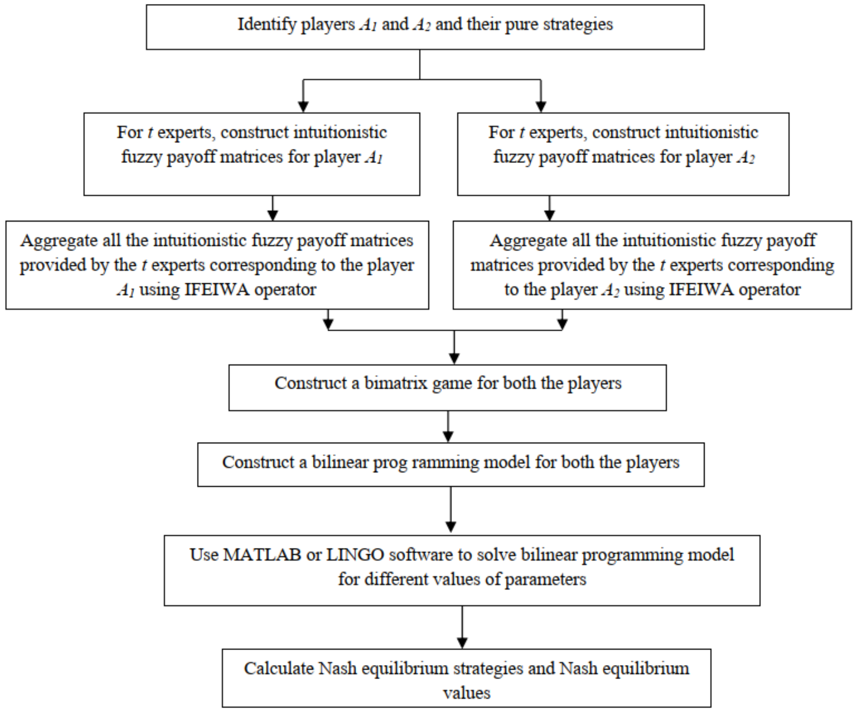

Figure 1: Flow chart of the proposed method

where $\xi_1(\lambda_1)$ and $\xi_2(\lambda_2)$ are decision variables.

Step 4: We use MATLAB or LINGO software to solve the bilinear programming model formulated in Step 3 and obtain the value of the bimatrix game with the optimal strategies of both the players.

We have drawn Figure 1 to show the flow-chart of the above-discussed method.

6 Applicability of the proposed method

In this section, we consider a numerical example to show the application of the developed method in solving real-world competitive problems.

Example 6.1. *Let us assume that A_1 and A_2 be the two major Life Insurance companies based in India. Since India's insurance industry presently has very low penetration and is rapidly growing. Besides looking to gain market share, the strategies adopted by different companies also aim to expand the overall market size. The situation can be described as a two-person non-zero-sum game. Further, assume that company A_1 has three strategies to implement: (i) to launch an aggressive insurance need awareness advertisement campaign under the company's banner (γ_1), (ii) to open more branches to reach large population branches (γ_2), and (iii) to add a variety of insurance policies in the company product portfolio (γ_3). On the other hand, company A_2 has three strategies as well: (i) to improve the company website to show the company's reputation in a better way (δ_1) , (ii) to make ease of doing business for customers (δ_2) (iii) to open more branches with effective customer care services to solve complaints timely (δ_3).*

These strategies adopted by both companies have great potential to expand the overall market size besides gaining market share from each other. Due to the volatile nature of the industry and the complex economic environment, both companies view the situation as highly uncertain. Therefore, they prefer to get opinions from two experts E_1 and E_2 on the possible impact of different strategies suggested by their marketing teams. Further, the companies have assigned a weight of 0.8 and 0.2, respectively, to the experts E_1 and E_2 according to their knowledge and experience in the insurance industry.

After making a critical analysis of the market, the intuitionistic fuzzy payoff matrices for company A_1 and company A_2 are provided by the experts E_1 and E_2 as listed below:

For player A_1

$$\hat{P}^{(E_1)} = \begin{array}{c} \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{array} \begin{array}{c} \begin{array}{ccc} \delta_1 & \delta_2 & \delta_3 \end{array} \\ \begin{pmatrix} \langle 0.70, 0.20\rangle & \langle 0.60, 0.30\rangle & \langle 0.70, 0.30\rangle \\ \langle 0.60, 0.15\rangle & \langle 0.80, 0.10\rangle & \langle 0.90, 0.05\rangle \\ \langle 0.85, 0.05\rangle & \langle 0.50, 0.40\rangle & \langle 0.40, 0.40\rangle \end{pmatrix} \end{array}, \quad and \quad \hat{P}^{(E_2)} = \begin{array}{c} \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{array} \begin{array}{c} \begin{array}{ccc} \delta_1 & \delta_2 & \delta_3 \end{array} \\ \begin{pmatrix} \langle 0.85, 0.10\rangle & \langle 0.70, 0.20\rangle & \langle 0.60, 0.20\rangle \\ \langle 0.50, 0.10\rangle & \langle 0.90, 0.05\rangle & \langle 0.80, 0.10\rangle \\ \langle 0.70, 0.10\rangle & \langle 0.60, 0.20\rangle & \langle 0.50, 0.20\rangle \end{pmatrix} \end{array}.$$

For player A_2

$$\hat{Q}^{(E_1)} = \begin{array}{c} \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{array} \begin{array}{c} \begin{array}{ccc} \delta_1 & \delta_2 & \delta_3 \end{array} \\ \begin{pmatrix} \langle 0.60, 0.30\rangle & \langle 0.70, 0.25\rangle & \langle 0.75, 0.10\rangle \\ \langle 0.50, 0.40\rangle & \langle 0.60, 0.30\rangle & \langle 0.75, 0.10\rangle \\ \langle 0.70, 0.20\rangle & \langle 0.60, 0.20\rangle & \langle 0.60, 0.30\rangle \end{pmatrix} \end{array}, \quad and \quad \hat{Q}^{(E_2)} = \begin{array}{c} \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{array} \begin{array}{c} \begin{array}{ccc} \delta_1 & \delta_2 & \delta_3 \end{array} \\ \begin{pmatrix} \langle 0.50, 0.40\rangle & \langle 0.60, 0.20\rangle & \langle 0.85, 0.05\rangle \\ \langle 0.60, 0.30\rangle & \langle 0.65, 0.25\rangle & \langle 0.60, 0.37\rangle \\ \langle 0.75, 0.10\rangle & \langle 0.70, 0.10\rangle & \langle 0.80, 0.10\rangle \end{pmatrix} \end{array}.$$

Step 1: $\hat{P}^{(E_k)}$and $\hat{Q}^{(E_k)}$ $(k = 1, 2)$ are the intuitionistic fuzzy payoff matrices of company A_1 and company A_2, respectively, as provided by the expertsE_1 and E_2.

Step 2: Utilize Eq. (2) and Eq. (3) to aggregate all the intuitionistic fuzzy payoff matrices given by experts $E_1$and E_2 for companies A_1 and A_2 with weights of 0.8 and 0.2, respectively. We obtain the following aggregated intuitionistic fuzzy payoff matrices corresponding to the companies A_1 and A_2:

$$\bar{\hat{P}} = \begin{array}{c} \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{array} \begin{array}{c} \begin{array}{ccc} \delta_1 & \delta_2 & \delta_3 \end{array} \\ \begin{pmatrix} \langle 0.74, 0.17\rangle & \langle 0.62, 0.27\rangle & \langle 0.68, 0.31\rangle \\ \langle 0.58, 0.14\rangle & \langle 0.83, 0.09\rangle & \langle 0.88, 0.06\rangle \\ \langle 0.83, 0.06\rangle & \langle 0.52, 0.36\rangle & \langle 0.42, 0.36\rangle \end{pmatrix} \end{array}, \quad \text{and} \quad \bar{\hat{Q}} = \begin{array}{c} \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{array} \begin{array}{c} \begin{array}{ccc} \delta_1 & \delta_2 & \delta_3 \end{array} \\ \begin{pmatrix} \langle 0.58, 0.32\rangle & \langle 0.68, 0.25\rangle & \langle 0.77, 0.09\rangle \\ \langle 0.52, 0.38\rangle & \langle 0.61, 0.29\rangle & \langle 0.72, 0.17\rangle \\ \langle 0.71, 0.18\rangle & \langle 0.62, 0.18\rangle & \langle 0.65, 0.25\rangle \end{pmatrix} \end{array},$$

Step 3: Convert the aggregated intuitionistic fuzzy matrices into bilinear programming model of both the companies A_1 and A_2 as follows:

$$\max \left\{ \begin{array}{l} [(0.74 + 0.58) + \lambda_1(1 - 0.74 - 0.17) + \lambda_2(1 - 0.58 - 0.32)]p_1q_1 \\ +[(0.62 + 0.68) + \lambda_1(1 - 0.62 - 0.27) + \lambda_2(1 - 0.68 - 0.25)]p_1q_2 \\ +[(0.68 + 0.77) + \lambda_1(1 - 0.68 - 0.31) + \lambda_2(1 - 0.77 - 0.09)]p_1q_3 \\ +[(0.58 + 0.52) + \lambda_1(1 - 0.58 - 0.14) + \lambda_2(1 - 0.52 - 0.38)]p_2q_1 \\ +[(0.83 + 0.61) + \lambda_1(1 - 0.83 - 0.09) + \lambda_2(1 - 0.61 - 0.29)]p_2q_2 \\ +[(0.88 + 0.72) + \lambda_1(1 - 0.88 - 0.06) + \lambda_2(1 - 0.72 - 0.17)]p_2q_3 \\ +[(0.83 + 0.71) + \lambda_1(1 - 0.83 - 0.06) + \lambda_2(1 - 0.71 - 0.18)]p_3q_1 \\ +[(0.52 + 0.62) + \lambda_1(1 - 0.52 - 0.36) + \lambda_2(1 - 0.62 - 0.18)]p_3q_2 \\ +[(0.42 + 0.65) + \lambda_1(1 - 0.42 - 0.36) + \lambda_2(1 - 0.65 - 0.25)]p_3q_3 \\ -\xi_1(\lambda_1) - \xi_2(\lambda_2) \end{array} \right\}, \tag{5}$$

$$s.t. \left\{ \begin{array}{l} \{[0.74 + (1 - 0.74 - 0.17)\lambda_1]\, q_1 + [0.62 + (1 - 0.62 - 0.27)\lambda_1]\, q_2 + [0.68 + (1 - 0.68 - 0.31)\lambda_1]\, q_3\} \le \xi_1(\lambda_1), \\ \{[0.58 + (1 - 0.58 - 0.14)\lambda_1]\, q_1 + [0.83 + (1 - 0.83 - 0.09)\lambda_1]\, q_2 + [0.88 + (1 - 0.88 - 0.06)\lambda_1]\, q_3\} \le \xi_1(\lambda_1), \\ \{[0.83 + (1 - 0.83 - 0.06)\lambda_1]\, q_1 + [0.52 + (1 - 0.52 - 0.36)\lambda_1]\, q_2 + [0.42 + (1 - 0.42 - 0.36)\lambda_1]\, q_3\} \le \xi_1(\lambda_1), \\ \{[0.58 + (1 - 0.58 - 0.32)\lambda_2]\, p_1 + [0.52 + (1 - 0.52 - 0.38)\lambda_2]\, p_2 + [0.71 + (1 - 0.71 - 0.18)\lambda_2]\, p_3\} \le \xi_2(\lambda_2), \\ \{[0.68 + (1 - 0.68 - 0.25)\lambda_2]\, p_1 + [0.61 + (1 - 0.61 - 0.29)\lambda_2]\, p_2 + [0.62 + (1 - 0.62 - 0.18)\lambda_2]\, p_3\} \le \xi_2(\lambda_2), \\ \{[0.77 + (1 - 0.77 - 0.09)\lambda_2]\, p_1 + [0.72 + (1 - 0.72 - 0.17)\lambda_2]\, p_2 + [0.65 + (1 - 0.65 - 0.25)\lambda_2]\, p_3\} \le \xi_2(\lambda_2), \\ p_1 + p_2 + p_3 = 1,\ q_1 + q_2 + q_3 = 1, \\ \xi_1(\lambda_1) \ge 0,\ \xi_2(\lambda_2) \ge 0, \\ p_i \ge 0, q_j \ge 0\, (i = 1, 2, 3; j = 1, 2, 3) \end{array} \right.$$

The above bilinear programming model can be written as the following

$$\max \left\{ \begin{array}{l} [1.32 + (0.09)\lambda_1 + (0.1)\lambda_2]p_1q_1 + [1.3 + (0.11)\lambda_1 + (0.07)\lambda_2]p_1q_2 \\ +[1.45 + (0.01)\lambda_1 + (0.14)\lambda_2]p_1q_3 + [1.1 + (0.28)\lambda_1 + (0.1)\lambda_2]p_2q_1 \\ +[1.44 + (0.08)\lambda_1 + (0.1)\lambda_2]p_2q_2 + [1.6 + (0.06)\lambda_1 + (0.11)\lambda_2]p_2q_3 \\ +[1.54 + (0.17)\lambda_1 + (0.11)\lambda_2]p_3q_1 + [1.14 + (0.12)\lambda_1 + (0.2)\lambda_2]p_3q_2 \\ +[1.07 + (0.22)\lambda_1 + (0.15)\lambda_2]p_3q_3 - \xi_1(\lambda_1) - \xi_2(\lambda_2) \end{array} \right\} \tag{6}$$

s.t.

$$
\begin{aligned}
&[0.74 + (0.09)\lambda_1]\, q_1 + [0.62 + (0.11)\lambda_1]\, q_2 + [0.68 + (0.01)\lambda_1]\, q_3 \le \xi_1(\lambda_1),\\
&[0.58 + (0.28)\lambda_1]\, q_1 + [0.83 + (0.08)\lambda_1]\, q_2 + [0.88 + (0.06)\lambda_1]\, q_3 \le \xi_1(\lambda_1),\\
&[0.83 + (0.11)\lambda_1]\, q_1 + [0.52 + (0.12)\lambda_1]\, q_2 + [0.42 + (0.22)\lambda_1]\, q_3 \le \xi_1(\lambda_1),\\
&[0.58 + (0.1)\lambda_2]\, p_1 + [0.52 + (0.1)\lambda_2]\, p_2 + [0.71 + (0.11)\lambda_2]\, p_3 \le \xi_2(\lambda_2),\\
&[0.68 + (0.07)\lambda_2]\, p_1 + [0.61 + (0.1)\lambda_2]\, p_2 + [0.62 + (0.2)\lambda_2]\, p_3 \le \xi_2(\lambda_2),\\
&[0.77 + (0.14)\lambda_2]\, p_1 + [0.72 + (0.11)\lambda_2]\, p_2 + [0.65 + (0.1)\lambda_2]\, p_3 \le \xi_2(\lambda_2),\\
&\quad p_1 + p_2 + p_3 = 1,\ q_1 + q_2 + q_3 = 1,\\
&\quad \xi_1(\lambda_1) \ge 0,\ \xi_2(\lambda_2) \ge 0,\\
&\quad p_i \ge 0;\ i = 1, 2, 3,\ q_j \ge 0;\ j = 1, 2, 3.
\end{aligned}
$$

Step 4: Now, solve the above bilinear programming model to obtain the nash equilibrium solution. Taking different values of parameters,$\lambda_1, \lambda_2 \in [0, 1]$, we use MATLAB 7.0 software to solve the above given bilinear programming model of both players. The obtained optimal solutions are summarized in Table 1.

Table 1: Optimal solution and expected payoffs of the fuzzy bimatrix game given Eq. (6)

λ_1	λ_2	p^{*T}	$\xi_1^*(\lambda_1)$	q^{*T}	$\xi_2^*(\lambda_2)$
0	0	$(0.24, 0, 0.76)$	0.72	$(0.74, 0, 0.26)$	0.68
0	0.1	$(0.24, 0, 0.76)$	0.72	$(0.74, 0, 0.26)$	0.69
0	0.5	$(0.24, 0, 0.76)$	0.73	$(0.74, 0, 0.26)$	0.73
0	0.9	$(0.11, 0, 0.89)$	0.73	$(0.74, 0, 0.26)$	0.79
0.5	0	$(0, 0.23, 0.77)$	0.78	$(0.70, 0, 0.30)$	0.67
0.5	0.3	$(0, 0.24, 0.76)$	0.78	$(0.70, 0, 0.30)$	0.72
0.5	0.9	$(0, 1, 0)$	0.91	$(0, 0, 1)$	0.82
0.9	0	$(0, 0, 1)$	0.93	$(1, 0, 0)$	0.71
0.9	0.7	$(0, 0, 1)$	0.93	$(1, 0, 0)$	0.79
0.3	0	$(0, 1, 0)$	0.89	$(0, 0, 1)$	0.72
0.8	0	$(0, 0, 1)$	0.92	$(1, 0, 0)$	0.71
0.3	0.9	$(0, 1, 0)$	0.89	$(0, 0, 1)$	0.82
1.0	1.0	$(0, 0, 1)$	0.94	$(1, 0, 0)$	0.82

Thus, we obtain the optimum strategies p^* of company A_1 and optimum strategies q^* of company A_2 as well as values of the fuzzy bimatrix game. We can obtain the nash equilibrium values and nash equilibrium strategies of both companies A_1 and A_2 for different values of the parameters $\lambda_1, \lambda_2 \in [0, 1]$ .For example, if $\lambda_1 = 0$and $\lambda_2 = 0.1$,then the nash equilibrium strategies p^* of company, A_1 is (0.24,0,0.76) and nash equilibrium strategies $q^*$of company, A_2 is (0.74,0,0.26), and the corresponding nash equilibrium values of company A_1 and A_2 are 0.72 and 0.69, respectively. Similarly if $\lambda_1 = 1$and $\lambda_2 = 1$,then the nash equilibrium strategies p^* of company, A_1 is (0,0,1) and nash equilibrium strategies $q^*$of company, A_2 is (1,0,0), and the corresponding nash equilibrium values of company A_1 and A_2 are 0.94 and 0.82, respectively. So, the result obtained at $\lambda_1 = 0$ and $\lambda_2 = 0$, reveals that company A_1 will use its δ_1 strategy with a probability of 0.24 and δ_3 strategy with a probability of 0.76. On the other hand, company A_2 will use its γ_1 strategy with a probability of 0.74 and γ_3 strategy with a probability of 0.26.

7 Conclusions

This paper has made a detailed study of fuzzy bimatrix games with intuitionistic fuzzy payoffs and pointed out some limitations of the existing solution method developed by Li [17]. In this work, we have proposed a method to solve the fuzzy bimatrix game problem with payoffs represented by intuitionistic fuzzy numbers considering the payoff values of more than one expert. Our advanced method overcomes the existing shortcomings of Li's method. A real-world practical numerical example has been given to show the applicability and superiority of the proposed method. In future work, we will explore the application of this method in manufacturing industries' competitive decision problems.

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