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Original paper

On the first-order autonomous interval-valued difference equations under gH-difference

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Abstract

The theory of interval-valued difference equations under gH-difference is an interesting topic, since it can be applied to study numerical solutions to interval-valued or fuzzy-valued differential equations. In this paper, we estimate the number of solutions to a class of first-order interval-valued difference equations under gH-difference, which reveals the complexity of the stability analysis in this area, as well as the difficulty for prediction and control problems. Then, based on the relative positions of initial values and equilibrium points, we provide sufficient conditions for the existence of convergent solutions. We also provide examples to illustrate the validity of our results.

Keywords: Interval-valued difference equations, gH-difference, equilibrium points, convergence.

1 Introduction

Difference equations appear as natural descriptions of discrete mathematical models, for example, in population dynamics, discrete state transition problems, economic problems, and so on [3]. The theory of difference equations has also been used in various fields such as numerical analysis, control theory, and computer science. In relation with the universal non-probabilistic uncertainty, fuzzy- and interval-valued difference equations have been introduced to study discrete mathematical models and problems.

In 1999, a model of carbon dioxide in blood was studied using the theory of fuzzy difference equations [4]. The authors considered a fixed membership as a threshold, and reduced the fuzzy difference equations proposed to crisp difference equations. In 2008, finance models in the form of fuzzy difference equations were discussed in [1]. We can also find results concerning the logistic model formulated as a fuzzy difference equation, see [9, 24]. In 2014, the existence of positive solutions to the following fuzzy logistic difference equation

$$x_{n+1} = A + Bx_{n-1}e^{-x_n},\ \ n = 0, 1, \dots$$

was considered in [24]. In 2018, two different fuzzy logistic difference equations were discussed in [9]. One of these equations admitted a right-hand term including a Hukuhara difference.

In recent years, it has been appearing an increasing interest in investigating fuzzy Riccati or Riccati-type difference equations as follows (see, for instance, [8, 10, 13, 14]):

$$\begin{aligned} x_{n+1} &= A + \tfrac{B}{x_n}, \\ x_{n+1} &= \tfrac{A+x_n}{B+x_n}, \\ x_{n+1} &= A + \tfrac{x_n}{x_{n-m}}. \end{aligned}$$

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Some other very recent results can be found, e.g., in [16, 17, 23], where periodicity and other dynamical properties are analyzed for fuzzy difference equations of higher order.

On the other hand, in [6], the authors have studied linear fractional interval-valued difference equations, and exact solutions have been obtained using Picard's iteration.

As we have already mentioned, difference equations are also often applied to calculate the numerical solutions to differential equations [2], but there are few results available on the numerical solutions to fuzzy- or interval-valued differential equations under gH-differentiability. In fact, based on the definition of the gH-derivative

$$\lim_{h\to 0} \frac{x(t+h) \ominus_g x(t)}{h} = x'(t),$$

the numerical solution to the fuzzy- or interval-valued differential equation $x' = f(t, x)$ can be approximated as

$$x(t+h) \ominus_g x(t) = h \cdot f(t, x(t)), \tag{1}$$

where $h > 0$ is the step size. This problem has not been addressed in previous researches in the field of fuzzy difference (or differential) equations, and is investigated for the first time in this paper.

We have recently studied interval-valued differential equations with switching points in our works [18, 19, 20]. Especially in [19], it was illustrated that there exist infinitely many gH-differentiable solutions to interval-valued differential equations, which are obtained by choosing particular single switching points or different combinations of switching points. It is natural to consider the behavior of numerical solutions to interval-valued differential equations under the notion of gH-differentiability.

In this paper, we consider the existence, length, and convergence of the solutions to the following class of nonlinear autonomous interval-valued difference equations

$$\begin{cases} x(n+1) \ominus_g x(n) = f(x(n)), \ n = 0, 1, 2, \ldots, \\ x(0) = x_0 \in K_C, \end{cases} \tag{2}$$

where $\{x(n)\}$ represents a sequence of real intervals, and $f : K_C \to K_C$ is a continuous interval-valued mapping.

The present paper is organized as follows. In Section 2, we recall basic definitions and facts about intervals and interval-valued functions. In Section 3, we provide results about the existence and the estimation of the length of the solutions to (2). The convergence of the solutions is discussed in Section 4. Finally, conclusions are given in Section 5.

2 Preliminaries

Let $\mathbb{R}$ be the set of real numbers, and K_C be the family of all non-empty compact convex subsets of $\mathbb{R}$ [5]. For every $u, v \in K_C$, we denote $u + v = \{a + b \mid a \in u, b \in v\}$, $u - v = u + (-1)v = \{a - b \mid a \in u, b \in v\}$, and $ku = \{ka \mid a \in u\}$ for $k \in \mathbb{R}$.

Definition 2.1. [7] *Let $u, v, w \in K_C$. The interval $w = u \ominus v$ is called the Hukuhara difference of u and v, if $u = v + w$.*

Definition 2.2. [15] *For $u, v \in K_C$, the generalized Hukuhara difference (gH-difference, for short) of u and v is defined as follows:*

$$u \ominus_g v = \begin{cases} u \ominus v, & \text{if } u \ominus v \text{ exists}, \\ (-1)(v \ominus u), & \text{if } v \ominus u \text{ exists}. \end{cases}$$

It is well known that $u \ominus_g v$ exists for every $u, v \in K_C$, and $u \ominus v$ exists only if $len(u) \geq len(v)$, where $len(u) = u_+ - u_-$. In addition, if $u \ominus v$ exists, then $u \ominus v = [u_- - v_-, u_+ - v_+]$.

In the following lemmas, we introduce several properties of the operations on intervals.

Lemma 2.3. *Let $u, v, w \in K_C$. The following properties are valid:*

(i) $len(u + v) = len(u) + len(v)$.

(ii) *If $u \ominus v$ exists, then* $len(u \ominus v) = len(u) - len(v)$.

(iii) *If $u \ominus v$ exists, then $u \ominus (-1)v$ exists, and* $(u \ominus (-1)v) \subseteq (u + v)$.

Proof. By direct calculations, we can check that (i) and (ii) hold. Therefore, we can omit their proofs. Here, we prove (iii).

Indeed, $len(u) \geq len(v)$ and $len(v) = len((-1)v)$ provide that $u \ominus (-1)v$ exists. Besides,

$$u \ominus (-1)v = [u_- + v_+, u_+ + v_-] \subseteq [u_- + v_-, u_+ + v_+] = u + v.$$

□

Lemma 2.4. [15] *Let $u, v \in K_C$. Then the gH-difference of u and v can be written as*

$$u \ominus_g v = [\min\{u_- - v_-, u_+ - v_+\}, \max\{u_- - v_-, u_+ - v_+\}].$$

For every $u, v \in K_C$, $H(u, v) = \max\{|u_- - v_-|, |u_+ - v_+|\}$ is known as the Pompeiu-Hausdorff metric on K_C. Then (K_C, H) is a complete and locally compact metric space. Now, we introduce some partial orderings on (K_C, H).

Lemma 2.5. *Let $u, v, w \in K_C$. The following properties are valid:*

(i) *If $v \subseteq u$, then $(-1)v \subseteq (-1)u$, and $(v + w) \subseteq (u + w)$.*

(ii) *$(v + w) \subseteq u \Longleftrightarrow u \ominus w$ exists, and $v \subseteq (u \ominus w)$.*

Proof. First, for (i), $v \subseteq u$ provides that $u_- \leq v_- \leq v_+ \leq u_+$, then $-u_- \geq -v_- \geq -v_+ \geq -u_+$ holds, which implies that $(-1)v \subseteq (-1)u$. Furthermore, for every $w \in K_C$, we have $u_- + w_- \leq v_- + w_- \leq v_+ + w_+ \leq u_+ + w_+$, that is, $(v + w) \subseteq (u + w)$.

Now, for (ii), based on Lemma 2.3(i), $(v + w) \subseteq u$ guarantees that $len(v) + len(w) = len(v + w) \leq len(u)$, thus $u \ominus w$ exists. On the other hand,

$$u_- \leq v_- + w_- \leq v_+ + w_+ \leq u_+ \Longleftrightarrow u_- - w_- \leq v_- \leq \ v_+ \leq u_+ - w_+,$$

so that $(v + w) \subseteq u \Longleftrightarrow v \subseteq (u \ominus w)$. □

Remark 2.6. *Given $f : K_C \to K_C$, the relation $f(u) \subseteq f(v)$ may not hold for $u \subseteq v$. For example, let $f(u) = [0, 1/u_+]$ for $u \in K_C$ and $u_+ \neq 0$, then $f([0, 1/2]) = [0, 2]$, and $f([0, 1]) = [0, 1]$, $f([0, 1/2]) \nsubseteq f([0, 1])$. If f is extended from a continuous real-valued function $\tilde{f} \in C(\mathbb{R}, \mathbb{R})$, that is, for every $u \in K_C$, $f(u) = \left[\min_{t \in u} \tilde{f}(t), \max_{t \in u} \tilde{f}(t)\right]$, then clearly $f(u) \subseteq f(v)$ holds for every $u \subseteq v$. Indeed, assuming that $u \subseteq v$, then v contains at least the same elements as in u (and eventually more), so that*

$$\min_{t \in v} \tilde{f}(t) \leq \min_{t \in u} \tilde{f}(t), \text{ and } \max_{t \in u} \tilde{f}(t) \leq \max_{t \in v} \tilde{f}(t),$$

which implies that $f(u) \subseteq f(v)$. For example, let $\tilde{f}(t) = 3t + 9$ for $t \in \mathbb{R}$, then $f(v) = [3v_- + 9, 3v_+ + 9]$ for $v \in K_C$. Let $u \subseteq v$, then $v_- \leq u_- \leq u_+ \leq v_+$ implies that $f(u) \subseteq f(v)$.

We define a partial ordering in K_C in the following way: Given $u, v \in K_C$, we say that $v \preceq u$, if and only if $v_- \leq u_-$, $v_+ \leq u_+$. Suppose that $v \preceq u$, we define $[v, u]_{\preceq} = \{w \in K_C \mid v \preceq w \text{ and } w \preceq u\}$. In addition, $v \prec u$ means $v \preceq u$ and $v \neq u$. In the following lemma, we introduce several properties of the partial ordering $\preceq$.

Lemma 2.7. *Let $u, v, w \in K_C$. The following properties are valid.*

(i) *If $u \preceq v$, then $(-1)v \preceq (-1)u$.*

(ii) *$u \preceq v \Longleftrightarrow u + w \preceq v + w$.*

(iii) *If $w \ominus v$ exists, then $(u + v) \preceq w \Longleftrightarrow u \preceq (w \ominus v)$, $(w \ominus v) \preceq u \Longleftrightarrow w \preceq u + v$.*

Proof. By direct calculations, we can check that (i) and (ii) hold, here we omit it.

In addition, $(u + v) \preceq w$ provides that $u_- + v_- \leq w_-$, $u_+ + v_+ \leq w_+$, then $u_- \leq w_- - v_-$, $u_+ \leq w_+ - v_+$, which implies that $u \preceq (w \ominus v)$ holds. Similarly, we can prove that $(w \ominus v) \preceq u \Longleftrightarrow w \preceq u + v$. □

Lemma 2.8. (Zorn's Lemma [11, 12]) *Let S be a partially ordered set. If every (totally) ordered subset of S has an upper bound, then S contains a maximal element.*

3 Existence and length estimation of solutions

In this section, we consider the existence and the estimation for the length of the solutions to (2). Suppose that $\{x_n\}$ is a solution to (2), then

$$f(x(n)) = \begin{cases} x(n+1) \ominus x(n), & \text{if } x(n+1) \ominus x(n) \text{ exists}, \\ (-1)(x(n) \ominus x(n+1)), & \text{if } x(n) \ominus x(n+1) \text{ exists}. \end{cases} \tag{3}$$

Let $A_1(x) = x + f(x)$, $A_2(x) = x \ominus (-1)f(x)$, and

$$A_i^n(x) = \overbrace{A_i(A_i(\cdots(A_i}^{n}(x)))),\ i = 1, 2.$$

For every $x_0 \in K_C$, we can check that $\{A_1^n(x_0)\}$ is a solution to (2). If $A_2^n(x_0)$ is well-defined for every $n \geq 1$, then $\{A_2^n(x_0)\}$ is also a solution to (2).

Definition 3.1. *$\{x_n\} \subset K_C$ is said to be a type-I solution to* (2)*, if $x(n+1) = A_1(x(n))$ for $n = 0, 1, 2, \ldots$. On the other hand, $\{x_n\}$ is said to be a type-II solution to* (2)*, if $x(n) \ominus (-1)f(x(n))$ exists, and $x(n+1) = A_2(x(n))$ holds for $n = 0, 1, 2, \ldots$.*

In general, problem (2) has always a solution of type-I for every given initial value $x_0 \in K_C$, and the existence of the type-II solution depends on the existence of the gH-differences $x(n) \ominus (-1)f(x(n))$ for $n \geq 0$. Besides, problem (2) might have many other different solutions. For example, if there exists $m \geq 1$ such that $x(m) \ominus (-1)f(x(m))$ exists, then

$$x(n+1) = \begin{cases} A_1(x(n)),\ n \neq m, \\ A_2(x(n)),\ n = m \end{cases}$$

is also a solution to (2).

Now, we consider the number of the solutions to (2).

Theorem 3.2. *Suppose that $f : K_C \to K_C$ is continuous. The following conclusions are valid:*

(i) If $f(x_0) = \{0\}$, then (2) *has a unique solution $x(n) = x_0$ for $n \geq 1$.*

(ii) If $len(f(x)) > len(x)$ for every $x \in K_C$, then (2) *has a unique solution, which is of type-I.*

(iii) If $0 < len(f(x)) \leq len(x)$ for every $x \in K_C$, then (2) *has 2^N different solutions for $0 \leq n \leq N$.*

Proof. (i) If $f(x_0) = \{0\}$, then $x(1) \ominus_g x(0) = f(x_0) = \{0\}$, that is, $x(1) = x(0) = x_0$. Similarly, we can obtain that $x(n) = x_0$, for every $n \geq 1$.
(ii) For every $x_0 \in K_C$, $\{A_1^n(x_0)\}$ is the type-I solution to (2). Suppose that $\{x_n\}$ is the other solution to (2), then there should exist $m \geq 1$ such that $x(m) \ominus x(m+1)$ exists, and $f(x(m)) = (-1)(x(m) \ominus x(m+1))$, which implies that

$$x(m+1) = x(m) \ominus (-1)f(x(m)). \tag{4}$$

However, $len(f(x)) > len(x)$ guarantees that the gH-difference in (4) does not exist. Therefore, problem (2) has a unique solution, which is the type-I solution.
(iii) For every $x \in K_C$, $len(f(x)) \leq len(x)$ implies that A_1 and A_2 are both well-defined on K_C, and $len(f(x)) > 0$ provides that $A_1(x) \neq A_2(x)$. In fact, we can check that $A_1(x) = A_2(x) \Leftrightarrow f(x) + (-1)f(x) = \{0\} \Leftrightarrow len(f(x)) = 0$.

In this case, problem (2) has two solutions $\{x_0,\ A_1(x_0)\}$ and $\{x_0,\ A_2(x_0)\}$ for $0 < n \leq 1$, and has four solutions $\{x_0,\ A_1(x_0),\ A_1(A_1(x_0))\}$, $\{x_0,\ A_1(x_0),\ A_2(A_1(x_0))\}$, $\{x_0,\ A_2(x_0),\ A_1(A_2(x_0))\}$, and $\{x_0,\ A_2(x_0),\ A_2(A_2(x_0))\}$ for $0 < n \leq 2$. In conclusion, problem (2) has 2^N different solutions for $0 \leq n \leq N$. □

Consider the numerical solutions to the interval-valued differential equation $x' = f(x)$ for $t \in [0, T]$ under the gH-differentiability. According to Theorem 3.2(iii), if $0 < len(f(x)) \leq len(x)$ for every $x \in K_C$, then the corresponding difference equation $x(t+h) \ominus_g x(t) = h \cdot f(x(t))$ has 2^N solutions, where $h > 0$ is the step size, $N = \left[\frac{T}{h}\right]$. Let $h \to 0$, the number of the solutions might tend to infinity.

Theorem 3.3. *Suppose that f is extended from a continuous real-valued function. Then, every solution of* (2) *satisfies $x(n) \subseteq A_1^n(x_0)$, $n \geq 1$.*

Proof. For every $u \in K_C$, Lemma 2.3(iii) guarantees that $A_2(u) \subseteq A_1(u)$, if $A_2(u)$ exists.

(i) If $x(1) = A_1(x_0)$ and $x(2) = A_1(x(1))$, then $x(2) = A_1^2(x_0)$.

(ii) If $x(1) = A_1(x_0)$ and $x(2) = A_2(x(1))$, then $x(2) = A_2(A_1(x_0)) \subseteq A_1^2(x_0)$.

(iii) If $x(1) = A_2(x_0)$ and $x(2) = A_1(x(1))$, then $x(1) \subseteq A_1(x_0)$ implies that $x(2) = A_1(x(1)) \subseteq A_1^2(x_0)$.

(iv) If $x(1) = A_2(x_0)$ and $x(2) = A_2(x(1))$, then we have $x(1) \subseteq A_1(x_0)$, $x(2) \subseteq A_1(x(1)) \subseteq A_1^2(x_0)$.

Applying mathematical induction, we can prove that $x(n) \subseteq A_1^n(x_0)$ holds for all $n \geq 0$. □

Remark 3.4. *If f is extended from a continuous real-valued function $\tilde{f}$ as in Remark 2.6, we can check that $A_1(u) \subseteq A_1(v)$ holds for $u, v \in K_C$ with $u \subseteq v$. Indeed, if $u \subseteq v$, then $v_- \leq u_- \leq u_+ \leq v_+$, and*

$$A_1(u) = \left[u_- + \min_{t\in u} \tilde{f}(t), u_+ + \max_{t\in u} \tilde{f}(t)\right] \subseteq \left[v_- + \min_{t\in v} \tilde{f}(t), v_+ + \max_{t\in v} \tilde{f}(t)\right] = A_1(v).$$

However, for $u \subseteq v$, $A_2(u) \subseteq A_2(v)$ may be true or not. For example, let $f(x) = \frac{1}{2}x$, $u = \{0\}$, and $v = [-1, 4]$, then $A_2(u) = \{0\}$, and $A_2(v) = \left[1, \frac{7}{2}\right]$, so that $A_2(u) \nsubseteq A_2(v)$.

Theorem 3.3 implies that we can estimate the range of solutions to (2) using the type-I solution.

Example 3.5. *Consider the interval-valued difference equation*

$$x(n+1) \ominus_g x(n) = a \cdot x(n), \;\; x(0) = x_0, \;\; a \in \mathbb{R}. \tag{5}$$

Based on Theorem 3.2, problem (5) has a unique solution as $x(n+1) = x(n) + a \cdot x(n)$ for $|a| > 1$ and 2^N solutions for $0 < |a| \leq 1$, and $0 \leq n \leq N$. For example, let $a = \frac{1}{2}$ and $x_0 = [1, 2]$, then problem (5) has exactly 2^3 solutions for $0 \leq n \leq 3$, and the type-I solution is $\left\{[1, 2], \left[\frac{3}{2}, 3\right], \left[\frac{9}{4}, \frac{9}{2}\right], \left[\frac{27}{8}, \frac{27}{4}\right]\right\}$. Moreover, we can check that every solution satisfies $x(n) \subseteq A_1^n([1, 2])$ for $n = 1, 2, 3$. See Figure 1.

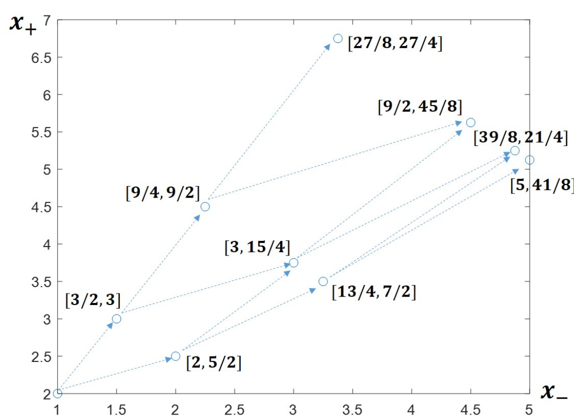

Figure 1: Problem (5) has 2^3 solutions for $0 \leq n \leq 3$.

For the length of the solutions to (2), we have the following conclusion.

Theorem 3.6. *Suppose that $\alpha \cdot len(x) \leq len(f(x)) \leq \beta \cdot len(x)$ for all $x \in K_C$, where $0 \leq \alpha \leq \beta$. Then, the following properties are valid:*

(i) For every $n \geq 0$, the type-I solution to (2) satisfies $(1 + \alpha)^n \cdot len(x_0) \leq len(x(n)) \leq (1 + \beta)^n \cdot len(x_0)$.

(ii) For every $n \geq 0$, any solution to (2) satisfies $(\max\{0, 1 - \beta\})^n \cdot len(x_0) \leq len(x(n)) \leq (1 + \beta)^n \cdot len(x_0)$.

Proof. According to Lemma 2.3(i), for every $x \in K_C$, $len(A_1(x)) = len(x + f(x)) = len(x) + len(f(x))$. Thus, we have

$$(1+\alpha)len(x) \leq len(A_1(x)) \leq (1+\beta)len(x). \tag{6}$$

If $A_2(x)$ is well-defined, then Lemma 2.3(ii) provides that $len(A_2(x)) = len(x \ominus (-1)f(x)) = len(x) - len(f(x))$. Hence,

$$\max\{0, 1-\beta\} \cdot len(x) \leq len(A_2(x)) \leq \max\{0, 1-\alpha\} \cdot len(x). \tag{7}$$

(i) Regarding (6), the type-I solution $\{A_1^n(x_0)\}$ to (2) satisfies $(1+\alpha)^n \cdot len(x_0) \leq len(x(n)) \leq (1+\beta)^n \cdot len(x_0)$ for $n \geq 0$.
(ii) Let $\{x(n)\}$ be a solution to (2). For every $n \geq 0$, $x(n+1) = A_1(x(n))$, if $x(n) \ominus (-1)f(x(n))$ does not exist, and $x(n+1) = A_1(x(n))$ or $A_2(x(n))$, if $x(n) \ominus (-1)f(x(n))$ exists. According to (6) and (7), we have

$$(1+\alpha)^i(\max\{0, 1-\beta\})^j \cdot len(x_0) \leq len(x(n)) \leq (1+\beta)^i(\max\{0, 1-\alpha\})^j \cdot len(x_0), \tag{8}$$

where $i + j = n$.

Finally, $0 \leq \alpha \leq \beta$ and $1 - \beta \leq 1 - \alpha \leq 1 + \alpha$ provide that $(1+\alpha)^i \geq (\max\{0, 1-\beta\})^i$, and $1 - \alpha \leq 1 + \alpha \leq 1 + \beta$ provides that $(\max\{0, 1-\alpha\})^j \leq (1+\beta)^j$. By applying these results to (8), we have

$$(\max\{0, 1-\beta\})^n \cdot len(x_0) \leq len(x(n)) \leq (1+\beta)^n \cdot len(x_0).$$

□

Remark 3.7. *In this section, we have shown that the initial value problem for interval-valued difference equations* (2) *might have* 2^N *different solutions for* $0 \leq n \leq N$*, which is a situation quite different with respect to the real-valued difference equation* $\Delta x(n) = f(x(n))$*, and with respect to the fuzzy-valued or interval-valued difference equation of the form* $x(n+1) = f(x(n))$*. Basically, due to the multiplicity of solutions, it's impossible to discuss the stability of an interval-valued difference system as a whole under the gH-difference. Alternatively, we can focus on the existence of convergent solutions and the stability of specific solutions, for example, the type-I and type-II solutions.*

Besides, in the study of fuzzy differential equations, the solutions obtained through different types of derivatives have been connected with different drawbacks. These difficulties have been solved through the introduction of new differences and the corresponding differentiability concepts. On the other hand, the variability of solutions that appeared in the continuous case with each new definition of derivative was sometimes seen as an enrichening perspective, in the sense that it is possible to select the appropriate approach depending on the particular properties of the phenomena of study. This has been seen as a positive fact for interval-valued and fuzzy differential equations, where the introduction of switching points also gives way to a possibly infinite number of solutions, but, on the other hand, the approach allowed imposing some control effects on the solutions.

In a similar way, we have illustrated the enormous variability in all the possible solutions to interval difference equations under the gH-difference. This behavior corresponds exactly to its continuous counterpart when introducing switching points at certain instants where the type of derivative is changed. In this sense, this study is mathematically consistent with the research made in the continuous case by many authors.

In this context, from the perspective of prediction of the evolution for a certain system, our study shows a quite complicated behavior for the solutions, and it is indeed difficult to deal with the enormous possibilities of solutions in terms of stability, which are an infinite number in the long term. However, in terms of the control of the system, this variability might provide a collection of possible choices or procedures to follow in order to keep the values of the solutions between some estimates, or to impose some other type of control. Taking into account the restrictions, we can obtain particular solutions that adopt a certain behavior once a determined iteration has been achieved.

On the other hand, for the continuous case, it has been shown in [22] *that the simplest optimal control cannot be achieved for linear systems by using the gH-derivative or strongly generalized derivative.*

The outcome of the use of gH-derivative for interval-difference equations could be catastrophic in the long-term if we do not combine this variability with the imposition of certain types of restrictions on the solutions. Leaving all the possible paths open would lead to the impossibility of prediction and control of the solutions.

In addition, looking at graphs such as the one in Figure 1, this approach somehow resembles the behavior of the solutions to continuous problems from the perspective of differential inclusions, where the attainable sets at time T *can be put into correspondence with the solutions to difference equations under the gH-difference up to time* N*. This analogy might bring some light into the interest and usefulness of this type of solutions, despite the inherent difficulty of the large number of solutions provided.*

4 Convergence of solutions

In this section, we consider the convergent behavior of the solutions near an equilibrium point.

Definition 4.1. *$x^* \in K_C$ is said to be an equilibrium point of (2), if $f(x^*) = \{0\}$.*

Theorem 4.2. *Suppose that $f : K_C \to K_C$ is continuous, and that there exists $x^* \in K_C$ such that $f(x^*) = \{0\}$. Moreover, assume that $len(f(x)) \neq 0$ and $f(x) \subseteq (x^* \ominus_g x)$ hold on the set $\{x \in K_C \mid x \subset x^* \text{ or } x \supset x^*\}$. The following conclusions are valid:*

(i) For every $x_0 \in \{x \in K_C \mid x \subset x^*\}$, the type-I solution to (2) converges to x^*.

(ii) For every $x_0 \in \{x \in K_C \mid x \supset x^*\}$, the type-II solution to (2) converges to x^*.

Proof. (i) Let $x \in K_C$ with $x \subset x^*$, then $x^* \ominus_g x = x^* \ominus x$. Based on Lemma 2.5(ii) and $f(x) \subseteq (x^* \ominus_g x)$, we get that $A_1(x) = x + f(x) \subseteq x^*$, which implies that $A_1^n(x_0) \subseteq x^*$, and $len(A_1^n(x_0)) \leq len(x^*)$ for every $x_0 \subset x^*$ and $n \geq 1$.

On the other hand, the definition of A_1 and Lemma 2.3(i) provide that

$$len(A_1^n(x_0)) = len(A_1^{n-1}(x_0)) + len(f(A_1^{n-1}(x_0))),\ n \geq 1, \tag{9}$$

then we have $len(A_1^{n-1}(x_0)) \leq len(A_1^n(x_0)) \leq len(x^*)$ for every $n \geq 1$, which guarantees that there exists $d \in [len(x_0), len(x^*)]$ such that $\lim_{n\to+\infty} len(A_1^n(x_0)) = d$. Considering (9), we get that

$$\lim_{n\to+\infty} len(f(A_1^n(x_0))) = 0. \tag{10}$$

Now, we prove that $A_1^n(x_0) \to x^*$ as $n \to +\infty$, that is, for each $\varepsilon > 0$, there exists $N > 0$ such that $H(A_1^n(x_0), x^*) \leq \varepsilon$ for $n > N$. If this is not true, then there should exist $\varepsilon_0 > 0$ and an infinite subsequence $\{A_1^{n_k}(x_0)\} \subseteq \{A_1^n(x_0)\}$ such that $H(A_1^{n_k}(x_0), x^*) > \varepsilon_0$.

As a bounded subset of the complete metric space (K_C, H), $\{A_1^{n_k}(x_0)\}$ has at least one convergent subsequence. For convenience, let us say that $\{A_1^{n_k}(x_0)\}$ itself is convergent and $\lim_{n_k\to+\infty} A_1^{n_k}(x_0) = \tilde{x}$. Considering $A_1^{n_k}(x_0) \subseteq x^*$ and $H(A_1^{n_k}(x_0), x^*) > \varepsilon_0$, we have $\tilde{x} \subset x^*$. However, (10) implies that $len(f(\tilde{x})) = 0$, which contradicts the hypothesis $len(f(x)) \neq 0$ on the set $\{x \in K_C \mid x \subset x^* \text{ or } x \supset x^*\}$. Consequently, $\{A_1^n(x_0)\}$, the type-I solution to (2), converges to x^*.

(ii) Let $x \in K_C$ and $x \supset x^*$, then $x^* \ominus_g x = (-1)(x \ominus x^*)$. Lemma 2.5(i), (ii) and $f(x) \subseteq (x^* \ominus_g x)$ provides that $A_2(x)$ is well-defined, and $x^* \subseteq (x \ominus (-1)f(x)) = A_2(x)$ holds for every $x \in \{x \in K_C \mid x \supset x^*\}$. Thus, for every $x_0 \in \{x \in K_C \mid x \supset x^*\}$ and $n \geq 1$, we have $A_2^n(x_0) \supseteq x^*$.

Based on Lemma 2.3(ii), $len(A_2^n(x_0)) = len(A_2^{n-1}(x_0)) - len(f(A_2^{n-1}(x_0)))$ for $n \geq 1$, that is, $len(A_2^n(x_0))$ decreases with n, and $len(x^*) \leq len(A_2^n(x_0)) \leq len(x_0)$. Thus, there exists $d \in [len(x^*), len(x_0)]$ such that $\lim_{n\to+\infty} len(A_2^n(x_0)) = d$, which implies that

$$\lim_{n\to+\infty} len(f(A_2^n(x_0))) = 0. \tag{11}$$

Additionally, $A_2^n(x_0) \supseteq x^*$ and $len(A_2^n(x_0)) \leq len(x_0)$ guarantee that $x_-^* - len(x_0) \leq [A_2^n(x_0)]_- \leq [A_2^n(x_0)]_+ \leq x_+^* + len(x_0)$, which implies that $\{A_2^n(x_0)\}$ is bounded. Applying the same method in the proof of (i), we can prove that $A_2^n(x_0) \to x^*$ as $n \to +\infty$. □

Example 4.3. *Consider the difference equation*

$$x(n+1) \ominus_g x(n) = \frac{1}{2}([1,4] \ominus_g x(n)),\ x(0) = x_0. \tag{12}$$

Let $f(x) = \frac{1}{2}([1,4] \ominus_g x)$, $x^* = [1,4]$, we can check that $f(x^*) = \{0\}$, and $len(f(x)) \neq 0$ holds for every $x \neq x^*$. According to Theorem 4.2, the type-I solution to (12) with $x_0 \subset x^*$ and the type-II solution to (12) with $x_0 \supset x^*$ converge to x^*.

For example, the type-I solution to problem (12) with $x_0 = [2,3]$ can be written as $x_I(n) = \frac{1}{2^n} \cdot [2,3] + \sum_{i=1}^{n} \frac{1}{2^i} \cdot [1,4]$, the type-II solution with $x_0 = [0,5]$ can be written as $x_{II}(n) = \frac{1}{2^n} \cdot [0,5] + \sum_{i=1}^{n} \frac{1}{2^i} \cdot [1,4]$. Let $n \to +\infty$, both of the solutions converge to $[1,4]$, see Figure 2.

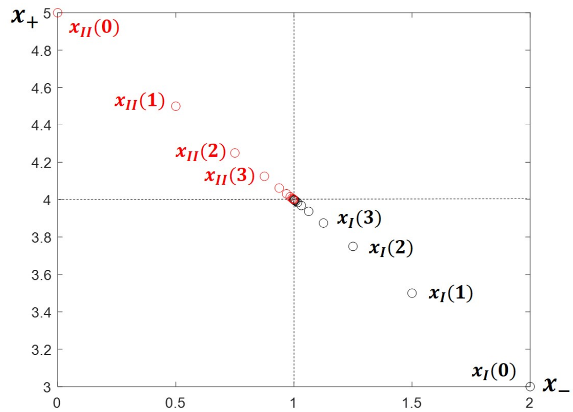

Figure 2: Type-I and type-II solutions to (12) converge to $[1, 4]$.

Corollary 4.4. *Suppose that $f : K_C \to K_C$ is continuous, with $f(\{0\}) = \{0\}$. If $len(f(x)) \neq 0$ and $f(x) \subseteq (-1)x$ hold on the set $\{x \in K_C \mid x \neq \{0\},\ 0 \in x\}$, then the type-II solution to (2) converges to $\{0\}$ for every $x_0 \in \{x \in K_C \mid 0 \in x\}$.*

Proof. Here $x^* = \{0\}$, $x^* \ominus_g x = (-1)(x \ominus x^*) = (-1)x$. According to Theorem 4.2(ii), the type-II solution to (2) converges to $\{0\}$. □

Example 4.5. *Consider the difference equation*

$$x(n+1) \ominus_g x(n) = \left(-\frac{1}{3}\right) \sin(x(n)),\ x(0) = x_0. \tag{13}$$

Let $x \in Y = \{x \in K_C \mid -1 \leq x_- \leq 0 \leq x_+ \leq 1\}$ and $f(x) = (-\frac{1}{3})\sin(x)$, then $f(\{0\}) = \{0\}$. For every $x \in Y$, we can check that $f(x) = \left[-\frac{1}{3}\sin(x_+), -\frac{1}{3}\sin(x_-)\right]$, and $(-1)x_+ \leq [f(x)]_- \leq [f(x)]_+ \leq (-1)x_-$ holds for $x \neq \{0\}$, which implies that $f(x) \subseteq (-1)x$.

For every $x_0 \in Y$, we can check that the type-II solution to (13) exists, and can be written as $x(n+1) = \left[(x(n))_- - \frac{1}{3}\sin((x(n))_-), (x(n))_+ - \frac{1}{3}\sin((x(n))_+)\right]$ for $n \geq 0$. See Figure 3.

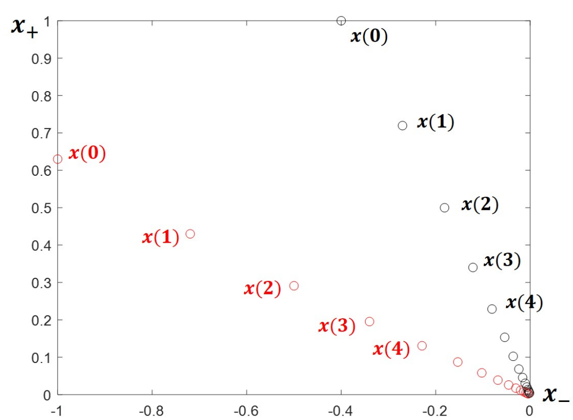

Figure 3: Type-II solutions to (13) with $x_0 = [-1, \frac{5}{8}]$, and $x_0 = \left[-\frac{2}{5}, 1\right]$, converge to $\{0\}$.

In the description above, we study the convergence of type-I solutions with $x_0 \subseteq x^*$ and type-II solutions with $x_0 \supseteq x^*$. Next, we consider $x_0 \preceq x^*$ and $x_0 \succeq x^*$.

Theorem 4.6. *Suppose that $f : K_C \to K_C$ is continuous and that there exist $x^* \in K_C$ and $r > 0$ such that $f(x^*) = \{0\}$, and $len(f(x)) \le len(x)$ holds on $[x^* - \{r\}, x^* + \{r\}]_{\preceq}$. The following conclusions are valid:*

(*i*) *If $x^* \ominus_g x \preceq f(x) \prec \{0\}$ for every $x \in [x^*, x^* + \{r\}]_{\preceq}$ with $x \neq x^*$, then problem (2) with $x_0 \in [x^*, x^* + \{r\}]_{\preceq}$ has at least one solution converging to x^*.*

(*ii*) *If $\{0\} \prec f(x) \preceq x^* \ominus_g x$ for every $x \in [x^* - \{r\}, x^*]_{\preceq}$ with $x \neq x^*$, then problem (2) with $x_0 \in [x^* - \{r\}, x^*]_{\preceq}$ has at least one solution converging to x^*.*

Proof. Let $x(0) = x_0$ and

$$x(n+1) = \begin{cases} A_1(x(n)), \text{ if } len(x(n)) \le len(x^*), \\ A_2(x(n)), \text{ if } len(x(n)) > len(x^*), \end{cases} \tag{14}$$

where $n = 0, 1, 2, ...$

(i) Let $x_0 \in [x^*, x^* + \{r\}]_{\preceq}$. For every $n \ge 0$, if $len(x(n)) \le len(x^*)$, then we have $x^* \ominus_g x(n) = (x^* \ominus x(n)) \preceq f(x(n)) \prec \{0\}$. Based on Lemma 2.7(ii) and (iii), we have that $x^* \preceq x(n) + f(x(n)) \preceq x(n)$ holds.

If $len(x(n)) > len(x^*)$, then $x^* \ominus_g x(n) = (-1)(x(n) \ominus x^*)$. According to $(x^* \ominus_g x(n)) \preceq f(x(n))$ and Lemma 2.7(i), (iii), we have $x^* \preceq x(n) \ominus (-1)f(x(n))$. In addition, $f(x(n)) \prec \{0\}$ provides that $[f(x(n))]_- \le [f(x(n))]_+ \le 0$, thus, $x(n) \preceq x(n) + (-1)f(x(n))$. Based on Lemma 2.7(iii), we have $x(n) \ominus (-1)f(x(n)) \preceq x(n)$.

As a result, the sequence in (14) satisfies $x_0 \succeq x(1) \succeq x(2) \succeq \cdots \succeq x(n) \succeq \cdots \succeq x^*$. According to Lemma 5, there exists $\tilde{x} \in [x^*, x^* + \{r\}]_{\preceq}$ such that $\lim_{n\to+\infty} x(n) = \tilde{x}$. If there exists $N > 0$ such that $len(x(n)) > len(x^*)$ for every $n > N$, then $x(n+1) = x(n) \ominus (-1)f(x(n))$. Let $n \to +\infty$, we get that $\tilde{x} = \tilde{x} \ominus (-1)f(\tilde{x})$, which implies that $f(\tilde{x}) = \{0\}$. Considering $f(x) \prec \{0\}$ on $\{x \in [x^*, x^* + \{r\}]_{\preceq} \mid x \neq x^*\}$, we have $\tilde{x} = x^*$.

Assume that there exists $\{x(n_k)\} \subseteq \{x(n)\}$ such that $len(x(n_k)) \le len(x^*)$, then $x(n_k + 1) = x(n_k) + f(x(n_k))$. Let $n_k \to +\infty$, we get that $\tilde{x} = \tilde{x} + f(\tilde{x})$, which implies that $\tilde{x} = x^*$.

(ii) Let $x_0 \in [x^* - \{r\}, x^*]_{\preceq}$. If $len(x(n)) \le len(x^*)$, then we have $\{0\} \prec f(x(n)) \preceq (x^* \ominus_g x(n)) = x^* \ominus x(n)$. Lemma 2.7(ii) guarantees that $x(n) \prec x(n) + f(x(n)) \preceq x^*$.

If $len(x(n)) > len(x^*)$, then $f(x(n)) \preceq (x^* \ominus_g x(n)) = (-1)(x(n) \ominus x^*)$. According to Lemma 2.7(i), (ii) and (iii), we have $x(n) \ominus (-1)f(x(n)) \preceq x^*$. Besides, $\{0\} \prec f(x(n))$ implies that $0 \le [f(x(n))]_- \le [f(x(n))]_+$, thus $x(n) \preceq x(n) \ominus (-1)f(x(n))$ holds.

Now, we have $x(n) \preceq x(n+1) \preceq x^*$, which implies that $x_0 \preceq x(1) \preceq x(2) \preceq \cdots \preceq x(n) \preceq \cdots \preceq x^*$. Using the same method as in the proof of (i), we can prove that $\lim_{n\to+\infty} x(n) = x^*$. □

Remark 4.7. *In Theorem 3.2, we have shown that, for the interval difference equations under gH-difference, there might exist many different solutions. Generally, we can not guarantee that all the solutions converge to an equilibrium point. For example, the type-I solution to (2) with $len(x(0)) > len(x^*)$ satisfies $len(x(n+1)) \ge len(x(n))$, and does not converge to x^*. Furthermore, the type-II solution to (2) with $len(x(0)) < len(x^*)$, provided that it exists, does not converge to x^* either.*

Now we provide an example to show that Theorem 4.6 is feasible.

Example 4.8. *Consider the difference equation*

$$x(n+1) \ominus_g x(n) = \frac{1}{3} \sin([5, 9] \ominus_g x(n)), \ x(0) = x_0. \tag{15}$$

Here $f(x) = \frac{1}{3} \sin([5, 9] \ominus_g x)$, $f([5 + k\pi, 9 + k\pi]) = \{0\}$, for $k = 0, \pm 1, \pm 2, \ldots$.

Let $x^* = [5, 9]$ and $r = 1$. For every $x \in [x^* - \{r\}, x^* + \{r\}]_{\preceq}$, we can check that $len(x^*) - 2r \le len(x) \le len(x^*) + 2r$ and $x^* \ominus_g x \subseteq [-1, 1]$ hold. Thus, $len(f(x)) \le \frac{1}{3} len(\sin([-1, 1])) < \frac{2}{3}$, which implies that $len(f(x)) < len(x)$ for $x \in [x^* - \{r\}, x^* + \{r\}]_{\preceq}$. In addition, the definition of f guarantees that the conditions in Theorem 4.6(i) and (ii) are all satisfied.

Let $x_0 = [5.9, 10]$, $[5.1, 10]$, $[4.1, 8]$, $[4, 8.8]$, and

$$x(n+1) = \left[x_-(n) + \frac{\sin(5 - x_-(n))}{3}, x_+(n) + \frac{\sin(9 - x_+(n))}{3} \right],$$

for $n = 0, 1, 2, \ldots$. We can check that the sequences $\{x(n)\}$ that are derived from the previous scheme are all solutions to (15), and converge to $x^* = [5, 9]$ (see Figure 4).

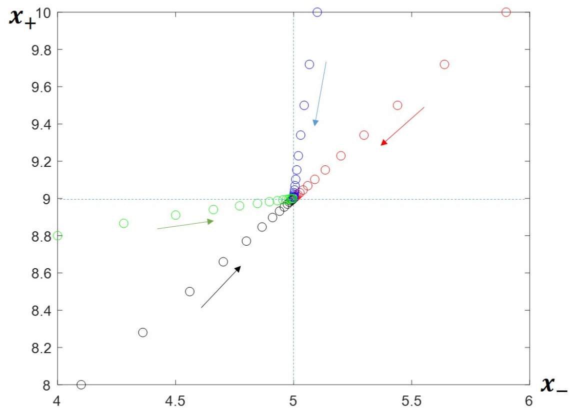

Figure 4: The solutions to (15) with different initial values converge to $x^* = [5, 9]$.

Based on Theorem 4.6, we introduce the following results without proof.

Corollary 4.9. *Suppose that $f : K_C \to K_C$ is continuous, with $f(\{0\}) = \{0\}$. Assume that there exists $r > 0$ such that $len(f(x)) \leq len(x)$ holds for every $x \in [\{-r\}, \{r\}]_{\preceq}$. The following conclusions are valid:*

(i) If $(-1)x \preceq f(x) \prec \{0\}$ for every $x \in [\{0\}, \{r\}]_{\preceq}$ with $x \neq \{0\}$, then problem (2) with $x_0 \in [\{0\}, \{r\}]_{\preceq}$ has at least one solution converging to $\{0\}$.

(ii) If $\{0\} \prec f(x) \preceq (-1)x$ for every $x \in [\{-r\}, \{0\}]_{\preceq}$ with $x \neq \{0\}$, then problem (2) with $x_0 \in [\{-r\}, \{0\}]_{\preceq}$ has at least one solution converging to $\{0\}$.

Example 4.10. *Consider the difference equation*

$$x(n+1) \ominus_g x(n) = \left(-\frac{1}{3}\right) \sin(x(n)), \;\; x(0) = x_0. \tag{16}$$

Let $f(x) = \left(-\frac{1}{3}\right)\sin(x)$. For every $x \subseteq [-\pi, \pi]$, $f(x)$ can be written as $f(x) = [-\frac{\sin a}{3}, -\frac{\sin b}{3}]$, where $a, b \in x$ are such that $\min\limits_{t\in x}\left(-\frac{\sin t}{3}\right) = -\frac{\sin a}{3}$ and $\max\limits_{t\in x}\left(-\frac{\sin t}{3}\right) = -\frac{\sin b}{3}$. Thus, we have $len(f(x)) = \frac{1}{3}|\sin b - \sin a| \leq |b - a| \leq len(x)$.

On the other hand, for every $r \in [-\pi, 0) \cup (0, \pi]$, we can check that $0 \leq \frac{\sin(r)}{3r} < 1$, which implies that the conditions in Corollary 4.9(i) and (ii) are all satisfied for every $r \in (0, \pi)$.

Let $r = \frac{9\pi}{10}$, $x_0 = [1.5, 1.6]$, $[0.5, 2.7]$, $[-2.8, -1]$, $[-2, -1.5]$, and

$$x(n+1) = \left[(x(n))_- - \frac{1}{3}\left(\sin(x(n))\right)_-, (x(n))_+ - \frac{1}{3}\left(\sin(x(n))\right)_+\right],$$

for $n = 0, 1, 2, \ldots$. We can check that the corresponding sequences $\{x(n)\}$ are all type-II solutions to (16), and converge to $\{0\}$ (see Figure 5).

5 Conclusions

In connection with the definition of the gH-derivative, first-order nonlinear fuzzy-valued or interval-valued differential equations under gH-differentiability can be written as a difference equation by approximation of the instantaneous rate of change by the averate rate of change on an interval of a certain length. Thus, the theory of difference equations written in the form $x(n+1) = f(n, x(n))$ is not enough to explain adequately the behavior of the numerical solutions. In this paper, we have shown that the problem (2) might have 2^N different solutions for $0 \leq n \leq N$, some of which are convergent to an equilibrium point under certain conditions, while others are not.

Consequently, it seems impossible to consider the stability of interval-valued difference equations under gH-differentiability. We can focus, instead, on considering the behavior of specific solutions, for example, the type-I and type-II solutions,

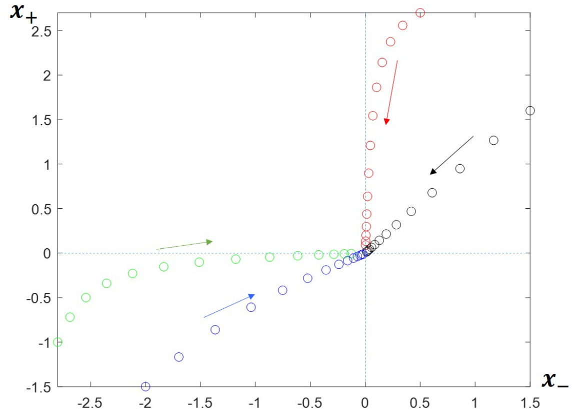

Figure 5: The type-II solutions to (16) converge to $\{0\}$.

or discuss the existence of solutions with specific properties. In this paper, we have provided sufficient conditions for the existence of convergent solutions.

The results presented in this paper are helpful to understand the diversity and stability of solutions to interval-valued differential equations under gH-differentiability. Since the study of interval systems is considered as a necessary step towards the analysis of fuzzy systems [21], we think that our results can also be used to investigate the dynamical properties of the solutions to fuzzy differential equations under the notion of gH-differentiability.

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