Iranian Journal of Fuzzy Systems

Volume 20, Number 2, (2023), pp. 57-67

Original paper

Completeness of L-quasi-uniform convergence spaces

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Abstract

The aim of this paper is to study the completeness of L-quasi-uniform convergence spaces and L-quasi-uniform spaces. Firstly, we describe L-quasi-uniform convergence spaces as enriched categories. Then we give two kinds of completeness of L-quasi-uniform convergence spaces and show that Lawvere completeness implies Cauchy completeness. Finally, we use the Cauchy completeness of L-quasi-uniform convergence spaces to define the Cauchy completeness of L-quasi-uniform spaces, and show that Cauchy completeness is equivalent to Lawvere completeness in L-quasi-uniform spaces.

Keywords: L-quasi-uniform convergence space, L-quasi-uniform space, enriched category, Cauchy completeness, Lawvere completeness.

1 Introduction

Uniformity plays an important role in the research and applications of topology. But it is known that the category of uniform spaces with uniformly continuous mappings as morphisms is not Cartesian closed. This leads to the concept of uniform convergence spaces first proposed by Cook and Fischer [7]. Later it was studied in a slightly modified form proposed by Wyler [29], since in this case natural function spaces exist, which had been demonstrated by Lee [23]. Further generalizations of uniform limit spaces were obtained by omitting some of their defining axioms, e.g., semiuniform limit spaces and semiuniform convergence spaces (these names go back to Wyler [29]). For the latticed-valued case, different filters produce different lattice-valued uniform convergence spaces, and many kinds of lattice-valued uniform convergence spaces are introduced and studied (see [8, 10, 11, 12, 19, 20, 21, 25, 26]). For example, Jäger and Burton [20] introduced stratified L-uniform convergence spaces. Later, Craig and Jäger generalized the lattice context of these spaces from complete Heyting algebras to the case of enriched lattices in [8]. Reid and Richardson defined and studied $\top$-uniform convergence spaces in [26]. The L-quasi-uniform convergence space to be studied in this paper is just the asymmetric case of stratified L-uniform convergence space in [20].

Completeness theory is an important content of (quasi-)uniform spaces and different types of filters are the key tools to study the completeness of (quasi-)uniform spaces. For example, Lowen studied the completeness of fuzzy uniform spaces by using prefilters in [24]; In [17], Höhle studied the $\top$-completeness of probabilistic uniform spaces with the help of $\top$-filters; J. Gutiérrez García and M. A. De Prada Vicente in [14] studied the completeness of Hutton [0,1]-quasi-uniform spaces based on tight and stratified L-filters. When discussing asymmetric quasi-uniform spaces, pair filters become important tools to study their completeness. For example, in [30], Yue and Fang have studied a kind of completeness of probabilistic quasi-uniform spaces based on pair $\top$-filters. Wang and Yue studied the Cauchy completeness of fuzzy quasi-uniform spaces drawing support from pair saturated prefilters in [28]. It can be seen that the completeness theory has been well studied in (quasi-)uniform spaces. But there are still many problems on completeness of lattice-valued quasi-uniform spaces worthy of further study. For example, the Cauchy completeness of L-quasi-uniform space has not been studied yet. Considering the close relationship between L-quasi-uniform space and L-quasi-uniform convergence space, in this paper, we study the Cauchy completeness of L-quasi-uniform convergence space by using pair L-filters, and then the Cauchy completeness of L-quasi-uniform space is studied.

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Received: April 2022; Revised: October 2022; Accepted: November 2022.

https://doi.org/10.22111/IJFS.2023.7556

Since Lawvere creatively regarded the generalized metric space as an enriched category in [22], enriched category has become a powerful tool to study topological structures (see [2, 3, 4, 5, 6, 15, 16, 22, 31]). For example, Hofmann and Reis regarded the (generalized) probabilistic (quasi-)metric space as an enriched category and studied its completeness in [15]. Wang and Yue regarded L-quasi-uniform space as an enriched category and studied its Lawvere completeness by using the enriched category theory in [27]. So in this paper, we want to regard L-quasi-uniform convergence space as an enriched category and to study its Lawvere completeness.

In this paper, the completeness of L-quasi-uniform convergence spaces will be studied from two perspectives. One is to define Cauchy complete directly by using pair L-filters, the other is to define Lawvere complete based on enriched category theory. As an application of Cauchy completeness in L-quasi-uniform convergence spaces to L-quasi-uniform spaces, we show that L-quasi-uniform space is Cauchy complete if and only if it is Lawvere complete if and only if the generated L-quasi-uniform convergence space is Cauchy complete.

2 Preliminaries

In this paper, $(L, \leq, \wedge)$ is a complete Heyting algebra, that is, $(L, \leq)$ is a complete lattice with the top element $\top(= \wedge\emptyset)$ and the bottom element $\bot(= \vee\emptyset)$, and the infinite distributive law $\alpha \wedge \left(\bigvee_{j\in J} \beta_j\right) = \bigvee_{j\in J} \alpha \wedge \beta_j$ holds. At this point, there exists a binary operation $\rightarrow: L \times L \rightarrow L$ defined by

$$\alpha \rightarrow \beta = \bigvee\{\gamma \in L \mid \alpha \wedge \gamma \leq \beta\},$$

called the implication (operation). Further, $\wedge$ and $\rightarrow$ form an adjoint pair in the sense of $\alpha \wedge \gamma \leq \beta \Longleftrightarrow \gamma \leq \alpha \rightarrow \beta$ for all α, β, $\gamma \in L$.

An L-subset on a set X is a map from X to L, and the family of all L-subsets on X will be denoted by L^X, called the L-power set of X. By $\bot_X$ and $\top_X$, we denote the constant L-subsets on X taking the value $\bot$ and $\top$, respectively. Except for the top element and bottom element, we do not distinguish an element $\alpha \in L$ and the constant function $\alpha_X : X \rightarrow L$ such that $\alpha_X(x) = \alpha$ for all $x \in X$. All algebraic operations on L can be extended to L^X pointwise. For example, $(A \vee B)(x) = A(x) \vee B(x)$, $(A \wedge B)(x) = A(x) \wedge B(x)$ for all $A, B \in L^X$ and $x \in X$.

For a give set X, $S_X(-,-) : L^X \times L^X \rightarrow L$ given by $S_X(A, B) = \bigwedge_{x\in X}(A(x) \rightarrow B(x))$ for $A, B \in L^X$, is interpreted as the degree to which A is a subset of B. $S_X(-,-)$ is usually called fuzzy inclusion order or subsethood degree of L-subsets. For convenience, $S_X(-,-)$ is simplified by $S(-,-)$ in this paper.

Lemma 2.1. [1] *For $A, B, C, D \in L^X$, $S(-,-)$ satisfies the following statements:*
(1) $A \leq B \Longleftrightarrow S(A, B) = \top$;
(2) $S(A, B) \wedge S(C, D) \leq S(A \wedge C, B \wedge D)$;
(3) $A \leq B$ *means* $S(C, A) \leq S(C, B)$ *and* $S(B, D) \leq S(A, D)$.

Here is a review of the definition of fuzzy relation and the relationship between it and general map.

An L-relation $r : X \nrightarrow Y$ from X to Y is a map $r : X \times Y \rightarrow L$.

In fact, a map $f : X \rightarrow Y$ can be seen as an L-relation $f : X \nrightarrow Y$:

$$f(x, y) = \begin{cases} \top, & y = f(x), \\ \bot, & \text{others}. \end{cases}$$

and its dual L-relation $f^\circ : Y \nrightarrow X$ induced by $f : X \rightarrow Y$ is as follows:

$$f^\circ(y, x) = \begin{cases} \top, & y = f(x), \\ \bot, & \text{others}. \end{cases}$$

Next, we review some concepts about stratified L-filters and L-quasi-uniform spaces.

Definition 2.2. [18] *Let X be a nonempty set. A map $\mathscr{F} : L^X \rightarrow L$ satisfying the following conditions:*
($\mathscr{F}$0) $\mathscr{F}(\top_X) = \top$;
($\mathscr{F}$1) $A \leq B$ *implies* $\mathscr{F}(A) \leq \mathscr{F}(B)$ *for all* $A, B \in L^X$;
($\mathscr{F}$2) $\mathscr{F}(A_1) \wedge \mathscr{F}(A_2) \leq \mathscr{F}(A_1 \wedge A_2)$ *for all* $A_1, A_2 \in L^X$,
is called an L-filter on X. If in addition, L-filter $\mathscr{F}$ satisfies:
(S) $\alpha \wedge \mathscr{F}(A) \leq \mathscr{F}(\alpha \wedge A)$ *for all* $\alpha \in L$ *and* $A \in L^X$,
then $\mathscr{F}$ is called a stratified L-filter on X. If in addition, stratified L-filter $\mathscr{F}$ satisfies:
(T) $\mathscr{F}(\alpha) = \alpha$ *for all* $\alpha \in L$,
then $\mathscr{F}$ is called a tight and stratified L-filter on X.

The set of all stratified L-filters on X is denoted by $\mathscr{F}_L^S(X)$. The partial order relation $\preceq$ on $\mathscr{F}_L^S(X)$ is defined as:

$$\forall \mathscr{F}, \mathscr{G} \in \mathscr{F}_L^S(X), \mathscr{F} \preceq \mathscr{G} \Longleftrightarrow \mathscr{G} \leq \mathscr{F}.$$

Example 2.3. [9, 18, 27] *Let X, Y be nonempty sets.*

(1) *For each $x \in X$, the map $\mathscr{F}_x : L^X \to L$ defined by $\mathscr{F}_x(A) := A(x)$ for all $A \in L^X$, is a stratified L-filter on X.*

(2) *For each $A \in L^{X\times Y}$, the map $[A] : L^{X\times Y} \to L$ defined by $[A](B) := S(A, B)$ for all $B \in L^{X\times Y}$, is a stratified L-filter on $X \times Y$.*

Definition 2.4. [13] *Let X be a nonempty set. A stratified L-filter $\mathscr{U} : L^{X\times X} \to L$ on $X \times X$ satisfying the following conditions:*

($\mathscr{U}0$) $\mathscr{U}(U) \leq \bigwedge_{x\in X} U(x, x)$ *for all* $U \in L^{X\times X}$;

($\mathscr{U}1$) $\mathscr{U}(U) \leq \bigvee_{W\circ V\leq U} \mathscr{U}(W) \wedge \mathscr{U}(V)$ *for all* $U \in L^{X\times X}$, *where* $W \circ V(x, y) = \bigvee_{z\in X} V(x, z) \wedge W(z, y)$,

is called an L-quasi-uniform structure, and the pair $(X, \mathscr{U})$ an L-quasi-uniform space.

The definition of L-quasi-uniform convergence space is given below.

Definition 2.5. [11, 20] *Let X be a nonempty set. A map $\Gamma : \mathscr{F}_L^S(X \times X) \to L$ satisfying the following conditions:*

($\Gamma1$) $\Gamma([1_X]) = \top$;

($\Gamma2$) $S(\mathscr{F}, \mathscr{G}) \leq \Gamma(\mathscr{F}) \to \Gamma(\mathscr{G})$;

($\Gamma3$) $\Gamma(\mathscr{F}) \wedge \Gamma(\mathscr{G}) \leq \Gamma(\mathscr{F} \wedge \mathscr{G})$;

($\Gamma4$) $\Gamma(\mathscr{F}) \wedge \Gamma(\mathscr{G}) \leq \Gamma(\mathscr{F} \circ \mathscr{G})$, *where* $\mathscr{F} \circ \mathscr{G}(A) = \bigvee_{U\circ V\leq A} \mathscr{F}(U) \wedge \mathscr{G}(V)$ *for all* $A \in L^{X\times X}$,

is called an L-quasi-uniform convergence structure, and the pair (X, Γ) an L-quasi-uniform convergence space.

Remark 2.6. *In* [13], $\bigvee_{U\circ V\leq A} \mathscr{F}(U) \wedge \mathscr{G}(V) = \bigvee_{U,V} \mathscr{F}(U) \wedge \mathscr{G}(V) \wedge S(U \circ V, A)$ *has been verified.*

3 L-quasi-uniform convergence spaces as enriched categories

In this section, it will be explained that L-quasi-uniform convergence spaces can be regarded as enriched categories.

Definition 3.1. *If a map $\Phi : \mathscr{F}_L^S(X \times Y) \to L$ satisfies* ($\Gamma2$) *and* ($\Gamma3$), *then Φ is called a prorelation from X to Y, denoted by $\Phi : X \multimap\!\!\!\to Y$.*

Remark 3.2. (1) *Let $\Phi : \mathscr{F}_L^S(X \times Y) \to L$ and $\Psi : \mathscr{F}_L^S(Y \times Z) \to L$ be two prorelations. Define the composition $\Psi \circ \Phi$ as follows:*

$$\forall \mathscr{W} \in \mathscr{F}_L^S(X \times Z), \Psi \circ \Phi(\mathscr{W}) = \bigvee_{\psi,\phi} \Phi(\phi) \wedge \Psi(\psi) \wedge S(\psi \circ \phi, \mathscr{W}),$$

where $\psi \circ \phi(A) = \bigvee_{B\circ C\leq A} \psi(B) \wedge \phi(C)$ for all $A \in L^{X\times Z}$. It is routine to check that $\Psi \circ \Phi$ is a prorelation from X to Z. In fact, for all $\mathscr{F}, \mathscr{G} \in \mathscr{F}_L^S(X \times Z)$,

($\Gamma2$): *$S(\mathscr{F}, \mathscr{G}) \wedge \Psi \circ \Phi(\mathscr{F}) \leq \Psi \circ \Phi(\mathscr{G})$ can be obtained by the following formulas*

$$\begin{aligned} S(\mathscr{F}, \mathscr{G}) \wedge \Psi \circ \Phi(\mathscr{F}) &= S(\mathscr{F}, \mathscr{G}) \wedge \bigvee_{\psi,\phi} \Phi(\phi) \wedge \Psi(\psi) \wedge S(\psi \circ \phi, \mathscr{F}) \\ &\leq \bigvee_{\psi,\phi} \Phi(\phi) \wedge \Psi(\psi) \wedge S(\psi \circ \phi, \mathscr{G}) \\ &= \Psi \circ \Phi(\mathscr{G}). \end{aligned}$$

($\Gamma3$): *$\Psi \circ \Phi(\mathscr{F}) \wedge \Psi \circ \Phi(\mathscr{G}) \leq \Psi \circ \Phi(\mathscr{F} \wedge \mathscr{G})$ can be obtained by the following formulas*

$$\begin{aligned} \Psi \circ \Phi(\mathscr{F}) \wedge \Psi \circ \Phi(\mathscr{G}) &= \bigvee_{\psi_1,\phi_1} \Phi(\phi_1) \wedge \Psi(\psi_1) \wedge S(\psi_1 \circ \phi_1, \mathscr{F}) \wedge \bigvee_{\psi_2,\phi_2} \Phi(\phi_2) \wedge \Psi(\psi_2) \wedge S(\psi_2 \circ \phi_2, \mathscr{G}) \\ &\leq \bigvee_{\psi_1,\phi_1,\psi_2,\phi_2} \Phi(\phi_1 \wedge \phi_2) \wedge \Psi(\psi_1 \wedge \psi_2) \wedge S((\psi_1 \circ \phi_1) \wedge (\psi_2 \circ \phi_2), \mathscr{F} \wedge \mathscr{G}) \\ &\leq \bigvee_{\psi_1,\phi_1,\psi_2,\phi_2} \Phi(\phi_1 \wedge \phi_2) \wedge \Psi(\psi_1 \wedge \psi_2) \wedge S((\psi_1 \wedge \psi_2) \circ (\phi_1 \wedge \phi_2), \mathscr{F} \wedge \mathscr{G}) \\ &\leq \Psi \circ \Phi(\mathscr{F} \wedge \mathscr{G}). \end{aligned}$$

(2) *Let $\Phi : \mathscr{F}_L^S(X\times Y)\to L$ be a prorelation, then $\Phi\circ[[1_X]]=\Phi$ and $[[1_Y]]\circ\Phi=\Phi$ hold. In fact, take $\Phi\circ[[1_X]]=\Phi$ as an example. For all $\mathscr{F}\in\mathscr{F}_L^S(X\times Y)$,*

$$\begin{aligned}\Phi\circ[[1_X]](\mathscr{F}) &= \bigvee_{\mathscr{H},\mathscr{G}} \Phi(\mathscr{H})\wedge S([1_X],\mathscr{G})\wedge S(\mathscr{H}\circ\mathscr{G},\mathscr{F})\\ &= \bigvee_{\mathscr{H}} \Phi(\mathscr{H})\wedge S(\mathscr{H}\circ[1_X],\mathscr{F})\\ &= \Phi(\mathscr{F}).\end{aligned}$$

(3) *Let $\Phi : \mathscr{F}_L^S(X\times Y)\to L$, $\Psi : \mathscr{F}_L^S(Y\times Z)\to L$, $\Theta : \mathscr{F}_L^S(Z\times W)\to L$ be prorelations. It is routine to check that $\Theta\circ(\Psi\circ\Phi)=(\Theta\circ\Psi)\circ\Phi$. In fact, for all $\mathscr{W}\in\mathscr{F}_L^S(X\times W)$,*

$$\Theta\circ(\Psi\circ\Phi)(\mathscr{W})=\bigvee_{\mathscr{H},\phi}\bigvee_{\mathscr{G},\mathscr{F}}\Theta(\mathscr{H})\wedge\Psi(\mathscr{G})\wedge\Phi(\mathscr{F})\wedge S(\mathscr{H}\circ\phi,\mathscr{W})\wedge S(\mathscr{G}\circ\mathscr{F},\phi),$$

$$(\Theta\circ\Psi)\circ\Phi(\mathscr{W})=\bigvee_{\psi,\mathscr{F}_1}\bigvee_{\mathscr{H}_1,\mathscr{G}_1}\Theta(\mathscr{H}_1)\wedge\Psi(\mathscr{G}_1)\wedge\Phi(\mathscr{F}_1)\wedge S(\mathscr{H}_1\circ\mathscr{G}_1,\psi)\wedge S(\psi\circ\mathscr{F}_1,\mathscr{W}).$$

On the one side, for all $\psi\in\mathscr{F}_L^S(Y\times W),\mathscr{F}_1\in\mathscr{F}_L^S(X\times Y),\mathscr{G}_1\in\mathscr{F}_L^S(Y\times Z),\mathscr{H}_1\in\mathscr{F}_L^S(Z\times W)$,

$$\begin{aligned}&\Theta(\mathscr{H}_1)\wedge\Psi(\mathscr{G}_1)\wedge\Phi(\mathscr{F}_1)\wedge S(\mathscr{H}_1\circ\mathscr{G}_1,\psi)\wedge S(\psi\circ\mathscr{F}_1,\mathscr{W})\\ \le\ &\Theta(\mathscr{H}_1)\wedge\Psi(\mathscr{G}_1)\wedge\Phi(\mathscr{F}_1)\wedge S(\mathscr{H}_1\circ\mathscr{G}_1\circ\mathscr{F}_1,\psi\circ\mathscr{F}_1)\wedge S(\psi\circ\mathscr{F}_1,\mathscr{W})\\ \le\ &\Theta(\mathscr{H}_1)\wedge\Psi(\mathscr{G}_1)\wedge\Phi(\mathscr{F}_1)\wedge S(\mathscr{H}_1\circ\mathscr{G}_1\circ\mathscr{F}_1,\mathscr{W})\\ =\ &\Theta(\mathscr{H}_1)\wedge\Psi(\mathscr{G}_1)\wedge\Phi(\mathscr{F}_1)\wedge S(\mathscr{H}_1\circ\phi,\mathscr{W})\wedge S(\mathscr{G}_1\circ\mathscr{F}_1,\phi)\quad (let\ \mathscr{G}_1\circ\mathscr{F}_1=\phi)\\ \le\ &\bigvee_{\mathscr{H},\phi}\bigvee_{\mathscr{G},\mathscr{F}}\Theta(\mathscr{H})\wedge\Psi(\mathscr{G})\wedge\Phi(\mathscr{F})\wedge S(\mathscr{H}\circ\phi,\mathscr{W})\wedge S(\mathscr{G}\circ\mathscr{F},\phi).\end{aligned}$$

The other aspect is similar.

A map $\varphi : (X,\Gamma)\to(Y,\Lambda)$ is called CL-uniform continuous if $\Gamma(\mathscr{F})\le\Lambda((\varphi\times\varphi)^{\Rightarrow}(\mathscr{F}))$ for all $\mathscr{F}\in\mathscr{F}_L^S(X\times X)$, where $(\varphi\times\varphi)^{\Rightarrow}(\mathscr{F})(B)=\mathscr{F}((\varphi\times\varphi)^{\leftarrow}(B))=\mathscr{F}(\varphi^\circ\circ B\circ\varphi)$ for all $B\in L^{Y\times Y}$.

Lemma 3.3. *Let (X,Γ), (Y,Λ) be two L-quasi-uniform convergence spaces. For the map $f : (X,\Gamma)\to(Y,\Lambda)$, the following statements are equivalent:*
(1) *f is CL-uniform continuous;*
(2) $[[f]]\circ\Gamma\le\Lambda\circ[[f]]$;
(3) $\Gamma\circ[[f^\circ]]\le[[f^\circ]]\circ\Lambda$.

Proof. It can be seen from [27] that for all $\mathscr{F}\in\mathscr{F}_L^S(X\times X)$, $[f]\circ\mathscr{F}(D)=\mathscr{F}(f^\circ\circ D)$ for all $D\in L^{X\times Y}$ and $\mathscr{F}\circ[f^\circ](A)=\mathscr{F}(A\circ f)$ for all $A\in L^{Y\times X}$ hold. Then, it follows that

$$[f]\circ\mathscr{H}\circ[f^\circ](B)=[f]\circ\mathscr{H}(B\circ f)=\mathscr{H}(f^\circ\circ B\circ f)=(f\times f)^{\Rightarrow}(\mathscr{H})(B),$$

for all $B\in L^{Y\times Y}$.

(1) $\Rightarrow$ (2) For all $\mathscr{F}\in\mathscr{F}_L^S(X\times Y)$,

$$\begin{aligned}\Lambda\circ[[f]](\mathscr{F}) &= \bigvee_{\mathscr{G},\mathscr{H}}\Lambda(\mathscr{G})\wedge S([f],\mathscr{H})\wedge S(\mathscr{G}\circ\mathscr{H},\mathscr{F})\\ &= \bigvee_{\mathscr{G}}\Lambda(\mathscr{G})\wedge S(\mathscr{G}\circ[f],\mathscr{F})\\ &\ge \Lambda(\mathscr{F}\circ[f^\circ])\wedge S(\mathscr{F}\circ[f^\circ]\circ[f],\mathscr{F})\\ &= \Lambda(\mathscr{F}\circ[f^\circ]),\end{aligned}$$

and

$$\begin{aligned}
[[f]]\circ\Gamma(\mathscr{F}) &= \bigvee_{\mathscr{G},\mathscr{H}} S([f],\mathscr{G})\wedge\Gamma(\mathscr{H})\wedge S(\mathscr{G}\circ\mathscr{H},\mathscr{F})\\
&\leq \bigvee_{\mathscr{G},\mathscr{H}} S([f],\mathscr{G})\wedge\Lambda((f\times f)^{\Rightarrow}(\mathscr{H}))\wedge S(\mathscr{G}\circ\mathscr{H},\mathscr{F})\\
&= \bigvee_{\mathscr{H}} S([f]\circ\mathscr{H},\mathscr{F})\wedge\Lambda((f\times f)^{\Rightarrow}(\mathscr{H}))\\
&\leq \bigvee_{\mathscr{H}} S([f]\circ\mathscr{H}\circ[f^{\circ}],\mathscr{F}\circ[f^{\circ}])\wedge\Lambda((f\times f)^{\Rightarrow}(\mathscr{H}))\\
&= \bigvee_{\mathscr{H}} S((f\times f)^{\Rightarrow}(\mathscr{H}),\mathscr{F}\circ[f^{\circ}])\wedge\Lambda((f\times f)^{\Rightarrow}(\mathscr{H}))\\
&\leq \Lambda(\mathscr{F}\circ[f^{\circ}])\\
&\leq \Lambda\circ[[f]](\mathscr{F}).
\end{aligned}$$

(2) $\Rightarrow$ (1) For all $\mathscr{F}\in\mathscr{F}_L^S(X\times Y), [[f]]\circ\Gamma(\mathscr{F})=\bigvee_{\mathscr{G},\mathscr{H}} S([f],\mathscr{G})\wedge\Gamma(\mathscr{H})\wedge S(\mathscr{G}\circ\mathscr{H},\mathscr{F})$. Let $\mathscr{F}=[f]\circ\mathscr{H}$ and $\mathscr{G}=[f]$, then $[[f]]\circ\Gamma([f]\circ\mathscr{H})\geq\Gamma(\mathscr{H})$. According to the condition (2),

$$\begin{aligned}
\Gamma(\mathscr{H}) &\leq [[f]]\circ\Gamma([f]\circ\mathscr{H})\\
&\leq \Lambda\circ[[f]]([f]\circ\mathscr{H})\\
&= \bigvee_{\mathscr{M},\mathscr{N}} \Lambda(\mathscr{M})\wedge S([f],\mathscr{N})\wedge S(\mathscr{M}\circ\mathscr{N},[f]\circ\mathscr{H})\\
&= \bigvee_{\mathscr{M}} \Lambda(\mathscr{M})\wedge S(\mathscr{M}\circ[f],[f]\circ\mathscr{H})\\
&\leq \bigvee_{\mathscr{M}} \Lambda(\mathscr{M})\wedge S(\mathscr{M}\circ[f]\circ[f^{\circ}],[f]\circ\mathscr{H}\circ[f^{\circ}])\\
&\leq \bigvee_{\mathscr{M}} \Lambda(\mathscr{M})\wedge S(\mathscr{M},[f]\circ\mathscr{H}\circ[f^{\circ}])\\
&\leq \Lambda([f]\circ\mathscr{H}\circ[f^{\circ}])\\
&= \Lambda((f\times f)^{\Rightarrow}(\mathscr{H})).
\end{aligned}$$

Thus f is CL-uniform continuous. Therefore, (1) $\Leftrightarrow$ (2). And it is easy to see that (2) and (3) are equivalent. □

Let $f:(X,\Gamma)\to(Y,\Lambda)$, $g:(Y,\Lambda)\to(Z,\Sigma)$ be two CL-uniform continuous maps, then $[[g]]\circ[[f]]=[[g\circ f]]$ holds. In fact, in [11], Fang has proved that $S(\mathscr{F},\mathscr{G})\wedge S(\mathscr{H},\mathscr{K})\leq S(\mathscr{F}\circ\mathscr{H},\mathscr{G}\circ\mathscr{K})$ for $\mathscr{F},\mathscr{G},\mathscr{H},\mathscr{K}\in\mathscr{F}_L^S(X\times X)$. Then, for all $\mathscr{F}\in\mathscr{F}_L^S(X\times Z)$,

$$\begin{aligned}
[[g]]\circ[[f]](\mathscr{F}) &= \bigvee_{\mathscr{G},\mathscr{H}} S([g],\mathscr{G})\wedge S([f],\mathscr{H})\wedge S(\mathscr{G}\circ\mathscr{H},\mathscr{F})\\
&= S([g]\circ[f],\mathscr{F})\\
&= S([g\circ f],\mathscr{F})\\
&= [[g\circ f]](\mathscr{F}).
\end{aligned}$$

Therefore, $[[g]]\circ[[f]]=[[g\circ f]]$. Then $[[f]]\circ[[f^{\circ}]]=[[f\circ f^{\circ}]]\leq[[1_Y]]$ and $[[f^{\circ}]]\circ[[f]]=[[f^{\circ}\circ f]]\geq[[1_X]]$ hold.

Based on discussion above, an L-quasi-uniform convergence structure Γ on X can be seen as a prorelation $\Gamma:\mathscr{F}_L^S(X\times X)\to L$ satisfies the following conditions:

$$[[1_X]]\leq\Gamma,\ \ \Gamma\circ\Gamma\leq\Gamma.$$

Similarly, a CL-uniform continuous map $f:(X,\Gamma)\to(Y,\Lambda)$ can be seen as a map $f:X\to Y$ such that

$$[[f]]\circ\Gamma\leq\Lambda\circ[[f]].$$

4 Two kinds of completeness of L-quasi-uniform convergence spaces

In this section, we give two kinds of completeness of L-quasi-uniform convergence spaces. First, we introduce the Cauchy completeness of L-quasi-uniform convergence spaces by using pair L-filters.

Definition 4.1. *$(\mathscr{F},\mathscr{G})$ is called a pair L-filter if $\mathscr{F}: L^X \to L, \mathscr{G}: L^X \to L$ are stratified L-filters on X and fulfills $\mathscr{F}(B)\wedge\mathscr{G}(C) \leq \bigvee_{x\in X}(B(x)\wedge C(x))$ for all $B, C \in L^X$.*

Definition 4.2. *Let (X,Γ) be an L-quasi-uniform convergence space and $(\mathscr{F},\mathscr{G})$ be a pair L-filter.*

(1) $(\mathscr{F},\mathscr{G})$ is called a Cauchy pair L-filter if $\Gamma(\mathscr{F}\times\mathscr{G}) = \top$, where

$$\mathscr{F}\times\mathscr{G}(A) = \bigvee_{B\times C\leq A}\mathscr{F}(C)\wedge\mathscr{G}(B) = \bigvee_{B,C}\mathscr{F}(C)\wedge\mathscr{G}(B)\wedge S(B\times C, A),$$

for all $A\in L^{X\times X}$ and $B\times C(x,y) = B(x)\wedge C(y)$ for all $x,y\in X$.

(2) $(\mathscr{F},\mathscr{G})$ is convergent to x if $\Gamma(\mathscr{F}\times[x]) = \top$ and $\Gamma([x]\times\mathscr{G}) = \top$.

Remark 4.3. (1) *It is routine to check that $\mathscr{F}\times\mathscr{G}$ is an L-filter on $X\times X$. We only show that $\mathscr{F}\times\mathscr{G}$ is stratified. In fact, for all $a\in L, A\in L^{X\times X}$, $a\wedge(\mathscr{F}\times\mathscr{G})(A) = a\wedge\bigvee_{C\times B\leq A}\mathscr{F}(B)\wedge\mathscr{G}(C)$ and $(\mathscr{F}\times\mathscr{G})(a\wedge A) = \bigvee_{E\times D\leq a\wedge A}\mathscr{F}(D)\wedge\mathscr{G}(E)$. When $C\times B\leq A$, we observe that $a\wedge(C\times B) = (a\wedge C)\times B\leq a\wedge A$. Let $E = a\wedge C$ and $D = B$, then*

$$\bigvee_{E\times D\leq a\wedge A}\mathscr{F}(D)\wedge\mathscr{G}(E)\geq\mathscr{G}(a\wedge C)\wedge\mathscr{F}(B)\geq a\wedge\mathscr{F}(B)\wedge\mathscr{G}(C).$$

(2) *If a pair L-filter $(\mathscr{F},\mathscr{G})$ is convergent to x, then $(\mathscr{F},\mathscr{G})$ is a Cauchy pair L-filter. In fact, it is easy to check that $\mathscr{F}\times[x]$ and $[x]\times\mathscr{G}$ are stratified L-filters on $X\times X$. It follows from* (Γ4) *that $\top = \Gamma(\mathscr{F}\times[x])\wedge\Gamma([x]\times\mathscr{G})\leq\Gamma((\mathscr{F}\times[x])\circ([x]\times\mathscr{G}))$. Thus we only need to show that $(\mathscr{F}\times[x])\circ([x]\times\mathscr{G})\leq\mathscr{F}\times\mathscr{G}$.*

First, it can be clear that $(\mathscr{F}\times[x])(B) = \mathscr{F}(B(x,-))$ for all $B\in L^{X\times X}$. In fact,

$$(\mathscr{F}\times[x])(B) = \bigvee_{E\times D\leq B}\mathscr{F}(D)\wedge E(x)\leq\bigvee_{E\times D\leq B}\mathscr{F}(D\wedge E(x))\leq\mathscr{F}(B(x,-)).$$

Conversely, let $D = B(x,-)$ and $E = x_\top$, then the other direction of the equation holds. Similarly, $([x]\times\mathscr{G})(C) = \mathscr{G}(C(-,x))$ for all $C\in L^{X\times X}$.

Then, for all $A\in L^{X\times X}$, $(\mathscr{F}\times[x])\circ([x]\times\mathscr{G})(A) = \bigvee_{B\circ C\leq A}(\mathscr{F}\times[x])(B)\wedge([x]\times\mathscr{G})(C)$, when $B\circ C\leq A$,

$$\begin{aligned}(\mathscr{F}\times[x])(B)\wedge([x]\times\mathscr{G})(C) &= \mathscr{F}(B(x,-))\wedge\mathscr{G}(C(-,x))\\ &\leq \bigvee_{N\times M\leq B\circ C}\mathscr{F}(M)\wedge\mathscr{G}(N)\\ &= (\mathscr{F}\times\mathscr{G})(B\circ C)\\ &\leq (\mathscr{F}\times\mathscr{G})(A).\end{aligned}$$

Therefore, $\Gamma(\mathscr{F}\times\mathscr{G}) = \top$, in other words, $(\mathscr{F},\mathscr{G})$ is a Cauchy pair L-filter.

(3) *It follows from* (Γ2) *that $\Gamma(\mathscr{F}\times\mathscr{G}) = \top\Leftrightarrow[\mathscr{F}\times\mathscr{G}]\leq\Gamma$.*

Definition 4.4. *An L-quasi-uniform convergence space (X,Γ) is called Cauchy complete if each Cauchy pair L-filter $(\mathscr{F},\mathscr{G})$ is convergent.*

Now we introduce Lawvere completeness of L-quasi-uniform convergence spaces by categorical method since we have described L-quasi-uniform convergence spaces as enriched categories.

Definition 4.5. *Let (X,Γ), (Y,Λ) be two L-quasi-uniform convergence spaces. A prorelation $\Phi: (X,\Gamma)\circ\!\!\rightarrow(Y,\Lambda)$ is said to be a CLU-module if it satisfies*

$$\Phi\circ\Gamma\leq\Phi,\ \Lambda\circ\Phi\leq\Phi.$$

For each L-quasi-uniform convergence space (X,Γ), $\Gamma: (X,\Gamma)\circ\!\!\rightarrow(X,\Gamma)$ itself is a CLU-module. For CLU-mudule $\Phi: (X,\Gamma)\circ\!\!\rightarrow(Y,\Lambda)$, since $\Phi = \Phi\circ[[1_X]]\leq\Phi\circ\Gamma$ is always true, then $\Phi\circ\Gamma = \Phi$ holds. Similarly, $\Lambda\circ\Phi = \Phi$. It is easy to see that the composition of CLU-modules is still a CLU-module and Γ acts as the identity of the composition.

Definition 4.6. *For two CLU-modules $\Phi: (X,\Gamma)\circ\!\!\rightarrow(Y,\Lambda)$, $\Psi: (Y,\Lambda)\circ\!\!\rightarrow(X,\Gamma)$, if $\Psi\circ\Phi\geq\Gamma$ and $\Phi\circ\Psi\leq\Lambda$ hold, then Φ is called the left adjoint of Ψ or Ψ is called the right adjoint of Φ, denoted by $\Phi\dashv\Psi$.*

Each given CL-uniform continuous map $f:(X,\Gamma)\to(Y,\Lambda)$ can determine a pair of CLU-modules $f_*:=\Lambda\circ[[f]]:(X,\Gamma)\multimap(Y,\Lambda)$, $f^*:=[[f^\circ]]\circ\Lambda:(Y,\Lambda)\multimap(X,\Gamma)$, and $f_*\dashv f^*$.

(1) It is easy to see that f_* and f^* are prorelations.

(2) From the fact that f is a CL-uniform continuous map: $f_*\circ\Gamma=\Lambda\circ[[f]]\circ\Gamma\le\Lambda\circ\Lambda\circ[[f]]=\Lambda\circ[[f]]=f_*$; $\Lambda\circ f_*=\Lambda\circ\Lambda\circ[[f]]=\Lambda\circ[[f]]=f_*$. Therefore, f_* is a CLU-module. Similarly, f^* is a CLU-module.

(3) $f^*\circ f_*=[[f^\circ]]\circ\Lambda\circ\Lambda\circ[[f]]=[[f^\circ]]\circ\Lambda\circ[[f]]\ge\Gamma\circ[[f^\circ]]\circ[[f]]\ge\Gamma$; $f_*\circ f^*=\Lambda\circ[[f]]\circ[[f^\circ]]\circ\Lambda\le\Lambda\circ[[1_Y]]\circ\Lambda=\Lambda$. Thus $f_*\dashv f^*$.

In particular, let (X,Γ) be an L-quasi-uniform convergence space and $\mathbb{P}:\mathscr{F}_L^S(\{*\}\times\{*\})\to L,\mathscr{F}\mapsto\mathbb{P}(\mathscr{F})=\top$ be the unique L-quasi-uniform convergence structure on the singleton $\{*\}$, denoted by $1=(\{*\},\mathbb{P})$. The CL-uniform continuous map $x:1\to X(*\mapsto x,\,x\in X)$ defines two adjoint CLU-modules $x_*\dashv x^*:X\multimap 1$, where $x_*=\Gamma\circ[[x]]$ and $x^*=[[x^\circ]]\circ\Gamma$. In fact, for L-quasi-uniform structure $\mathscr{P}:L^{\{*\}\times\{*\}}\to L, A\mapsto\mathscr{P}(A)=A(*,*)$,

$$\forall\mathscr{F}\in\mathscr{F}_L^S(\{*\}\times\{*\}),[\mathscr{P}](\mathscr{F})=S(\mathscr{P},\mathscr{F})=\bigwedge_{A\in L^{\{*\}\times\{*\}}}(A(*,*)\to\mathscr{F}(A))\ge\bigwedge_{A\in L^{\{*\}\times\{*\}}}(A(*,*)\to[(*,*)](A))=\top.$$

Therefore, $\mathbb{P}=[\mathscr{P}]$.

Definition 4.7. *An L-quasi-uniform convergence space (X,Γ) is said to be Lawvere complete if for each adjoint CLU-module $(\Phi:1\multimap X)\dashv(\Psi:X\multimap 1)$, there exists $x\in X$ such that $\Phi=x_*,\Psi=x^*$.*

For the relationship between Lawvere completeness and Cauchy completeness of L-quasi-uniform convergence space, we have the following result.

Theorem 4.8. *Let (X,Γ) be an L-quasi-uniform convergence space. If (X,Γ) is Lawvere complete, then (X,Γ) is Cauchy complete.*

Proof. Let $(\mathscr{F},\mathscr{G})$ be a Cauchy pair L-filter in (X,Γ). Set $\tilde{\mathscr{F}}:L^{1\times X}\to L$ be defined as $\tilde{\mathscr{F}}(\tilde{A})=\mathscr{F}(A)$ for all $\tilde{A}\in L^{1\times X}$ and $\tilde{\mathscr{G}}:L^{X\times 1}\to L$ be defined as $\tilde{\mathscr{G}}(\tilde{B})=\mathscr{G}(B)$ for all $\tilde{B}\in L^{X\times 1}$, where $\tilde{A}(*,x)=A(x)$ and $\tilde{B}(x,*)=B(x)$. And set $\Phi=\Gamma\circ[\tilde{\mathscr{F}}],\Psi=[\tilde{\mathscr{G}}]\circ\Gamma$. It is easy to check that Φ and Ψ are CLU-modules. We need to check that $\Phi\dashv\Psi$. It follows from $(\mathscr{F},\mathscr{G})$ is a pair L-filter that for all $A\in L^{1\times 1}$,

$$\begin{aligned}\tilde{\mathscr{G}}\circ\tilde{\mathscr{F}}(A)&=\bigvee_{\tilde{B}\circ\tilde{C}\le A}\tilde{\mathscr{G}}(\tilde{B})\wedge\tilde{\mathscr{F}}(\tilde{C})\\&=\bigvee_{\bigvee_{x\in X}B(x)\wedge C(x)\le A(*,*)}\mathscr{G}(B)\wedge\mathscr{F}(C)\\&\le\bigvee_{\bigvee_{x\in X}B(x)\wedge C(x)\le A(*,*)}\bigvee_{x\in X}B(x)\wedge C(x)\\&\le A(*,*)\\&=\mathscr{P}(A).\end{aligned}$$

Therefore, $\tilde{\mathscr{G}}\circ\tilde{\mathscr{F}}\le\mathscr{P}$. Then it follows from $[[1_X]]\le\Gamma$ that $\Psi\circ\Phi=[\tilde{\mathscr{G}}]\circ\Gamma\circ\Gamma\circ[\tilde{\mathscr{F}}]\ge[\tilde{\mathscr{G}}\circ\tilde{\mathscr{F}}]\ge[\mathscr{P}]=\mathbb{P}$. We notice that $\tilde{\mathscr{F}}\circ\tilde{\mathscr{G}}(D)=\bigvee_{\tilde{B}\circ\tilde{C}\le D}\tilde{\mathscr{F}}(\tilde{B})\wedge\tilde{\mathscr{G}}(\tilde{C})=\bigvee_{C\times B\le D}\mathscr{F}(B)\wedge\mathscr{G}(C)=\mathscr{F}\times\mathscr{G}(D)$ for all $D\in L^{X\times X}$. On account of $(\mathscr{F},\mathscr{G})$ is a Cauchy pair L-filter, it follows that $[\mathscr{F}\times\mathscr{G}]\le\Gamma$. Then $[\tilde{\mathscr{F}}\circ\tilde{\mathscr{G}}]\le\Gamma$. So $\Phi\circ\Psi=\Gamma\circ[\tilde{\mathscr{F}}]\circ[\tilde{\mathscr{G}}]\circ\Gamma\le\Gamma\circ\Gamma\circ\Gamma=\Gamma$. From above we know that $\Phi\dashv\Psi$. So there exists $x\in X$ such that $\Phi=\Gamma\circ[\tilde{\mathscr{F}}]=x_*=\Gamma\circ[[(*,x)]]$, $\Psi=[\tilde{\mathscr{G}}]\circ\Gamma=x^*=[[(x,*)]]\circ\Gamma$. Then we have to check $\Gamma(\mathscr{F}\times[x])=\top$ and $\Gamma([x]\times\mathscr{G})=\top$. It follows from $[[1_X]]\le\Gamma$ that $[\tilde{\mathscr{F}}]\le\Gamma\circ[\tilde{\mathscr{F}}]$. Hence $\Gamma\circ[[(*,x)]](\tilde{\mathscr{F}})=\Gamma\circ[\tilde{\mathscr{F}}](\tilde{\mathscr{F}})\ge[\tilde{\mathscr{F}}](\tilde{\mathscr{F}})=\top$. And then

$$\Gamma\circ[[(*,x)]](\tilde{\mathscr{F}})=\bigvee_{\mathscr{M},\tilde{\mathscr{N}}}\Gamma(\mathscr{M})\wedge S([(*,x)],\tilde{\mathscr{N}})\wedge S(\mathscr{M}\circ\tilde{\mathscr{N}},\tilde{\mathscr{F}})=\bigvee_{\mathscr{M}}\Gamma(\mathscr{M})\wedge S(\mathscr{M}\circ[(*,x)],\tilde{\mathscr{F}}).$$

So for all $\mathscr{M}\in\mathscr{F}_L^S(X\times X)$,

$$\begin{aligned}S(\mathscr{M}\circ[(*,x)],\tilde{\mathscr{F}})&\le S((\mathscr{M}\circ[(*,x)])\circ[(x,*)],\tilde{\mathscr{F}}\circ[(x,*)])\\&=S(\mathscr{M}\circ[(x,x)],\mathscr{F}\times[x])\\&\le S(\mathscr{M}\circ[1_X],\mathscr{F}\times[x])\\&\le\Gamma(\mathscr{M})\to\Gamma(\mathscr{F}\times[x]).\end{aligned}$$

It is easy to see that $\Gamma\circ[[(*,x)]](\tilde{\mathscr{F}})\le\Gamma(\mathscr{F}\times[x])$. Then $\Gamma(\mathscr{F}\times[x])=\top$. Similarly, $\Gamma([x]\times\mathscr{G})=\top$. Therefore, (X,Γ) is Cauchy complete. $\square$

Remark 4.9. *In Theorem 4.8, we just show that Lawvere completeness implies Cauchy completeness, we want to know whether Cauchy completeness implies Lawvere completeness. We leave it as a question. Since a Cauchy pair L-filter $(\mathscr{F},\mathscr{G})$ is used to discuss Cauchy completeness of L-quasi-uniform convergence space, in order to maintain consistency, the pair adjoint CLU-module in the definition of Lawvere completeness is restricted to the form of $[\mathscr{F}]\dashv[\mathscr{G}]$. To be specific, an L-quasi-uniform convergence space (X,Γ) is said to be weak Lawvere complete if for each adjoint CLU-module $([\mathscr{F}]:1\multimap X)\dashv([\mathscr{G}]:X\multimap 1)$, there exists $x\in X$ such that $[\mathscr{F}]=x_*,[\mathscr{G}]=x^*$.*

Let $(X,\mathscr{U})$ be an L-quasi-uniform space. Then it is easy to see that $(X,[\mathscr{U}])$ will be an L-quasi-uniform convergence space, where $[\mathscr{U}](\mathscr{F})=S(\mathscr{U},\mathscr{F})$ for all $\mathscr{F}\in\mathscr{F}_L^S(X\times X)$. And we have the following result.

Theorem 4.10. *Let $(X,\mathscr{U})$ be an L-quasi-uniform space and $(X,[\mathscr{U}])$ be the corresponding L-quasi-uniform convergence space. Then $(X,[\mathscr{U}])$ is weak Lawvere complete if and only if $(X,[\mathscr{U}])$ is Cauchy complete.*

5 Applications to the completeness of L-quasi-uniform spaces

In [27], Wang and Yue regarded L-quasi-uniform spaces as enriched categories and studied the Lawvere completeness of L-quasi-uniform spaces by using the enriched category theory. Some relevant conclusions and the definition of Lawvere complete are given below.

For given L-quasi-uniform spaces $(X,\mathscr{U})$ and $(Y,\mathscr{V})$, a stratified L-filter $\phi:L^{X\times Y}\to L$ is said to be an LU-module if it satisfies $\phi\circ\mathscr{U}\preceq\phi$, $\mathscr{V}\circ\phi\preceq\phi$, denoted by $\phi:(X,\mathscr{U})\multimap(Y,\mathscr{V})$.

For two LU-modules $\phi:(X,\mathscr{U})\multimap(Y,\mathscr{V})$, $\psi:(Y,\mathscr{V})\multimap(X,\mathscr{U})$, if $\psi\circ\phi\succeq\mathscr{U}$, $\phi\circ\psi\preceq\mathscr{V}$ hold, then ϕ is called the left adjoint of ψ or ψ is called the right adjoint of ϕ, denoted by $\phi\dashv\psi$.

Each given L-uniform continuous map $f:(X,\mathscr{U})\to(Y,\mathscr{V})$ can determine a pair of LU-modules $f_*:=\mathscr{V}\circ[f]:(X,\mathscr{U})\multimap(Y,\mathscr{V})$ and $f^*:=[f^\circ]\circ\mathscr{V}:(Y,\mathscr{V})\multimap(X,\mathscr{U})$. And $f_*\dashv f^*$.

In particular, let $(X,\mathcal{U})$ be an L-quasi-uniform space and $\mathcal{P}:*\nrightarrow *$ be the unique L-quasi-uniform structure on the singleton $\{*\}$ (in fact, $\mathscr{P}:L^{\{*\}\times\{*\}}\to L\ \ A\mapsto\mathscr{P}(A)=A(*,*)$), denoted by $1=(\{*\},\mathcal{P})$. The L-uniform continuous map $x:1\to X(*\mapsto x,\,x\in X)$ defines two adjoint LU-modules $x_*\dashv x^*:X\multimap 1$, where $x_*=\mathcal{U}\circ[x]$ and $x^*=[x^\circ]\circ\mathcal{U}$.

An L-quasi-uniform space $(X,\mathcal{U})$ is said to be Lawvere complete if for each adjoint LU-module $(\phi:1\multimap X)\dashv(\psi:X\multimap 1)$, there exists $x\in X$ such that $\phi=x_*,\psi=x^*$.

As an application of Cauchy completeness of L-quasi-uniform convergence space, in this section, we will define the Cauchy completeness of L-quasi-uniform space when $\Gamma=[\mathscr{U}]$. And we will show that an L-quasi-uniform space is Cauchy complete if and only if it is Lawvere complete.

Let $(X,\mathscr{U})$ be an L-quasi-uniform space and $(X,[\mathscr{U}])$ be the corresponding L-quasi-uniform convergence space. If $(\mathscr{F},\mathscr{G})$ is a Cauchy pair L-filter in L-quasi-uniform convergence space $(X,[\mathscr{U}])$, then it follows that $[\mathscr{U}](\mathscr{F}\times\mathscr{G})=\top$. This is equivalent to $\mathscr{U}\le\mathscr{F}\times\mathscr{G}$. Therefore, we have the following definition.

Definition 5.1. *A pair L-filter $(\mathscr{F},\mathscr{G})$ is called Cauchy pair L-filter in L-quasi-uniform space $(X,\mathscr{U})$ if $\mathscr{U}\le\mathscr{F}\times\mathscr{G}$.*

For each $x\in X$, the map $N_x^L:L^X\to L$ defined by

$$\forall A\in L^X, N_x^L(A)=\bigvee_{U(x,-)\le A}\mathscr{U}(U),$$

and the map $N_x^R:L^X\to L$ defined by

$$\forall A\in L^X, N_x^R(A)=\bigvee_{U(-,x)\le A}\mathscr{U}(U),$$

are stratified L-filters on X. In fact, for L-quasi-uniform space $(X,\mathscr{U})$, it holds that $N_x^L=\mathscr{U}\circ[x]$ and $N_x^R=[x^\circ]\circ\mathscr{U}$. What is more, (N_x^L,N_x^R) is a minimal Cauchy pair L-filter, i.e., for Cauchy pair L-filter $(\mathscr{F},\mathscr{G})$ satisfying $(\mathscr{F},\mathscr{G})\le(N_x^L,N_x^R)$ (equivalently $\mathscr{F}\le N_x^L$ and $\mathscr{G}\le N_x^R$), it holds that $(\mathscr{F},\mathscr{G})=(N_x^L,N_x^R)$.

Proposition 5.2. *A pair L-filter $(\mathscr{F},\mathscr{G})$ is convergent to x in $(X,[\mathscr{U}])$ if and only if $(\mathscr{F},\mathscr{G})\ge(N_x^L,N_x^R)$ holds in $(X,\mathscr{U})$.*

Proof. Suppose $(\mathscr{F},\mathscr{G}) \geq (N_x^L, N_x^R)$ holds in $(X,\mathscr{U})$. Then $N_x^L \times [x] \leq \mathscr{F} \times [x]$. For all $A \in L^{X\times X}$,

$$\begin{aligned} N_x^L \times [x](A) &= \bigvee_{B\times C\leq A} N_x^L(C) \wedge B(x) \\ &= \bigvee_{B\times C\leq A} \bigvee_{U(x,-)\leq C} \mathscr{U}(U) \wedge B(x) \\ &\geq \mathscr{U}(A) \ \ (U=A, C=A(x,-), B=x_\top). \end{aligned}$$

Therefore, $\mathscr{U} \leq N_x^L \times [x] \leq \mathscr{F} \times [x]$. This is to say that $[\mathscr{U}](\mathscr{F}\times[x]) = \top$. Similarly, $[\mathscr{U}]([x]\times\mathscr{G}) = \top$. Then $(\mathscr{F},\mathscr{G})$ is convergent to x in $(X,[\mathscr{U}])$.

Conversely, suppose $(\mathscr{F},\mathscr{G})$ converges to x in $(X,[\mathscr{U}])$. Then $[\mathscr{U}](\mathscr{F}\times[x]) = \top$ and $[\mathscr{U}]([x]\times\mathscr{G}) = \top$, in other words, $\mathscr{U} \leq \mathscr{F}\times[x]$ and $\mathscr{U} \leq [x]\times\mathscr{G}$. For all $A \in L^X$,

$$\begin{aligned} N_x^L(A) &= \bigvee_{U(x,-)\leq A} \mathscr{U}(U) \\ &\leq \bigvee_{U(x,-)\leq A} (\mathscr{F}\times[x])(U) \\ &= \bigvee_{U(x,-)\leq A} \mathscr{F}(U(x,-)) \\ &\leq \mathscr{F}(A). \end{aligned}$$

Therefore, $N_x^L \leq \mathscr{F}$. Similarly, $N_x^R \leq \mathscr{G}$. □

Definition 5.3. *A pair L-filter $(\mathscr{F},\mathscr{G})$ is convergent to x in L-quasi-uniform space $(X,\mathscr{U})$ if $(\mathscr{F},\mathscr{G}) \geq (N_x^L, N_x^R)$.*

Definition 5.4. *An L-quasi-uniform space $(X,\mathscr{U})$ is called Cauchy complete if each Cauchy pair L-filter $(\mathscr{F},\mathscr{G})$ is convergent.*

Corollary 5.5. *$(X,\mathscr{U})$ is Cauchy complete if and only if $(X,[\mathscr{U}])$ is Cauchy complete.*

According to the above proposition and corollary, the definition of Cauchy completeness of L-quasi-uniform space is reasonable. And then, we prove that the Cauchy completeness and Lawvere completeness of L-quasi-uniform spaces are equivalent.

Lemma 5.6. *Let $(X,\mathscr{U})$ be an L-quasi-uniform space. If $(\mathscr{F},\mathscr{G})$ is a Cauchy pair L-filter, then there exists a unique minimal Cauchy pair L-filter $(\mathscr{F}_0,\mathscr{G}_0)$ such that $(\mathscr{F}_0,\mathscr{G}_0) \leq (\mathscr{F},\mathscr{G})$. In fact,*

$$\forall A \in L^X, \mathscr{F}_0(A) = \bigvee_{U(F)\leq A} \mathscr{U}(U) \wedge \mathscr{F}(F), \mathscr{G}_0(A) = \bigvee_{U^{-1}(G)\leq A} \mathscr{U}(U) \wedge \mathscr{G}(G),$$

where $U(F)(x) = \bigvee_{y\in X} F(y) \wedge U(y,x), U^{-1}(G)(x) = \bigvee_{y\in X} G(y) \wedge U(x,y)$.

Theorem 5.7. *Let $(X,\mathscr{U})$ be an L-quasi-uniform space. Then $(X,\mathscr{U})$ is Cauchy complete if and only if $(X,\mathscr{U})$ is Lawvere complete.*

Proof. Let $(X,\mathscr{U})$ be Cauchy complete and $\phi \dashv \psi$ be a pair adjoint LU-module in $(X,\mathscr{U})$, where $\phi\colon(\{*\},\mathscr{P}) \,\circ\!\!\!\rightarrow (X,\mathscr{U})$, $\psi\colon(X,\mathscr{U}) \,\circ\!\!\!\rightarrow (\{*\},\mathscr{P})$. Then $\psi\circ\phi \succeq \mathscr{P}, \phi\circ\psi \preceq \mathscr{U}$. Set $\mathscr{F}_\phi(A) = \phi(\tilde{A})$ and $\mathscr{G}_\psi(B) = \psi(\tilde{B})$ for all $A \in L^X$, where $\tilde{A}(*,x) = A(x)$ and $\tilde{B}(x,*) = B(x)$. Let $\mathscr{P}(V) = V(*,*) = \alpha$. On account of $\phi\dashv\psi$, it follows that

$$\forall V \in L^{\{*\}\times X}, \psi\circ\phi(V) = \bigvee_{\tilde{C}\circ\tilde{B}\leq V} \psi(\tilde{C}) \wedge \phi(\tilde{B}) = \bigvee_{\bigvee_{x\in X}(B(x)\wedge C(x))\leq\alpha} \mathscr{G}_\psi(C) \wedge \mathscr{F}_\phi(B) \leq \alpha,$$

$$\forall U \in L^{X\times X}, \mathscr{U}(U) \leq \phi\circ\psi(U) = \bigvee_{\tilde{C}\circ\tilde{B}\leq U} \phi(\tilde{C}) \wedge \psi(\tilde{B}) = \bigvee_{B\times C\leq U} \mathscr{F}_\phi(C) \wedge \mathscr{G}_\psi(B) = (\mathscr{F}_\phi \times \mathscr{G}_\psi)(U).$$

Therefore, $\mathscr{G}_\psi(C) \wedge \mathscr{F}_\phi(B) \leq \bigvee_{x\in X}(B(x)\wedge C(x))$ for all $B, C \in L^X$ and $\mathscr{U} \leq \mathscr{F}_\phi \times \mathscr{G}_\psi$. Then from above, we can say that $(\mathscr{F}_\phi,\mathscr{G}_\psi)$ is a Cauchy pair L-filter. Since $(X,\mathscr{U})$ is Cauchy complete, there exists $x \in X$ such that $(\mathscr{F}_\phi,\mathscr{G}_\psi) \geq (N_x^L, N_x^R)$. And on account of $\mathscr{U}\circ\phi = \phi, \psi\circ\mathscr{U} = \psi$, it is easy to see that

$$\forall A \in L^X, \mathscr{F}_\phi(A) = \bigvee_{U(B)\leq A} \mathscr{U}(U) \wedge \mathscr{F}_\phi(B), \mathscr{G}_\psi(A) = \bigvee_{U^{-1}(G)\leq A} \mathscr{U}(U) \wedge \mathscr{G}_\psi(G).$$

Therefore, it follows from Lemma 5.6 that $(\mathscr{F}_\phi,\mathscr{G}_\psi)$ is the unique minimal Cauchy pair L-filter. Then $(\mathscr{F}_\phi,\mathscr{G}_\psi)=(N_x^L,N_x^R)=(x_*,x^*)$. This is to say that $(X,\mathscr{U})$ is Lawvere complete.

Conversely, let $(X,\mathscr{U})$ be Lawvere complete and $(\mathscr{F},\mathscr{G})$ be a Cauchy pair L-filter in $(X,\mathscr{U})$. Let $\Phi_{\mathscr{F}}:L^{\{*\}\times X}\to L$ be defined as $\Phi_{\mathscr{F}}(\tilde{A})=\mathscr{F}(A)$ for all $\tilde{A}\in L^{\{*\}\times X}$ and $\Psi_{\mathscr{G}}:L^{X\times\{*\}}\to L$ be defined as $\Psi_{\mathscr{G}}(\tilde{B})=\mathscr{G}(B)$ for all $\tilde{B}\in L^{X\times\{*\}}$, where $\tilde{A}(*,x)=A(x)$ and $\tilde{B}(x,*)=B(x)$. Then it follows from Lemma 5.6 that $(\mathscr{F}_{\mathscr{U}\circ\Phi_{\mathscr{F}}},\mathscr{G}_{\Psi_{\mathscr{G}}\circ\mathscr{U}})$ is a minimal Cauchy pair L-filter. It is routine to check that $\mathscr{U}\circ\Phi_{\mathscr{F}}$ and $\Psi_{\mathscr{G}}\circ\mathscr{U}$ are LU-modules. Since $(\mathscr{F},\mathscr{G})$ is a Cauchy pair L-filter in $(X,\mathscr{U})$, then $\Phi_{\mathscr{F}}\circ\Psi_{\mathscr{G}}\geq\mathscr{U},\Psi_{\mathscr{G}}\circ\Phi_{\mathscr{F}}\leq\mathscr{P}$. Therefore,

$$\Psi_{\mathscr{G}}\circ\mathscr{U}\circ\mathscr{U}\circ\Phi_{\mathscr{F}}=\Psi_{\mathscr{G}}\circ\mathscr{U}\circ\Phi_{\mathscr{F}}\leq\Psi_{\mathscr{G}}\circ\Phi_{\mathscr{F}}\leq\mathscr{P},$$

$$\mathscr{U}\circ\Phi_{\mathscr{F}}\circ\Psi_{\mathscr{G}}\circ\mathscr{U}\geq\mathscr{U}\circ\mathscr{U}\circ\mathscr{U}=\mathscr{U}.$$

This is to say that $\mathscr{U}\circ\Phi_{\mathscr{F}}\dashv\Psi_{\mathscr{G}}\circ\mathscr{U}$. Since $(X,\mathscr{U})$ is Lawvere complete, there exists $x\in X$ such that

$$\mathscr{U}\circ\Phi_{\mathscr{F}}=x_*=\Phi_{N_x^L},\ \Psi_{\mathscr{G}}\circ\mathscr{U}=x^*=\Psi_{N_x^R}.$$

Hence, $(\mathscr{F},\mathscr{G})\geq(N_x^L,N_x^R)$. Therefore, $(X,\mathscr{U})$ is Cauchy complete. □

From Theorem 4.10, Corollary 5.5 and Theorem 5.7, it follows that $(X,[\mathscr{U}])$ is Cauchy complete if and only if $(X,[\mathscr{U}])$ is weak Lawvere complete if and only if $(X,\mathscr{U})$ is Cauchy complete if and only if $(X,\mathscr{U})$ is Lawvere complete.

6 Conclusions

In this paper, we study the Lawvere completeness of L-quasi-uniform convergence spaces based on enriched category theory. And we study the Cauchy completeness of L-quasi-uniform convergence spaces by using pair L-filters. Further, we prove that L-quasi-uniform spaces are Cauchy complete if and only if the generated L-quasi-uniform convergence spaces are Cauchy complete, and this conclusion is also applicable to Lawvere complete. When the L-quasi-uniform convergence space is generated by an L-quasi-uniform space, the weak Lawvere completeness and Cauchy completeness of the L-quasi-uniform convergence space are equivalent.

G. Preuss has introduced the concept of classical uniform limit space and proposed its Cauchy completeness in [25]. When $L=2=\{\bot,\top\}$, the L-quasi-uniform convergence spaces will reduce to the quasi-uniform limit spaces, and the Cauchy completeness of quasi-uniform limit spaces will imply the Cauchy completeness introduced in [25].

The relationship between the completeness of L-quasi-uniform convergence space and its corresponding symmetric L-uniform convergence space is also an interesting question, we leave it for further study. The completion of L-quasi-uniform spaces and L-quasi-uniform convergence spaces is a very important part in convenient topology, we also leave it for future study. The compactness of L-quasi-uniform convergence spaces is a topic worthy of study, and the relationship between compactness and Lawvere completeness of L-quasi-uniform convergence spaces is also worthy of further study.

Acknowledgement

The authors thank the referees gratefully for their valuable comments and suggestions which improved the paper significantly. This work is supported by the National Natural Science Foundation of China (11971448).

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