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Conditional distributivity of continuous triangular norms over 2-uninorms

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Abstract

Conditional distributivity of aggregation functions, which has received wide attention from the researchers, is vital for many different fields, for example, integration theory, utility theory and so on. This article is mainly devoted to dealing with the conditional distributivity of continuous t-norms over 2-uninorms. As the first step for investigating the conditional distributivity of 2-uninorms, we give the complete characterization of all pairs $(T, \mathcal{H})$ fulfilling this property. Compared to the case of distributivity of continuous t-norms over 2-uniorms, which leads to the 2-uninorm must be idempotent, the results obtained in this paper demonstrate that conditional distributivity and distributivity on this topic, are not equivalent.

Keywords: Conditional distributivity, 2-uninorms, uninorms, continuous triangular norms.

1 Introduction

There are many real life situations where the process of merging and combining a certain number of data into a single representative value is required. Mathematical functions carrying out this process are known as aggregation operations. The axiomatic characterizations of aggregation operators have been a research hot spot in recent years because purely theoretical researches of this subject can lead to new possibilities of applications as seen in [2, 3, 9]. Although the majority of applications focused on continuous t-norms because of their straightforward characterizations, the researchers soon realized that relaxation or the replacement of some axioms can enhance their performance in real-life applications. This leads to several generalizations including left-continuous t-norms, uninorms, nullnorms, and overlap functions, among others. Replacement of the unit interval, which is a bounded chain, by more general structures yields t-norms on bounded posets and bounded lattices. Generalization of the position of the neutral element or the annihilator of a t-norm yields the definition of uninorms and nullnorms, respectively. Recently, bringing uninorms and nullnorms together, a new family aggregation functions named 2-uninorms was introduced by Akella [1]. This concept shares the same idea with the ordinal sum of t-norms and generalizes uninorms in such a way that the global neutral element is replaced by two local neutral elements $e^{\dagger}$ and $e^{\ddagger}$. Since then, many scholars have devoted themselves to the study of the structural characterization of 2-uninorms. Specifically, members of some classes of 2-uninorms have been completely characterized under some appropriate continuity conditions. In 2018, five mutually exclusive classes of 2-uninorm were characterized by using the absorbing element and the nature (conjunctive/disjunctive) of the underlying uninorms [30]. The most eye-catching is that a systematic analysis involving the 2-uninorms with continuous underlying functions were obtained by decomposing into the form of ordinal sum [18, 19, 20, 21].

In addition to the axiomatic characterization mentioned above, another research direction of aggregation operator is considered from its application aspect. One of the application properties is conditional distributivity. Investigations of this property are directed towards finding solutions for different families of aggregation operators such as t-conorms, t-norms, nullnorms, uninorms, semi-t-operators, 2-uninorms, etc. (see [4, 14, 15, 16, 17, 22, 27]). As far as we know,

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the significance of the conditional distributivity of different aggregation functions stems from its role in different fields, for example, possibilistic utility theory, integration theory and pseudo-additive measures. Specifically:

1. In possibilistic utility theory, whose basic ingredient is possibilistic mixtures, it is proved that the distributive pairs of aggregation functions are essential for modelling some specific problems, for example, generalized mixtures [6].

2. In several types of integrals, e.g., $(\mathbb{S}, \mathbb{U})$-integrals, Benvenuti integrals, Murofushi integrals and pan-integrals, the (conditional) distributivity of the pseudo-multiplications over the pseudo-additions is indispensable [9].

3. In theory of pseudo-additive measures characterized by t-conorms, to make it practically useful, the conditionally distributive pairs of t-conorms and t-norm must be used [7, 8].

Due to the wide application of conditional distributivity, a wave of research on this topic was initiated and many results were obtained [4, 5, 10, 13, 14, 16, 17, 22, 24, 27]. Since we are going to deal with 2-uninorms in this work, over here, we only recall some results of conditional distributivity related to 2-uninorms. Indeed, to our best knowledge, only the conditional distributivity of very special 2-uninorms (T-uninorms or uni-nullnorms) has been studied and the proper 2-uninorms on this topic is still a blank. Specifically speaking, in 2018, Jočić and his partners researched the conditional distributivity of a T-uninorm over a uninorm from $U_{\max} \cup U_{\min}$ in [11]. The obtained results showed that the inner operator U either degenerates into an idempotent uninorm, or its underlying t-norm is idempotent and the underlying t-conorm is an ordinal sum of S_L. Subsequently, Wang [26] studied the conditional distributivity of a uni-nullnorm over a triangular norm, a triangular conorm and a uninorm from $U_{\max} \cup U_{\min}$ with continuous underlying operators.

It can be seen from the above research process of conditional distributivity that when we deal with the conditional distributivity of 2-uninorms, the 2-uninorms considered are still based on having two special local neutral elements ($e^{\dagger}$ and 1), which leads to the certain limitation of this function in the actual application process. Considering the wide application of conditional distributivity and the importance of aggregation operators as we have already mentioned above, the goal of this article is to fill the gap mentioned above. In other words, as a supplement of this topic from the theoretical perspective, we will continue to investigate the conditional distributivity of 2-uninorms, but the 2-uninorms considered need not to have two special local neutral elements. Specifically, we will investigate the conditional distributivity of a continuous t-norms over a 2-uninorm. However, as the first step for relaxing the restriction of the local neutral elements, we assume a sensible assumption: the 2-uninorms attached with some appropriate continuity conditions. We gave the complete characterization of all pairs $(T, \mathcal{H})$ fulfilling this property and the obtained results demonstrate that although distributivity and conditional distributivity are two different research topic, the following two perspectives show that they are closely related. One is that if the conditional distributivity equation has a solution, then so has the distributivity equation and vice versa. Another is that if every point in $(e^{\ddagger}, 1)$ is an idempotent element of $\mathcal{H}$, then distributivity and conditional distributivity on this topic are not equivalent, otherwise, they are not equivalent.

The layout of this work is as follows. We review some basic definitions and conclusions about 2-uninorms and the conditional distributivity used in this paper in Section 2. The conditional distributivity of continuous triangular norms over 2-uninorms is given in Section 3. The last section is a summary of this article.

2 Preliminaries

We assume that the reader is proficient in the facts about t-norms and t-conorms (which are dual to t-norms) and for more details about them please see [12]. Over here, we only recall the facts involving uninorms and 2-uninorms.

Definition 2.1. [28] *A uninorm is a binary operator* $U\colon [0,1]^2 \to [0,1]$ *which is commutative, associative, non-decreasing in each variable and there exists some element* $e \in [0,1]$*, called* neutral element*, such that* $U(x, e) = x$ *for all* $x \in [0,1]$.

Any uninorm acts as a t-norm T in $[0, e]^2$ and as a t-conorm S in $[e, 1]^2$. Moreover, its value is bounded by min and max in the set of $[0,1]^2 \setminus \{[0, e]^2 \cup [e, 1]^2\}$.

Usually, a uninorm with neutral element e and underlying t-norm T and underlying t-conorm S is denoted by $U \equiv \langle T, e, S\rangle$. For any uninorm we have $U(1, 0) \in \{1, 0\}$. A uninorm U is called *disjunctive* if $U(1, 0) = 1$ and called *conjunctive* if $U(1, 0) = 0$. It should be noted that the class of uninorm $U_{\min}$ ($U_{\max}$) is the family of uninorms, which are expressed by min (max). Further, an idempotent uninorm U from $U_{\min}$ ($U_{\max}$) and has neutral element e, is denoted by $U_e^{\min}$ ($U_e^{\max}$).

Definition 2.2. [1, 30] *Let $k \in (0,1), e^\dagger \in [0,k]$ and $e^\ddagger \in [k,1]$. A binary operator $\mathcal{H} : [0,1]^2 \to [0,1]$ is called a 2-uninorm if it is associative, commutative, non-decreasing and satisfies*

(*i*) $\mathcal{H}(x, e^\dagger) = x$, *for arbitrary* $x \in [0,k]$.

(*ii*) $\mathcal{H}(x, e^\ddagger) = x$, *for arbitrary* $x \in [k,1]$.

All 2-uninorms with the fixed parameters $e^\dagger$, k, $e^\ddagger$ are denoted by $U_{k(e^\dagger,e^\ddagger)}$.

Lemma 2.3. [1, 30] *Let $\mathcal{H} \in U_{k(e^\dagger,e^\ddagger)}$ be a 2-uninorm.*

(*i*) *The binary functions $U^\dagger$ and $U^\ddagger$ on the region of $[0,1]^2$ given by*

$$U^\dagger(x,y) = \frac{\mathcal{H}(kx,ky)}{k},$$

and

$$U^\ddagger(x,y) = \frac{\mathcal{H}(k+(1-k)x, k+(1-k)y) - k}{1-k},$$

are uninorms with neutral element $\frac{e^\dagger}{k}$ and $\frac{e^\ddagger - k}{1-k}$, respectively. Namely, $\mathcal{H}$ works as the uninorms $U^\dagger$ and the uninorms $U^\ddagger$ in region $[0,k]^2$ and in region $[k,1]^2$, respectively.

(*ii*) *It holds that $\mathcal{H}(0,1) \in \{0,1,k\}$, $\mathcal{H}(k,0) \in \{0,k\}$ and $\mathcal{H}(1,k) \in \{1,k\}$.*

Based on Lemma 2.3, 2-uninorms can be divided into the following five exclusive classes and the parameters $e^\dagger$, k and $e^\ddagger$ in each class have the corresponding order relationship [30].

(i) The family of 2-uninorms with $\mathcal{H}(1,0) = k$ is denoted by $\mathcal{C}^k$. When $e^\dagger$, k and $e^\ddagger$ are taking some specific values, we obtain several already known classes of aggregation operations. For example, a 2-uninorm $\mathcal{H}$ with $0 < e^\dagger < k < e^\ddagger = 1$ is called a uni-nullnorm.

(ii) The family of 2-uninorms with $\mathcal{H}(1,0) = 0$ and $\mathcal{H}(k,1) = k$ is denoted by $\mathcal{C}^0_k$. For this family of 2-uninorms, it holds that $0 < e^\dagger \le k < e^\ddagger \le 1$.

(iii) The family of 2-uninorms with $\mathcal{H}(1,0) = 0$ and $\mathcal{H}(k,1) = 1$ is denoted by $\mathcal{C}^0_1$. For this family of 2-uninorms, it holds that $0 < e^\dagger \le k \le e^\ddagger < 1$.

(iv) The family of 2-uninorms with $\mathcal{H}(1,0) = 1$ and $\mathcal{H}(k,0) = 0$ is denoted by $\mathcal{C}^1_0$. Similarly, we have $0 < e^\dagger \le k \le e^\ddagger < 1$ for this family of 2-uninorms.

(v) The family of 2-uninorms with $\mathcal{H}(1,0) = 1$ and $\mathcal{H}(k,0) = k$ is denoted by $\mathcal{C}^1_k$. Similarly, we have $0 \le e^\dagger < k \le e^\ddagger < 1$ for this family of 2-uninorms.

With some additional conditions, these five types of 2-uninorms have specific expressions, whose distributivity and migrativity have been widely studied [29].

Theorem 2.4. [1] *Let $\mathcal{H} : [0,1]^2 \to [0,1]$ be a binary function such that $\mathcal{H}(\cdot,0)$ is continuous except at the point $e^\dagger$ and $\mathcal{H}(\cdot,1)$ is continuous except at the point $e^\ddagger$. Then $\mathcal{H}$ is a 2-uninorm and $\mathcal{H} \in C^k$ iff $\mathcal{H}$ is expressed by*

$$\mathcal{H}(x,y) = \begin{cases} kU^\dagger(\frac{x}{k}, \frac{y}{k}) & \text{if } (x,y) \in [0,k]^2, \\ k + (1-k)U^\ddagger(\frac{x-k}{1-k}, \frac{y-k}{1-k}) & \text{if } (x,y) \in [k,1]^2, \\ k & \text{if } (x,y) \in [k,1] \times [0,k] \cup [0,k] \times [k,1], \end{cases} \tag{1}$$

where $U^\dagger \in U^{\max}_{\frac{e^\dagger}{k}}$ and $U^\ddagger \in U^{\min}_{\frac{e^\ddagger - k}{1-k}}$.

Theorem 2.5. [1] *Let $\mathcal{H} : [0,1]^2 \to [0,1]$ be a binary function such that $\mathcal{H}(\cdot,1)$ is continuous except at the points $e^\dagger$ and $e^\ddagger$. Then $\mathcal{H}$ is a 2-uninorm and $\mathcal{H} \in \mathcal{C}^0_k$ iff $\mathcal{H}$ is expressed by*

$$\mathcal{H}(x,y) = \begin{cases} kU^\dagger(\frac{x}{k}, \frac{y}{k}) & \text{if } (x,y) \in [0,k]^2, \\ k + (1-k)U^\ddagger(\frac{x-k}{1-k}, \frac{y-k}{1-k}) & \text{if } (x,y) \in [k,1]^2, \\ \min(x,y) & \text{if } (x,y) \in (k,1] \times [0,e^\dagger) \cup [0,e^\dagger) \times (k,1], \\ k & \text{if } (x,y) \in [k,1] \times [e^\dagger,k] \cup [e^\dagger,k] \times [k,1], \end{cases} \tag{2}$$

where $U^\dagger \in U^{\min}_{\frac{e^\dagger}{k}}$ and $U^\ddagger \in U^{\min}_{\frac{e^\ddagger - k}{1-k}}$.

Theorem 2.6. [1] *Let $\mathcal{H}:[0,1]^2\to[0,1]$ be a binary function such that $\mathcal{H}(\cdot,1)$ is continuous except at the point $e^{\dagger}$ and $\mathcal{H}(\cdot,e)$ is continuous except at the point $e^{\ddagger}$. Then $\mathcal{H}$ is a 2-uninorm and $\mathcal{H}\in\mathcal{C}_1^0$ iff $\mathcal{H}$ is given by*

$$\mathcal{H}(x,y)=\begin{cases} kU^{\dagger}(\frac{x}{k},\frac{y}{k}) & \text{if } (x,y)\in[0,k]^2,\\ k+(1-k)U^{\ddagger}(\frac{x-k}{1-k},\frac{y-k}{1-k}) & \text{if } (x,y)\in[k,1]^2,\\ \min(x,y) & \text{if } (x,y)\in(k,1]\times[0,e^{\dagger})\cup[0,e^{\dagger})\times(k,1],\\ \max(x,y) & \text{if } (x,y)\in(e^{\ddagger},1]\times[e^{\dagger},k)\cup[e^{\dagger},k)\times(e^{\ddagger},1],\\ k & \text{if } x,y\in[k,e^{\ddagger}]\times[e^{\dagger},k]\cup[e^{\dagger},k]\times[k,e^{\ddagger}], \end{cases}\tag{3}$$

where $U^{\dagger}\in U^{\min}_{\frac{e^{\dagger}}{k}}$ and $U^{\ddagger}\in U^{\max}_{\frac{e^{\ddagger}-k}{1-k}}$.

Theorem 2.7. [1] *Let $\mathcal{H}:[0,1]^2\to[0,1]$ be a binary function such that $\mathcal{H}(\cdot,0)$ is continuous except at the points $e^{\dagger}$ and $e^{\ddagger}$. Then $\mathcal{H}$ is a 2-uninorm and $\mathcal{H}\in C_k^1$ iff $\mathcal{H}$ is given by*

$$\mathcal{H}(x,y)=\begin{cases} kU^{\dagger}(\frac{x}{k},\frac{y}{k}) & \text{if } (x,y)\in[0,k]^2,\\ k+(1-k)U^{\ddagger}(\frac{x-k}{1-k},\frac{y-k}{1-k}) & \text{if } (x,y)\in[k,1]^2,\\ \max(x,y) & \text{if } (x,y)\in(e^{\ddagger},1]\times[0,k)\cup[0,k)\times(e^{\ddagger},1],\\ k & \text{if } (x,y)\in[k,e^{\ddagger}]\times[0,k]\cup[0,k]\times[k,e^{\ddagger}], \end{cases}\tag{4}$$

where $U^{\dagger}\in U^{\max}_{\frac{e^{\dagger}}{k}}$ and $U^{\ddagger}\in U^{\max}_{\frac{e^{\ddagger}-k}{1-k}}$.

Theorem 2.8. [1] *Let $\mathcal{H}:[0,1]^2\to[0,1]$ be a binary function such that $\mathcal{H}(\cdot,e^{\ddagger})$ is continuous except at the point $e^{\dagger}$ and $\mathcal{H}(\cdot,0)$ is continuous except at the point $e^{\ddagger}$. Then $\mathcal{H}$ is a 2-uninorm and $\mathcal{H}\in\mathcal{C}_0^1$ iff $\mathcal{H}$ is expressed by*

$$\mathcal{H}(x,y)=\begin{cases} kU^{\dagger}(\frac{x}{k},\frac{y}{k}) & \text{if } (x,y)\in[0,k]^2,\\ k+(1-k)U^{\ddagger}(\frac{x-k}{1-k},\frac{y-k}{1-k}) & \text{if } (x,y)\in[k,1]^2,\\ \min(x,y) & \text{if } (x,y)\in(k,e^{\ddagger}]\times[0,e^{\dagger})\cup[0,e^{\dagger})\times(k,e^{\ddagger}],\\ \max(x,y) & \text{if } (x,y)\in(e^{\ddagger},1]\times[0,k)\cup[0,k)\times(e^{\ddagger},1],\\ k & \text{if } (x,y)\in[k,e^{\ddagger}]\times[e^{\dagger},k]\cup[e^{\dagger},k]\times[k,e^{\ddagger}], \end{cases}\tag{5}$$

where $U^{\dagger}\in U^{\min}_{\frac{e^{\dagger}}{k}}$ and $U^{\ddagger}\in U^{\max}_{\frac{e^{\ddagger}-k}{1-k}}$.

3 Conditional distributivity of continuous triangular norms over 2-uninorms

First, let us review the definition of conditional distributivity of a t-norm over another aggregation operator.

Definition 3.1. [23] *A t-norm T is conditionally distributive over an aggregation operator $\mathcal{H}$ if for all $x,y,z\in[0,1]$ the following holds*

$$T(x,\mathcal{H}(y,z))=\mathcal{H}(T(x,y),T(x,z)),\qquad \textit{whenever } \mathcal{H}(y,z)<1.\tag{6}$$

In the following, we discuss the conditional distributivity of continuous t-norms over 2-uninorms, that is, the aggregation operator considered in Eq. (6) is a 2-uninorm. It should be noted that for a 2-uninorm $\mathcal{H}\in\mathcal{C}^k\cup\mathcal{C}_k^0$, it will be degenerated into a uni-nullnorm in the case of $e^{\ddagger}=1$ [25]. As we all know, the conditional distributivity of uni-nullnorm have been discussed in many literatures, for example, [26]. Therefore, in this paper, we consider the general setting that $e^{\ddagger}<1$. Moreover, unless otherwise specified, the underlying t-conorm of uninorm $U^{\ddagger}$ is attached with continuity throughout the article. Before discussing class by class, the following basic lemma is necessary.

Lemma 3.2. *Let T be a continuous t-norm and $\mathcal{H}$ be a 2-uninorm. If T is conditionally distributive over $\mathcal{H}$ then each point in $[0,e^{\ddagger}]$ is an idempotent element of $\mathcal{H}$.*

Proof. Take $z=y=e^{\ddagger}$ in Eq. (6). Then it holds that

$$T(x,e^{\ddagger})=T(x,\mathcal{H}(e^{\ddagger},e^{\ddagger}))=\mathcal{H}(T(x,e^{\ddagger}),T(x,e^{\ddagger})),$$

for arbitrary $x\in[0,1]$. That is, $T(x,e^{\ddagger})$ is an idempotent element of $\mathcal{H}$ for arbitrary $x\in[0,1]$. By using the fact that $\mathrm{Ran}(T(x,e^{\ddagger}))=[0,e^{\ddagger}]$, we know the conclusion is established. □

Base on the lemma above, let us begin with the case when $\mathcal{H}\in\mathcal{C}^k$.

3.1 The case: $\mathcal{H} \in \mathcal{C}^k$

Lemma 3.3. *Let T be a continuous t-norm and $\mathcal{H}$ be a 2-uninorm with the form Eq. (1). If T is conditionally distributive over $\mathcal{H}$ then $T(k,k)=k$ and $e^\dagger = 0$.*

Proof. Taking $x=k, z=1, y=0$ in Eq. (6), we obtain

$$T(k,k) = T(k,\mathcal{H}(0,1)) = \mathcal{H}(T(k,0),T(k,1)) = \mathcal{H}(0,k) = k.$$

Therefore, we obtain that $T(k,k)=k$.
To show $e^\dagger = 0$, let us suppose that $e^\dagger > 0$. Then by T is conditionally distributive over $\mathcal{H}$ we obtain that

$$x = T(x,k) = T(x,\mathcal{H}(0,1)) = \mathcal{H}(T(x,0),T(x,1)) = \mathcal{H}(0,x) = 0.$$

for arbitrary $x \in (0,e^\dagger)$. This is a contradiction. □

Theorem 3.4. *Let T be a continuous t-norm and $\mathcal{H}$ be a 2-uninorm with the form Eq. (1).*

(*i*) *If every point in $(e^\ddagger,1)$ is an idempotent element of $\mathcal{H}$, then T is conditionally distributive over $\mathcal{H}$ iff the following three items are true.*

(*a*) *It holds that $e^\dagger = 0$.*

(*b*) *T is expressed by*

$$T(x,y) = \begin{cases} kT_1(\frac{x}{k},\frac{y}{k}) & \text{if } (x,y)\in[0,k]^2,\\ k+(e^\ddagger-k)T_2(\frac{x-k}{e^\ddagger-k},\frac{y-k}{e^\ddagger-k}) & \text{if } (x,y)\in[k,e^\ddagger]^2,\\ e^\ddagger+(1-e^\ddagger)T_3(\frac{x-e^\ddagger}{1-e^\ddagger},\frac{y-e^\ddagger}{1-e^\ddagger}) & \text{if } (x,y)\in[e^\ddagger,1]^2,\\ \min(x,y) & \text{otherwise,}\end{cases} \tag{7}$$

where T_1, T_2, T_3 are continuous t-norms.

(*c*) *$\mathcal{H}$ is expressed by*

$$\mathcal{H}(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y)\in[0,k]^2\cup[e^\ddagger,1]^2,\\ k & \text{if } (x,y)\in[0,k]\times[k,1]\cup[k,1]\times[0,k],\\ \min(x,y) & \text{otherwise,}\end{cases} \tag{8}$$

(*ii*) *If c is the largest idempotent element of $\mathcal{H}$ in $[e^\ddagger,1)$, then T is conditionally distributive over $\mathcal{H}$ iff the following three items are true.*

(*a*) *It holds that $e^\dagger = 0$.*

(*b*) *$\mathcal{H}$ is given by*

$$\mathcal{H}(x,y) = \begin{cases} c+(1-c)S_L(\frac{x-c}{1-c},\frac{y-c}{1-c}) & \text{if } (x,y)\in[c,1]^2,\\ k & \text{if } (x,y)\in[0,k]\times[k,1]\cup[k,1]\times[0,k],\\ \max(x,y) & \text{if } (x,y)\in[0,k]^2\cup[e^\ddagger,c]\times[e^\ddagger,1]\cup[c,1]\times[e^\ddagger,c],\\ \min(x,y) & \text{otherwise,}\end{cases} \tag{9}$$

(*c*) *T is given by*

$$T(x,y) = \begin{cases} kT_1(\frac{x}{k},\frac{y}{k}) & \text{if } (x,y)\in[0,k]^2,\\ k+(e^\ddagger-k)T_2(\frac{x-k}{e^\ddagger-k},\frac{y-k}{e^\ddagger-k}) & \text{if } (x,y)\in[k,e^\ddagger]^2,\\ e^\ddagger+(c-e^\ddagger)T_3(\frac{x-e^\ddagger}{c-e^\ddagger},\frac{y-e^\ddagger}{c-e^\ddagger}) & \text{if } (x,y)\in[e^\ddagger,c]^2,\\ c+(1-c)T_P(\frac{x-c}{1-c},\frac{y-c}{1-c}) & \text{if } (x,y)\in[c,1]^2,\\ \min(x,y) & \text{otherwise,}\end{cases} \tag{10}$$

where T_1, T_2 and T_3 are continuous t-norms.

Proof. ($\Rightarrow$) Based on the results of Lemma 3.2 and Lemma 3.3, we already know that $T(k,k)=k$, $e^\dagger=0$ and every point in $[0,e^\ddagger]$ is an idempotent element of $\mathcal{H}$.

Now, we assert $T(e^\ddagger,e^\ddagger)=e^\ddagger$. If not, suppose that $e^\ddagger > T(e^\ddagger,e^\ddagger)$. Then there exists an open generating interval (a,b) of T, such that $e^\ddagger\in[a,b]$ and T is a continuous Archimedean t-norm on $[a,b]^2$. Therefore, we obtain that

$$T(e^\ddagger,\mathcal{H}(e^\ddagger,z)) = \mathcal{H}(T(e^\ddagger,e^\ddagger),T(e^\ddagger,z)),$$

for all $z \in (e^{\ddagger}, b)$. According to the fact $T(k,k) = k$, it holds that both $T(e^{\ddagger}, e^{\ddagger})$ and $T(e^{\ddagger}, z)$ are in $[k, e^{\ddagger}]$. Further, from $\mathcal{H}$ is idempotent in $[0, e^{\ddagger}]^2$, we deduce from previous equation that $T(e^{\ddagger}, z) = T(e^{\ddagger}, e^{\ddagger})$ for any $z \in (e^{\ddagger}, b)$. But this is impossible because T is a continuous Archimedean t-norm on $[a,b]^2$. Up to now, by T is continuous it holds that there are three continuous t-norms T_1, T_2 and T_3 such that T is the form of $T = (\langle 0, k, T_1\rangle, \langle k, e^{\ddagger}, T_2\rangle, \langle e^{\ddagger}, 1, T_3\rangle)$.

Finally, we consider the structures of $\mathcal{H}$ and T on $[e^{\ddagger}, 1]^2$. If $c \in (e^{\ddagger}, 1)$ is an idempotent element of $\mathcal{H}$, then for arbitrary $x \in (e^{\ddagger}, 1)$ it establishes that $T(x,c) = T(x, \mathcal{H}(c,c)) = \mathcal{H}(T(x,c), T(x,c))$ if T is conditionally distributive over $\mathcal{H}$. Namely, $T(x,c)$ is an idempotent element of $\mathcal{H}$ for arbitrary $x \in (e^{\ddagger}, 1)$. Therefore, the continuity of T_3 implies that $\mathcal{H}(x,x) = x$ for any $x \in [e^{\ddagger}, c)$. Aa a result, $\mathcal{H}$ and T is given by Eq. (8) and Eq. (7) respectively, which means that item (i) is established, or there is a largest idempotent element $c \in [e^{\ddagger}, 1)$ of $\mathcal{H}$ such that $\mathcal{H}$ restricted on the region of $[c,1]^2$ is a continuous Archimedean t-conorm S^*. Now, similar to the way of Theorem 5.21 in [12], we can obtain that $T(c,c) = c$ and $\mathcal{H}$ is expressed by Eq. (9), and T is given by Eq. (10). Therefore, the item (ii) is valid.

($\Leftarrow$) To see the sufficiency, over here, we only present the verifications of (ii). Let T be given by Eq. (10) and $\mathcal{H}$ be given by Eq. (9). In what follows we will verify that the conditional distributivity is established.

If $x, y, z \in [0,k]^2$, then $T|_{[0,k]^2} = T_1$ conditionally distributive over $\mathcal{H}|_{[0,k]^2} = \max$.

If $x, y, z \in [k, e^{\ddagger}]^2$, then $T|_{[k,e^{\ddagger}]^2} = T_2$ conditionally distributive over $\mathcal{H}|_{[k,e^{\ddagger}]^2} = \min$.

If $x, y, z \in [e^{\ddagger}, c]^2$, then $T|_{[e^{\ddagger},c]^2} = T_3$ conditionally distributive over $\mathcal{H}|_{[e^{\ddagger},c]^2} = \max$.

If $x, y, z \in [c,1]^2$, then it reduces to conditional distributivity of T_P over S_L.

When x, y, z does not belong to the above situations, then by the commutativity of $\mathcal{H}$, it is enough for us to consider the case of $y \le z$.

- If $x \le k$, then the following situations arise.
  - If $k \le y \le z$, then $T(x, \mathcal{H}(y,z)) = x = \mathcal{H}(x,x) = \mathcal{H}(T(x,y), T(x,z))$.
  - If $y \le k \le z$, then $T(x, \mathcal{H}(y,z)) = T(x,k) = x = \max(T(x,y), x) = \mathcal{H}(T(x,y), T(x,z))$.
- If $k < x \le e^{\ddagger}$, then we have the following discussions.
  - If $y \le z \le k$, then $T(x, \mathcal{H}(y,z)) = T(x,z) = z = \mathcal{H}(y,z) = \mathcal{H}(T(x,y), T(x,z))$.
  - If $e^{\ddagger} \le y \le z$, then $T(x, \mathcal{H}(y,z)) = \min(x, \mathcal{H}(y,z)) = x = \mathcal{H}(x,x) = \mathcal{H}(T(x,y), T(x,z))$.
  - If $y \le k \le z$, then $T(x, \mathcal{H}(y,z)) = T(x,k) = k = \mathcal{H}(y, T(x,z)) = \mathcal{H}(\min(x,y), T(x,z)) = \mathcal{H}(T(x,y), T(x,z))$.
  - If $k < y < e^{\ddagger} \le z$, then $T(x, \mathcal{H}(y,z)) = T(x,y) = \mathcal{H}(T(x,y), x) = \mathcal{H}(T(x,y), T(x,z))$.
- If $x > e^{\ddagger}$, then the following cases arise.
  - If $y \le z \le e^{\ddagger}$, then $T(x, \mathcal{H}(y,z)) = \mathcal{H}(y,z) = \mathcal{H}(T(x,y), T(x,z))$.
  - If $k < y < e^{\ddagger} \le z$, then $T(x, \mathcal{H}(y,z)) = y = \mathcal{H}(y, T(x,z)) = \mathcal{H}(T(x,y), T(x,z))$.
  - If $y = e^{\ddagger} \le z$, then $T(x, \mathcal{H}(y,z)) = T(x,z) = \mathcal{H}(y, T(x,z)) = \mathcal{H}(T(x,y), T(x,z))$.
  - If $y \le k$ and $e^{\ddagger} \le z$, then $T(x, \mathcal{H}(y,z)) = T(x,k) = k = \mathcal{H}(y, T(x,z)) = \mathcal{H}(T(x,y), T(x,z))$.

□

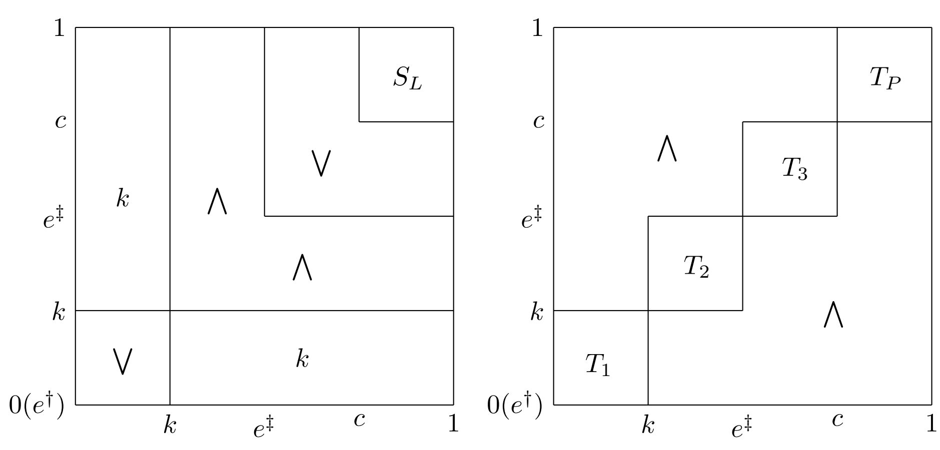

Fig.1 Structures of $\mathcal{H}$ (left) and T (right) in Theorem 3.4 (ii).

Remark 3.5. *Let T be a continuous t-norm and $\mathcal{H}$ be a 2-uninorm with the form Eq.* (5)*. From Theorem 3.1 in [29] and Theorem 3.4, one concludes that the distributivity and the conditional distributivity of T over $\mathcal{H}$ are two different but related research notions. To be specific,*

- *if every point in $(e^\ddagger, 1)$ is an idempotent element of $\mathcal{H}$, then these two notions are equivalent. Namely, T is conditionally distributive over $\mathcal{H}$ iff T is distributive over $\mathcal{H}$.*
- *if $c \in [e^\ddagger, 1)$ is the largest idempotent element of $\mathcal{H}$, then these two notions are not equivalent.*

3.2 The case: $\mathcal{H} \in \mathcal{C}_k^0 \cup \mathcal{C}_1^0 \cup \mathcal{C}_0^1 \cup \mathcal{C}_k^1$.

Now, we consider the case when $\mathcal{H}$ belongs to one of $\mathcal{C}_k^0, \mathcal{C}_1^0, \mathcal{C}_0^1, \mathcal{C}_k^1$. Similar to the case of $\mathcal{C}^k$, we also have $T(e^\ddagger, e^\ddagger) = e^\ddagger$ in this situation.

Lemma 3.6. *Let T be a continuous t-norm and $\mathcal{H}$ be a 2-uninorm with the form Eq.* (2) *or Eq.* (3) *or Eq.* (4) *or Eq.* (5)*. If T is conditionally distributive over $\mathcal{H}$ then $T(e^\ddagger, e^\ddagger) = e^\ddagger$.*

Proof. To see $T(e^\ddagger, e^\ddagger) = e^\ddagger$, we assume that $e^\ddagger > T(e^\ddagger, e^\ddagger)$. Then from $\mathrm{Ran}(T(e^\ddagger, \cdot)) = [0, e^\ddagger]$ we know that there exists a point z' such that $\max(k, T(e^\ddagger, e^\ddagger)) \geq T(e^\ddagger, y)$ for all $z' \geq y$ and $T(e^\ddagger, y) > \max(k, T(e^\ddagger, e^\ddagger))$ for all $y > z'$. Now, let us take $z \in (z', 1)$. From $T(e^\ddagger, e^\ddagger) < e^\ddagger$ and the assumption that T is conditionally distributive over $\mathcal{H}$, we obtain

$$T(e^\ddagger, z) = T(e^\ddagger, (\mathcal{H}(e^\ddagger, z)) = \mathcal{H}(T(e^\ddagger, e^\ddagger), T(e^\ddagger, z)).$$

According to the fact that $T(e^\ddagger, z) \in [k, e^\ddagger]$, it holds that $T(e^\ddagger, z) = \mathcal{H}(T(e^\ddagger, e^\ddagger), T(e^\ddagger, z)) \in \{k, T(e^\ddagger, e^\ddagger)\}$ for all $z \in (z', 1)$. But this is impossible because by the definition of z' we obtain $T(e^\ddagger, y) > \max(k, T(e^\ddagger, e^\ddagger))$ for all $z > z'$. Therefore, it must be that $T(e^\ddagger, e^\ddagger) = e^\ddagger$. □

3.2.1 The case: $\mathcal{H} \in \mathcal{C}_k^0$

Theorem 3.7. *Let T be a continuous t-norm and $\mathcal{H}$ be a 2-uninorm with the form Eq.* (2)*.*

(i) If $T(k, k) > e^\dagger$ and every point in $(e^\ddagger, 1)$ is an idempotent element of $\mathcal{H}$, then T is conditionally distributive over $\mathcal{H}$ iff $\mathcal{H}$ and T are given by

$$\mathcal{H}(x, y) = \begin{cases} \max(x, y) & \text{if } (x, y) \in [e^\dagger, k]^2 \cup [e^\ddagger, 1]^2, \\ k & \text{if } (x, y) \in [e^\dagger, k] \times [k, 1] \cup [k, 1] \times [e^\dagger, k], \\ \min(x, y) & \text{otherwise,} \end{cases} \tag{11}$$

and

$$T(x, y) = \begin{cases} e^\dagger T_1(\frac{x}{e^\dagger}, \frac{y}{e^\dagger}) & \text{if } (x, y) \in [0, e^\dagger]^2, \\ e^\dagger + (k - e^\dagger) T_2(\frac{x - e^\dagger}{k - e^\dagger}, \frac{y - e^\dagger}{k - e^\dagger}) & \text{if } (x, y) \in [e^\dagger, k]^2, \\ k + (e^\ddagger - k) T_3(\frac{x - k}{e^\ddagger - k}, \frac{y - k}{e^\ddagger - k}) & \text{if } (x, y) \in [k, e^\ddagger]^2, \\ e^\ddagger + (1 - e^\ddagger) T_4(\frac{x - e^\ddagger}{1 - e^\ddagger}, \frac{y - e^\ddagger}{1 - e^\ddagger}) & \text{if } (x, y) \in [e^\ddagger, 1]^2, \\ \min(x, y) & \text{otherwise,} \end{cases} \tag{12}$$

where T_1, T_2, T_3 and T_4 are continuous t-norms.

(ii) If $T(k, k) > e^\dagger$ and c is the largest idempotent element of $\mathcal{H}$ in $[e^\ddagger, 1)$, then T is conditionally distributive over $\mathcal{H}$ iff $\mathcal{H}$ and T have the following expressions.

$$\mathcal{H}(x, y) = \begin{cases} c + (1 - c) S_L(\frac{x - c}{1 - c}, \frac{y - c}{1 - c}) & \text{if } (x, y) \in [c, 1]^2, \\ \max(x, y) & \text{if } (x, y) \in [e^\dagger, k]^2 \cup [e^\ddagger, c] \times [e^\ddagger, 1] \cup [c, 1] \times [e^\ddagger, c], \\ k & \text{if } (x, y) \in [e^\dagger, k] \times [k, 1] \cup [k, 1] \times [e^\dagger, k], \\ \min(x, y) & \text{otherwise,} \end{cases} \tag{13}$$

and

$$T(x, y) = \begin{cases} e^\dagger T_1(\frac{x}{e^\dagger}, \frac{y}{e^\dagger}) & \text{if } (x, y) \in [0, e^\dagger]^2, \\ e^\dagger + (k - e^\dagger) T_2(\frac{x - e^\dagger}{k - e^\dagger}, \frac{y - e^\dagger}{k - e^\dagger}) & \text{if } (x, y) \in [e^\dagger, k]^2, \\ k + (e^\ddagger - k) T_3(\frac{x - k}{e^\ddagger - k}, \frac{y - k}{e^\ddagger - k}) & \text{if } (x, y) \in [k, e^\ddagger]^2, \\ e^\ddagger + (c - e^\ddagger) T_4(\frac{x - e^\ddagger}{c - e^\ddagger}, \frac{y - e^\ddagger}{c - e^\ddagger}) & \text{if } (x, y) \in [e^\ddagger, c]^2, \\ c + (1 - c) T_P(\frac{x - c}{1 - c}, \frac{y - c}{1 - c}) & \text{if } (x, y) \in [c, 1]^2, \\ \min(x, y) & \text{otherwise,} \end{cases} \tag{14}$$

where T_1, T_2, T_3, T_4 are continuous t-norms.

(*iii*) *If $T(k,k)\le e^{\dagger}$ and every point in $(e^{\ddagger},1)$ is an idempotent element of $\mathcal{H}$, then T is conditionally distributive over $\mathcal{H}$ iff $\mathcal{H}$ and T are given by*

$$\mathcal{H}(x,y)=\begin{cases}\max(x,y) & \text{if } (x,y)\in[e^{\dagger},k]^2\cup[e^{\ddagger},1]^2,\\ \min(x,y) & \text{otherwise,}\end{cases} \tag{15}$$

and

$$T(x,y)=\begin{cases}e^{\ddagger}T_1(\frac{x}{e^{\ddagger}},\frac{y}{e^{\ddagger}}) & \text{if } (x,y)\in[0,e^{\ddagger}]^2,\\ e^{\ddagger}+(1-e^{\ddagger})T_2(\frac{x-e^{\ddagger}}{1-e^{\ddagger}},\frac{y-e^{\ddagger}}{1-e^{\ddagger}}) & \text{if } (x,y)\in[e^{\ddagger},1]^2,\\ \min(x,y) & \text{otherwise,}\end{cases} \tag{16}$$

where T_1 and T_2 are continuous t-norms.

(*iv*) *If $T(k,k)\le e^{\dagger}$ and c is the largest idempotent element of $\mathcal{H}$ in $[e^{\ddagger},1)$, then T is conditionally distributive over $\mathcal{H}$ iff $\mathcal{H}$ and T have the following expressions.*

$$\mathcal{H}(x,y)=\begin{cases}c+(1-c)S_L(\frac{x-c}{1-c},\frac{y-c}{1-c}) & \text{if } (x,y)\in[c,1]^2,\\ \max(x,y) & \text{if } (x,y)\in[e^{\ddagger},c]\times[e^{\ddagger},1]\cup[c,1]\times[e^{\ddagger},c],\\ \min(x,y) & \text{otherwise,}\end{cases} \tag{17}$$

and

$$T(x,y)=\begin{cases}e^{\ddagger}T_1(\frac{x}{e^{\ddagger}},\frac{y}{e^{\ddagger}}) & \text{if } (x,y)\in[0,e^{\ddagger}]^2,\\ e^{\ddagger}+(c-e^{\ddagger})T_2(\frac{x-e^{\ddagger}}{c-e^{\ddagger}},\frac{y-e^{\ddagger}}{c-e^{\ddagger}}) & \text{if } (x,y)\in[e^{\ddagger},c]^2,\\ c+(1-c)T_P(\frac{x-c}{1-c},\frac{y-c}{1-c}) & \text{if } (x,y)\in[c,1]^2,\\ \min(x,y) & \text{otherwise,}\end{cases} \tag{18}$$

where T_1 and T_2 is a continuous t-norms.

Proof. (⇒) Firstly, let us consider the case of $T(k,k)>e^{\dagger}$. Let us first show that $T(k,k)=k$ and $T(e^{\dagger},e^{\dagger})=e^{\dagger}$. Indeed, from the expression of $\mathcal{H}$ and the fact that T is conditionally distributive over $\mathcal{H}$, we obtain

$$T(k,k)=T(k,\mathcal{H}(k,1))=\mathcal{H}(T(k,k),T(k,1))=\mathcal{H}(T(k,k),k)=k.$$

Moreover, we also have

$$e^{\dagger}=T(e^{\dagger},k)=T(e^{\dagger},\mathcal{H}(k,e^{\dagger}))=\mathcal{H}(T(e^{\dagger},k),T(e^{\dagger},e^{\dagger}))=\min(T(e^{\dagger},k),T(e^{\dagger},e^{\dagger}))=T(e^{\dagger},e^{\dagger}).$$

This means that T has the form of an ordinal sum of three continuous t-norms determined by the points of $e^{\dagger}$ and k. Therefore, similar to Theorem 3.4. We obtain the following two statements.

- The structures of $\mathcal{H}$ and T are expressed by Eq. (11) and Eq. (12), respectively, under the condition that $T(k,k)>e^{\dagger}$ and every point in $(e^{\ddagger},1)$ is an idempotent element of $\mathcal{H}$.
- The structures of $\mathcal{H}$ and T are expressed by Eq. (13) and Eq. (14), respectively, under the condition that $T(k,k)>e^{\dagger}$ and c is the largest idempotent element of $\mathcal{H}$ in $[e^{\ddagger},1)$.

As for the case of $T(k,k)\le e^{\dagger}$, then there is some $x_0\in[k,e^{\ddagger})$ satisfying $T(x_0,x_0)=e^{\dagger}$. As a consequence, $e^{\dagger}=T(x_0,x_0)=T(x_0,\mathcal{H}(x_0,1))=\mathcal{H}(T(x_0,x_0),T(x_0,1))=\mathcal{H}(e^{\dagger},x_0)=k$. Similar to the case of $T(k,k)>e^{\dagger}$, we obtain that the structures of $\mathcal{H}$ and T on the region of $[e^{\ddagger},1]^2$. Namely,

- The structures of $\mathcal{H}$ and T are given by Eq. (15) and Eq. (16), respectively, under the condition that $T(k,k)\le e^{\dagger}$ and every point in $(e^{\ddagger},1)$ is an idempotent element of $\mathcal{H}$.
- The structures of $\mathcal{H}$ and T are given by Eq. (17) and Eq. (18), respectively, under the condition that $T(k,k)\le e^{\dagger}$ and c is the largest idempotent element of $\mathcal{H}$ in $[e^{\ddagger},1)$.

(⇐) To see the sufficiency, over here, we only present the detail proof of the case of (ii). Because (i) is a special case of (ii), the proof is completely similar to (ii). Let T be given by Eq. (14) and $\mathcal{H}$ be given by Eq. (13). In what follows we will verify that the conditional distributivity is established.

If $x,y,z\in[0,e^{\dagger}]^2$, then $T|_{[0,e^{\dagger}]^2}=T_1$ conditionally distributive over $\mathcal{H}|_{[0,e^{\dagger}]^2}=\min$;
If $x,y,z\in[e^{\dagger},k]^2$, then $T|_{[e^{\dagger},k]^2}=T_2$ conditionally distributive over $\mathcal{H}|_{[e^{\dagger},k]^2}=\max$;
If $x,y,z\in[k,e^{\ddagger}]^2$, then $T|_{[k,e^{\ddagger}]^2}=T_3$ conditionally distributive over $\mathcal{H}|_{[k,e^{\ddagger}]^2}=\min$;

If $x, y, z \in [e^{\ddagger}, c]^2$, then $T|_{[e^{\ddagger},c]^2} = T_4$ conditionally distributive over $\mathcal{H}|_{[e^{\ddagger},c]^2} = \max$;
If $x, y, z \in [c, 1]^2$, then it will be reduce to conditional distributivity of T_P over S_L;
When x, y, z does not belong to the above situations, then by the commutativity of $\mathcal{H}$, it is enough for us to consider the case of $y \le z$.

- If $x \le e^{\dagger}$, then we get the following situations.
  - If $e^{\dagger} \le y \le z$, then $T(x, \mathcal{H}(y,z)) = \min(x, \mathcal{H}(y,z)) = x = \mathcal{H}(x,x) = \mathcal{H}(T(x,y), T(x,z))$.
  - If $y < e^{\dagger} \le z$, then $T(x, \mathcal{H}(y,z)) = T(x,y) = \min(T(x,y), T(y,z)) = \mathcal{H}(T(x,y), T(x,z))$.
- If $e^{\dagger} < x \le k$, then the following discussions appear.
  - If $y \le z \le e^{\dagger}$, then $T(x, \mathcal{H}(y,z)) = T(x,y) = y = \min(y,z) = \min(T(x,y), T(x,z)) = \mathcal{H}(T(x,y), T(x,z))$.
  - If $k \le y \le z$, then $T(x, \mathcal{H}(y,z)) = \min(x, \mathcal{H}(y,z)) = x = \mathcal{H}(x,x) = \mathcal{H}(T(x,y), T(x,z))$.
  - If $y < e^{\dagger} \le z$, then $T(x, \mathcal{H}(y,z)) = T(x,y) = \min(T(x,y), T(x,z)) = \mathcal{H}(T(x,y), T(x,z))$.
  - If $y = e^{\dagger}$ and $k \le z$, then $T(x, \mathcal{H}(y,z)) = T(x,k) = x = \mathcal{H}(y,x) = \mathcal{H}(T(x,y), T(x,z))$.
  - If $y = e^{\dagger} \le z \le k$, then $T(x, \mathcal{H}(y,z)) = T(x,z) = \max(T(x,y), T(x,z)) = \mathcal{H}(T(x,y), T(x,z))$.
  - If $e^{\dagger} < y \le k \le z$, then $T(x, \mathcal{H}(y,z)) = T(x,k) = x = T(x,z) = \max(T(x,y), T(x,z)) = \mathcal{H}(T(x,y), T(x,z))$.
- If $k < x \le e^{\ddagger}$, then the following situations arise.
  - If $y \le z \le k$, then $T(x, \mathcal{H}(y,z)) = \mathcal{H}(y,z) = \mathcal{H}(T(x,y), T(x,z))$.
  - If $e^{\ddagger} \le y \le z$, then $T(x, \mathcal{H}(y,z)) = x = \mathcal{H}(x,x) = \mathcal{H}(T(x,y), T(x,z))$.
  - If $y < e^{\dagger}$ and $k \le z$, then $T(x, \mathcal{H}(y,z)) = T(x,y) = \min(T(x,y), T(x,z)) = \mathcal{H}(T(x,y), T(x,z))$.
  - If $e^{\dagger} \le y \le k \le z$, then $T(x, \mathcal{H}(y,z)) = T(x,k) = k = \mathcal{H}(T(x,y), T(x,z))$.
  - If $k < y < e^{\ddagger} \le z$, then $T(x, \mathcal{H}(y,z)) = T(x,y) = \min(T(x,y), T(x,z)) = \mathcal{H}(T(x,y), T(x,z))$.
  - If $y = e^{\ddagger} \le z$, then $T(x, \mathcal{H}(y,z)) = T(x,z) = x = \mathcal{H}(x,x) = \mathcal{H}(T(x,y), T(x,z))$.
- If $e^{\ddagger} < x$, then the following situations arise.
  - If $y \le z \le e^{\ddagger}$, then $T(x, \mathcal{H}(y,z)) = \mathcal{H}(y,z) = \mathcal{H}(T(x,y), T(x,z))$.
  - If $y < e^{\dagger}$ and $e^{\ddagger} \le z$ or $(k < y < e^{\ddagger} \le z)$, then $T(x, \mathcal{H}(y,z)) = T(x,y) = \mathcal{H}(T(x,y), T(x,z))$.
  - If $y = e^{\ddagger}$ and $e^{\ddagger} \le z$, then $T(x, \mathcal{H}(y,z)) = T(x,z) = \mathcal{H}(T(x,y), T(x,z))$.
  - If $e^{\dagger} \le y \le k$ and $e^{\ddagger} \le z$, then $T(x, \mathcal{H}(y,z)) = T(x,k) = k = \mathcal{H}(T(x,y), T(x,z))$.

□

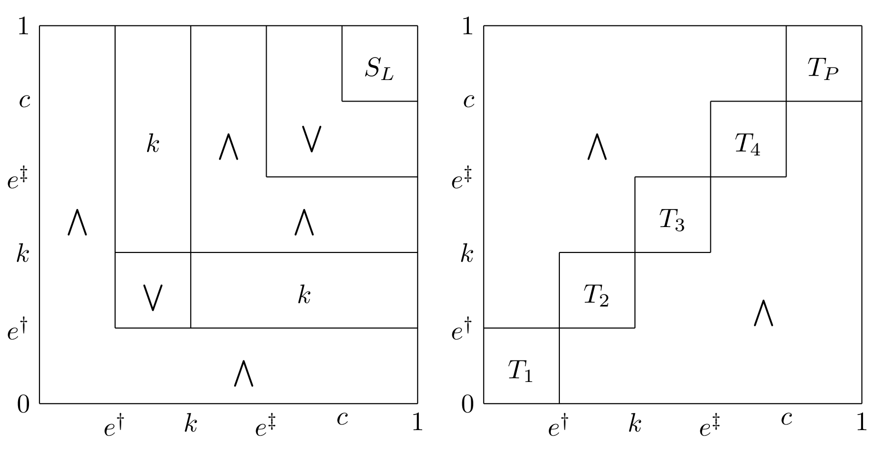

Fig.2 Structures of $\mathcal{H}$ (left) and T (right) in Theorem 3.7 (ii).

Remark 3.8. *Let T be a continuous t-norm and $\mathcal{H}$ be a 2-uninorm with the form Eq. (2). From Theorem 3.2 in [29] and Theorem 3.7, one concludes that the distributivity and the conditional distributivity of T over $\mathcal{H}$ are two different but related research notions. To be specific,*

- *if every point in* $(e^{\ddagger}, 1)$ *is an idempotent element of* $\mathcal{H}$, *then these two notions are equivalent. Namely,* T *is conditionally distributive over* $\mathcal{H}$ *iff* T *is distributive over* $\mathcal{H}$.

- *if* $c \in [e^{\ddagger}, 1)$ *is the largest idempotent element of* $\mathcal{H}$, *then these two notions are not equivalent.*

3.2.2 The case: $\mathcal{H} \in \mathcal{C}_1^0$

In this case, interestingly, the conditional distributivity will force the 2-uninorm $\mathcal{H}$ to degenerate into a uninorm.

Lemma 3.9. *Let* T *be a continuous t-norm and* $\mathcal{H}$ *be a 2-uninorm with the form Eq.* (3). *If* T *is conditionally distributive over* $\mathcal{H}$ *then* $k = e^{\ddagger}$ *and* $\mathcal{H}$ *is the uninorm* $U_{e^{\ddagger}}^{\min}$.

Proof. From Lemma 3.6 we already know $T(e^{\ddagger}, e^{\ddagger}) = e^{\ddagger}$. Consequently, taking a fixed $c \in (e^{\ddagger}, 1)$, it establishes that $e^{\ddagger} = T(e^{\ddagger}, c) = T(e^{\ddagger}, \mathcal{H}(k, c)) = \mathcal{H}(T(e^{\ddagger}, k), T(e^{\ddagger}, c)) = \mathcal{H}(k, e^{\ddagger}) = k$. Therefore, $\mathcal{H}$ is the uninorm $U_{e^{\ddagger}}^{\min}$. □

Theorem 3.10. *Let* T *be a continuous t-norm and* $\mathcal{H}$ *be a 2-uninorm with the form Eq.* (3).

(*i*) *If every point in* $(e^{\ddagger}, 1)$ *is an idempotent element of* $\mathcal{H}$, *then* T *is conditionally distributive over* $\mathcal{H}$ *iff* $\mathcal{H}$ *is given by* $U_{e^{\ddagger}}^{\min}$ *and* T *is given by*

$$T(x,y) = \begin{cases} e^{\dagger}T_1(\frac{x}{e^{\dagger}}, \frac{y}{e^{\dagger}}) & \text{if } (x,y) \in [0, e^{\dagger}]^2, \\ e^{\dagger} + (e^{\ddagger} - e^{\dagger})T_2(\frac{x-e^{\dagger}}{e^{\ddagger}-e^{\dagger}}, \frac{y-e^{\dagger}}{e^{\ddagger}-e^{\dagger}}) & \text{if } (x,y) \in [e^{\dagger}, e^{\ddagger}]^2, \\ e^{\ddagger} + (1 - e^{\ddagger})T_3(\frac{x-e^{\ddagger}}{1-e^{\ddagger}}, \frac{y-e^{\ddagger}}{1-e^{\ddagger}}) & \text{if } (x,y) \in [e^{\ddagger}, 1]^2, \\ \min(x,y) & \text{otherwise,} \end{cases} \tag{19}$$

where T_1, T_2 *and* T_3 *are continuous t-norms.*

(*ii*) *If* c *is the largest idempotent element of* $\mathcal{H}$ *in* $[e^{\ddagger}, 1)$, *then* T *is conditionally distributive over* $\mathcal{H}$ *iff* $\mathcal{H}$ *and* T *have the following expressions.*

$$\mathcal{H}(x,y) = \begin{cases} c + (1-c)S_L(\frac{x-c}{1-c}, \frac{y-c}{1-c}) & \text{if } (x,y) \in [c,1]^2, \\ \max(x,y) & \text{if } (x,y) \in [e^{\dagger}, c] \times [e^{\dagger}, 1] \cup [e^{\dagger}, 1] \times [e^{\dagger}, c], \\ \min(x,y) & \text{otherwise,} \end{cases} \tag{20}$$

and

$$T(x,y) = \begin{cases} e^{\dagger}T_1(\frac{x}{e^{\dagger}}, \frac{y}{e^{\dagger}}) & \text{if } (x,y) \in [0, e^{\dagger}]^2, \\ e^{\dagger} + (e^{\ddagger} - e^{\dagger})T_2(\frac{x-e^{\dagger}}{e^{\ddagger}-e^{\dagger}}, \frac{y-e^{\dagger}}{e^{\ddagger}-e^{\dagger}}) & \text{if } (x,y) \in [e^{\dagger}, e^{\ddagger}]^2, \\ e^{\ddagger} + (c - e^{\ddagger})T_3(\frac{x-e^{\ddagger}}{c-e^{\ddagger}}, \frac{y-e^{\ddagger}}{c-e^{\ddagger}}) & \text{if } (x,y) \in [e^{\ddagger}, c]^2, \\ c + (1-c)T_P(\frac{x-c}{1-c}, \frac{y-c}{1-c}) & \text{if } (x,y) \in [c, 1]^2, \\ \min(x,y) & \text{otherwise,} \end{cases} \tag{21}$$

where T_1, T_2 *and* T_3 *are continuous t-norms.*

Proof. (⇒) According to Lemma 3.9, we have $k = e^{\ddagger}$. Therefore, in this case, the conditional distributivity of continuous t-norm T over $\mathcal{H}$ is reduced to the conditional distributivity of T over $U_{\min}$. Further, using the result obtained in Lemma 3.2 that $\mathcal{H}(e^{\dagger}, e^{\dagger}) = e^{\dagger}$, we prove that $T(e^{\dagger}, e^{\dagger}) = e^{\dagger}$. If not, suppose that $e^{\dagger} > T(e^{\dagger}, e^{\dagger})$, then there is an open generating interval (a, b) of T satisfying $e^{\dagger} \in (a, b)$ and T is a continuous Archimedean t-norm on $[a, b]^2$. Taking $z \in (e^{\dagger}, b)$, we obtain $T(e^{\dagger}, z) = T(e^{\dagger}, \mathcal{H}(e^{\dagger}, z)) = \mathcal{H}(T(e^{\dagger}, e^{\dagger}), T(e^{\dagger}, z)) = T(e^{\dagger}, e^{\dagger})$, that is $T(e^{\dagger}, z) = T(e^{\dagger}, e^{\dagger})$ for all $z \in (e^{\dagger}, b)$. But this contradicts the property of continuous and Archimedean t-norms. Therefore, it must be that $T(e^{\dagger}, e^{\dagger}) = e^{\dagger}$. Up to now, together with Lemma 3.6 we have proven that both $e^{\dagger}$ and $e^{\ddagger}$ are idempotent elements of T. Finally, By the same arguments similar to Theorem 3.4, we obtain that item (i) and item (ii) are true.

(⇐) Similar to the verifications of Theorem 3.4, one has that T is conditionally distributive over $\mathcal{H}$. □

Remark 3.11. *Let* T *be a continuous t-norm and* $\mathcal{H}$ *be a 2-uninorm with the form Eq.* (3). *From Theorem 3.3 in [29] and Theorem 3.10, one concludes that the distributivity and the conditional distributivity of* T *over* $\mathcal{H}$ *are two different but related research notions. To be specific,*

- *if every point in* $(e^{\ddagger}, 1)$ *is an idempotent element of* $\mathcal{H}$, *then these two notions are equivalent. Namely,* T *is conditionally distributive over* $\mathcal{H}$ *iff* T *is distributive over* $\mathcal{H}$.

- *if* $c \in [e^{\ddagger}, 1)$ *is the largest idempotent element of* $\mathcal{H}$, *then these two notions are not equivalent.*

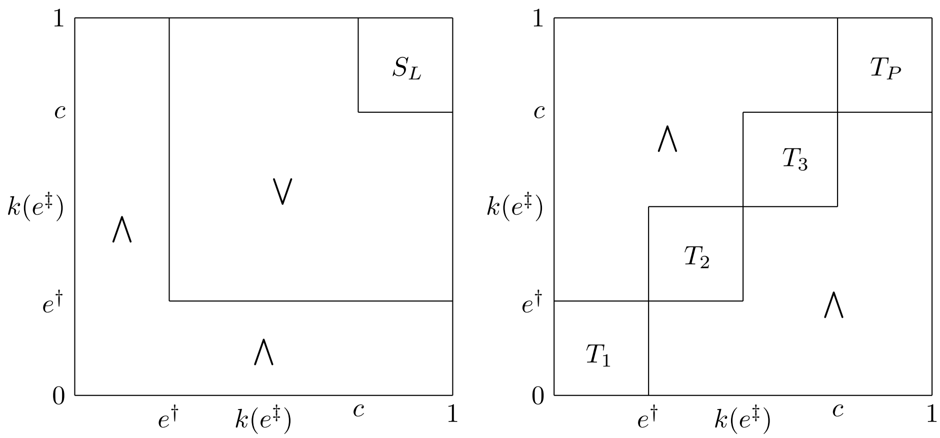

Fig.3 Structures of $\mathcal{H}$ (left) and T (right) in Theorem 3.10 (ii).

3.2.3 The case: $\mathcal{H} \in \mathcal{C}_0^1$

In this situation, contrary to the previous cases, conditional distributivity has no solution.

Theorem 3.12. *Let T be a continuous t-norm and $\mathcal{H}$ be a 2-uninorm with the form Eq.* (5)*. Then T is not conditionally distributive over $\mathcal{H}$.*

Proof. We suppose that T is conditionally distributive over $\mathcal{H}$. Then by Lemma 3.6 we know that $T(e^\ddagger, e^\ddagger) = e^\ddagger$. Based on this fact, let us prove that $e^\ddagger = k$. Indeed, by taking arbitrary fixed $\varsigma \in (e^\ddagger, 1)$, it is deduced by the conditional distributivity that $e^\ddagger = T(e^\ddagger, \varsigma) = T(e^\ddagger, \mathcal{H}(k, \varsigma)) = \mathcal{H}(T(e^\ddagger, k), T(e^\ddagger, \varsigma)) = \mathcal{H}(k, e^\ddagger) = k$. Moreover, by a same way to Theorem 3.10 we obtain $T(e^\dagger, e^\dagger) = e^\dagger$. Now, for arbitrary $(y, z) \in (0, e^\dagger) \times (k, 1)$, we get $e^\dagger = T(e^\dagger, z) = T(e^\dagger, \mathcal{H}(y, z)) = \mathcal{H}(T(e^\dagger, y), T(e^\dagger, z)) = \mathcal{H}(y, e^\dagger) = y$, which is a contradiction. Therefore, our assumption is not valid. □

Remark 3.13. *Let T be a continuous t-norm and $\mathcal{H}$ be a 2-uninorm with the form Eq.* (5)*. From Theorem 3.4 in* [29] *and Theorem 3.12, one concludes that both the distributivity and the conditional distributivity of T over $\mathcal{H}$ are not established. To be specific, in this situation, we have*

- *T is not distributive over $\mathcal{H}$.*
- *T is not conditionally distributive over $\mathcal{H}$.*

3.2.4 The case: $\mathcal{H} \in \mathcal{C}_k^1$

Theorem 3.14. *Let T be a continuous t-norm and $\mathcal{H}$ be a 2-uninorm with the form Eq.* (5)*.*

(*i*) *If $e^\dagger \neq 0$ then T is not conditionally distributive over $\mathcal{H}$.*

(*ii*) *If $e^\dagger = 0$ then T is conditionally distributive over $\mathcal{H}$ iff one of the following cases is satisfied:*

(*a*) $\mathcal{H} = S_M$.

(*b*) *There exists some $c \in [e^\ddagger, 1)$ such that the structure of $\mathcal{H}$ and T are expressed by*

$$\mathcal{H}(x,y) = \begin{cases} c + (1-c)S_L(\frac{x-c}{1-c}, \frac{y-c}{1-c}) & \text{if } (x,y) \in [c,1]^2, \\ \max(x,y) & \text{otherwise,} \end{cases} \tag{22}$$

and

$$T(x,y) = \begin{cases} cT_1'(\frac{x}{c}, \frac{y}{c}) & \text{if } (x,y) \in [0,c]^2, \\ c + (1-c)T_P(\frac{x-c}{1-c}, \frac{y-c}{1-c}) & \text{if } (x,y) \in [c,1]^2, \\ \min(x,y) & \text{otherwise,} \end{cases} \tag{23}$$

where T_1' is a continuous t-norm.

Proof. ($\Rightarrow$) According to Lemma 3.6 one concludes that $T(e^{\ddagger}, e^{\ddagger}) = e^{\ddagger}$. Setting arbitrary fixed $c \in (e^{\ddagger}, 1)$, we obtain $e^{\ddagger} = T(e^{\ddagger}, c) = T(e^{\ddagger}, \mathcal{H}(k, c)) = \mathcal{H}(T(e^{\ddagger}, k), T(e^{\ddagger}, c)) = \mathcal{H}(k, e^{\ddagger}) = k$. If $e^{\dagger} \neq 0$, then the conditional distributivity of continuous t-norm T over $\mathcal{H}$ is reduced to T over $U_{\max}$. Similar to Theorem 3.12, one concludes that T is not conditionally distributive over $\mathcal{H}$ in this situation.

If $e^{\dagger} = 0$, then the conditional distributivity of continuous t-norm T over $\mathcal{H}$ is reduced to T over a continuous t-conorm S. Therefore, in this case, by Theorem 5.21 in [12], the item (ii) is true.

($\Leftarrow$) It is straightforward by using Theorem 5.21 in [12]. □

Remark 3.15. *Let T be a continuous t-norm and $\mathcal{H}$ be a 2-uninorm with the form Eq.* (4)*. From Theorem 3.5 in* [29] *and Theorem 3.14, one concludes that both the distributivity and the conditional distributivity of T over $\mathcal{H}$ lead to $e^{\dagger} = 0$. And in the case of $e^{\dagger} = 0$, both the distributivity and the conditional distributivity of continuous t-norm T over $\mathcal{H}$ are reduced to that of T over a continuous t-conorm S. Interestingly, the results obtained in this paper demonstrate that although distributivity and conditional distributivity are two different research topics, the following two perspectives show that they are closely related. One is that if the distributivity equation has a solution, then so has the conditional distributivity equation and vice versa. Another is that if every point in $(e^{\ddagger}, 1)$ is an idempotent element of $\mathcal{H}$, then conditional distributivity and distributivity in this case are equivalent, otherwise, they are not equivalent.*

The following table is a comparison between distributivity and conditional distributivity on this topic. Here, $\surd$ means that $\mathcal{H}$ is idempotent on $(e^{\dagger}, 1)$, and $\times$ means that $\mathcal{H}$ not is idempotent on $(e^{\dagger}, 1)$.

Table 1: comparison between distributivity and conditional distributivity .

		idempotent in $(e^{\ddagger}, 1)$	distributivity VS conditional distributivity	more detail
$\mathcal{C}^k$		$\surd$	equivalent	See Remark 3.5
		$\times$	not equivalent	
$\mathcal{C}_k^0$		$\surd$	equivalent	See Remark 3.8
		$\times$	not equivalent	
$\mathcal{C}_1^0$		$\surd$	equivalent	See Remark 3.11
		$\times$	not equivalent	
$\mathcal{C}_0^1$			not distributive and not conditionally distributive	See Remark 3.13
$\mathcal{C}_k^1$	$e^{\dagger} = 0$	$\surd$	equivalent	See Remark 3.15
		$\times$	not equivalent	
	$e^{\dagger} > 0$		not distributive and not conditionally distributive	

4 Conclusions

We discussed the conditional distributivity of a continuous t-norms over a 2-uninorm. We gave the complete solution of all pairs $(T, \mathcal{H})$ fulfilling this property and the reached conclusions demonstrating that although distributivity and conditional distributivity are two different research topic, the following two perspectives show that they are closely related. One is that if the distributivity equation has a solution, then so has the conditional distributivity equation and vice versa. Another is that if every point in $(e^{\ddagger}, 1)$ is an idempotent element of $\mathcal{H}$, then distributivity and conditional distributivity on this topic are equivalent, otherwise, they are not equivalent. In the future, we will continue to study the conditional distributivity on this topic, but the 2-uninorms will be relaxed to have only continuous underlying operators.

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