Iranian Journal of Fuzzy Systems

Volume 20, Number 2, (2023), pp. 83-102

Original paper

A dual probabilistic linguistic MARCOS approach based on generalized Dombi operator for decision-making

A. Saha[1], A. R. Mishra[2], P. Rani[3], T. Senapati[4] and R. R. Yager[5]

[1,3] *Department of Engineering Mathematics, College of Engineering, Koneru Lakshmaiah Education Foundation, Vaddeswaram-522302, AP, India*

[2] *Department of Mathematics, Government College Raigaon, Satna, M P-485441, India*

[4] *School of Mathematics and Statistics, Southwest University, Beibei-400715, Chongqing, China*

[4] *Department of Mathematics, Padima Janakalyan Banipith, Kukrakhupi, Jhargram, 721517, India*

[5] *Machine Intelligence Institute, Iona College, New Rochelle, NY 10801, USA*

abhijit84.math@gmail.com, arunodaya87@outlook.com, pratibha138@gmail.com, math.tapan@gmail.com, yager@panix.com

Abstract

Dual probabilistic linguistic term sets (DPLTSs) are more powerful compare to probabilistic linguistic term sets, probabilistic hesitant fuzzy sets, hesitant fuzzy sets and intuitionistic fuzzy sets for the reason that they deal with both belongingness grades and non-belongingness grades along with their respective probabilities. On the other hand, the generalized Dombi operators have higher flexibility due to inclusion of two parameters. MARCOS (Measurements alternatives and ranking according to compromise solution) technique was developed by utilizing the utility degrees of options using the ideal and anti-ideal solutions. Here, we combine the merits of generalized Dombi operator and MARCOS and propose a DPL-MARCOS approach under dual probabilistic linguistic setting. In this methodology, the concepts of consistency and similarity between the experts are used to calculate their weights of subjective and objective types, respectively. For aggregating experts' preferences, we propose dual probabilistic linguistic- generalized Dombi weighted averaging aggregation operator. A biomass feedstock selection problem is furnished to show the applicability of our technique. We have considered coconut shell, coffee husk and sugarcane baggage as alternatives. The result shows that coffee husk is the most suitable option. The sensitivity assessment of parameter values reveals that our technique is stable. The comparative study proves that our model is more significant and realistic compare to the existing ones.

Keywords: Dual probabilistic linguistic term set, generalized Dombi weighted averaging operator, biomass, MARCOS, decision-making.

1 Introduction

For now-a-days, it becomes the most difficult task for an expert (decision-maker) to exhibit his/her preference values in terms of crisp values due to vagueness of human thinking. Accordingly, assessment information found from an expert becomes inaccurate or inexact. The "linguistic term sets (LTSs)", proposed by Herrera and Martinez [11] can handle qualitative evaluation information by imposing "linguistic variables (LVs)" and thus minimizes the inaccuracy of evaluation data. In case of a LTS, an expert is allowed to use only one "linguistic term (LT)" for particular information. But in reality, an expert cannot put his assessment with a single LT, due to hesitancy among various LTs. To tackle this type of scenario, Rodriguez et al. [21] developed the notion of a "hesitant fuzzy linguistic term set (HFLTS)", by allowing expert's to put his/her preferences with various possible LTs. Isik and Kaya [12] proposed one methodology that integrates 2-tuple LFM and linguistic fuzzy modifiers with HFLTS to overcome the accuracy problem and obtain a more sensitive and flexible decision procedure. However, HFLTS bears the drawback of assigning equal importance to all the LTs. In reality, the importance of the LTs shouldn't be ignored, because experts may prefer various degrees of

Corresponding Author: T. Senapati

Received: July 2021; Revised: September 2022; Accepted: November 2022.

https://doi.org/10.22111/IJFS.2023.7558

Table 1: List of abbreviations

Abbreviation	**Meaning**
DPL	Dual probabilistic linguistic
DPLE	DPL element
MCDM	Multi-criteria decision-making
DE	Decision expert
DPLGDWAA	DPL generalized Dombi weighted averaging aggregation
Λ_n	Set of natural numbers upto n
TOPSIS	Technique for Order Performance by Similarity to Ideal Solution
COPRAS	Complex proportional assessment
MAIRCA	Multi Attribute Ideal-Real Comparative Analysis
MARCOS	Measurements alternatives and ranking according to compromise solution

importance with respect to several possible LTs. For example, if an expert claims that a life insurance plan is 'moderately bad' and 'bad' with probabilities 80%, and 20% respectively; the assessment information contains LTs as well as their probabilistic information. To elude the issue, Pang et al. [19] presented the "probabilistic linguistic term sets (PLTSs)" where LTs are described along with their associated probabilities. Thus, PLTSs are advanced to the "hesitant fuzzy sets (HFSs)", LTSs and HFLTs. Recently, several concepts have been discussed on PLTSs [14, 15, 16, 17, 18, 20, 23, 31, 33] as pointed out by Xie et al. [29], PLTSs do not portray the assessment information from the "non-belongingness grade (NG)" viewpoint. To address this scenario, the notion of "dual probabilistic linguistic term set (DPLTS)" was presented by Xie et al. [29]. The DPLTSs deal with the assessment information with the "belongingness grade (BG)" and the NG. Xie et al. [29] proposed averaging aggregation operator to treat with "multi-criteria decision-making (MCDM)" problems with DPLTSs setting. Xie et al. [30] discussed an MCDM model using the incomplete DPL preference relations.

1.1 Research gaps

The below mentioned issues have been found under DPL environment:

1. The combination of objective and subjective weighting of "decision experts (DEs)" was not taken into consideration in existing researches [29, 30] on DPLTSs. The subjective weighting completely reflects DE's experimental opinions, but objective weighting indifference subjective preferences of a DE. To obtain a reasonable result, objective and subjective weighting of DEs ought to be fused.

2. In real situations, all criteria don't have equal importance. The weight of criteria needs to be assessed very logically. In the decision-making methodology developed by Xie et al. [29], criteria weights were given arbitrarily during the aggregation of criteria values. Consequently, the final ranking gets influenced.

3. As DPL-algebraic operator [29] is lacking in flexibility during aggregation, a flexible operator should be opted for aggregating data.

4. Various applied tools, namely the TOPSIS, COPRAS, MAIRCA, MARCOS and others have been utilized under "fuzzy sets (FSs)", "intuitionistic fuzzy sets (IFSs)", HFSs, PHLTSs and DPHFSs for aiming of obtaining prioritizations of options. But, ranking of options by a renowned MCDM method on DPLTSs is missing in the existing study.

1.2 Our contribution

According to the aforesaid challenges and to evade them, the key contribution is given by

1. A new procedure for DE's weight is presented. The objective and subjective weighting approaches of DEs are used based on the idea of "consistency harmonious weight index (CHWI)" and assessment similarity of DEs, respectively.

2. The "Grey correlation coefficient (GCC)" [5] is applied for computing the criteria weights. The benefit of choosing grey correlation coefficient is that it is capable of reflecting the significance of each criterion.

3. We propose DPL-generalized Dombi operational laws. Consequently, "generalized Dombi operators (GDO) [7]" are more effective and generalized. Next, the "DPL generalized Dombi weighted averaging (DPLGDWA)" operator is discussed.

4. Compare to other well-known existing MCDM methods (MAIRCA, TODIM, TOPSIS and COPRAS), MARCOS method [27] has the advantage of fusing of outcomes of the ratio and reference point approach. In this line, a MARCOS approach is introduced by applying the aforesaid DPLGDWA operator to treat the MCDM concerns on DPLTSs. To show the applicability of the proposed technique, biomass feedstock selection based case study is presented and solved with the developed technique. After that we furnish the impacts of parameters upon ranking order in order to verify the stability of our technique.

1.3 Arrangement of the paper

In Section 2, the overview of DPL term sets and generalized Dombi operations is provided. The creation of DPL-generalized Dombi operational laws and their properties are covered in Section 3. The associated DPL generalized Dombi weighted averaging aggregation (DPLGDWAA) operator along with it's essential properties are also furnished. The extension of MARCOS tool under DPL setting is provided in Section 4. Section 5 defines the investigated problem. Section 6 shows the outcomes and it's analysis. Some concluding remarks and future scopes are provided at the end.

2 Basic concepts

In this line, we present some elementary ideas of the study.

Definition 2.1. [32] *A discrete linguistic term set* $\Gamma = \{\ell_\alpha : \alpha = -T, \dots, 0, \dots, T\}$*is a set (*$\ell_\alpha$ *signifies a LV and* T *symbolizes a positive integer) which holds the axioms as follows:*
1. $Neg(\ell_\alpha) = \ell_\delta$ *provided* $\alpha + \delta = 2T$.
2. $\ell_\alpha \leq \ell_\varsigma$ *if* $\alpha \leq \varsigma$.

The reversible transformation functions [9] $\tilde{\xi} : [-T, T] \to [0, 1]$ and $\tilde{\xi}^{-1} : [0, 1] \to [-T, T]$ are defined as

$$\tilde{\xi}\left(\ell_\alpha\right) = \frac{\alpha}{2T} + \frac{1}{2}\ (\alpha = -T, \dots, 0, \dots, T)\,,\ \tilde{\xi}^{-1}\left(\beta\right) = \ell_{(2\beta - 1)T}\left(\beta \in [0, 1]\right).$$

Definition 2.2. [29] *Assume* $U = \{y_i : i \in \Lambda_n\}$ *a fixed set and* $\Gamma = \{\ell_\alpha : \alpha = -T, \dots, 0, \dots, T\}$*be an LTS. Then a DPLTS* $Y_\ell(\Xi)$ *on* U *is defined as*

$$Y_\ell(\Xi) = \{< y_i, \mu_\ell(\Xi)(y_i), \nu_\ell(\Xi)(y_i) >: y_i \in U\},$$

where $\mu_\ell(\Xi)(x_i) = \{\ell_{\phi^{i(a)}}(\Xi^{(a)}) : \ell_{\phi^{i(a)}} \in \Gamma, \Xi^{(a)} \geq 0, \sum_a \Xi^{(a)} \leq 1\}$ *and* $\nu_\ell(\Xi)(y_i) = \{\ell_{\theta^{i(b)}}(\Xi^{(b)}) : \ell_{\theta^{i(b)}} \in \Gamma,\ \Xi^{(b)} \geq 0, \sum_b \Xi^{(b)} \leq 1\}$ *signify the linguistic BG and NG of* $y_i \in U$ *with probabilities* $\Xi^{(a)}$*and* $\Xi^{(b)}$*respectively.* $\phi^{i(a)}$ *and* $\theta^{i(b)}$*being the subscripts of* $\ell_{\phi^{i(a)}}$*and* $\ell_{\theta^{i(b)}}$*respectively.*

If the DPLTS $Y_\ell(\Xi)$comprises only single value, then it eases to a "DPL element (DPLE)" and is represented by $\delta_\ell\left(\Xi\right) = \left\langle \left\{\ell_{\phi^{i(a)}}\left(\Xi^{(a)}\right)\right\}, \left\{\ell_{\theta^{i(b)}}\left(\Xi^{(b)}\right)\right\}\right\rangle$ where $\ell_{\phi^{i(a)}}, \ell_{\theta^{i(b)}} \in \Gamma$ and $0 \leq \Xi^{(a)},\ \Xi^{(b)} \leq 1$.

Definition 2.3. [29] *Let* $\delta_\ell(\Xi) =< \{\ell_{\phi^{i(a)}}(\Xi^{(a)})\}, \{\ell_{\theta^{i(b)}}(\Xi^{(b)})\} >$ *be a DPLE. Then the linguistic score value of* $\delta_\ell(\Xi)$ *is given by*

$$Sc(\delta_\ell(\Xi)) = \ell_{\alpha - \beta} \text{ where } \alpha = \frac{\sum_a \phi^{i(a)} \times \Xi^{(a)}}{\sum_a \Xi^{(a)}},\ \beta = \frac{\sum_b \theta^{i(b)} \times \Xi^{(b)}}{\sum_b \Xi^{(b)}}. \tag{1}$$

The notion of accuracy value [29] should be considered for comparison when linguistic score values of DPLEs becomes identical.

Definition 2.4. [29] *Consider* $\delta_\ell(\Xi) =< \{\ell_{\phi^{i(a)}}(\Xi^{(a)})\}, \{\ell_{\theta^{i(b)}}(\Xi^{(b)})\} >$ *be a DPLE. Then the accuracy value of* $\delta_\ell\left(\Xi\right)$ *is given by*

$$Ac\left(\delta_\ell(\Xi)\right) = \frac{\sqrt{\sum_a ((\phi^{i(a)} - \alpha)\Xi^{(a)})^2}}{\sum_a \Xi^{(a)}} + \frac{\sqrt{\sum_b ((\theta^{i(b)} - \beta)\Xi^{(b)})^2}}{\sum_b \Xi^{(b)}}. \tag{2}$$

Definition 2.5. [29] *Let us take two DPLEs* $\delta_\ell^{(1)}(\Xi)$ and $\delta_\ell^{(2)}(\Xi)$. *Then, the comparison procedure is described as*

A. If $Sc(\delta_\ell^{(1)}(\Xi)) > Sc(\delta_\ell^{(2)}(\Xi))$, *then* $\delta_\ell^{(1)}(\Xi) \succ \delta_\ell^{(2)}(\Xi)$,

B. If $Sc(\delta_\ell^{(1)}(\Xi)) > Sc(\delta_\ell^{(2)}(\Xi))$, *then*

- *(a) If* $Ac(\delta_\ell^{(1)}(\Xi)) > Ac(\delta_\ell^{(2)}(\Xi))$, *then* $\delta_\ell^{(1)}(\Xi) \succ \delta_\ell^{(2)}(\Xi)$,
- *(b) If* $Ac(\delta_\ell^{(1)}(\Xi)) = Ac(\delta_\ell^{(2)}(\Xi))$, *then* $\delta_\ell^{(1)}(\Xi) = \delta_\ell^{(2)}(\Xi)$.

Definition 2.6. [7] *The generalized Dombi operators* $GD_t^k(p_1, p_2)$ [7] *are defined as*

$$GD_t^k(p_1,p_2) = \left(1 + \left(\frac{1}{t}\left(\prod_{j=1}^{2}\Phi_t^k(p_j) - 1\right)\right)^{-\frac{1}{k}}\right)^{-1} \quad \text{or} \quad GD_t^k(p_1,p_2) = \left(1 + \left(\frac{1}{t}\left(\prod_{j=1}^{2}\Theta_t^k(p_j) - 1\right)\right)^{\frac{1}{k}}\right)^{-1},$$

where $\Phi_t^k(p_j) = 1 + t\left(\frac{p_j}{1-p_j}\right)^k$, $\Theta_t^k(p_j) = 1 + t\left(\frac{1-p_j}{p_j}\right)^k$ $(p_j \in (0,1],\ j = 1,2)$ *with* $t > 0$ *and* $k \geq 1$.

3 Generalized Dombi operations and associated averaging aggregation operator

For sake of minimizing the steps of calculations during aggregating data, we propose the technique of adjustment of probabilities demonstrated below.

Example 3.1. *Consider the linguistic term set* $\Gamma = \{\ell_\alpha : \alpha = -3, \ldots, 0, \ldots, 3\}$. *Suppose we have two DPLEs* $\delta_\ell^{(1)}(\Xi) =< \{\wp_2(0.2), \wp_3(0.8)\}, \{\wp_{-1}(0.5), \wp_{-2}(0.5)\} >$ *and* $\delta_\ell^{(2)}(\Xi) =< \{\wp_0(1)\}, \{\wp_{-1}(0.4), \wp_2(0.6)\} >$. *Then upon adjustment we have:* $\tilde{\delta}_\ell^{(1)}(\Xi) =< \{\ell_2(0.2), \ell_3(0.8)\}, \{\ell_{-1}(0.4), \ell_{-1}(0.1), \ell_{-2}(0.5)\} >$ *and* $\tilde{\delta}_\ell^{(2)}(\Xi) = \tilde{\delta}_\ell^{(2)}(\Xi) =< \{\ell_0(0.2), \ell_0(0.8)\}, \{\ell_{-1}(0.4), \ell_2(0.1), \ell_2(0.5)\} >$. *The procedure for adjustment is shown below (Fig. 1).*

$$\begin{array}{lcc} & \delta_\ell^{(1)}(\Xi) & \delta_\ell^{(2)}(\Xi) \\ \text{Membership degree (original)}: & \ell_2(0.2)\quad \ell_3(0.8) & \ell_0(1) \\ & \downarrow \qquad\quad \downarrow & \swarrow\ \searrow \\ \text{Membership degree (adjusted)}: & \ell_2(0.2)\quad \ell_3(0.8) & \ell_0(0.2)\quad \ell_0(0.8) \end{array}$$

$$\begin{array}{lcccc} \text{Non-membership degree (original)}: & \ell_{-1}(0.5) & \ell_{-2}(0.5) & \ell_{-1}(0.4) & \ell_2(0.6) \\ & \swarrow\ \searrow & \downarrow & \downarrow & \swarrow\ \searrow \\ \text{Non-membership degree (adjusted)}: & \ell_{-1}(0.4)\quad \ell_{-1}(0.1) & \ell_{-2}(0.5) & \ell_{-1}(0.4) & \ell_2(0.1)\quad \ell_2(0.5) \end{array}$$

Figure 1: Adjustment procedure of DPLEs

Next, we propose DPL Generalized Dombi operations.

Definition 3.2. *Let* $\tilde{\delta}_\ell^{(j)}(\Xi) = \left\langle \{\ell_{\phi^{j(a)}}(\Xi^{(a)})\}, \{\ell_{\theta^{j(b)}}(\Xi^{(b)})\}\right\rangle$, $j = 1,2$ *be two DPLEs and* $\lambda > 0$. *Also let* $\tilde{\xi}_\phi^j = \tilde{\xi}(\ell_{\phi^{j(a)}})$ *and* $\tilde{\xi}_\theta^j = \tilde{\xi}(\ell_{\theta^{j(b)}})$ $(j = 1,2)$. *Then DPLGD operations are defined as*

(I)

$$\tilde{\delta}_\ell^{(1)}(\Xi)\tilde{\oplus}\tilde{\delta}_\ell^{(2)}(\Xi) = \left\langle \tilde{\xi}^{-1}\left(\left(1 + \left(\frac{1}{t}\left(\prod_{j=1}^{2}\Phi_t^k(\tilde{\xi}_\phi^j) - 1\right)\right)^{-\frac{1}{k}}\right)^{-1}\right)(\Xi^{(a)}), \right.$$
$$\left. \tilde{\xi}^{-1}\left(\left(1 + \left(\frac{1}{t}\left(\prod_{j=1}^{2}\Theta_t^k(\tilde{\xi}_\theta^j) - 1\right)\right)^{\frac{1}{k}}\right)^{-1}\right)(\Xi^{(b)}) \right\rangle. \quad (3)$$

(*II*)

$$\tilde{\delta}_\ell^{(1)}(\Xi)\tilde{\otimes}\tilde{\delta}_\ell^{(2)}(\Xi) = \Bigg\langle \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{2}\Theta_t^k(\tilde{\xi}_\phi^j)-1\right)\right)^{\frac{1}{k}}\right)^{-1}\right)(\Xi^{(a)}),$$
$$\tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{2}\Phi_t^k(\tilde{\xi}_\theta^j)-1\right)\right)^{-\frac{1}{k}}\right)^{-1}\right)(\Xi^{(b)})\Bigg\rangle. \quad (4)$$

(*III*)

$$\lambda\tilde{\delta}_\ell^{(1)}(\Xi) = \Bigg\langle \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left((\Phi_t^k(\tilde{\xi}_\phi^j))^\lambda-1\right)\right)^{-\frac{1}{k}}\right)^{-1}\right)(\Xi^{(a)}),$$
$$\tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left((\Theta_t^k(\tilde{\xi}_\theta^j))^\lambda-1\right)\right)^{\frac{1}{k}}\right)^{-1}\right)(\Xi^{(b)})\Bigg\rangle. \quad (5)$$

(*IV*)

$$(\tilde{\delta}_\ell^{(1)}(\Xi))^\lambda = \Bigg\langle \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left((\Theta_t^k(\tilde{\xi}_\phi^j))^\lambda-1\right)\right)^{\frac{1}{k}}\right)^{-1}\right)(\Xi^{(a)}),$$
$$\tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left((\Phi_t^k(\tilde{\xi}_\theta^j))^\lambda-1\right)\right)^{-\frac{1}{k}}\right)^{-1}\right)(\Xi^{(b)})\Bigg\rangle. \quad (6)$$

Theorem 3.3. *Consider two DPLEs* $\tilde{\delta}_\ell^{(j)}(\Xi) = \left\langle\{\ell_{\phi^{j(a)}}(\Xi^{(a)})\},\{\ell_{\theta^{j(b)}}(\Xi^{(b)})\}\right\rangle,\ j=1,2$. *Then, for any* $\lambda,\lambda_1,\lambda_2>0$, *we have*

(*i*) $\tilde{\delta}_\ell^{(1)}(\Xi)\tilde{\oplus}\tilde{\delta}_\ell^{(2)}(\Xi) = \tilde{\delta}_\ell^{(2)}(\Xi)\tilde{\oplus}\tilde{\delta}_\ell^{(1)}(\Xi)$,

(*ii*) $\tilde{\delta}_\ell^{(1)}(\Xi)\tilde{\otimes}\tilde{\delta}_\ell^{(2)}(\Xi) = \tilde{\delta}_\ell^{(2)}(\Xi)\tilde{\otimes}\tilde{\delta}_\ell^{(1)}(\Xi)$,

(*iii*) $\lambda(\tilde{\delta}_\ell^{(1)}(\Xi)\tilde{\oplus}\tilde{\delta}_\ell^{(2)}(\Xi)) = (\lambda\tilde{\delta}_\ell^{(1)}(\Xi))\tilde{\oplus}(\lambda\tilde{\delta}_\ell^{(2)}(\Xi))$,

(*iv*) $(\tilde{\delta}_\ell^{(1)}(\Xi)\tilde{\otimes}\tilde{\delta}_\ell^{(2)}(\Xi))^\lambda = (\tilde{\delta}_\ell^{(1)}(\Xi))^\lambda\tilde{\otimes}(\tilde{\delta}_\ell^{(2)}(\Xi))^\lambda$,

(*v*) $(\lambda_1+\lambda_2)\tilde{\delta}_\ell^{(1)}(\Xi) = (\lambda_1\tilde{\delta}_\ell^{(1)}(\Xi))\tilde{\oplus}(\lambda_2\tilde{\delta}_\ell^{(1)}(\Xi))$,

(*vi*) $(\tilde{\delta}_\ell^{(1)}(\Xi))^{\lambda_1+\lambda_2} = (\tilde{\delta}_\ell^{(1)}(\Xi))^{\lambda_1}\tilde{\otimes}(\tilde{\delta}_\ell^{(1)}(\Xi))^{\lambda_2}$.

Proof. Given in supplementary material. □

Definition 3.4. *Consider a collection of DPLEs* $\tilde{\delta}_\ell^{(j)}(\Xi) = \left\langle\{\ell_{\phi^{j(a)}}(\Xi^{(a)})\},\{\ell_{\theta^{j(b)}}(\Xi^{(b)})\}\right\rangle,\ j\in\Lambda_n$. *Then the DPLGDWAA operator is defined along these lines:*

$$DPLGDWAA(\tilde{\delta}_\ell^{(1)}(\Xi),\tilde{\delta}_\ell^{(2)}(\Xi),\ldots,\tilde{\delta}_\ell^{(3)}(\Xi)) = \tilde{\bigoplus_{j=1}^{n}}(w_j\tilde{\delta}_\ell^{(j)}(\Xi)), \quad (7)$$

where $w_j(>0)$ *is weight of* $\tilde{\delta}_\ell^{(j)}(\Xi)$ *with* $\sum_{j=1}^n w_j = 1$.

Theorem 3.5. *Consider a collection of adjusted DPLEs* $\tilde{\delta}_\ell^{(j)}(\Xi) = \left\langle\{\ell_{\phi^{j(a)}}(\Xi^{(a)})\},\{\ell_{\theta^{j(b)}}(\Xi^{(b)})\}\right\rangle,\ j\in\Lambda_n$. *Then the aggregation of DPLEs is again a DPLE and*

$$DPLGDWAA(\tilde{\delta}_\ell^{(1)}(\Xi), \tilde{\delta}_\ell^{(2)}(\Xi), \ldots, \tilde{\delta}_\ell^{(n)}(\Xi)) = \\ \left\langle \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^n(\Phi_t^k(\tilde{\xi}_\phi^j))^{w_j}-1\right)\right)^{-\frac{1}{k}}\right)^{-1}\right)(\Xi^{(a)}),\ \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^n(\Theta_t^k(\tilde{\xi}_\theta^j))^{w_j}-1\right)\right)^{\frac{1}{k}}\right)^{-1}\right)(\Xi^{(b)})\right\rangle. \tag{8}$$

Proof. Added in the supplementary material. □

The specific instances of the $DPLGDWAA$ operator are as follows:

Case-I: If $k = 1, t = 1$ then the $DPLGDWAA$ operator reduces to the DPL weighted averaging aggregation ($DPLWAA$) operator as shown below:

$$DPLWAA(\tilde{\delta}_\ell^{(1)}(\Xi), \tilde{\delta}_\ell^{(2)}(\Xi), \ldots, \tilde{\delta}_\ell^{(n)}(\Xi)) = \left\langle \tilde{\xi}^{-1}\left(1-\prod_{j=1}^n(1-\tilde{\xi}_\phi^j)^{w_j}\right)(\Xi^{(a)}),\ \tilde{\xi}^{-1}\left(\prod_{j=1}^n(\tilde{\xi}_\theta^j)^{w_j}\right)(\Xi^{(b)})\right\rangle.$$

Case-II: If $k = 1, t = 2$ then the operators $DPLGDWAA$ and DPL-Einstein weighted averaging aggregation (*DPLE-WAA*) operator become identical as shown below:

$$DPLEWAA(\tilde{\delta}_\ell^{(1)}(\Xi), \tilde{\delta}_\ell^{(2)}(\Xi), \ldots, \tilde{\delta}_\ell^{(n)}(\Xi)) \\ = \left\langle \tilde{\xi}^{-1}\left(\frac{\prod_{j=1}^n(1+\tilde{\xi}_\phi^j)^{w_j}-\prod_{j=1}^n(1-\tilde{\xi}_\phi^j)^{w_j}}{\prod_{j=1}^n(1+\tilde{\xi}_\phi^j)^{w_j}+\prod_{j=1}^n(1-\tilde{\xi}_\phi^j)^{w_j}}\right)(\Xi^{(a)}),\ \tilde{\xi}^{-1}\left(\frac{2\prod_{j=1}^n(\tilde{\xi}_\theta^j)^{w_j}}{\prod_{j=1}^n(2-\tilde{\xi}_\theta^j)^{w_j}+\prod_{j=1}^n(\tilde{\xi}_\theta^j)^{w_j}}\right)(\Xi^{(b)})\right\rangle.$$

Case-III: If $k = 1$ then the operators $DPLGDWAA$and DPL-Hamachar weighted averaging aggregation ($DPLHWAA$) operator coincides as shown below:

$$DPLHWAA(\tilde{\delta}_\ell^{(1)}(\Xi), \tilde{\delta}_\ell^{(2)}(\Xi), \ldots, \tilde{\delta}_\ell^{(n)}(\Xi)) \\ = \left\langle \tilde{\xi}^{-1}\left(\frac{\prod_{j=1}^n(1+(t-1)\tilde{\xi}_\phi^j)^{w_j}-\prod_{j=1}^n(1-\tilde{\xi}_\phi^j)^{w_j}}{\prod_{j=1}^n(1+(t-1)\tilde{\xi}_\phi^j)^{w_j}+(t-1)\prod_{j=1}^n(1-\tilde{\xi}_\phi^j)^{w_j}}\right)(\Xi^{(a)}), \tilde{\xi}^{-1}\left(\frac{t\prod_{j=1}^n(\tilde{\xi}_\theta^j)^{w_j}}{\prod_{j=1}^n(1+(t-1)(1-\tilde{\xi}_\theta^j))^{w_j}+(t-1)\prod_{j=1}^n(\tilde{\xi}_\theta^j)^{w_j}}\right)(\Xi^{(b)})\right\rangle.$$

The axioms based on Theorem 2 are given as follows:

Axiom 1 (Idempotency): Consider a collection of adjusted DPLEs $\tilde{\delta}_\ell^{(j)}(\Xi) = \langle\{\ell_{\phi^{j(a)}}(\Xi^{(a)})\}, \{\ell_{\theta^{j(b)}}(\Xi^{(b)})\}\rangle,\ j \in \Lambda_n$ with $\tilde{\delta}_\ell^{(j)}(\Xi) = \tilde{\delta}_\ell^{(L)}(\Xi)\ \forall j$ (L is a fixed positive integer). Then

$$DPLGDWAA(\tilde{\delta}_\ell^{(1)}(\Xi), \tilde{\delta}_\ell^{(2)}(\Xi), \ldots, \tilde{\delta}_\ell^{(3)}(\Xi)) = \tilde{\delta}_\ell^{(L)}(\Xi).$$

Axiom 2 (Monotonicity): Consider two collections of adjusted DPLEs $\tilde{\delta}_\ell^{(j)}(\Xi) = \langle\{\ell_{\phi^{j(a)}}(\Xi^{(a)})\}, \{\ell_{\theta^{j(b)}}(\Xi^{(b)})\}\rangle$ and $\tilde{\delta'}_\ell^{(j)}(\Xi) = \left\langle\{\ell'_{\phi^{j(a)}}(\Xi^{(a)})\}, \{\ell'_{\theta^{j(b)}}(\Xi^{(b)})\}\right\rangle$, $j \in \Lambda_n$ satisfying $\forall j$, $\ell_{\phi^{j(a)}} \leq \ell'_{\phi^{j(a)}}$ and $\ell_{\theta^{j(b)}} \geq \ell'_{\theta^{j(b)}}$. Then

$$DPLGDWAA(\tilde{\delta}_\ell^{(1)}(\Xi), \tilde{\delta}_\ell^{(2)}(\Xi), \ldots, \tilde{\delta}_\ell^{(3)}(\Xi)) \prec DPLGDWAA(\tilde{\delta'}_\ell^{(1)}(\Xi), \tilde{\delta'}_\ell^{(2)}(\Xi) \ldots, \tilde{\delta'}_\ell^{(3)}(\Xi)).$$

Axiom 3 (Boundedness): Consider a collection of adjusted DPLEs $\tilde{\delta}_\ell^{(j)}(\Xi) = \langle\{\ell_{\phi^{j(a)}}(\Xi^{(a)})\}, \{\ell_{\theta^{j(b)}}(\Xi^{(b)})\}\rangle, j \in \Lambda_n$ If $\tilde{\delta}_\ell^{(j-)}(\Xi) = \left\langle\{\min_a \ell_{\phi^{j(a)}}(\Xi^{(a)})\}, \{\max_b \ell_{\theta^{j(b)}}(\Xi^{(b)})\}\right\rangle$ and $\tilde{\delta}_\ell^{(j+)}(\Xi) = \left\langle\{\max_a \ell_{\phi^{j(a)}}(\Xi^{(a)})\}, \{\min_b \ell_{\theta^{j(b)}}(\Xi^{(b)})\}\right\rangle$. Then

$$\tilde{\delta}_\ell^{(j-)}(\Xi) \prec DPLGDWAA(\tilde{\delta}_\ell^{(1)}(\Xi), \tilde{\delta}_\ell^{(2)}(\Xi), \ldots, \tilde{\delta}_\ell^{(3)}(\Xi)) \prec \tilde{\delta}_\ell^{(j+)}(\Xi).$$

4 DPL-MARCOS methodology for group decision-making

The MARCOS tool [27] is a compromise solution-based ranking technique that considers anti-ideal and ideal solutions for determination of utility values. Recently, MARCOS method has been applied in various fields, namely manpower resources assessment in transport industries [26], stacker's selection in logistics systems [28], supplier selection [1], road traffic risk analysis [25], selection of cold chain logistics distribution center [22], and ranking of zero-emission last-mile delivery assessment [24]. In this line, we propose DPL-MARCOS approach and the steps of approach are given by

Step -1: Suppose that the experts D_r $(r \in \Lambda_l)$ assesses m various possibilities A_i $(i \in \Lambda_m)$ DPL setting based on n distinct criteria C_j $(j \in \Lambda_n)$. Assume that the initial evaluation of the DE Dr is represented by $\Re_r = \left[\delta_\ell^{(ijr)}(\Xi)\right]_{m\times n} = \left[\left\langle \{\ell_{\phi^{ijr(a)}}(\Xi^{(a)})\}, \{\ell_{\theta^{ijr(b)}}(\Xi^{(b)})\}\right\rangle\right]_{m\times n},\ i \in \Lambda_m, j \in \Lambda_n, r \in \Lambda_l$ ($\Gamma = \{\ell_\alpha : \alpha = -T, \ldots, 0, \ldots, T\}$ being the LTS).

Step-2: Adjust the DPLEs occurred in the assessment matrices $\Re_r = \left[\delta_\ell^{(ijr)}(\Xi)\right]_{m\times n}, i \in \Lambda_m, j \in \Lambda_n, r \in \Lambda_l$. Suppose the adjusted DPLEs are denoted by $\tilde{\delta}_\ell^{(ijr)}(\Xi) =< \{\ell'_{\phi^{ijr(a)}}(\Xi^{(a)})\}, \{\ell'_{\theta^{ijr(b)}}(\Xi^{(b)})\} > (i \in \Lambda_m, j \in \Lambda_n, r \in \Lambda_l)$.

Step-3: Determination of experts' weights.

If ϖ_r stands for the weight of rth expert D_r $(r \in \Lambda_l)$. Let the consistency of the expert D_r be ϖ'_r and the homogeneity between the DEs D_r and D_q be ϖ''_r . Cheng et al. [3] defined ϖ'_r to reveal the "consistency harmonious weight index (CHWI)" is given by

$$CHWI\,(D_r) = \sum_{j=1}^{n} \frac{n}{\beta_r^{(j)}},\ r \in \Lambda_l, \tag{9}$$

where $\beta_r^{(j)} = \sum_{i=1}^m \sum_{s=1}^m \zeta\left(\delta_\ell^{(ijr)}(\Xi)\right) \times \zeta\left(\delta_\ell^{(sji)}(\Xi)\right),\ i \in \Lambda_m, j \in \Lambda_n, r \in \Lambda_l,$

$$\zeta(\delta_\ell^{(ijr)}(\Xi)) = \sum_a \tilde{\xi}_\phi^{ijr} \times \Xi^{(a)} + \sum_b \tilde{\xi}_\theta^{ijr} \times \Xi^{(b)}.$$

Han and Li [10] defined that if $CHWI(D_r)$=1, then the evaluation matrix $\Re_r$ is consistent. Corresponding to the consistency of the evaluation matrix, ϖ'_r can be expressed by

$$\varpi'_r = \frac{CHWI(D_r)}{\sum_{r=1}^{l} CHWI(D_r)} \qquad (r \in \Lambda_l) \tag{10}$$

In order to describe ϖ''_r , an index should be defined to reflect the similarity between $\Re_r$and $\Re_q$. The derived vector of $\Re_r$can be expressed as

$$\begin{aligned} Vec(\Re_r) \quad = \quad & \Big(\delta_\ell^{(11r)}(\Xi), \delta_\ell^{(21r)}(\Xi), \ldots, \delta_\ell^{(m1r)}(\Xi)\ , \delta_\ell^{(12r)}(\Xi), \delta_\ell^{(22r)}(\Xi), \ldots, \delta_\ell^{(m2r)}(\Xi), \\ & \ldots, \delta_\ell^{(1nr)}(\Xi), \delta_\ell^{(2nr)}(\Xi), \ldots, \delta_\ell^{(mnr)}(\Xi)\Big)\ \ (r \in \Lambda_l). \end{aligned}$$

Let α_{rq}be the angle between $\Re_r$and $\Re_q$. Then α_{rq} is represented as

$$\alpha_{rq} = \frac{\langle Vec(\Re_r), Vec(\Re_q)\rangle}{\|Vec(\Re_r)\| \times \|Vec(\Re_q)\|},\ r, q \in \Lambda_l \tag{11}$$

where $\langle Vec(\Re_r), Vec(\Re_q)\rangle$ is defined as $\langle Vec(\Re_r), Vec(\Re_q)\rangle = \sum_{i=1}^m \sum_{j=1}^n \left(\zeta(\delta_\ell^{(ijr)}(\Xi)) + \zeta(\delta_\ell^{(ijq)}(\Xi))\right)$ and notation $\|\,.\,\|$ signifies the positive square root of $\langle ., .\rangle$.

Suppose $\alpha_r = \sum_{q=1, q\neq r}^{l} \alpha_{rq}$. The $\alpha_r$shows the similarity with different evaluation matrix and is normalized to find the weight value as

$$\varpi''_r = \frac{\alpha_r}{\sum_{r=1}^{l} \alpha_r}, r \in \Lambda_l. \tag{12}$$

Incorporating the subjective weighting $\varpi'_r,\ r \in \Lambda_l$ and the objective weighting $\varpi''_r,\ r \in \Lambda_l$, the integrated weight of DEs' is given by

$$\varpi_r = \lambda\varpi'_r + (1-\lambda)\varpi''_r,\ r \in \Lambda_l, \tag{13}$$

where high values of λ (>0.5), subjective weight is preferred; and for low values of λ (<0.5), objective weight is preferred. However, for λ=0.5, both type of weights have equal significance.

Step-4: Construct the "aggregated DPL decision-matrix (A-DPL-DM)".

Here, the $DPLGDWAA$ operator is utilized to construct the A-DPL-DM $\left[\tilde{\delta}_{\ell}^{(ij)}(\Xi)\right]_{m\times n}$ described as

$$\begin{aligned}\tilde{\delta}_{\ell}^{(ij)}(\Xi) &= DPLGDWAA(\tilde{\delta}_{\ell}^{(ij1)}(\Xi), \tilde{\delta}_{\ell}^{(ij2)}(\Xi), \ldots, \tilde{\delta}_{\ell}^{(ijl)}(\Xi)) \\ &= \left\langle \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left(\prod_{r=1}^{l}(\Phi_t^k(\tilde{\xi}_{\phi}^{ijr}))^{\varpi_r}-1\right)\right)^{-\frac{1}{k}}\right)^{-1}\right)(\Xi^{(a)}),\right. \\ &\quad \left. \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left(\prod_{r=1}^{l}(\Theta_t^k(\tilde{\xi}_{\theta}^{ijr}))^{\varpi_r}-1\right)\right)^{\frac{1}{k}}\right)^{-1}\right)(\Xi^{(b)})\right\rangle. \end{aligned} \tag{14}$$

Step-5: Determine the criteria weights.

Suppose $\left[\tilde{\delta}_{\ell}^{(ij)}(\Xi)\right]_{m\times n} = \left[\langle\{\ell_{\phi^{ij(a)}}(\Xi^{(a)})\}, \{\ell_{\theta^{ij(b)}}(\Xi^{(b)})\}\rangle\right]_{m\times n}$, $i\in\Lambda_m, j\in\Lambda_n$. The reference matrix $\left[\tilde{\delta}_{\ell}^{(j)}(\Xi)\right]_{1\times n}$ is described as

$$\tilde{\delta}_{\ell}^{(j)}(\Xi) = \left\langle\{\max_i\max_a \ell_{\phi^{ij(a)}}(1)\}, \{\min_i\min_b \ell_{\theta^{ij(b)}}(1)\}\right\rangle, \; j\in\Lambda_n. \tag{15}$$

Eq. (16) gives the "correlation coefficient (CRC)" between the reference matrix $\left[\tilde{\delta}_{\ell}^{(j)}(\Xi)\right]_{1\times n}$ and the A-DPL-DM $\left[\tilde{\delta}_{\ell}^{(ij)}(\Xi)\right]_{m\times n}$.

$$\Upsilon_{ij}^{\rho} = \frac{\min_i\min_j \tilde{D}(\tilde{\delta}_{\ell}^{(ij)}(\Xi), \tilde{\delta}_{\ell}^{(j)}(\Xi)) + \rho\max_i\max_j \tilde{D}(\tilde{\delta}_{\ell}^{(ij)}(\Xi), \tilde{\delta}_{\ell}^{(j)}(\Xi))}{\tilde{D}(\tilde{\delta}_{\ell}^{(ij)}(\Xi), \tilde{\delta}_{\ell}^{(j)}(\Xi)) + \rho\max_i\max_j \tilde{D}(\tilde{\delta}_{\ell}^{(ij)}(\Xi), \tilde{\delta}_{\ell}^{(j)}(\Xi))}, \tag{16}$$

where $\tilde{D}(\tilde{\delta}_{\ell}^{(ij)}(\Xi), \tilde{\delta}_{\ell}^{(j)}(\Xi))$ is the distance between DPLEs $\tilde{\delta}_{\ell}^{(ij)}(\Xi)$ and $\tilde{\delta}_{\ell}^{(j)}(\Xi)$ as

$$\tilde{D}(\tilde{y}_{\wp}^{(ij)}(\mho), \tilde{y}_{\wp}^{(j)}(\mho)) = \sqrt{\frac{1}{2}\left(\sum_u \left(\mho^{(u)}\times\left|\tilde{g}(\wp_{\xi^{ij(u)}})-\tilde{g}(\wp_{\xi^{j(u)}})\right|\right)+\sum_v\left(\mho^{(v)}\times\left|\tilde{g}(\wp_{\vartheta^{ij(v)}})-\tilde{g}(\wp_{\vartheta^{j(v)}})\right|\right)\right)}. \tag{17}$$

The parameter ρ (known as the distinguishing coefficient) lies in the range [0, 1] and is inversely proportional to the correlation coefficient. The grey correlation level (GCL) is formulated as

$$\pi_j^{\rho} = \frac{1}{m}\sum_{i=1}^{m}\Upsilon_{ij}^{\rho}, \; j\in\Lambda_n. \tag{18}$$

Now, weight w_j of attribute C_j, $j\in\Lambda_n$ is obtained as

$$w_j = \frac{\pi_j^{\rho}}{\sum_{j=1}^{n}\pi_j^{\rho}}, \; j\in\Lambda_n. \tag{19}$$

Step-6: Form the "extended aggregated decision-matrix (EA-DM)" $\Re^*$.

Here, the concepts of "ideal solution (ID)" and "anti-ideal solution (AID)" are utilized to extend the matrix. The AID and ID is a choice with worst and best characteristic respectively. We present the AID and ID as

$$\tilde{\delta}_{\ell}^{(j-)}(\Xi) = \begin{cases}\left\langle\{\min_i\min_a \ell_{\phi^{ij(a)}}(1)\}, \{\max_i\max_b \ell_{\theta^{ij(b)}}(1)\}\right\rangle, & \text{if } C_j\in Q_B \\ \left\langle\{\max_i\max_a \ell_{\phi^{ij(a)}}(1)\}, \{\min_i\min_b \ell_{\theta^{ij(b)}}(1)\}\right\rangle, & \text{if } C_j\in Q_C\end{cases} \tag{20}$$

$$\tilde{\delta}_{\ell}^{(j+)}(\Xi) = \begin{cases} \left\langle \{\max_i \max_a \ell_{\phi^{ij(a)}}(1)\}, \{\min_i \min_b \ell_{\theta^{ij(b)}}(1)\} \right\rangle, & \text{if } C_j \in Q_B \\ \left\langle \{\min_i \min_a \ell_{\phi^{ij(a)}}(1)\}, \{\max_i \max_b \ell_{\theta^{ij(b)}}(1)\} \right\rangle, & \text{if } C_j \in Q_C \end{cases} \tag{21}$$

Here, Q_B, Q_C denotes the set of all beneficial and non-beneficial attribute respectively.
The EA-DM $\Re^*$ is given by

$$\begin{array}{c} \\ AID \\ A_1 \\ A_2 \\ \vdots \\ A_m \\ ID \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \dots & C_n \end{array} \\ \begin{pmatrix} y_{\wp}^{(1)(-)}(\mho) & y_{\wp}^{(2)(-)}(\mho) & \dots & y_{\wp}^{(1)(n)}(\mho) \\ \tilde{y}_{\wp}^{(11)}(\mho) & \tilde{y}_{\wp}^{(12)}(\mho) & \dots & \tilde{y}_{\wp}^{(1n)}(\mho) \\ \tilde{y}_{\wp}^{(21)}(\mho) & \tilde{y}_{\wp}^{(22)}(\mho) & \dots & \tilde{y}_{\wp}^{(2n)}(\mho) \\ \dots & \dots & \dots & \dots \\ \tilde{y}_{\wp}^{(m1)}(\mho) & \tilde{y}_{\wp}^{(m2)}(\mho) & \dots & \tilde{y}_{\wp}^{(mn)}(\mho) \\ y_{\wp}^{(1)(+)}(\mho) & y_{\wp}^{(2)(+)}(\mho) & \dots & y_{\wp}^{(n)(+)}(\mho) \end{pmatrix} \end{array}.$$

Step-7: Make the normalized EA-DM $\Re^{*N} = \left[\hat{\delta}_{\ell}^{(ij)}(\Xi)\right]_{m\times n} = \left[\left\langle \{\hat{\ell}_{\phi^{ij(a)}}(\Xi^{(a)})\}, \{\hat{\ell}_{\theta^{ij(b)}}(\Xi^{(b)})\} \right\rangle\right]_{m\times n}$, where

$$\hat{\delta}_{\ell}^{(ij)}(\Xi) = \begin{cases} \left\langle \left\{\ell_{\tilde{\xi}_{\phi}^{ij}/\tilde{\xi}_{\phi}^{j+}}(\Xi^{(a)})\right\}, \left\{\ell_{\tilde{\xi}_{\theta}^{ij}/\tilde{\xi}_{\theta}^{j+}}(\Xi^{(b)})\right\} \right\rangle, & \text{if } C_j \in Q_B \\ \left\langle \left\{\ell_{\tilde{\xi}_{\phi}^{j+}/\tilde{\xi}_{\phi}^{ij}}(\Xi^{(a)})\right\}, \left\{\ell_{\tilde{\xi}_{\theta}^{j+}/\tilde{\xi}_{\theta}^{ij}}(\Xi^{(b)})\right\} \right\rangle, & \text{if } C_j \in Q_C \end{cases}, \ i \in \Lambda_m, j \in \Lambda_n \tag{22}$$

Step-8: Form the weighted normalized EA-DM $\Re^{*NW} = \left[\hat{\delta'}_{\ell}^{(ij)}(\Xi)\right]_{m\times n}$, where

$$\hat{\delta'}_{\ell}^{(ij)}(\Xi) = \left\langle \{\ell_{w_j \times \tilde{\xi'}_{\phi}^{ij(a)}}(\Xi^{(a)})\}, \{\ell_{w_j \times \tilde{\xi'}_{\theta}^{ij(b)}}(\Xi^{(b)})\} \right\rangle, \ i \in \Lambda_m, j \in \Lambda_n.$$

Suppose $\hat{\delta'}_{\ell}^{(ij)}(\Xi) = \left\langle \{\hat{\ell}'_{\phi^{ij(a)}}(\Xi^{(a)})\}, \{\hat{\ell}'_{\theta^{ij(b)}}(\Xi^{(b)})\} \right\rangle$, $i \in \Lambda_m, j \in \Lambda_n$..
Step-9: Determine the utility degrees (UDs) of options.
Eqs. (23) and (24) present the UD of ith option corresponding to AID and ID respectively.

$$\hbar_i^- = \frac{\sum_{j=1}^n \tilde{\xi}(Sc(\hat{\delta'}_{\ell}^{(ij)}(\Xi)))}{\sum_{j=1}^n \tilde{\xi}(Sc(\hat{\delta'}_{\ell}^{(j-)}(\Xi)))}, \ i \in \Lambda_m, \tag{23}$$

$$\hbar_i^+ = \frac{\sum_{j=1}^n \tilde{\xi}\left(Sc(\hat{\delta'}_{\ell}^{(ij)}(\Xi))\right)}{\sum_{j=1}^n \tilde{\xi}\left(Sc(\hat{\delta'}_{\ell}^{(j+)}(\Xi))\right)}, \ i \in \Lambda_m. \tag{24}$$

Step 10: Obtain the "utility values (UVs)" based on the AID and ID.
The UVs using the AID and ID are estimated as

$$\lambda_i^- = \frac{\hbar_i^+}{\hbar_i^- + \hbar_i^+}, i \in \Lambda_m, \tag{25}$$

$$\lambda_i^+ = \frac{\hbar_i^-}{\hbar_i^- + \hbar_i^+}, \ i \in \Lambda_m. \tag{26}$$

Step 11: Assess the "overall utility values (OUV)". The OUV of $A_i, i \in \Lambda_m$ is computed by

$$\lambda_i = \frac{\hbar_i^+ + \hbar_i^-}{1 + \frac{1-\lambda_i^+}{\lambda_i^+} + \frac{1-\lambda_i^-}{\lambda_i^-}}, \ i \in \Lambda_m. \tag{27}$$

Step 12: Prioritize the options $A_i, i \in \Lambda_m$ using the decreasing OUVs and choose the suitable one.

Table 2: Initial assessment matrix

		C_1	C_2	C_3
D_1	A_1	$<\{\wp_{-2}(0.3), \wp_{-1}(0.4), \wp_1(0.3)\}, \{\wp_{-1}(0.3), \wp_0(0.7)\}>$	$<\{\wp_1(0.6), \wp_2(0.4)\}, \{\wp_{-2}(0.6), \wp_1(0.4)\}>$	$<\{\wp_{-1}(0.7), \wp_1(0.3)\}, \{\wp_0(0.1), \wp_2(0.9)\}>$
	A_2	$<\{\wp_{-1}(0.1), \wp_0(0.9)\}, \{\wp_{-2}(0.5), \wp_{-1}(0.3), \wp_1(0.2)\}>$	$<\{\wp_{-2}(0.7), \wp_1(0.3)\}, \{\wp_1(0.4), \wp_2(0.6)\}>$	$<\{\wp_0(0.1),\wp_2(0.9)\}, \{\wp_{-1}(0.9),\wp_1(0.1)\}>$
	A_3	$<\{\wp_1(1)\}, \{\wp_0(0.3), \wp_2(0.7)\}>$	$<\{\wp_{-1}(0.3), \wp_0(0.7)\}, \{\wp_{-2}(0.5), \wp_{-1}(0.3), \wp_1(0.2)\}>$	$<\{\wp_{-2}(0.9), \wp_{-1}(0.1)\}, \{\wp_{-1}(0.1), \wp_0(0.9)\}>$
D_2	A_1	$<\{\wp_{-1}(0.7), \wp_1(0.3)\}, \{\wp_{-2}(0.7), \wp_{-1}(0.3)\}>$	$<\{\wp_{-2}(0.6), \wp_{-1}(0.4)\}, \{\wp_2(1)\}>$	$<\{\wp_{-2}(0.2), \wp_{-1}(0.5), \wp_1(0.3)\}, \{\wp_1(1)\}>$
	A_2	$<\{\wp_0(0.1), \wp_2(0.9)\}, \{\wp_{-1}(0.8), \wp_1(0.2)\}>$	$<\{\wp_{-2}(0.5), \wp_{-1}(0.3), \wp_1(0.2)\}, \{\wp_0(1)\}>$	$<\{\wp_{-1}(0.1), \wp_0(0.9)\}, \{\wp_{-2}(0.1), \wp_{-1}(0.8), \wp_1(0.1)\}>$
	A_3	$<\{\wp_{-2}(0.7), \wp_{-1}(0.3)\}, \{\wp_{-1}(0.6), \wp_1(0.4)\}>$	$<\{\wp_2(1)\}, \{\wp_{-2}(0.6), \wp_{-1}(0.4)\}>$	$<\{\wp_1(1)\}, \{\wp_{-2}(0.1), \wp_{-1}(0.8), \wp_1(0.1)\}>$
D_3	A_1	$<\{\wp_0(0.3), \wp_2(0.7)\}, \{\wp_{-2}(0.4), \wp_{-1}(0.6)\}>$	$<\{\wp_{-2}(0.4), \wp_{-1}(0.5), \wp_1(0.1)\}, \{\wp_2(1)\}>$	$<\{\wp_{-1}(0.2), \wp_1(0.4), \wp_0(0.4)\}, \{\wp_1(1)\}>$
	A_2	$<\{\wp_{-2}(0.8), \wp_{-1}(0.2)\}, \{\wp_0(0.5), \wp_2(0.5)\}>$	$<\{\wp_2(1)\}, \{\wp_{-2}(0.3), \wp_{-1}(0.3), \wp_1(0.4)\}>$	$<\{\wp_1(1)\}, \{\wp_{-1}(0.1), \wp_0(0.9)\}>$
	A_3	$<\{\wp_{-1}(0.7), \wp_1(0.3)\}, \{\wp_{-1}(0.6), \wp_1(0.4)\}>$	$<\{\wp_0(1)\}, \{\wp_{-1}(1)\}>$	$<\{\wp_{-2}(0.2), \wp_{-1}(0.7), \wp_1(0.1)\}, \{\wp_{-2}(0.1), \wp_{-1}(0.8), \wp_1(0.1)\}>$

5 Case study

5.1 Analysis phase

The most important long-term source of carbon that is generally dispersed in many different locations is biomass. Systems that generate electricity from biomass represent significant and astute investments in terms of economic development, particularly in nations still in the process of industrialization. The "heating value" (HV), "moisture content" (MC), and "fixed carbon" (FC) are considered to be the primary feedstock properties that influence the credibility of the biogas. These features are accompanied by the various characteristics of biomass. The essential components of biomass, which are oxygen, carbon, and hydrogen, have a significant effect on HV. For disintegration, materials with low HV and low MC should be chosen wherever possible. The FC material of biomass has a big effect on how much energy it can store and release. Suppose that coconut shell (A_1), coffee husk (A_2) and sugarcane bagasse (A_3) are three available feedstock's that need to assessed by three experts under DPL setting based on three evaluation attributes- HV (C_1), MC (C_2), and FC (C_3).

5.2 Solution

The solution of the above mentioned problem using our developed methodology includes the steps mentioned below:
Step-1: The LTS $\Gamma = \{\ell_\alpha : \alpha = -3, -2, -1, 0, 1, 2, 3\}$($\wp_{-3}$=very bad, $\wp_{-2}$=bad, $\wp_{-1}$=moderately bad, $\wp_0$=fair, $\wp_1$=good, $\wp_2$=very good, and $\wp_3$=very very good) is used here for initial assessment, the results of which are provided as the DPL-DM $\Re_k = \left[\tilde{y}_{\wp}^{(ijk)}(\mho)\right]_{3\times 3} = \left[\left\langle\{\wp_{\xi^{ijk(u)}}(\mho^{(u)})\}, \{\wp_{\vartheta^{ijk(v)}}(\mho^{(v)})\}\right\rangle\right]_{3\times 3}, i \in \Lambda_3, j \in \Lambda_3, k \in \Lambda_3$ (Table (2)).
Step-2: The adjusted forms of $\Re_r = \left[\delta_{\ell}^{(ijr)}(\Xi)\right]_{3\times 3}, \ i \in \Lambda_3, j \in \Lambda_3, r \in \Lambda_3$ are furnished in Table 10 of the supplementary material.

Table 3: Experts' weights

	C_1	C_2	C_3	**CHWI**	**Subjective weight of DEs**	α_r	**Objective weight of DEs**	**Final weights of DEs**
D_1	$\beta_1^{(1)} =$ 9.181111	$\beta_1^{(2)}=$ 8.883611	$\beta_1^{(3)} =$ 9.227222	0.989583	$\varpi'_1=$ 0.314818	$\alpha_1=$ 1.922718063	$\varpi''_1 =$ 0.333612	$\varpi_1=$ 0.324215
D_2	$\beta_2^{(1)}=$ 7.179722	$\beta_2^{(2)}=$ 8.2225	$\beta_2^{(3)}=$ 8.125833	1.151888	$\varpi'_2=$ 0.366453	$\alpha_2=$ 1.898333293	$\varpi''_2 =$ 0.329381	$\varpi_2=$ 0.347917
D_3	$\beta_3^{(1)}=$ 8.284444	$\beta_3^{(2)}=$ 9.684166	$\beta_3^{(3)}=$ 9.091944	1.001870	$\varpi'_3=$ 0.318727	$\alpha_3=$ 1.942276513	$\varpi''_3 =$ 0.337006	$\varpi_3=$ 0.327866

Step-3: Eqs. (9) and (10) give DEs' subjective weights, Eqs. (11) and (12) give DEs' objective weights. Eq. (13) (taking λ=0.5) determines DEs' final weights in Table (3).

Step-4: The aggregated DPL matrix (Taking $t = 3, k = 2$) is presented in Table 4.

Table 4: The A-DPL-DM

	C_1	C_2	C_3
A_1	$<\{\wp_{-0.044}(0.3), \wp_{0.457}(0.2), \wp_{0.457}(0.2), \wp_{0.710}(0.3)\}, \{\wp_{-0.872}(0.3), \wp_{-0.820}(0.1), \wp_{-0.665}(0.3), \wp_{-0.427}(0.3)\}>$	$<\{\wp_{0.088}(0.4), \wp_{0.123}(0.2), \wp_{0.449}(0.3), \wp_{0.584}(0.1)\}, \{\wp_{-0.402}(0.4), \wp_{-0.402}(0.2), \wp_{0.206}(0.3), \wp_{0.206}(0.1)\}>$	$<\{\wp_{-0.154}(0.2),\wp_{0.163}(0.4), \wp_{-0.002}(0.1), \wp_{0.425}(0.3)\}, \{\wp_{0.004}(0.1), \wp_{0.150}(0.3), \wp_{0.150}(0.3), \wp_{0.150}(0.3)\}>$
A_2	$<\{\wp_{-0.036}(0.1), \wp_{0.528}(0.5), \wp_{0.528}(0.2), \wp_{0.547}(0.2)\}, \{\wp_{-0.635}(0.5), \wp_{-0.347}(0.1), \wp_{-0.347}(0.2), \wp_{0.151}(0.2)\}>$	$<\{\wp_{0.411}(0.5), \wp_{0.436}(0.2), \wp_{0.589}(0.1), \wp_{0.710}(0.2)\}, \{\wp_{-0.509}(0.3), \wp_{-0.240}(0.1), \wp_{-0.210}(0.2), \wp_{0.036}(0.4)\}>$	$<\{\wp_{0.233}(0.1), \wp_{0.627}(0.1), \wp_{0.627}(0.5), \wp_{0.627}(0.3)\}, \{\wp_{-0.732}(0.1), \wp_{-0.425}(0.4), \wp_{-0.425}(0.4), \wp_{0.002}(0.1)\}>$
A_3	$<\{\wp_{0.123}(0.4), \wp_{0.123}(0.1), \wp_{0.123}(0.2), \wp_{0.355}(0.3)\}, \{\wp_{-0.427}(0.3), \wp_{-0.350}(0.2), \wp_{-0.350}(0.1), \wp_{0.150}(0.4)\}>$	$<\{\wp_{0.548}(0.2), \wp_{0.548}(0.1), \wp_{0.589}(0.1), \wp_{0.589}(0.6)\}, \{\wp_{-0.869}(0.5), \wp_{-0.732}(0.1), \wp_{-0.521}(0.2), \wp_{-0.374}(0.2)\}>$	$<\{\wp_{0.119}(0.2), \wp_{0.153}(0.4), \wp_{0.153}(0.3), \wp_{0.374}(0.1)\}, \{\wp_{-0.872}(0.1), \wp_{-0.427}(0.3), \wp_{-0.427}(0.5), \wp_{0.004}(0.1)\}>$

Step-5: Utilizing Eqs. (15)-(19) (Taking $\rho = 0.5$) criteria weights are determined in Table (5).

Table 5: Weights of criteria

	Distance values			$\Upsilon_{ji}^{0.5}$			$\pi_j^{0.5}$	w_j
	A_1	A_2	A_3	A_1	A_2	A_3		
C_1	0.209799	0.244174	0.318392	0.875959	0.804167	0.68326	0.787794	0.333191
C_2	0.311915	0.271	0.162038	0.692344	0.755824	1	0.816055	0.345144
C_3	0.350482	0.203843	0.274262	0.641553	0.889722	0.750338	0.760537	0.321663

Steps 6-8: From Eq. (22), the normalized EA-DM is formed. Next, we construct the weighted EA-DM in Table 6.

Table 6: Weighted EA-DM

	C_1	C_2	C_3
AID	$<\{\wp_{-1.407}(1)\},\{\wp_{-1.001}(1)\}>$	$<\{\wp_{-1.276}(1)\},\{\wp_{-0.929}(1)\}>$	$<\{\wp_{-1.486}(1)\},\{\wp_{-1.070}(1)\}>$
A_1	$<\{\wp_{-1.407}(0.3), \wp_{-1.136}(0.2), \wp_{-1.136}(0.2), \wp_{-1.001}(0.3)\}, \{\wp_{-1.650}(0.3), \wp_{-1.617}(0.1), \wp_{-1.519}(0.3), \wp_{-1.368}(0.3)\}>$	$<\{\wp_{-1.276}(0.4), \wp_{-1.256}(0.2), \wp_{-1.074}(0.3), \wp_{-0.999}(0.1)\}, \{\wp_{-1.322}(0.4), \wp_{-1.322}(0.2), \wp_{-0.929}(0.3), \wp_{-0.929}(0.1)\}>$	$<\{\wp_{-1.486}(0.2),\wp_{-1.316}(0.4), \wp_{-1.405}(0.1), \wp_{-1.177}(0.3)\}, \{\wp_{-1.159}(0.1), \wp_{-1.070}(0.3), \wp_{-1.070}(0.3), \wp_{-1.070}(0.3)\}>$
A_2	$<\{\wp_{-1.403}(0.1), \wp_{-1.098}(0.5), \wp_{-1.098}(0.2), \wp_{-1.088}(0.2)\}, \{\wp_{-1.499}(0.5), \wp_{-1.317}(0.1), \wp_{-1.317}(0.2), \wp_{-1.001}(0.2)\}>$	$<\{\wp_{-1.095}(0.5), \wp_{-1.081}(0.2), \wp_{-0.996}(0.1), \wp_{-0.929}(0.2)\}, \{\wp_{-1.391}(0.3), \wp_{-1.217}(0.1), \wp_{-1.198}(0.2), \wp_{-1.039}(0.4)\}>$	$<\{\wp_{-1.279}(0.1), \wp_{-1.070}(0.1), \wp_{-1.070}(0.5), \wp_{-1.070}(0.3)\}, \{\wp_{-1.610}(0.1), \wp_{-1.423}(0.4), \wp_{-1.423}(0.4), \wp_{-1.160}(0.1)\}>$
A_3	$<\{\wp_{-1.316}(0.4), \wp_{-1.316}(0.1), \wp_{-1.316}(0.2), \wp_{-1.191}(0.3)\}, \{\wp_{-1.368}(0.3), \wp_{-1.319}(0.2), \wp_{-1.319}(0.1), \wp_{-1.001}(0.4)\}>$	$<\{\wp_{-1.019}(0.2), \wp_{-1.019}(0.1), \wp_{-0.996}(0.1), \wp_{-0.996}(0.6)\}, \{\wp_{-1.624}(0.5), \wp_{-1.535}(0.1), \wp_{-1.399}(0.2), \wp_{-1.304}(0.2)\}>$	$<\{\wp_{-1.340}(0.2), \wp_{-1.322}(0.4), \wp_{-1.322}(0.3), \wp_{-1.204}(0.1)\}, \{\wp_{-1.697}(0.1), \wp_{-1.424}(0.3), \wp_{-1.424}(0.5), \wp_{-1.159}(0.1)\}>$
ID	$<\{\wp_{-1.001}(1)\},\{\wp_{-1.650}(1)\}>$	$<\{\wp_{-0.929}(1)\},\{\wp_{-1.624}(1)\}>$	$<\{\wp_{-1.070}(1)\},\{\wp_{-1.697}(1)\}>$

Steps- 9 to 11: Determine the UVs for alternatives.

Eq. (23) and Eq. (24) are used to calculate the UDs related to AID and ID, respectively. The UVs are formulated by Eq. (25) and (26) that has connection with AID and ID, respectively. Finally, overall UVs of alternatives are computed utilizing Eq. (27). Table 7 describes the detailed results.

Table 7: UDs, UVs and OUVs of options

	$\hbar_i^-$	$\hbar_i^+$	$\bar{\lambda}_i^-$	$\bar{\lambda}_i^+$	$\bar{\lambda}_i$
A1	-0.07574698	0.044925636	-1.457614484	2.457614484	0.024095103
A2	-0.58814704	0.348830805	-1.457614484	2.457614484	0.187089487
A3	-0.46315702	0.274699058	-1.457614484	2.457614484	0.147330182

Step-12: We obtain $A_2 \succ A_3 \succ A_1$ ("$\succ$" means "superior to") as preference order with A_2as the most suitable option.

6 Discussion

Here we present three sub-sections to illustrate usefulness of the presented DPL-MARCOS as follows:

6.1 Impact of the parameters on OUVs

To study the influence of parameters k and t on OUVs and prioritization, we use the DPLGDWAO for diverse values of k and t lying in interval [1, 8] and [1, 5], respectively. Afterwards, we depict the OUVs of options in Fig. 2.

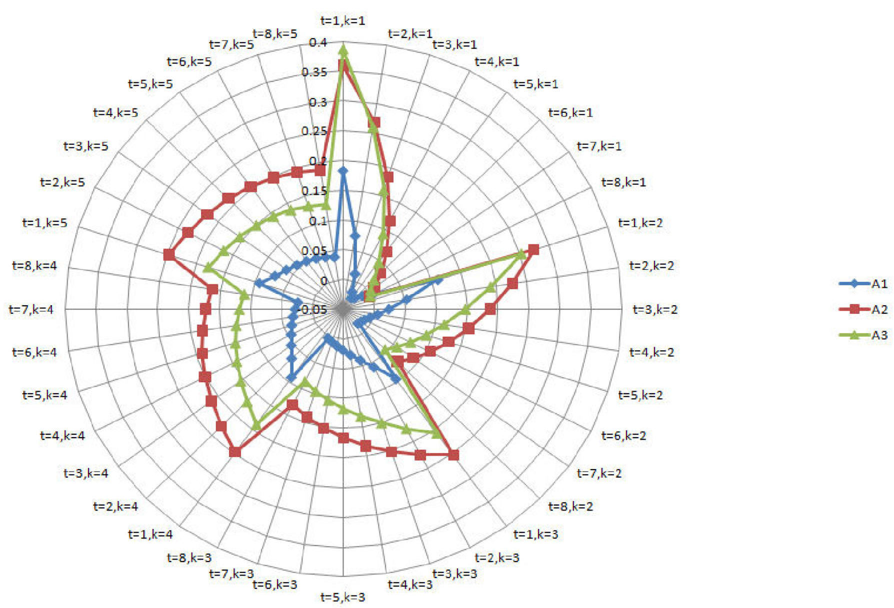

Figure 2: The OUVs of options for diverse parameter values

Analyzing the utility values of $A_1, A_2, A_3$for distinct values of k and t lying in the interval [1, 8] and [1, 5] respectively, we can conclude that in almost all the cases, A_1 is third ranked, A_2 is first ranked and A_3 is second ranked. The analysis spectacles that the option $A2$ is more appropriate to the other options. Moreover, for a fixed value of t, we find that the decreasing nature of the OUVs (except very few cases) of the alternatives A_1, A_2, A_3 inferred from the DPLGDWA operator for increasing of k lying in 1 to 5. Interestingly, in contrast, for a fixed value of k, the utility of A_1, A_2, A_3 derived from the DPLGDWAA operator decreases (except very few cases) with the increase of the value of t that ranges from 1 to 8.

6.2 Sensitivity analysis of criteria weights

In this subsection, we assess the impact of an attribute weights on the presented approach. This assessment is considered the weights eleven attributes, such as CWS1, CWS2,..., CWS11 (see Table 8). The considered attributes' weights are obtained using diverse values of $\rho \in [0,1]$. Next, we estimate the OUVs of options that are reflected in Fig. 3. Scrutinizing the prioritizations of options, we find that option $A1$ is 3rd choice in each case, $A2$ is 1st choice in each case, and $A3$ is 2nd choice in each case. The assessment spectacles that the option $A2$ is the beast option over the other options. Also, we estimate the "Spearman's rank correlation coefficient (SRCC)" [8] for each weight sets and the average degree of SRCC is 1.00, which illustrates the "strong association" [8] of priority of options. Consequently, the prioritization of options obtained by the presented approach is dependable and reliable.

Table 8: Criteria weights set

	CWS1 (ρ=0)	**CWS2** (ρ=0.1)	**CWS3** (ρ=0.2)	**CWS4** (ρ=0.3)	**CWS5** (ρ=0.4)	**CWS6** (ρ=0.5)	**CWS7** (ρ=0.6)	**CWS8** (ρ=0.7)	**CWS9** (ρ=0.8)	**CWS10** (ρ=0.9)	**CWS11** (ρ=1)
C1	0.3290	0.3307	0.3317	0.3324	0.3328	0.3331	0.3334	0.3335	0.3336	0.3337	0.3338
C2	0.3582	0.3537	0.3505	0.3482	0.3465	0.3451	0.3440	0.3431	0.3423	0.3416	0.3411
C3	0.3126	0.3155	0.3176	0.3192	0.3205	0.3216	0.3225	0.3233	0.3239	0.3245	0.3250

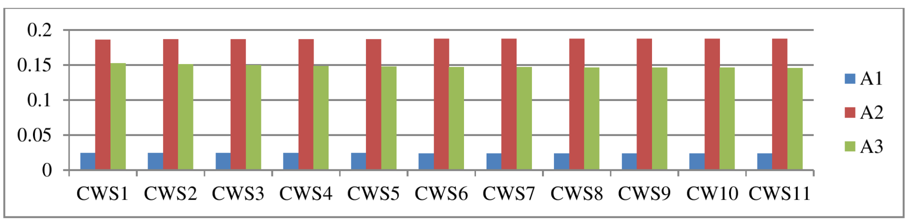

Figure 3: The OUVs of options for diverse weight sets of criteria

6.3 Comparative study: Presented vs. extant

In this line, we show a comparison from theoretical and numerical aspects. To compare the presented approach with the extant approaches on DPLTSs, we consider an extant method such as Xie et al.'s method [29]. The outcomes are discussed in Table 9.

Table 9: Comparison with presented and extant approaches

Aspects	**Presented**	**Xie et al. [29] method**
Information	"Dual probabilistic linguistic (DPL)"	"Dual probabilistic linguistic (DPL)"
Decision-making type	Group	Group
DEs weight	Integration of subjective and objective weighting process	Assumed
Criteria weights	"Gray correlation coefficient (GCC)"	Assumed
Aggregation type	DPLGDWA	DPLWA
Whether tackle hesitancy in prioritization	Yes	Yes
Whether treat with probabilistic data	Yes	Yes
Model utilization	MARCOS	Score-based DPLWA
Sensitivity of attribute weights	Investigated	Not investigated
Priority order	$A_2 \succ A_3 \succ A_1$	$A_2 \succ A_3 \succ A_1$

The advantages of the presented method are described as

1. The described method makes advantage of DPL-based metadata, which is an influential form of data that may be utilized to treat integrity of the information. In practice, DPLTSs improve the robustness of the already available MCDM models by taking into account NGs and the instantaneous occurrence of probabilistic and non-stochastic ambiguity in real-world contexts. Therefore, DPLTSs are preferable to PLTSs in every way.

2. The developed DPLGDWA operator is utilized "reversible transformation function (RTF)". One of the advantages of RTFs is that they may be utilized to characterize the semantics of RTFs, which then allows for the semantic to be allocated to LTs in a variety of different contexts.

3. The DPLGDWA operator can efficiently combine the DPLEs with maximum applicability because it contains parameters 't' and 'k' of which can be selected based on actual decision desires.

4. To make a more scientific decision, we have combined both subjective and objective weighting of DEs to get a more reasonable result.

5. In the current method, we employ MARCOS, which has the following advantages: (i) assigning ID and AID ratings; (ii) evaluating the link between alternatives and ID/AID ratings; and (iii) demonstrating the UD of every alternative in relation to ID and AID. Consequently, the DPL-MARCOS yields better comprehensible and applicable results.

7 Conclusions

The dual probabilistic linguistic term sets (DPLTSs) deals with both belongingness grades and non-belongingness grades along with their respective probabilities. As a result, the DPLTSs are more powerful compare to probabilistic linguistic term sets, probabilistic hesitant fuzzy sets, hesitant fuzzy sets and intuitionistic fuzzy sets. The existing researches on dual probabilistic linguistic term sets haven't paid attention on expert's weight determination (subjective and objective nature). So, the evaluation outcomes get distorted. The existing aggregation operators for fusing DPLEs have circumscribed to the algebraic operators and lacked the flexibility. So, to overcome these issues, in this paper, we have developed a new methodology using new aggregation operator under DPL setting. Motivated by the concept of "generalized Dombi operations (GDOs)", we have introduced new AOs on DPLEs and discussed various elegant features of proposed AOs. In this line, we have presented the DPLGDWA operator and several interesting axioms of the AOs. The some renowned AOs such as algebraic, Einstein, Hamachar operators can be construed as particular case of DPLGDWA operators. Next, MARCOS approach is introduced by applying the aforesaid DPLGDWA operator to treat the MCDM concerns on DPLTSs. The weights of criteria are thus established via the "grey correlation technique (GCT)", which reveals the nearness between the attributes and the ID/AID ratings. The DEs' weights are estimated by integrating the subjective and objective weighting process with the idea of CHWI and assessment similarity of DEs, respectively. To improve applicability, we conducted a case study of biomass energy evaluation. The comparative and sensitivity assessments determine that the presented approach can be successfully utilized in group decision-making issues in DPLTSs setting.

The managerial implications of our research are mentioned below:

(i) An integration of objective and subjective weighting diminishes the impact of prioritization spikes from DEs and the systematically estimation of DEs' weights decreases imprecisions and biases in MCDM.

(ii) This study enhances the theoretic-base of DPLTSs by recommending the operations, AOs and recognizing their elegant axioms analytically.

(iii) The sensitivity and comparative assessments illustrate the prioritizations of options by the presented approach are more consistent with the extant techniques. From now, the presented model is creative to tackle DE's assessments in biomass feedstock selection.

In future, fuzzy parameterized soft set theory [2, 4, 6, 13] can be combined with DPLTSs to develop new structures for treating with MCDM problems. Moreover, some new AOs namely Hamy mean, Maclaurin Symmetric mean operators and others can be proposed with DPLTSs. Some decision-making tools can be presented for giving the outcomes to the MCDM problems. Besides, consensus based group decision-making methodology can be developed to mitigate the biasness of experts under DPL setting.

Acknowledgement

The authors wish to express their appreciation for several excellent suggestions for improvements in this paper made by the referees.

References

[1] I. Badi, D. Pamucar, *Supplier selection for steel-making company by using combined GREY-MARCOS methods*, Decision Making: Applications in Management and Engineering, **3** (2020), 37-47.

[2] N. Cagman, I. Deli, *Products of FP-soft sets and their applications*, Hacettepe Journal of Mathematics and Statistics, **41** (2012), 365-374.

[3] X. Cheng, J. Gu, Z. S. Xu, *Venture capital group decision-making with interaction under probabilistic linguistic environment*, Knowledge-Based Systems, **140** (2018), 82-91.

[4] I. Deli, N. Cagman, *Means of FP-soft sets and their applications*, Hacettepe Journal of Mathematics and Statistics, **41** (2012), 615-625.

[5] J. L. Deng, *Control problems on grey systems*, Systems and Control Letters, **1** (1982), 288-294.

[6] A. Dey, T. Senapati, M. Pal, G. Chen, *Pythagorean fuzzy soft RMS approach to decision making and medical diagnosis*, Afrika Matematika, **33**(97) (2022). DOI:10.1007/s13370-022-01031-7.

[7] J. Dombi, *The generalized Dombi operator family and the multiplicative utility function*, In: V. E. Balas, J. Fodor, A. R. Várkonyi-Kóczy, (eds) Soft Computing Based Modeling in Intelligent Systems. Studies in Computational Intelligence, vol 196. Springer, Berlin, Heidelberg. DOI:10.1007/978-3-642-00448-3-6.

[8] M. K. Ghorabaee, E. K. Zavadskas, M. Amiri, A. Esmaeili, *Multi-criteria evaluation of green suppliers using an extended WASPAS method with interval type-2 fuzzy sets*, Journal of Cleaner Production, **137** (2016), 213-219.

[9] X. J. Gou, Z. S. Xu, H. C. Liao, *Multi-criteria decision making based on Bonferroni means with hesitant fuzzy linguistic information*, Soft Computing, **21** (2017), 6515-6529.

[10] X. L. Han, S. R. Li, *The priority method in view of consistency harmonious weight index*, Systems Engineering-Theory Methodology, **3** (1994), 41-45.

[11] F. Herrera, L. Martinez. *A 2-tuple fuzzy linguistic representation model for computing with words*, IEEE Transactions on Fuzzy Systems, **8** (2000), 746-752.

[12] G. Isik, I. Kaya, *A novel hesitant fuzzy linguistic term sets approach and its application on acceptance sampling plans*, Iranian Journal of Fuzzy Systems, **19**(6) (2022), 175-192.

[13] C. Jana, T. Senapati, K. P. Shum, M. Pal, *Bipolar fuzzy soft subalgebras and ideals of BCK/BCI-algebras based on bipolar fuzzy points*, Journal of Intelligent and Fuzzy Systems, **37**(2) (2019), 2785-2795.

[14] A. Kobina, D. Liang, X. He, *Probabilistic linguistic power aggregation operators for multi-criteria group decision making*, Symmetry, **9** (2017), 1-21.

[15] B. Li, Z. S. Xu, Y. X. Zhang, *A probabilistic linguistic evaluation based multi-stage medical system selection process related to referral system*, Expert Systems with Applications, **170** (2020), 114523. DOI:10.1016/j.eswa.2020.114523.

[16] H. Liao, X. Mi, Z. Xu, *A survey of decision-making methods with probabilistic linguistic information?: Bibliometrics, preliminaries, methodologies, applications and future directions*, Fuzzy Optimization and Decision Making, **19** (2010), 81-134.

[17] H. Liao, X. Wu, *DNMA: A double normalization-based multiple aggregation method for multi-expert multi-criteria decision making*, Omega, **94** (2020), 102058. DOI:10.1016/j.omega.2019.04.001.

[18] P. Liu, F. Teng, *Probabilistic linguistic TODIM method for selecting products through online product reviews*, Information Science, **485** (2019), 441-455.

[19] Q. Pang, H. Wang, Z. S. Xu, *Probabilistic linguistic term sets in multi-attribute group decision making*, Information Science, **369** (2016), 128-143.

[20] S. Ramadass, R. Krishankumar, K. S. Ravichandran, H. Liao, S. Kar, *Evaluation of cloud vendors from probabilistic linguistic information with unknown/partial weight values*, Applied Soft Computing, **97** (2020), 106801.

[21] R. M. Rodriguez, L. Martinez, F. Herrera, *Hesitant fuzzy linguistic term sets for decision making*, IEEE Transactions on Fuzzy Systems, **20** (2012), 109-119.

[22] Y. Rong, L. Yu, W. Niu, Y. Liu, T. Senapati, A. R. Mishra, *MARCOS approach based upon cubic Fermatean fuzzy set and its application in evaluation and selecting cold chain logistics distribution center*, Engineering Applications of Artificial Intelligence, **116** (2022), 105401. DOI:10.1016/j.engappai.2022.105401.

[23] A. Saha, T. Senapati, R. R. Yager, *Hybridizations of generalized Dombi operators and Bonferroni mean operators under dual probabilistic linguistic environment for group decision-making*, International Journal of Intelligent Systems, **36**(11) (2021), 6645-6679.

[24] A. Saha, V. Simic, T. Senapati, D. Dabic-Liletic, A. Ala, *A dual hesitant fuzzy sets-based methodology for advantage prioritization of zero-emission last-mile delivery solutions*, IEEE Transactions on Fuzzy Systems, (2022). DOI: 10.1109/TFUZZ.2022.3164053.

[25] M. Stankovic, Z. Stevic, D. K. Das, M. Subotic, D. Pamucar, *A new fuzzy MARCOS method for road traffic risk analysis*, Mathematics, **8** (2020), 1-18.

[26] Z. Stevic, N. Brkovic, *A novel integrated FUCOM-MARCOS model for evaluation of human resources in a transport company*, Logistics, **4** (2020), 1-14.

[27] Z. Stevic, D. Pamucar, A. Puska, P. Chatterjee, *Sustainable supplier selection in healthcare industries using a new MCDM method: Measurement alternatives and ranking according to compromise solution (MARCOS)*, Computers and Industrial Engineering, **140** (2020), 106231.

[28] A. Ulutas, D. Karabasevic, G. Popovic, D. Stanujkic, P. T. Nguyen, C. Karakoy, *Development of a novel integrated CCSD-ITARA-MARCOS decision-making approach for stackers selection in a logistics system*, Mathematics, **8** (2020), 1-15.

[29] W. Xie, Z. S. Xu, Z. Ren, *Dual probabilistic linguistic term set and its application on the multi-criteria group decision-making problems*, in: 2017, IEEE International Conference on Industrial Engineering and Engineering Management, (2017), 1469-1474.

[30] W. Xie, Z. S. Xu, Z. Ren, E. H. Viedma, *A new multi-criteria decision-making model based on incomplete dual probabilistic linguistic preference relations*, Applied Soft Computing, **91** (2020), 106237.

[31] G. L. Xu, S. P. Wan, J. Y. Dong, *An entropy-based method for probabilistic linguistic group decision making and its application of selecting car sharing platforms*, Informatica, **31**(3) (2020), 621-658.

[32] L. A. Zadeh, *The concept of a linguistic variable and its application to approximate reasoning-I*, Information Science, **8** (1975), 199-249.

[33] X. Zhang, X. Xing, *Probabilistic linguistic VIKOR method to evaluate green supply chain initiatives*, Sustainability, **9** (2017), 1-18.

SUPPLEMENTARY MATERIAL

Proof of Theorem-1:

(i) and (ii) are straight forward.

(iii) By definition of generalized Dombi operations on DPLEs, we have,

$$
\begin{aligned}
&\lambda(\tilde{\delta}_\ell^{(1)}(\Xi)\tilde{\oplus}\tilde{\delta}_\ell^{(2)}(\Xi)(\mho))\\
&=\lambda\left\langle \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{2}\Phi_t^k(\tilde{\xi}_\phi^j)-1\right)\right)^{-\frac{1}{k}}\right)^{-1}\right)(\Xi^{(a)}),\ \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{2}\Theta_t^k(\tilde{\xi}_\theta^j)-1\right)\right)^{\frac{1}{k}}\right)^{-1}\right)(\Xi^{(b)})\right\rangle\\
&=\left\langle \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left(\left(1+t\left(\frac{\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{2}\Phi_t^k(\tilde{\xi}_\phi^j)-1\right)\right)^{-\frac{1}{k}}\right)^{-1}}{1-\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{2}\Phi_t^k(\tilde{\xi}_\phi^j)-1\right)\right)^{-\frac{1}{k}}\right)^{-1}}\right)^{k}\right)^{\lambda}-1\right)\right)^{-\frac{1}{k}}\right)^{-1}\right)(\Xi^{(a)}),\right.\\
&\qquad\left.\tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left(\left(1+t\left(\frac{1-\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{2}\Theta_t^k(\tilde{\xi}_\theta^j)-1\right)\right)^{\frac{1}{k}}\right)^{-1}}{\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{2}\Theta_t^k(\tilde{\xi}_\theta^j)-1\right)\right)^{\frac{1}{k}}\right)^{-1}}\right)^{k}\right)^{\lambda}-1\right)\right)^{\frac{1}{k}}\right)^{-1}\right)(\Xi^{(b)})\right\rangle\\
&=\left\langle \tilde{\xi}^{-1}\left(1+\left(\frac{1}{t}\left(\left(1+t\times\frac{1}{t}\left(\prod_{j=1}^{2}\Phi_t^k(\tilde{\xi}_\phi^j)-1\right)\right)^{\lambda}-1\right)\right)^{-\frac{1}{k}}\right)^{-1}(\Xi^{(a)}),\right.\\
&\qquad\left.\tilde{\xi}^{-1}\left(1+\left(\frac{1}{t}\left(\left(1+t\times\frac{1}{t}\left(\prod_{j=1}^{2}\Theta_t^k(\tilde{\xi}_\theta^j)-1\right)\right)^{\lambda}-1\right)\right)^{\frac{1}{k}}\right)^{-1}(\Xi^{(b)})\right\rangle\\
&=\left\langle \tilde{\xi}^{-1}\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{2}(\Phi_t^k(\tilde{\xi}_\phi^j))^{\lambda}-1\right)\right)^{-\frac{1}{k}}\right)^{-1}(\Xi^{(a)}),\right.\\
&\qquad\left.\tilde{\xi}^{-1}\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{2}(\Theta_t^k(\tilde{\xi}_\theta^j))^{\lambda}-1\right)\right)^{\frac{1}{k}}\right)^{-1}(\Xi^{(b)})\right\rangle.
\end{aligned}
$$

On the other hand, by generalized Dombi operation laws, we have,

$$
\begin{aligned}
&(\lambda\tilde{\delta}_\ell^{(1)}(\Xi))\tilde{\oplus}(\lambda\tilde{\delta}_\ell^{(2)}(\Xi))\\
&=\left\langle \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left((\Phi_t^k(\tilde{\xi}_\phi^1))^{\lambda}-1\right)\right)^{-\frac{1}{k}}\right)^{-1}\right)(\Xi^{(a)}),\ \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left((\Theta_t^k(\tilde{\xi}_\theta^1))^{\lambda}-1\right)\right)^{\frac{1}{k}}\right)^{-1}\right)(\Xi^{(b)})\right\rangle\\
&\quad\tilde{\oplus}\left\langle \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left((\Phi_t^k(\tilde{\xi}_\phi^2))^{\lambda}-1\right)\right)^{-\frac{1}{k}}\right)^{-1}\right)(\Xi^{(a)}),\ \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left((\Theta_t^k(\tilde{\xi}_\theta^2))^{\lambda}-1\right)\right)^{\frac{1}{k}}\right)^{-1}\right)(\Xi^{(b)})\right\rangle\\
&=\left\langle \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{2}\left(1+t\left(\frac{\left(1+\left(\frac{1}{t}((\Phi_t^k(\tilde{\xi}_\phi^j))^{\lambda}-1)\right)^{-\frac{1}{k}}\right)^{-1}}{1-\left(1+\left(\frac{1}{t}((\Phi_t^k(\tilde{\xi}_\phi^j))^{\lambda}-1)\right)^{-\frac{1}{k}}\right)^{-1}}\right)^{k}\right)-1\right)\right)^{-\frac{1}{k}}\right)^{-1}\right)(\Xi^{(a)}),\right.\\
&\qquad\left.\tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{2}\left(1+t\left(\frac{1-\left(1+\left(\frac{1}{t}((\Theta_t^k(\tilde{\xi}_\theta^j))^{\lambda}-1)\right)^{\frac{1}{k}}\right)^{-1}}{\left(1+\left(\frac{1}{t}((\Theta_t^k(\tilde{\xi}_\theta^j))^{\lambda}-1)\right)^{\frac{1}{k}}\right)^{-1}}\right)^{k}\right)-1\right)\right)^{\frac{1}{k}}\right)^{-1}\right)(\Xi^{(b)})\right\rangle\\
&=\left\langle \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{2}\left(1+t\left(\frac{1}{t}\left((\Phi_t^k(\tilde{\xi}_\phi^j))^{\lambda}-1\right)\right)\right)-1\right)\right)^{-\frac{1}{k}}\right)^{-1}\right)(\Xi^{(a)}),\right.\\
&\qquad\left.\tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{2}\left(1+t\left(\frac{1}{t}\left((\Theta_t^k(\tilde{\xi}_\theta^j))^{\lambda}-1\right)\right)\right)-1\right)\right)^{\frac{1}{k}}\right)^{-1}\right)(\Xi^{(b)})\right\rangle\\
&=\left\langle \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{2}(\Phi_t^k(\tilde{\xi}_\phi^j))^{\lambda}-1\right)\right)^{-\frac{1}{k}}\right)^{-1}\right)(\Xi^{(a)}),\ \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{2}(\Theta_t^k(\tilde{\xi}_\theta^j))^{\lambda}-1\right)\right)^{\frac{1}{k}}\right)^{-1}\right)(\Xi^{(b)})\right\rangle.
\end{aligned}
$$

Hence $\lambda(\tilde{\delta}_\ell^{(1)}(\Xi)\tilde{\oplus}\tilde{\delta}_\ell^{(2)}(\Xi))=(\lambda\tilde{\delta}_\ell^{(1)}(\Xi))\tilde{\oplus}(\lambda\tilde{\delta}_\ell^{(2)}(\Xi))$.

(iv)-(vi) Similar to (iii).

Proof of Theorem-2:
First part easily follows from Definition 3.4. To prove the remaining part, the principle of mathematical induction on 'n' is utilized.
For n=2, based on generalized Dombi operational laws, we have,

$$DPLGDWAA(\tilde{\delta}_{\ell}^{(1)}(\Xi), \tilde{\delta}_{\ell}^{(2)}(\Xi))$$
$$= (w_1\tilde{\delta}_{\ell}^{(1)}(\Xi))\tilde{\oplus}(w_2\tilde{\delta}_{\ell}^{(2)}(\Xi))$$
$$= \left\langle \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left((\Phi_t^k(\tilde{\xi}_\phi^1))^{w_1}-1\right)\right)^{-\frac{1}{k}}\right)^{-1}\right)(\Xi^{(a)}),\right.$$
$$\left.\tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left((\Theta_t^k(\tilde{\xi}_\theta^1))^{w_1}-1\right)\right)^{\frac{1}{k}}\right)^{-1}\right)(\Xi^{(b)})\right\rangle$$
$$\tilde{\oplus}\left\langle \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left((\Phi_t^k(\tilde{\xi}_\phi^2))^{w_2}-1\right)\right)^{-\frac{1}{k}}\right)^{-1}\right)(\Xi^{(a)}),\right.$$
$$\left.\tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left((\Theta_t^k(\tilde{\xi}_\theta^2))^{w_2}-1\right)\right)^{\frac{1}{k}}\right)^{-1}\right)(\Xi^{(b)})\right\rangle$$
$$= \left\langle \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{2}\left(1+t\left(\frac{\left(1+\left(\frac{1}{t}((\Phi_t^k(\tilde{\xi}_\phi^j))^{w_j}-1)\right)^{-\frac{1}{k}}\right)^{-1}}{1-\left(1+\left(\frac{1}{t}((\Phi_t^k(\tilde{\xi}_\phi^j))^{w_j}-1)\right)^{-\frac{1}{k}}\right)^{-1}}\right)^{k}\right)-1\right)\right)^{-\frac{1}{k}}\right)^{-1}\right)(\Xi^{(a)}),\right.$$
$$\left.\tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{2}\left(1+t\left(\frac{1-\left(1+\left(\frac{1}{t}((\Theta_t^k(\tilde{\xi}_\theta^j))^{w_j}-1)\right)^{\frac{1}{k}}\right)^{-1}}{\left(1+\left(\frac{1}{t}((\Theta_t^k(\tilde{\xi}_\theta^j))^{w_j}-1)\right)^{\frac{1}{k}}\right)^{-1}}\right)^{k}\right)-1\right)\right)^{\frac{1}{k}}\right)^{-1}\right)(\Xi^{(b)})\right\rangle$$
$$= \left\langle \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{2}\left(1+t\times\frac{1}{t}\left((\Phi_t^k(\tilde{\xi}_\phi^j))^{w_j}-1\right)\right)-1\right)\right)^{-\frac{1}{k}}\right)^{-1}\right)(\Xi^{(a)}),\right.$$
$$\left.\tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{2}\left(1+t\times\frac{1}{t}\left((\Theta_t^k(\tilde{\xi}_\theta^j))^{w_j}-1\right)\right)-1\right)\right)^{\frac{1}{k}}\right)^{-1}\right)(\Xi^{(b)})\right.$$
$$= \left\langle \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{2}(\Phi_t^k(\tilde{\xi}_\phi^j))^{w_j}-1\right)\right)^{-\frac{1}{k}}\right)^{-1}\right)(\Xi^{(a)}),\right.$$
$$\left.\tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{2}(\Theta_t^k(\tilde{\xi}_\theta^j))^{w_j}-1\right)\right)^{\frac{1}{k}}\right)^{-1}\right)(\Xi^{(b)})\right\rangle.$$

Thus Eq. (8) is satisfied for n=2. Suppose Eq. (8) is true for $n=r$ i.e;

$$DPLGDWAA(\tilde{\delta}_{\ell}^{(1)}(\Xi), \tilde{\delta}_{\ell}^{(2)}(\Xi), ..., \tilde{\delta}_{\ell}^{(r)}(\Xi))$$
$$= \left\langle \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{r}(\Phi_t^k(\tilde{\xi}_\phi^j))^{w_j}-1\right)\right)^{-\frac{1}{k}}\right)^{-1}\right)(\Xi^{(a)}),\ \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{r}(\Theta_t^k(\tilde{\xi}_\theta^j))^{w_j}-1\right)\right)^{\frac{1}{k}}\right)^{-1}\right)(\Xi^{(b)})\right\rangle$$

Now for $n=r+1$, we have,

$$DPLGDWAA(\tilde{\delta}_{\ell}^{(1)}(\Xi), \tilde{\delta}_{\ell}^{(2)}(\Xi), ..., \tilde{\delta}_{\ell}^{(r+1)}(\Xi))$$
$$= \left\langle \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{r}(\Phi_t^k(\tilde{\xi}_\phi^j))^{w_j}-1\right)\right)^{-\frac{1}{k}}\right)^{-1}\right)(\Xi^{(a)})\ \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{r}(\Theta_t^k(\tilde{\xi}_\theta^j))^{w_j}-1\right)\right)^{\frac{1}{k}}\right)^{-1}\right)(\Xi^{(b)})\right\rangle$$
$$\tilde{\oplus}\left\langle \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}((\Phi_t^k(\tilde{\xi}_\phi^{r+1}))^{w_{r+1}}-1)\right)^{-\frac{1}{k}}\right)^{-1}\right)(\Xi^{(a)}),\ \tilde{\xi}^{-1}\left(\left(1+\left(\frac{1}{t}((\Theta_t^k(\tilde{\xi}_\theta^{r+1}))^{w_{r+1}}-1)\right)^{\frac{1}{k}}\right)^{-1}\right)(\Xi^{(b)})\right\rangle$$

$$= \Bigg\langle \tilde{\xi}^{-1} \Bigg(\Bigg(\Bigg(1 + \Bigg(\frac{1}{t} \Bigg(1 + t \Bigg(\frac{\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{r}(\Phi_t^k(\tilde{\xi}_\phi^j))^{w_j}-1\right)\right)^{-\frac{1}{k}}\right)^{-1}}{1-\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{r}(\Phi_t^k(\tilde{\xi}_\phi^j))^{w_j}-1\right)\right)^{-\frac{1}{k}}\right)^{-1}} \Bigg)^k \Bigg) \times$$

$$\Bigg(1 + t \Bigg(\frac{\left(1+\left(\frac{1}{t}((\Phi_t^k(\tilde{\xi}_\phi^{r+1}))^{w_{r+1}}-1)\right)^{-\frac{1}{k}}\right)^{-1}}{1-\left(1+\left(\frac{1}{t}((\Phi_t^k(\tilde{\xi}_\phi^{r+1}))^{w_{r+1}}-1)\right)^{-\frac{1}{k}}\right)^{-1}} \Bigg)^k \Bigg) - 1 \Bigg) \Bigg)^{-\frac{1}{k}} \Bigg)^{-1} \Bigg) (\Xi^{(a)}),$$

$$\tilde{\xi}^{-1} \Bigg(\Bigg(1 + \Bigg(\frac{1}{t} \Bigg(1 + t \Bigg(\frac{1-\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{r}(\Theta_t^k(\tilde{\xi}_\theta^j))^{w_j}-1\right)\right)^{\frac{1}{k}}\right)^{-1}}{\left(1+\left(\frac{1}{t}\left(\prod_{j=1}^{r}(\Theta_t^k(\tilde{\xi}_\theta^j))^{w_j}-1\right)\right)^{\frac{1}{k}}\right)^{-1}} \Bigg)^k \Bigg) \times$$

$$\Bigg(1 + t \Bigg(\frac{1-\left(1+\left(\frac{1}{t}((\Theta_t^k(\tilde{\xi}_\theta^{r+1}))^{w_{r+1}}-1)\right)^{\frac{1}{k}}\right)^{-1}}{\left(1+\left(\frac{1}{t}((\Theta_t^k(\tilde{\xi}_\theta^{r+1}))^{w_{r+1}}-1)\right)^{\frac{1}{k}}\right)^{-1}} \Bigg)^k \Bigg) - 1 \Bigg) \Bigg)^{\frac{1}{k}} \Bigg)^{-1} \Bigg) (\Xi^{(b)}) \Bigg\rangle$$

$$= \Bigg\langle \tilde{\xi}^{-1} \left(\left(1 + \left(\frac{1}{t} \left(\prod_{j=1}^{r}(\Phi_t^k(\tilde{\xi}_\phi^j))^{w_j} \times (\Phi_t^k(\tilde{\xi}_\phi^{r+1}))^{w_{r+1}} - 1 \right) \right)^{-\frac{1}{k}} \right)^{-1} \right) (\Xi^{(a)}),$$

$$\tilde{\xi}^{-1} \left(\left(1 + \left(\frac{1}{t} \left(\prod_{j=1}^{r}(\Theta_t^k(\tilde{\xi}_\theta^j))^{w_j} \times (\Theta_t^k(\tilde{\xi}_\theta^{r+1}))^{w_{r+1}} - 1 \right) \right)^{\frac{1}{k}} \right)^{-1} \right) (\Xi^{(b)}) \Bigg\rangle$$

$$= \Bigg\langle \tilde{\xi}^{-1} \left(\left(1 + \left(\frac{1}{t} \left(\prod_{j=1}^{r+1} \left(\Phi_t^k(\tilde{\xi}_\phi^j)\right)^{w_j} - 1 \right) \right)^{-\frac{1}{k}} \right)^{-1} \right) (\Xi^{(a)}),$$

$$\tilde{\xi}^{-1} \left(\left(1 + \left(\frac{1}{t} \left(\prod_{j=1}^{r+1} \left(\Theta_t^k(\tilde{\xi}_\theta^j)\right)^{w_j} - 1 \right) \right)^{\frac{1}{k}} \right)^{-1} \right) (\Xi^{(b)}) \Bigg\rangle$$

Thus, Eq. (8) is also satisfied for $n{=}r{+}1$. Hence, by the principle of mathematical induction, Eq. (8) is true for all natural number n.

Table 10: Adjusted initial decision matrix

		C_1	C_2	C_3
D_1	A_1	$<\{\wp_{-2}(0.3), \wp_{-1}(0.2), \wp_{-1}(0.2), \wp_{1}(0.3)\}, \{\wp_{-1}(0.3), \wp_{0}(0.1), \wp_{0}(0.3), \wp_{0}(0.3)\}>$	$<\{\wp_{1}(0.4), \wp_{1}(0.2), \wp_{2}(0.3), \wp_{2}(0.1)\}, \{\wp_{-2}(0.4), \wp_{-2}(0.2), \wp_{1}(0.3), \wp_{1}(0.1)\}>$	$<\{\wp_{-1}(0.2), \wp_{-1}(0.4), \wp_{-1}(0.1), \wp_{1}(0.3)\}, \{\wp_{0}(0.1), \wp_{2}(0.3), \wp_{2}(0.3), \wp_{2}(0.3)\}>$
	A_2	$<\{\wp_{-1}(0.1), \wp_{0}(0.5), \wp_{0}(0.2), \wp_{0}(0.2)\}, \{\wp_{-2}(0.5), \wp_{-1}(0.1), \wp_{-1}(0.2), \wp_{1}(0.2)\}>$	$<\{\wp_{-2}(0.5), \wp_{-2}(0.2), \wp_{1}(0.1), \wp_{1}(0.2)\}, \{\wp_{1}(0.3), \wp_{1}(0.1), \wp_{2}(0.2), \wp_{2}(0.4)\}>$	$<\{\wp_{0}(0.1), \wp_{2}(0.1), \wp_{2}(0.5), \wp_{2}(0.3)\}, \{\wp_{-1}(0.1), \wp_{-1}(0.4), \wp_{-1}(0.4), \wp_{1}(0.1)\}>$
	A_3	$<\{\wp_{1}(0.4), \wp_{1}(0.1), \wp_{1}(0.2), \wp_{1}(0.3)\}, \{\wp_{0}(0.3), \wp_{2}(0.2), \wp_{2}(0.1), \wp_{2}(0.4)\}>$	$<\{\wp_{-1}(0.2), \wp_{-1}(0.1), \wp_{0}(0.1), \wp_{0}(0.6)\}, \{\wp_{-2}(0.5), \wp_{-1}(0.1), \wp_{-1}(0.2), \wp_{1}(0.2)\}>$	$<\{\wp_{-2}(0.2), \wp_{-2}(0.4), \wp_{-2}(0.3), \wp_{-1}(0.1)\}, \{\wp_{-1}(0.1), \wp_{0}(0.3), \wp_{0}(0.5), \wp_{0}(0.1)\}>$
D_2	A_1	$<\{\wp_{-1}(0.3), \wp_{-1}(0.2), \wp_{-1}(0.2), \wp_{1}(0.3)\}, \{\wp_{-2}(0.3), \wp_{-2}(0.1), \wp_{-2}(0.3), \wp_{-1}(0.3)\}>$	$<\{\wp_{-2}(0.4), \wp_{-2}(0.2), \wp_{-1}(0.3), \wp_{-1}(0.1)\}, \{\wp_{2}(0.4), \wp_{2}(0.2), \wp_{2}(0.3), \wp_{2}(0.1)\}>$	$<\{\wp_{-2}(0.2), \wp_{-1}(0.4), \wp_{-1}(0.1), \wp_{1}(0.3)\}, \{\wp_{1}(0.1), \wp_{1}(0.3), \wp_{1}(0.3), \wp_{1}(0.3)\}>$
	A_2	$<\{\wp_{0}(0.1), \wp_{2}(0.5), \wp_{2}(0.2), \wp_{2}(0.2)\}, \{\wp_{-1}(0.5), \wp_{-1}(0.1), \wp_{-1}(0.2), \wp_{1}(0.2)\}>$	$<\{\wp_{-2}(0.5), \wp_{-1}(0.2), \wp_{-1}(0.1), \wp_{1}(0.2)\}, \{\wp_{0}(0.3), \wp_{0}(0.1), \wp_{0}(0.2), \wp_{0}(0.4)\}>$	$<\{\wp_{-1}(0.1), \wp_{0}(0.1), \wp_{0}(0.5), \wp_{0}(0.3)\}, \{\wp_{-2}(0.1), \wp_{-1}(0.4), \wp_{-1}(0.4), \wp_{1}(0.1)\}>$
	A_3	$<\{\wp_{-2}(0.4), \wp_{-2}(0.1), \wp_{-2}(0.2), \wp_{-1}(0.3)\}, \{\wp_{-1}(0.3), \wp_{-1}(0.2), \wp_{-1}(0.1), \wp_{1}(0.4)\}>$	$<\{\wp_{2}(0.2), \wp_{2}(0.1), \wp_{2}(0.1), \wp_{2}(0.6)\}, \{\wp_{-2}(0.5), \wp_{-2}(0.1), \wp_{-1}(0.2), \wp_{-1}(0.2)\}>$	$<\{\wp_{1}(0.2), \wp_{1}(0.4), \wp_{1}(0.3), \wp_{1}(0.1)\}, \{\wp_{-2}(0.1), \wp_{-1}(0.3), \wp_{-1}(0.5), \wp_{1}(0.1)\}>$
D_3	A_1	$<\{\wp_{0}(0.3), \wp_{2}(0.2), \wp_{2}(0.2), \wp_{2}(0.3)\}, \{\wp_{-2}(0.3), \wp_{-2}(0.1), \wp_{-1}(0.3), \wp_{-1}(0.3)\}>$	$<\{\wp_{-2}(0.4), \wp_{-1}(0.2), \wp_{-1}(0.3), \wp_{1}(0.1)\}, \{\wp_{2}(0.4), \wp_{2}(0.2), \wp_{2}(0.3), \wp_{2}(0.1)\}>$	$<\{\wp_{-1}(0.2), \wp_{1}(0.4), \wp_{0}(0.1), \wp_{0}(0.3)\}, \{\wp_{1}(0.1), \wp_{1}(0.3), \wp_{1}(0.3), \wp_{1}(0.3)\}>$
	A_2	$<\{\wp_{-2}(0.1), \wp_{-2}(0.5), \wp_{-2}(0.2), \wp_{-1}(0.2)\}, \{\wp_{0}(0.5), \wp_{2}(0.1), \wp_{2}(0.2), \wp_{2}(0.2)\}>$	$<\{\wp_{2}(0.5), \wp_{2}(0.2), \wp_{2}(0.1), \wp_{2}(0.2)\}, \{\wp_{-2}(0.3), \wp_{-1}(0.1), \wp_{-1}(0.2), \wp_{1}(0.4)\}>$	$<\{\wp_{1}(0.1), \wp_{1}(0.1), \wp_{1}(0.5), \wp_{1}(0.3)\}, \{\wp_{-1}(0.1), \wp_{0}(0.4), \wp_{0}(0.4), \wp_{0}(0.1)\}>$
	A_3	$<\{\wp_{-1}(0.4), \wp_{-1}(0.1), \wp_{-1}(0.2), \wp_{1}(0.3)\}, \{\wp_{-1}(0.3), \wp_{-1}(0.2), \wp_{-1}(0.1), \wp_{1}(0.4)\}>$	$<\{\wp_{0}(0.2), \wp_{0}(0.1), \wp_{0}(0.1), \wp_{0}(0.6)\}, \{\wp_{-1}(0.5), \wp_{-1}(0.1), \wp_{-1}(0.2), \wp_{-1}(0.2)\}>$	$<\{\wp_{-2}(0.2), \wp_{-1}(0.4), \wp_{-1}(0.3), \wp_{1}(0.1)\}, \{\wp_{-2}(0.1), \wp_{-1}(0.3), \wp_{-1}(0.5), \wp_{1}(0.1)\}>$