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Original paper

The distributivity characterization of idempotent null-uninorms over two special aggregation operators

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Abstract

Recently, Zhao et al. [55] characterized the distributivity equations of null-uninorms with continuous and Archimedean underlying operators over overlap or grouping functions. Moreover, Liu et al. [34] studied the distributive laws of continuous t-norms over overlap functions. In this paper, we proceed with the distributivity characterization of idempotent null-uninorms over overlap or grouping functions. In order to do that, we introduce a class of weak overlap and grouping functions with weak coefficients, and obtain the full characterizations of overlap and grouping functions by considering the different values of underlying uninorms' associated functions of idempotent null-uninorms on the interval endpoints and comparing them with the weak coefficients. Obviously, idempotent null-uninorms generalize idempotent uninorms. Thus, the obtained results also generalize the distributivity of idempotent uninorms proposed as future work in [34].

Keywords: Distributivity equation, idempotent null-uninorms, associated function, idempotent uninorms, overlap function, grouping function.

1 Introduction

1.1 A short retrospect of the distributive laws between aggregation operators

We know that aggregation operators [4, 5, 23] perform an important role in the field of information aggregation. The research on them all along becomes the significant components in fuzzy theory. Among the research about various aggregation operators, except for the important axiomatic characterization of them, all kinds of functional equations [10] such as the distributivity [33, 42, 46], the modularity [45, 50, 54], the migrativity [32, 36, 39, 56] and the homogeneity [41] have formed a momentous research direction. The above functional equations are a forceful tool in the community of fuzzy sets and logic. For instance, the distributivity characterizations of pairs of aggregation operators are interesting because this problem is closely related to the so-called pseudo-analysis, where t-norms and t-conorms have been frequently used to model the pseudo-operations as pseudo-addition and pseudo-multiplication [21], and sometimes uninorms have been used [30, 47, 48] in this context. Moreover, the distributivity has also been proved to be useful in several fields, such as fuzzy logic framework and fuzzy quantifiers [31], neural networks and expert systems [35], integral theory [47, 48], utility theory [18, 26], decision making theory [18, 21], and others [6, 37]. Therefore, in recent years, there are lots of papers on distributivity equations between aggregation operators discussed in various literatures [46, 53]. But as for related problems on overlap and grouping functions there are still many unresolved issues.

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1.2 Briefly reviewing on null-uninorms and overlap functions

Null-uninorms, as a class of special 2-uninorms [1] and a combination of uninorms and nullnorms letting them share the same underlying t-norm, caught some authors' concerns [55, 52] since they were introduced in [49]. The significance of the research on null-uninorms is that many works on them are both the generalization of the corresponding ones on uninorms and nullnorms and the attempt for further extending into general 2-uninorms. In addition, overlap and grouping functions [2, 3, 7] as two special aggregation operators which are not necessarily associative or of having neutral elements, have become a hot topic. They may be remarkably different from other aggregation operators both associative and possessing neutral elements. On the other hand, they also have widespread application in many fields such as fuzzy preference modeling [8], classification [19], decision making [20], fuzzy detection [24] and image processing [27]. And universal concerns on them directly lead to the rapid development in theory and application in recent years [12]-[16].

1.3 Our motivation for the paper

We know, in [55] authors have only studied the distributivity equations on eight classes of particular null-uninorms whose underlying operators are continuous Archimedean over overlap or grouping functions. It is natural to extend the research to more general circumstances such as idempotent null-uninorms, null-uninorms with continuous underlying operators and even general 2-uninorms. Furthermore, in [34] authors have mentioned their future works on the distributivity characterizations of idempotent uninorms [11, 28, 44] over overlap functions. It is evident that our current research will answer and generalize the above-mentioned question. Just based on the above consideration, we have introduced the weak overlap and grouping functions with the weak coefficients which respectively generalize overlap and grouping functions and obtained the equivalent conditions for the distributivity equations through comparing the values of the weak coefficient and the associated function of idempotent null-uninorm's underlying uninorm. We find the overlap and grouping functions can always be represented as an ordinal sum [12, 51] when an idempotent null-uninorm is distributive over them. In this process, we reveal the sufficient and necessary conditions and full structures of the two aggregation operators and present some corresponding examples fulfilling relevant results.

This paper is arranged as follows. In Section 2, we mainly present some basic concepts and results related to t-norms, idempotent uninorms, idempotent null-uninorms, overlap and grouping functions. In Sections 3 and 4, we study the distributivity equations of idempotent null-uninorms over overlap and grouping functions simultaneously attaching some corresponding examples, respectively. Section 5 ends with some conclusions and future works.

2 Preliminaries

In this part, we only review some basic notations and results about t-norm, overlap function, grouping function, uninorm and null-uninorm and refer the reader to [7, 22, 29, 34] for further details.

Definition 2.1. [29] *A* triangular norm *(abbreviated as* t-norm*) is a function* $\mathbb{T} : [0,1]^2 \to [0,1]$ *that is commutative, associative, increasing for each variable and has the neutral element* 1*. A* triangular conorm *(abbreviated as* t-conorm*) is a function* $\mathbb{S} : [0,1]^2 \to [0,1]$ *that is commutative, associative, increasing for each variable and has the neutral element* 0*.*

The duality between these two functions can be as follows: one can obtain a dual t-conorm $\mathbb{S}$ from any t-norm $\mathbb{T}$ by the equation $\mathbb{S}(x,y) = 1-\mathbb{T}(1-x,1-y)$ and vice-versa. Furthermore, we know, the minimum t-norm $\mathbb{T}_M(x,y) = \min(x,y)$, product t-norm $\mathbb{T}_P(x,y) = xy$ and Lukasiewicz t-norm $\mathbb{T}_L(x,y) = \max(x+y-1,0)$ are typical examples of t-norms. And their dual t-conorms, respectively, are $\mathbb{S}_M(x,y) = \max(x,y)$, $\mathbb{S}_P(x,y) = x+y-xy$ and $\mathbb{S}_L(x,y) = \min(x+y,1)$.

Definition 2.2. [22] *A binary operator* $\mathbb{U}\colon [0,1]^2 \to [0,1]$ *is called a* uninorm *if it is commutative, associative, increasing and there exists a* neutral element $e \in [0,1]$ *such that* $\mathbb{U}(x,e) = x$ *for all* $x \in [0,1]$*.*

Obviously, a uninorm reduces to a t-norm when $e = 1$, or a t-conorm when $e = 0$. A uninorm with the neutral element $e \in (0,1)$ is called to be *proper*. We know that any proper uninorm $\mathbb{U}$ satisfies the following form

$$\mathbb{U}(x,y) = \begin{cases} e\mathbb{T}_{\mathbb{U}}(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)\mathbb{S}_{\mathbb{U}}(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ D_{\mathbb{U}}(x,y) & \text{otherwise}, \end{cases}$$

where $\mathbb{T}_{\mathbb{U}}$ and $\mathbb{S}_{\mathbb{U}}$ are respectively a underlying t-norm and a underlying t-conorm of $\mathbb{U}$, $D_{\mathbb{U}}$ is an increasing function fulfilling $\min(x,y) \le D_{\mathbb{U}}(x,y) \le \max(x,y)$.

Theorem 2.3. [17] *A function* $\mathbb{U} : [0,1]^2 \to [0,1]$ *is an idempotent uninorm with the neutral element* $e \in (0,1)$ *iff there exists an Id-symmetrical decreasing function* $\mathcal{A} : [0,1] \to [0,1]$ *with* $\mathcal{A}(e) = e$ *such that*

$$\mathbb{U}(x,y) = \begin{cases} \min(x,y) & \text{if } y < \mathcal{A}(x) \text{ or } (y = \mathcal{A}(x) \text{ and } x < \mathcal{A}(\mathcal{A}(x))), \\ \max(x,y) & \text{if } y > \mathcal{A}(x) \text{ or } (y = \mathcal{A}(x) \text{ and } x > \mathcal{A}(\mathcal{A}(x))), \\ x \text{ or } y & \text{if } y = \mathcal{A}(x) \text{ and } x = \mathcal{A}(\mathcal{A}(x)), \end{cases}$$

where $\mathcal{A}$ *is called the associated function of uninorm* $\mathbb{U}$.

We know, both t-norms and uninorms are important binary aggregation operators on $[0,1]$, which are defined to be mappings $\mathbb{A}\colon [0,1]^2 \to [0,1]$ increasing in each variable and satisfying $\mathbb{A}(0,0) = 0$ and $\mathbb{A}(1,1) = 1$.

In the sequel, let us present some related concepts and results about overlap and grouping functions.

Definition 2.4. [9] *A binary operator* $\mathbb{O}\colon [0,1]^2 \to [0,1]$ *is known as an* overlap function *if, for all* $x, y \in [0,1]$, $\mathbb{O}$ *holds the following properties:*

($\mathbb{O}$1) $\mathbb{O}$ *is commutative,*

($\mathbb{O}$2) $\mathbb{O}(x,y) = 0 \Leftrightarrow \min(x,y) = 0$,

($\mathbb{O}$3) $\mathbb{O}(x,y) = 1 \Leftrightarrow \min(x,y) = 1$,

($\mathbb{O}$4) $\mathbb{O}$ *is increasing,*

($\mathbb{O}$5) $\mathbb{O}$ *is continuous.*

Definition 2.5. [8] *A binary operator* $\mathbb{G}\colon [0,1]^2 \to [0,1]$ *is known as a* grouping function *if, for all* $x, y \in [0,1]$, $\mathbb{G}$ *holds the following properties:*

($\mathbb{G}$1) $\mathbb{G}$ *is commutative,*

($\mathbb{G}$2) $\mathbb{G}(x,y) = 0 \Leftrightarrow \max(x,y) = 0$,

($\mathbb{G}$3) $\mathbb{G}(x,y) = 1 \Leftrightarrow \max(x,y) = 1$,

($\mathbb{G}$4) $\mathbb{G}$ *is increasing,*

($\mathbb{G}$5) $\mathbb{G}$ *is continuous.*

From Definitions 2.4 and 2.5, we know that overlap and grouping functions are dual. In the following, we only offer some examples of overlap functions considering the duality between the above two functions.

Example 2.6. (1) *Evidently, any continuous positive t-norm belongs to the set of overlap functions. Besides, for any* $\delta > 0$, *the function* $\mathbb{O}_\delta\colon [0,1]^2 \to [0,1]$ *given by*

$$\mathbb{O}_\delta(x,y) = x^\delta y^\delta,$$

is an overlap function. And we notice, for any $\delta \neq 1$, $\mathbb{O}_\delta$ *neither is associative nor has neutral element 1. But the overlap function* $\mathbb{O}_{mM}$ *is non-associative and has neutral element 1, where*

$$\mathbb{O}_{mM}(x,y) = \min(x,y)\max(x^2,y^2) \text{ for any } x,y \in [0,1].$$

(2) *We all know that the unique idempotent t-norm is the Minimum, i.e.,* $\mathbb{T}_M(x,y) = \min(x,y)$. *But there are infinitely many idempotent overlap functions. For instance, for any* $p, q > 0$, *the binary Bonferroni mean* $\mathbb{O}(x,y) = (\frac{x^p y^q + x^q y^p}{2})^{\frac{1}{p+q}}$ *is an idempotent overlap function.*

Lemma 2.7. [40] *If an overlap function* $\mathbb{O}$ *is idempotent and has the neutral element 1, then* $\mathbb{O} = \mathbb{T}_M$.

Lemma 2.8. [40] *If a grouping function* $\mathbb{G}$ *is idempotent and has the neutral element 0, then* $\mathbb{G} = \mathbb{S}_M$.

Definition 2.9. [12] *Suppose Λ is a countable set of indexes, $(\mathbb{O}_\lambda)_{\lambda\in\Lambda}$ are some overlap functions and $((a_\lambda, b_\lambda))_{\lambda\in\Lambda}$ are nonempty, pairwise disjoint open subintervals of $[0,1]$. The ordinal sum of $(\mathbb{O}_\lambda)_{\lambda\in\Lambda}$ is the binary function $(\langle a_\lambda, b_\lambda, \mathbb{O}_\lambda\rangle)_{\lambda\in\Lambda}\colon [0,1]^2 \to [0,1]$, defined by*

$$(\langle a_\lambda, b_\lambda, \mathbb{O}_\lambda\rangle)_{\lambda\in\Lambda}(x,y) = \begin{cases} a_\lambda + (b_\lambda - a_\lambda)\mathbb{O}_\lambda(\frac{x-a_\lambda}{b_\lambda-a_\lambda}, \frac{y-a_\lambda}{b_\lambda-a_\lambda}) & \text{if } (x,y)\in[a_\lambda,b_\lambda]^2, \\ \min(f_\Lambda(x), f_\Lambda(y)) & \text{otherwise,} \end{cases}$$

where $f_\Lambda : [0,1] \to [0,1]$ is given by

$$f_\Lambda(x) = \begin{cases} a_\lambda + (b_\lambda - a_\lambda)\mathbb{O}_\lambda(\frac{x-a_\lambda}{b_\lambda-a_\lambda}, 1) & \text{if } x\in[a_\lambda,b_\lambda] \text{ for some } \lambda\in\Lambda, \\ x & \text{otherwise.} \end{cases}$$

Similarly, we can give the ordinal sum of a family of grouping functions $(\mathbb{G}_\lambda)_{\lambda\in\Lambda}$ as a binary function $(\langle a_\lambda, b_\lambda, \mathbb{G}_\lambda\rangle)_{\lambda\in\Lambda}\colon [0,1]^2 \to [0,1]$, defined by

$$(\langle a_\lambda, b_\lambda, \mathbb{G}_\lambda\rangle)_{\lambda\in\Lambda}(x,y) = \begin{cases} a_\lambda + (b_\lambda - a_\lambda)\mathbb{G}_\lambda(\frac{x-a_\lambda}{b_\lambda-a_\lambda}, \frac{y-a_\lambda}{b_\lambda-a_\lambda}) & \text{if } (x,y)\in[a_\lambda,b_\lambda]^2, \\ \max(h_\Lambda(x), h_\Lambda(y)) & \text{otherwise,} \end{cases}$$

where $h_\Lambda : [0,1] \to [0,1]$ is given by

$$h_\Lambda(x) = \begin{cases} a_\lambda + (b_\lambda - a_\lambda)\mathbb{G}_\lambda(\frac{x-a_\lambda}{b_\lambda-a_\lambda}, 0) & \text{if } x\in[a_\lambda,b_\lambda] \text{ for some } \lambda\in\Lambda, \\ x & \text{otherwise.} \end{cases}$$

In [34], authors introduced a class of continuous aggregation operators generalizing overlap functions and presented the distributivity conditions of $\mathbb{T}_M$ over them. In what follows, we give the relevant definitions and results.

Definition 2.10. [34] *For $w\in(0,1]$, consider next binary aggregation operator $\mathbb{O}_w\colon [0,1]^2\to[0,1]$ holding the following conditions:*

($\mathbb{O}_w1$) *$\mathbb{O}_w$ is commutative,*

($\mathbb{O}_w2$) *$\mathbb{O}_w$ is increasing,*

($\mathbb{O}_w3$) *$\mathbb{O}_w$ is continuous,*

($\mathbb{O}_w4$) $\mathbb{O}_w(x,y) = 0 \Leftrightarrow xy = 0$,

($\mathbb{O}_w5$) $\mathbb{O}_w(x,y) = 1 \Leftrightarrow (x,y)\in([w,1]\times\{1\})\cup(\{1\}\times[w,1])$.

We call the function $\mathbb{O}_w$ as a weak overlap function and $w\in(0,1]$ as the weak coefficient of $\mathbb{O}_w$. Note that $\mathbb{O}_1$ is just an overlap function.

Now we give other concerned notions and results on null-uninorms, idempotent null-uninorms.

Definition 2.11. [1] *Let $\mathbb{F}$ be a binary commutative operator on $[0,1]$. Then $\{e,f\}_a$ is called the* 2-neutral element *of $\mathbb{F}$ if $\mathbb{F}(e,x) = x$ for all $x\le a$ and $\mathbb{F}(f,x) = x$ for all $x\ge a$, where $0<a<1$, $e\in[0,a]$ and $f\in[a,1]$.*

Definition 2.12. [1] *A binary operator $\mathbb{F}$ with a 2-neutral element $\{e,f\}_a$ and being commutative, associative, increasing is called a* 2-uninorm.

Definition 2.13. [49] *A* null-uninorm *$\mathbb{F}$ is a 2-uninorm with a 2-neutral element $\{0,e\}_a$ and element a as an annihilator over $[0,e]$.*

Obviously, $\mathbb{F}$ is a uninorm when $a=0$, and a t-norm when $e=1$ and $a=0$, and a t-conorm when $e=a=0$ or $e=a=1$. A null-uninorm with the 2-neutral element $\{0,e\}_a$ is *proper* when $0<a<e<1$. Moreover, $\mathbb{F}$ also satisfies $\mathbb{F}(0,a) = \mathbb{F}(0,e) = \mathbb{F}(a,e) = a$.

Lemma 2.14. [49] *If $\mathbb{F}$ is a proper null-uninorm with the 2-neutral element $\{0,e\}_a$, then $\mathbb{F}(0,1) = \mathbb{F}(a,1)\in\{a,1\}$.*

For any proper null-uninorm $\mathbb{F}$, define binary operators $\mathbb{U}_\mathbb{F}(x,y) = \frac{\mathbb{F}((1-a)x+a,(1-a)y+a)-a}{1-a}$ for any $x,y\in[0,1]$, then $\mathbb{U}_\mathbb{F}$ called the underlying uninorm of $\mathbb{F}$ is a uninorm with a neutral element $\frac{e-a}{1-a}$.

Lemma 2.15. [55] *Let $\mathbb{F}$ be a proper idempotent null-uninorm. Then $\mathbb{F}$ fulfills one of the following formulas:*

$$\mathbb{F}(x,y)=\begin{cases}\min(x,y) & \text{if } (x,y)\in[a,e]^2, \text{ or}\\ & \text{if } (x,y)\in B(e) \text{ and } y<\mathcal{A}(x), \text{ or}\\ & \text{if } (x,y)\in B(e),\ y=\mathcal{A}(x) \text{ and } x<\mathcal{A}(\mathcal{A}(x)),\\ a & \text{if } (x,y)\in[0,a]\times[a,\beta]\cup[a,\beta]\times[0,a],\\ x \text{ or } y & \text{if } (x,y)\in B(e),\ y=\mathcal{A}(x) \text{ and } x=\mathcal{A}(\mathcal{A}(x)),\\ \max(x,y) & \text{otherwise,}\end{cases}$$

$$\mathbb{F}(x,y)=\begin{cases}\min(x,y) & \text{if } (x,y)\in[a,e]^2, \text{ or}\\ & \text{if } (x,y)\in B(e) \text{ and } y<\mathcal{A}(x), \text{ or}\\ & \text{if } (x,y)\in B(e),\ y=\mathcal{A}(x) \text{ and } x<\mathcal{A}(\mathcal{A}(x)),\\ a & \text{if } (x,y)\in[0,a]\times[a,\beta)\cup[a,\beta)\times[0,a],\\ x \text{ or } y & \text{if } (x,y)\in B(e),\ y=\mathcal{A}(x) \text{ and } x=\mathcal{A}(\mathcal{A}(x)),\\ \max(x,y) & \text{otherwise,}\end{cases}$$

where $\beta=\mathcal{A}(a)$, $\mathcal{A}$ is the associated function of $\mathbb{U}_{\mathbb{F}}$ and $B(e)=[a,e)\times(e,1]\cup(e,1]\times[a,e)$.

Finally, let us display the definition of the distributivity equation between binary operators and some corresponding results.

Definition 2.16. [35] *Let $\mathbb{A},\mathbb{B}\colon[0,1]^2\to[0,1]$ be two aggregation operators. We say that $\mathbb{A}$ is* distributive *over $\mathbb{B}$, if*

$$\mathbb{A}(x,\mathbb{B}(y,z))=\mathbb{B}(\mathbb{A}(x,y),\mathbb{A}(x,z)), \text{for all } x,y,z\in[0,1]. \tag{1}$$

Lemma 2.17. [43] *Any increasing binary function $\mathbb{F}$ on $[0,1]$ is always distributive over* max *and* min.

Lemma 2.18. [34] *Let $w\in(0,1]$ and $\mathbb{O}_w$ be a weak overlap function. Then $\mathbb{T}_M$ is distributive over $\mathbb{O}_w$ iff there is a continuous and increasing unary function f on $[0,1]$ satisfying the following conditions:*

(i) $f(x)=0\Leftrightarrow x=0$,

(ii) $f(x)=1\Leftrightarrow x\geq w$,

(iii) $f(x)\geq x$ *for any* $x\in[0,1]$,

such that $\mathbb{O}_w$ has the following formula

$$\mathbb{O}_w(x,y)=\begin{cases}f(x) & \text{if } (x,y)\in[0,w)\times[f(x),1],\\ f(y) & \text{if } (x,y)\in[f(y),1]\times[0,w),\\ \max(x,y) & \text{otherwise.}\end{cases} \tag{2}$$

From Lemma 2.18, we immediately get the following distributivity conditions of $\mathbb{T}_M$ over an overlap function $\mathbb{O}$.

Corollary 2.19. *Let $\mathbb{O}$ be an overlap function. Then $\mathbb{T}_M$ is distributive over $\mathbb{O}$ iff there is a continuous and increasing unary function f on $[0,1]$ satisfying the following conditions:*

(i) $f(x)=0\Leftrightarrow x=0$,

(ii) $f(x)=1\Leftrightarrow x=1$,

(iii) $f(x)\geq x$ *for any* $x\in[0,1]$,

such that $\mathbb{O}$ satisfies the following formula

$$\mathbb{O}(x,y)=\begin{cases}f(x) & \text{if } (x,y)\in[0,1)\times[f(x),1],\\ f(y) & \text{if } (x,y)\in[f(y),1]\times[0,1),\\ \max(x,y) & \text{otherwise.}\end{cases} \tag{3}$$

Lemma 2.20. [38] *If an idempotent uninorm $\mathbb{U}$ is distributive over an overlap function $\mathbb{O}$, then $\mathbb{O}$ is idempotent.*

3 Distributivity conditions of idempotent null-uninorms over overlap functions

In this part, we will study the distributivity equations of idempotent null-uninorms whose underlying uninorms' associated functions are $\mathcal{A}(x)$ over overlap functions through considering the different cases of the values of $\mathcal{A}(1)$.

Lemma 3.1. *Suppose that $\mathbb{F}$ is a proper null-uninorm and $\mathbb{O}$ is an overlap function. If $\mathbb{F}$ is distributive over $\mathbb{O}$, then $\mathbb{O}(x,x)=x$ for any $x\in[0,a]$. Moreover, if $\mathbb{O}(e,e)=e$, then $\mathbb{O}(x,x)=x$ for any $x\in[a,1]$.*

Proof. It can be verified similarly to Lemma 3.1 in [53] □

Lemma 3.2. *Suppose that $\mathbb{F}$ is an idempotent null-uninorm and $\mathbb{O}$ is an overlap function. If $\mathbb{F}$ is distributive over $\mathbb{O}$, then $\mathbb{O}$ is idempotent.*

Proof. It is evident from Lemma 3.1 that we only need to prove that $\mathbb{O}(e,e)=e$. Assume that $\mathbb{O}(e,e)>e$, then by the continuity of $\mathbb{O}$ and $\mathbb{O}(a,a)=a$ there exists $\mu\in(a,e)$ such that $\mathbb{O}(\mu,\mu)=e$. Further, for any $x\in(a,e)$ it holds that

$$x=\mathbb{F}(x,e)=\mathbb{F}(x,\mathbb{O}(\mu,\mu))=\mathbb{O}(\mathbb{F}(x,\mu),\mathbb{F}(x,\mu))=\mathbb{O}(x,x)\ or\ \mathbb{O}(\mu,\mu).$$

If $x=\mathbb{O}(x,x)$, then it implies that $\mathbb{O}(e,e)=\lim_{x\to e^-}\mathbb{O}(x,x)=\lim_{x\to e^-}x=e$. Otherwise, $x=\mathbb{O}(\mu,\mu)=e$. Both of them are contradictive. Assume that $\mathbb{O}(e,e)<e$, then there exists $\nu\in(e,1)$ such that $\mathbb{O}(\nu,\nu)=e$. Further, for any $y\in(e,1)$ it holds that

$$y=\mathbb{F}(y,e)=\mathbb{F}(y,\mathbb{O}(\nu,\nu))=\mathbb{O}(\mathbb{F}(y,\nu),\mathbb{F}(y,\nu))=\mathbb{O}(y,y)\ or\ \mathbb{O}(\nu,\nu).$$

That will similarly give rise to a contradiction. Thus, it indicates $\mathbb{O}(e,e)=e$. Further, we get that $\mathbb{O}$ is idempotent. □

Theorem 3.3. *Let $\mathbb{F}$ be an idempotent null-uninorm with the underlying uninorm's associated function $\mathcal{A}(1)=a$ and $\mathbb{O}$ be an overlap function. Then $\mathbb{F}$ is distributive over $\mathbb{O}$ iff $\mathbb{O}=\mathbb{T}_M$.*

Proof. (⇒) If $\mathcal{A}(1)=a$, then by Theorem 2.3 there are the following three cases:

Case 1. $\mathcal{A}(1)=a$ and $\mathcal{A}(a)<1$.

Case 2. $\mathcal{A}(a)=1$ and $\mathbb{F}(a,1)=1$.

Case 3. $\mathcal{A}(a)=1$ and $\mathbb{F}(a,1)=a$.

Obviously, under case 1 or 2 we have $\mathbb{F}(0,1)=\mathbb{F}(a,1)=\max(a,1)=1$, and $\mathbb{F}(a,1)=a$ under case 3. Now we separately consider them.

Case 1 or 2: If $\mathbb{F}$ is distributive over $\mathbb{O}$, then one has that

$$y=\mathbb{F}(y,0)=\mathbb{F}(y,\mathbb{O}(1,0))=\mathbb{O}(\mathbb{F}(y,1),\mathbb{F}(y,0))=\mathbb{O}(1,y)\ for\ any\ y\in[0,a].$$

This implies that $\mathbb{O}(x,y)=\min(x,y)$ for all $x\in[0,a]$ or $y\in[0,a]$ from the monotonicity and idempotency of $\mathbb{O}$. Furthermore, we proceed to verify that $\mathbb{O}|_{[a,1]^2}=\min$. First of all, we conclude that $\mathbb{O}(1,e)=e$. Otherwise, $\mathbb{O}$ satisfies $a=\mathbb{O}(1,a)<e<\mathbb{O}(1,e)$. Then there exists $\mu\in(a,e)$ such that $\mathbb{O}(1,\mu)=e$. This will lead to the following contradiction

$$\mu=\mathbb{F}(\mu,e)=\mathbb{F}(\mu,\mathbb{O}(1,\mu))=\mathbb{O}(\mathbb{F}(\mu,1),\mathbb{F}(\mu,\mu))=\mathbb{O}(1,\mu)=e.$$

Hence, for any $x\in[a,1]$ one has that

$$x=\mathbb{F}(x,e)=\mathbb{F}(x,\mathbb{O}(1,e))=\mathbb{O}(\mathbb{F}(x,1),\mathbb{F}(x,e))=\mathbb{O}(1,x).$$

It similarly implies that $\mathbb{O}(x,y)=\min(x,y)$ for any $x,y\in[a,1]$. Therefore, $\mathbb{O}(x,y)=\min(x,y)$ for any $x,y\in[0,1]$.

Case 3. In this case we have $\mathbb{F}(1,y)=\max(1,y)=1$ for any $y\in(a,1]$. Suppose that $\mathbb{F}$ is distributive over $\mathbb{O}$, then we assert that $\mathbb{O}(1,a)=a$. On the contrary, if $\mathbb{O}(1,a)>a$, then it means that

$$1=\mathbb{F}(1,\mathbb{O}(1,a))=\mathbb{O}(\mathbb{F}(1,1),\mathbb{F}(1,a))=\mathbb{O}(1,a).$$

That shows $a=1$ from ($\mathbb{O}$3), which is a contradiction. Further, we can similarly prove $\mathbb{O}(1,e)=e$ and $\mathbb{O}(x,y)=\min(x,y)$ for all $(x,y)\in[a,1]^2$ as the previous discussion. In addition, we have the following equation

$$x=\mathbb{F}(x,0)=\mathbb{F}(x,\mathbb{O}(0,e))=\mathbb{O}(\mathbb{F}(x,0),\mathbb{F}(x,e))=\mathbb{O}(x,a)\ for\ any\ x\in[0,a].$$

This means that $\mathbb{O}(x,y)=\min(x,y)$ for any $(x,y)\in[0,a]^2$.

Now for any $x \in [0,a]$, it follows that $x = \mathbb{O}(a,x) \le \mathbb{O}(1,x) \le \mathbb{O}(1,a) = a$. Further, we get that

$$\mathbb{O}(x,1) = \mathbb{F}(0,\mathbb{O}(x,1)) = \mathbb{O}(\mathbb{F}(0,x),\mathbb{F}(0,1)) = \mathbb{O}(x,a) = x,$$

which means $\mathbb{O}(x,y) = \min(x,y)$ for all $(x,y) \in [0,a] \times [a,1] \cup [a,1] \times [0,a]$.

To sum up, we also obtain that $\mathbb{O}(x,y) = \min(x,y)$ for any $x, y \in [0,1]$.

($\Leftarrow$) The reverse is clear from Lemma 2.17. □

What as follows, we will explore the related distributivity conditions under the circumstance of $\mathcal{A}(1) > a$, which shows that the overlap function is no longer trivial, but has the form of an ordinal sum.

Theorem 3.4. *Let $\mathbb{F}$ be an idempotent null-uninorm with the associated function $\gamma = \mathcal{A}(1) > a$ and $\mathbb{O}$ be an overlap function. Then $\mathbb{F}$ is distributive over $\mathbb{O}$ iff $\mathbb{O} = (\langle a, \gamma, \mathbb{O}_a\rangle)$, i.e., $\mathbb{O}$ has the form*

$$\mathbb{O}(x,y) = \begin{cases} a + (\gamma - a)\mathbb{O}_a(\frac{x-a}{\gamma-a}, \frac{y-a}{\gamma-a}) & \text{if } (x,y) \in [a,\gamma]^2, \\ a + (\gamma - a)\mathbb{O}_a(\frac{\min(x,y)-a}{\gamma-a}, 1) & \text{if } (x,y) \in [a,\gamma] \times [\gamma,1] \cup [\gamma,1] \times [a,\gamma], \\ \min(x,y) & \text{otherwise,} \end{cases} \tag{4}$$

where $\mathbb{O}_a$ fulfills Eq. (2) when $\mathbb{F}(\gamma,1) = \gamma$ or Eq. (3) when $\mathbb{F}(\gamma,1) = 1$.

Proof. ($\Rightarrow$) If $\gamma = \mathcal{A}(1) > a$, then $\mathbb{F}(1,a) = a$, but $\mathbb{F}(1,\gamma) = \gamma$ or 1. Hence, we consider the following two cases.

Case 1. When $\mathbb{F}(1,\gamma) = \gamma$, then it implies that $a < \gamma < e$ and $\mathbb{F}(1,y) = \max(1,y) = 1$ for any $y \in (\gamma, 1]$. Now we conclude that $\mathbb{O}(1,\gamma) = \gamma$. Otherwise, if $\mathbb{O}(1,\gamma) > \gamma$, then we have that

$$1 = \mathbb{F}(1,\mathbb{O}(1,\gamma)) = \mathbb{O}(\mathbb{F}(1,1),\mathbb{F}(1,\gamma)) = \mathbb{O}(1,\gamma),$$

which leads to $\gamma = 1$ from ($\mathbb{O}$3). That is contradictive with $\gamma < e$. Further, we continue to claim that $\mathbb{O}(1,e) = e$. Conversely, it implies that $\mathbb{O}(1,\gamma) < e < \mathbb{O}(1,e)$ and there exists $\mu \in (\gamma, e)$ such that $\mathbb{O}(1,\mu) = e$. But this will result in the following contradiction

$$\mu = \mathbb{F}(\mu,e) = \mathbb{F}(\mu,\mathbb{O}(1,\mu)) = \mathbb{O}(\mathbb{F}(\mu,1),\mathbb{F}(\mu,\mu)) = \mathbb{O}(1,\mu) = e.$$

Further, for any $y \in (\gamma,1)$, it shows that

$$y = \mathbb{F}(y,e) = \mathbb{F}(y,\mathbb{O}(1,e)) = \mathbb{O}(\mathbb{F}(y,1),\mathbb{F}(y,e)) = \mathbb{O}(1,y).$$

Thus, it implies that $\mathbb{O}(x,y) = \min(x,y)$ for any $(x,y) \in [\gamma,1]^2$. Moreover, for any $y \in [a,\gamma]$ we get that

$$\mathbb{O}(1,y) = \min(\gamma,\mathbb{O}(1,y)) = \mathbb{F}(\gamma,\mathbb{O}(1,y)) = \mathbb{O}(\mathbb{F}(\gamma,1),\mathbb{F}(\gamma,y)) = \mathbb{O}(\gamma,y).$$

Therefore, it means that $\mathbb{O}(x,y) = \mathbb{O}(\min(x,y),1)$ for any $(x,y) \in [a,\gamma] \times [\gamma,1] \cup [\gamma,1] \times [a,\gamma]$.

Case 2. When $\mathbb{F}(1,\gamma) = 1$, then it means that $\mathbb{F}(1,y) = 1$ for any $y \in [\gamma,1]$, $\mathbb{F}(1,y) = y$ for any $y \in (a,\gamma)$ and $\mathbb{F}(1,y) = a$ for any $y \in [0,a]$. At first, we have

$$a = \mathbb{F}(1,\mathbb{O}(1,0)) = \mathbb{O}(\mathbb{F}(1,1),\mathbb{F}(1,0)) = \mathbb{O}(1,a).$$

Based on this, we conclude that $\mathbb{O}(1,\gamma) = \gamma$. Otherwise, it means that $\mathbb{O}(1,a) < \gamma < \mathbb{O}(1,\gamma)$ and there is $\nu \in (a,\gamma)$ such that $\gamma = \mathbb{O}(1,\nu)$. Then it follows that

$$1 = \mathbb{F}(1,\gamma) = \mathbb{F}(1,\mathbb{O}(1,\nu)) = \mathbb{O}(\mathbb{F}(1,1),\mathbb{F}(1,\nu)) = \mathbb{O}(1,\nu) = \gamma,$$

which is a contradiction. If $\gamma = e$, then it shows $\mathbb{O}(1,e) = e$. Otherwise, $\gamma < e$, then in a way similar to case 1. we have $\mathbb{O}(1,e) = e$ and $\mathbb{O}(x,y) = \min(x,y)$ for any $(x,y) \in [\gamma,1]^2$. Furthermore, for any $x \in [a,\gamma)$, we get the following equation

$$\mathbb{F}(1,\mathbb{O}(x,\gamma)) = \mathbb{O}(\mathbb{F}(1,x),\mathbb{F}(1,\gamma)) = \mathbb{O}(x,1). \tag{5}$$

From Eq. (5) and $\mathbb{O}(x,\gamma) \le \mathbb{O}(\gamma,\gamma) = \gamma$, we claim that $\mathbb{O}(x,\gamma) < \gamma$. Suppose that $\mathbb{O}(x,\gamma) = \gamma$, then by Eq. (5) it means that $\mathbb{O}(x,1) = \mathbb{F}(1,\gamma) = 1$. Further, it implies that $x = 1$, which is a contradiction. Therefore, we obtain from Eq. (5) that $\mathbb{O}(x,\gamma) = \mathbb{O}(x,1)$ for any $x \in [a,\gamma]$. Namely, in this case we also have that $\mathbb{O}(x,y) = \mathbb{O}(\min(x,y),1)$ for any $(x,y) \in [a,\gamma] \times [\gamma,1] \cup [\gamma,1] \times [a,\gamma]$.

Now we begin to determine the structure of $\mathbb{O}$ on the domain $[a,\gamma]^2$. Let us consider a function $\varphi(x)=\frac{x-a}{\gamma-a}$, then for any $(x,y)\in[0,1]$ we can get two binary functions $\mathbb{T}_a(x,y)=\varphi(\mathbb{F}(\varphi^{-1}(x),\varphi^{-1}(y)))=\frac{\mathbb{F}((\gamma-a)x+a,(\gamma-a)y+a)-a}{\gamma-a}$ which obviously satisfies $\mathbb{T}_a=\mathbb{T}_M$, and $\mathbb{O}_a(x,y)=\frac{\mathbb{O}((\gamma-a)x+a,(\gamma-a)y+a)-a}{\gamma-a}$. Moreover, one has that $\mathbb{O}_a(x,y)=0\Leftrightarrow x=0$ or $y=0$ because from $\mathbb{O}(a,\gamma)=\mathbb{O}(a,1)=a=\mathbb{O}(a,a)$ we have $\mathbb{O}(a,y)=a$ for any $y\in[a,\gamma]$ and $\mathbb{O}(x,y)\geq\mathbb{O}(\min(x,y),\min(x,y))=\min(x,y)>a$ for any $x,y\in(a,\gamma]$. Further, this shows that $\mathbb{O}_a$ is a weak overlap function when $\mathbb{F}(1,\gamma)=\gamma$ due to $\mathbb{O}_a(1,1)=1$. But when $\mathbb{F}(1,\gamma)=1$, then $\mathbb{O}_a$ is an overlap function since $\mathbb{O}(x,\gamma)<\gamma$ for any $x\in[a,\gamma)$ which means $\mathbb{O}_a$ satisfies $\mathbb{O}_a(x,y)=1\Leftrightarrow x=y=1$.

Obviously, if $\mathbb{F}$ is distributive over $\mathbb{O}$, then $\mathbb{F}|[a,\gamma]^2$ is distributive over $\mathbb{O}|[a,\gamma]^2$. Thus, from Lemma 2.18 and Corollary 2.19, it implies that $\mathbb{O}_a$ fulfills Eq. (2) when $\mathbb{F}(1,\gamma)=\gamma$ or Eq. (3) when $\mathbb{F}(1,\gamma)=1$, respectively. In the end, we can also obtain that $\mathbb{O}(x,y)=\min(x,y)$ for any $(x,y)\in[0,1]^2\backslash[a,1]^2$ which is similar to Theorem 3.3. Hence, $\mathbb{O}$ has the form in Eq. (4).

($\Leftarrow$) On the contrary, if $\mathbb{O}$ satisfies Eq. (4), the sufficiency is easy to verify. So we omit it.

□

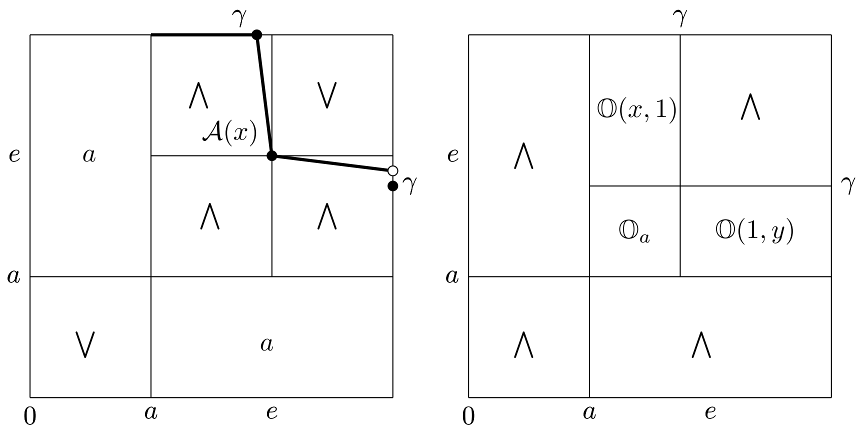

Figure 1: Structures of $\mathbb{F}$ (left) and $\mathbb{O}$ (right) in Theorem 3.4.

Example 3.5. *Consider the following idempotent null-uninorm $\mathbb{F}$ with $a=\frac{1}{3}$, $e=\frac{2}{3}$:*

$$\mathbb{F}(x,y)=\begin{cases}\min(x,y) & \text{if } (x,y)\in[\frac{1}{3},\frac{2}{3}]^2, \text{ or}\\ & \text{if } (x,y)\in B(\frac{2}{3}) \text{ and } y<\mathcal{A}(x), \text{ or}\\ & \text{if } (x,y)\in B(\frac{2}{3}),\ y=\mathcal{A}(x) \text{ and } x\leq\mathcal{A}(\mathcal{A}(x)),\\ \frac{1}{3} & \text{if } (x,y)\in[0,\frac{1}{3}]\times[\frac{1}{3},1]\cup[\frac{1}{3},1]\times[0,\frac{1}{3}],\\ \max(x,y) & \text{otherwise,}\end{cases}$$

where $B(\frac{2}{3})=[\frac{1}{3},\frac{2}{3}]\times[\frac{2}{3},1]\cup[\frac{2}{3},1]\times[\frac{1}{3},\frac{2}{3}]$ and its associated function with $\mathcal{A}(1)=\frac{1}{2}$ has the form

$$\mathcal{A}(x)=\begin{cases}1 & \text{if } x\in[\frac{1}{3},\frac{3}{5}),\\ 4-5x & \text{if } x\in[\frac{3}{5},\frac{2}{3}),\\ \frac{4}{5}-\frac{1}{5}x & \text{if } x\in[\frac{2}{3},1),\\ \frac{1}{2} & \text{if } x=1.\end{cases}$$

Let $\mathbb{O}$ be an overlap function satisfying

$$\mathbb{O}(x,y)=\begin{cases}\frac{1}{3}+\frac{1}{6}\mathbb{O}_a(6x-2,6y-2) & \text{if } (x,y)\in[\frac{1}{3},\frac{1}{2}]^2,\\ \frac{1}{3}+\frac{1}{6}\mathbb{O}_a(6\min(x,y)-2,1) & \text{if } (x,y)\in[\frac{1}{3},\frac{1}{2}]\times[\frac{1}{2},1]\cup[\frac{1}{2},1]\times[\frac{1}{3},\frac{1}{2}],\\ \min(x,y) & \text{otherwise,}\end{cases}$$

where $\mathbb{O}_a$ is a weak overlap function with the weak coefficient $w=\frac{2}{3}$ and the form

$$\mathbb{O}_a(x,y)=\begin{cases}\sqrt{\frac{3}{2}x} & \text{if } (x,y)\in[0,\frac{2}{3}]\times[\sqrt{\frac{3}{2}x},1],\\ \sqrt{\frac{3}{2}y} & \text{if } (x,y)\in[\sqrt{\frac{3}{2}y},1]\times[0,\frac{2}{3}],\\ \max(x,y) & \text{otherwise.}\end{cases}$$

Then from Theorem 3.4 we have that $\mathbb{F}$ is distributive over $\mathbb{O}$.

4 Distributivity conditions of idempotent null-uninorms over grouping functions

In this section, we will consider the distributivity conditions of idempotent null-uninorms over grouping functions. Before doing that, we need to define a new class of continuous aggregation operators called weak grouping functions which generalize grouping functions and present the distributivity of $\mathbb{S}_M$ over them. Besides, we need to give the distributivity solutions of an idempotent uninorm over this class of functions and utilize the results.

Definition 4.1. *Let $w \in [0,1)$, consider next binary aggregation operator $\mathbb{G}_w\colon [0,1]^2 \to [0,1]$ satisfying the following conditions:*

($\mathbb{G}_w1$) $\mathbb{G}_w$ *is commutative,*

($\mathbb{G}_w2$) $\mathbb{G}_w$ *is increasing,*

($\mathbb{G}_w3$) $\mathbb{G}_w$ *is continuous,*

($\mathbb{G}_w4$) $\mathbb{G}_w(x,y) = 1 \Leftrightarrow x = 1$ *or* $y = 1$,

($\mathbb{G}_w5$) $\mathbb{G}_w(x,y) = 0 \Leftrightarrow (x,y) \in ([0,w] \times \{0\}) \cup (\{0\} \times [0,w])$.

We call the function $\mathbb{G}_w$ as a weak grouping function and $w \in [0,1)$ the weak coefficient of $\mathbb{G}_w$. Note that $\mathbb{G}_0$ is just a grouping function. Similarly, according to the duality between overlap and grouping functions, we know the weak grouping functions $\mathbb{G}_w$ are also dual to the weak overlap functions $\mathbb{O}_{1-w}$ for $w \in [0,1)$. Thus, in the following, the related results on the weak grouping functions which are dual to the weak overlap functions will be omitted.

Proposition 4.2. *Let $w \in [0,1)$ and $\mathbb{G}_w$ be a weak grouping function. Then $\mathbb{S}_M$ is distributive over $\mathbb{G}_w$ iff there is a continuous and increasing function $h\colon [0,1] \to [0,1]$ satisfying the following conditions:*
(1) $h(x) = 1 \Leftrightarrow x = 1$,
(2) $h(x) = 0 \Leftrightarrow x \le w$,
(3) $h(x) \le x$ *for any* $x \in [0,1]$, *such that* $\mathbb{G}_w$ *has the following formula*

$$\mathbb{G}_w(x,y) = \begin{cases} h(x) & \text{if } (x,y) \in (w,1] \times [0,h(x)], \\ h(y) & \text{if } (x,y) \in [0,h(y)] \times (w,1], \\ \min(x,y) & \text{otherwise.} \end{cases} \tag{6}$$

Proof. It can be verified similarly to Lemma 2.18 by virtue of the duality between $\mathbb{G}_w$ and $\mathbb{O}_{1-w}$. So we omit it. □

Naturally, when putting $w = 0$ in Proposition 4.2, we get next distributivity conditions of $\mathbb{S}_M$ over a grouping function.

Corollary 4.3. *Let $\mathbb{G}$ be a grouping function. Then $\mathbb{S}_M$ is distributive over $\mathbb{G}$ iff there exists a continuous and increasing function h satisfying the following conditions:*
(1) $h(x) = 1 \Leftrightarrow x = 1$,
(2) $h(x) = 0 \Leftrightarrow x = 0$,
(3) $h(x) \le x$ *for any* $x \in [0,1]$, *such that* $\mathbb{G}$ *has the following formula*

$$\mathbb{G}(x,y) = \begin{cases} h(x) & \text{if } (x,y) \in (0,1] \times [0,h(x)], \\ h(y) & \text{if } (x,y) \in [0,h(y)] \times (0,1], \\ \min(x,y) & \text{otherwise.} \end{cases} \tag{7}$$

Lemma 4.4. *Suppose that $\mathbb{F}$ is a proper null-uninorm and $\mathbb{G}$ is a grouping function. If $\mathbb{F}$ is distributive over $\mathbb{G}$, then $\mathbb{G}(x,x) = x$ for any $x \in [0,a]$. Moreover, if $\mathbb{G}(e,e) = e$, then $\mathbb{G}(x,x) = x$ for any $x \in [a,1]$.*

Proof. The proof is similar to Lemma 3.1. □

Lemma 4.5. *Suppose that $\mathbb{F}$ is an idempotent null-uninorm and $\mathbb{G}$ is a grouping function. If $\mathbb{F}$ is distributive over $\mathbb{G}$, then $\mathbb{G}$ is idempotent.*

Proof. It is similar to Lemma 3.2. So we also omit it. □

Theorem 4.6. *Let $\mathbb{F}$ be an idempotent null-uninorm and $\mathbb{G}$ be a grouping function. If $\mathbb{F}$ is distributive over $\mathbb{G}$, then the following statements hold:*

(1) $\mathbb{G}|_{[0,a]^2}$ *is a grouping function when* $\mathbb{G}(0,a) = a$.

(2) $\mathbb{G}|_{[a,1]^2}$ *is a weak grouping function.*

Proof. (1) If $\mathbb{F}$ is distributive over $\mathbb{G}$, then $\mathbb{G}(a,a) = a$. Now through the linear transformation ϕ: $[0,a] \to [0,1]$ given by $\phi(x) = \frac{x}{a}$ we can define a binary function $\mathbb{G}_a(x,y) = \phi(\mathbb{G}(\phi^{-1}(x), \phi^{-1}(y))) = \frac{\mathbb{G}(ax,ay)}{a}$ for any $x, y \in [0,1]$. Evidently $\mathbb{G}_a$ is commutative, increasing, continuous and satisfies ($\mathbb{G}2$) because $\mathbb{G}_a(x,y) = 0 \Leftrightarrow x = y = 0$ which is equivalent to $\mathbb{G}(x,y) = 0 \Leftrightarrow x = y = 0$. Thus, we only need to check that $\mathbb{G}_a$ holds ($\mathbb{G}3$). At first, from $\mathbb{G}(0,a) = a = \mathbb{G}(a,a)$, we have that $\mathbb{G}(x,a) = a$ for any $x \in [0,a]$, which means that $\mathbb{G}_a(x,y) = 1$ when $x = 1$ or $y = 1$. In addition, for any $x, y \in [0,a)$ it follows that $\mathbb{G}(x,y) \le \mathbb{G}(\max(x,y), \max(x,y)) = \max(x,y) < a$, which means that $x = 1$ or $y = 1$ when $\mathbb{G}_a(x,y) = 1$. This shows that $\mathbb{G}_a$ satisfies the definition of a grouping function.

(2) If $\mathbb{F}$ is distributive over $\mathbb{G}$, we can define the function $\mathbb{G}_b(x,y) = \frac{\mathbb{G}(a+(1-a)x, a+(1-a)y)-a}{1-a}$ for any $x, y \in [0,1]$, and obviously $\mathbb{G}_b$ is commutative, increasing and continuous. Moreover, we have that $\mathbb{G}_b(x,y) = 1 \Leftrightarrow x = 1$ or $y = 1$ since $\mathbb{G}(x,y) = 1 \Leftrightarrow x = 1$ or $y = 1$. In the following, we need to prove that there exists $w \ge 0$ such that $\mathbb{G}_b$ satisfies ($\mathbb{G}_w5$). In fact, since $\mathbb{G}(x,y) \ge \mathbb{G}(\min(x,y), \min(x,y)) = \min(x,y) > a$ for any $x, y \in (a,1]$, this implies that $\mathbb{G}(x,y) > a$ for any $x, y \in (a,1]$, namely, if $\mathbb{G}_b(x,y) = 0$, then $x = 0$ or $y = 0$. Moreover, it is evident that $\mathbb{G}_b(0,0) = 0$ due to $\mathbb{G}(a,a) = a$. Therefore, $\mathbb{G}_b$ is a weak grouping function, i.e., $\mathbb{G}|_{[a,1]^2}(x,y) = a + (1-a)\mathbb{G}_b(x,y)$ is also a weak grouping function. □

Remark 4.7. *We notice from Theorem 4.6 that if an idempotent null-uninorm is distributive over a grouping function, then $\mathbb{G}|_{[0,a]^2}$ is a grouping function under the condition $\mathbb{G}(0,a) = a$ and $\mathbb{G}|_{[a,1]^2}$ is a weak grouping function. But we find that both of them must be grouping functions and $\mathbb{G}$ has the form of an ordinal sum when we distinguish the different forms of the underlying uninorm's associated functions $\mathcal{A}$ of idempotent null-uninorm. Specific results can be seen in Theorems 4.8 and 4.12. In the sequel, we discuss the distributivity conditions through distinguishing the different values of the associated function on interval endpoint $x = a$.*

Theorem 4.8. *Let $\mathbb{F}$ be an idempotent null-uninorm with the associated function $\mathcal{A}(a) = 1$ and $\mathbb{G}$ is a grouping function. Then $\mathbb{F}$ is distributive over $\mathbb{G}$ iff $\mathbb{G} = (\langle 0, a, \mathbb{G}_a \rangle)$, i.e., $\mathbb{G}$ has the form*

$$\mathbb{G}(x,y) = \begin{cases} a\mathbb{G}_a(\frac{x}{a}, \frac{y}{a}) & \text{if } (x,y) \in [0,a]^2, \\ \max(x,y) & \text{otherwise}, \end{cases} \tag{8}$$

where $\mathbb{G}_a$ is a grouping function satisfying Eq. (7).

Proof. ($\Rightarrow$) If $\mathcal{A}(a) = 1$, then it implies that $\mathbb{F}(x,a) = \min(x,a) = a$ for any $a \le x < 1$. Further, we get that $\mathbb{F}(0,x) = a$ for any $x \in [a,1)$ since $a = \mathbb{F}(0,a) \le \mathbb{F}(0,x) \le \mathbb{F}(a,x) = a$. Now, basing on Theorem 4.6, we will prove that $\mathbb{G}|_{[0,a]^2}$ and $\mathbb{G}|_{[a,1]^2}$ are both a grouping function. At first, we claim that $\mathbb{G}(0,e) \ge a$. Otherwise, from $\mathbb{G}(0,e) < a < 1 = \mathbb{G}(0,1)$ there is some $\mu \in (e,1)$ such that $\mathbb{G}(0,\mu) = a$. Then it follows that

$$a = \mathbb{F}(0,a) = \mathbb{F}(0, \mathbb{G}(0,\mu)) = \mathbb{G}(\mathbb{F}(0,0), \mathbb{F}(0,\mu)) = \mathbb{G}(0,a).$$

This implies that $\mathbb{G}(0,a) = a > \mathbb{G}(0,e)$, which is contradicts $\mathbb{G}(0,a) \le \mathbb{G}(0,e)$. Thus, we obtain that

$$\mathbb{G}(0,a) = \mathbb{G}(0, \mathbb{F}(0,e)) = \mathbb{G}(\mathbb{F}(0,0), \mathbb{F}(0,e)) = \mathbb{F}(0, \mathbb{G}(0,e)) \ge \mathbb{F}(0,a) = a.$$

Therefore, combining with $\mathbb{G}(0,a) \le \mathbb{G}(a,a) = a$, we have that $\mathbb{G}(0,a) = a$. Further, we can get $\mathbb{G}|_{[0,a]^2}$ is a grouping function by Theorem 4.6 (1).

In the following, we continue to verify that $\mathbb{G}|_{[a,1]^2}$ is also a grouping function. Obviously, we only need to deduce that the function $\mathbb{G}_b(x,y) = \frac{\mathbb{G}((1-a)x+a, (1-a)y+a)-a}{1-a} = 0$ implies $x = y = 0$ because $\mathbb{G}_b$ has been proved to be a weak grouping function in Theorem 4.6. That is, we need to verify that for any $x, y \in [0,1]$ $\mathbb{G}((1-a)x + a, (1-a)y + a) = a$ implies $x = y = 0$, which is equivalent to prove that $\mathbb{G}(a,y) > a$ for any $y > a$. Firstly, we claim that $\mathbb{G}(a,e) = e$. On the contrary, if $\mathbb{G}(a,e) < e$, then there exists $\nu \in (e,1)$ such that $\mathbb{G}(a,\nu) = e$. It follows that

$$\nu = \mathbb{F}(\nu,e) = \mathbb{F}(\nu, \mathbb{G}(a,\nu)) = \mathbb{G}(\mathbb{F}(\nu,a), \mathbb{F}(\nu,\nu)) = \mathbb{G}(a,\nu) = e,$$

which is a contradiction. Further, for any $y \in (a,1]$, it shows that

$$\mathbb{G}(a,y) = \mathbb{G}(\mathbb{F}(y,a), \mathbb{F}(y,e)) = \mathbb{F}(y, \mathbb{G}(a,e)) = \mathbb{F}(y,e) = y > a. \tag{9}$$

To sum up, $\mathbb{G}|_{[a,1]^2}$ is also an idempotent grouping function. Further, it also means from Eq. (9) that for any $y \in [0,1]$

$$\mathbb{G}_b(0,y) = \frac{\mathbb{G}(a,(1-a)y+a)-a}{1-a} = \frac{a+(1-a)y-a}{1-a} = y.$$

Namely, $\mathbb{G}_b$ is an idempotent grouping function with the neutral element 0. Thus, we have $\mathbb{G}_b = \mathbb{S}_M$ by Lemma 2.8.

In the end, let us determine the structure of $\mathbb{G}$ on the rest domain. For any $x \in [a,1]$, one has that $\mathbb{G}(x,0) \geq \mathbb{G}(a,0) = a$ and

$$\mathbb{G}(x,0) = \mathbb{F}(e,\mathbb{G}(x,0)) = \mathbb{G}(\mathbb{F}(e,x),\mathbb{F}(e,0)) = \mathbb{G}(x,a).$$

Hence, we get that for any $(x,y) \in [a,1] \times [0,a]$

$$\mathbb{G}(x,y) = \mathbb{G}(x,a) = a+(1-a)\mathbb{G}_b(\frac{x-a}{1-a},0) = a+(1-a)\mathbb{S}_M(\frac{x-a}{1-a},0) = a+(1-a)\frac{x-a}{1-a} = x = \max(x,y).$$

In a word, $\mathbb{G}$ has the form of Eq. (8), which obviously indicates $\mathbb{G} = (\langle 0,a,\mathbb{G}_a\rangle)$ according to Definition 2.9.

($\Leftarrow$) On the contrary, if $\mathbb{G}$ has the formula of Eq. (8), then it is easy to verify that $\mathbb{F}$ is distributive over $\mathbb{G}$. □

We notice from Theorem 4.8 that $\mathbb{G}|_{[a,1]^2} = \mathbb{S}_M$ when $\mathcal{A}(a) = 1$. But we find that the result will be different when $\mathcal{A}(a) < 1$. On the other hand, since $\mathbb{F}|_{[a,1]^2}$ is just an idempotent uninorm, before discussing the distributive conditions of $\mathbb{F}$ over $\mathbb{G}$ under this case we need to consider the connected result of an idempotent uninorm over a weak grouping function under the condition $\mathcal{A}(0) < 1$.

Lemma 4.9. *Suppose that $\mathbb{U}$ is an idempotent uninorm and $\mathbb{G}_w$ is a weak grouping function with the weak coefficient $w \in [0,1)$. If $\mathbb{U}$ is distributive over $\mathbb{G}_w$, then $\mathbb{G}_w$ is idempotent.*

Proof. The proof is similar to Lemma 2.20. □

Lemma 4.10. *Let $\mathbb{U}$ be an idempotent uninorm with its associated function $\alpha = \mathcal{A}(0) < 1$ and $\mathbb{G}_w$ be a weak grouping function with $w \in [0,1)$. Then $\mathbb{U}$ is distributive over $\mathbb{G}_w$ iff $w = 0$ and $\mathbb{G}_w$ fulfills the form*

$$\mathbb{G}_w(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [0,\alpha]^2, \\ \alpha + (1-\alpha)\mathbb{G}_\psi(\frac{x-\alpha}{1-\alpha},\frac{y-\alpha}{1-\alpha}) & \text{if } (x,y) \in [\alpha,1]^2, \\ \alpha + (1-\alpha)\mathbb{G}_\psi(\frac{\max(x,y)-\alpha}{1-\alpha},0) & \text{otherwise,} \end{cases} \tag{10}$$

where $\mathbb{G}_\psi$ is a weak grouping function fulfilling Eq. (6) when $\mathbb{U}(0,\alpha) = \alpha$, or a grouping function fulfilling Eq. (7) when $\mathbb{U}(0,\alpha) = 0$.

Proof. ($\Rightarrow$) If $\mathbb{U}$ is distributive over $\mathbb{G}_w$, then we firstly conclude that $w < \alpha$. Assume that $w \geq \alpha$. Let us divide the following two cases. When $\mathbb{U}(0,\alpha) = \alpha$, then from $0 = \mathbb{G}_w(0,w) < e < \mathbb{G}_w(0,1)$ and the continuity of $\mathbb{G}_w$ there exists $\mu \in (w,1)$ such that $e = \mathbb{G}_w(0,\mu)$. But that will lead to the following contradiction

$$0 = \mathbb{U}(0,e) = \mathbb{U}(0,\mathbb{G}_w(0,\mu)) = \mathbb{G}_w(\mathbb{U}(0,0),\mathbb{U}(0,\mu)) = \mathbb{G}_w(0,\mu) = e.$$

When $\mathbb{U}(0,\alpha) = 0$, it will similarly result in the contradiction. Thus, it must be $w < \alpha$. In the following, we will further get $w = 0$ and the specific structures of $\mathbb{G}_w$.

Case 1. When $\mathbb{U}(0,\alpha) = \alpha$, then we have that $\alpha > e$ and $0 = \mathbb{G}_w(0,w) < \mathbb{G}_w(0,\alpha) \leq \mathbb{G}_w(\alpha,\alpha) = \alpha$. Further, we claim that $\mathbb{G}_w(0,\alpha) = \alpha$. Otherwise, if $\mathbb{G}_w(0,\alpha) < \alpha$, then it follows that

$$0 = \mathbb{U}(0,\mathbb{G}_w(0,\alpha)) = \mathbb{G}_w(\mathbb{U}(0,0),\mathbb{U}(0,\alpha)) = \mathbb{G}_w(0,\alpha) > 0,$$

which is a contradiction. Further, for all $x \in [\alpha,1]$ it holds that

$$\mathbb{G}_w(0,x) = \max(\alpha,\mathbb{G}_w(0,x)) = \mathbb{U}(\alpha,\mathbb{G}_w(0,x)) = \mathbb{G}_w(\mathbb{U}(\alpha,0),\mathbb{U}(\alpha,x)) = \mathbb{G}_w(\alpha,x).$$

Hence, it implies that $\mathbb{G}_w(x,y) = \mathbb{G}_w(\max(x,y),\alpha)$ for all $(x,y) \in [0,\alpha] \times [\alpha,1] \cup [\alpha,1] \times [0,\alpha]$ by the commutativity and monotonicity of $\mathbb{G}_w$. In addition, we also assert that $\mathbb{G}_w(0,e) = e$. Otherwise, from $\mathbb{G}_w(0,e) < e < \alpha = \mathbb{G}_w(0,\alpha)$ there is some $\nu \in (e,\alpha)$ such that $e = \mathbb{G}_w(0,\nu)$. And this will result in the contradiction

$$\nu = \mathbb{U}(\nu,e) = \mathbb{U}(\nu,\mathbb{G}_w(0,\nu)) = \mathbb{G}_w(\mathbb{U}(\nu,0),\mathbb{U}(\nu,\nu)) = \mathbb{G}_w(0,\nu) = e.$$

Therefore, for any $x \in [0, \alpha)$ we have that

$$x = \mathbb{U}(x, e) = \mathbb{U}(x, \mathbb{G}_w(0, e)) = \mathbb{G}_w(\mathbb{U}(x, 0), \mathbb{U}(x, e)) = \mathbb{G}_w(0, x).$$

Hence, we get that $\mathbb{G}_w(0, x) = x$ for all $x \in [0, \alpha]$. This shows that $\mathbb{G}_w(x, y) = \max(x, y)$ for all $(x, y) \in [0, \alpha]^2$.

Case 2. When $\mathbb{U}(0, \alpha) = 0$, then for any $x \in (\alpha, 1]$ we have that

$$\mathbb{U}(0, \mathbb{G}_w(x, \alpha)) = \mathbb{G}_w(\mathbb{U}(0, x), \mathbb{U}(0, \alpha)) = \mathbb{G}_w(x, 0). \tag{11}$$

From $\mathbb{G}_w(x, \alpha) \geq \mathbb{G}_w(\alpha, \alpha) = \alpha$ and Eq. (11), we claim that $\mathbb{G}_w(x, \alpha) > \alpha$. Otherwise, according to $x > \alpha > w$ it implies that $\mathbb{G}_w(0, x) > 0$ and Eq. (11) becomes

$$0 = \mathbb{U}(0, \alpha) = \mathbb{U}(0, \mathbb{G}_w(x, \alpha)) = \mathbb{G}_w(x, 0) > 0,$$

which is a contradiction. Thus, from Eq. (11) we obtain that $\mathbb{G}_w(x, \alpha) = \mathbb{U}(0, \mathbb{G}_w(x, \alpha)) = \mathbb{G}_w(x, 0)$ for any $x \in (\alpha, 1]$. Of course, by the continuity of $\mathbb{G}_w$ we also have $\mathbb{G}_w(\alpha, 0) = \mathbb{G}_w(\alpha, \alpha) = \alpha$. Further, we can similarly get $\mathbb{G}_w(0, e) = e$ and $\mathbb{G}|_{[0,a]^2} = \mathbb{S}_M$ as case 1. In a word, $\mathbb{G}_w$ has the same structure on the domain $[0, 1]^2 \setminus [\alpha, 1]^2$ as case 1.

The above results in two cases indicate that $\mathbb{G}_w(x, y) = 0 \Leftrightarrow x = y = 0$ because of $\mathbb{G}_w|_{[0,\alpha]^2} = \mathbb{S}_M$, which means $w = 0$, i.e., $\mathbb{G}_w$ is actually a grouping function. Now let us determine the structure of $\mathbb{G}_w$ on the domain $[\alpha, 1]^2$. Define a function ψ: $[\alpha, 1] \to [0, 1]$ given by $\psi(x) = \frac{x-\alpha}{1-\alpha}$. Then we get two binary functions $\mathbb{S}_\psi(x, y) = \psi(\mathbb{U}(\psi^{-1}(x), \psi^{-1}(y)))$ which obviously satisfies $\mathbb{S}_\psi = \mathbb{S}_M$ and $\mathbb{G}_\psi(x, y) = \psi(\mathbb{G}_w(\psi^{-1}(x), \psi^{-1}(y)))$ for any $x, y \in [0, 1]$. Moreover, $\mathbb{G}_\psi$ is a weak grouping function when $\mathbb{U}(0, \alpha) = \alpha$. But when $\mathbb{U}(0, \alpha) = 0$, $\mathbb{G}_\psi$ must be a grouping function because in this case we have $\mathbb{G}_\psi(x, y) = \frac{\mathbb{G}_w(\alpha+(1-\alpha)x, \alpha+(1-\alpha)y)-\alpha}{1-\alpha} = 0 \Leftrightarrow x = y = 0$ from $\mathbb{G}_w(x, \alpha) > \alpha$ for any $x \in (\alpha, 1]$. Moreover, if $\mathbb{U}$ is distributive over $\mathbb{G}_w$, then $\mathbb{S}_\psi$ must be distributive over $\mathbb{G}_\psi$.

Thus, relying on the above results and Lemma 2.18 and Corollary 2.19, we obtain that $\mathbb{G}_w$ has the form in Eq. (10), where $\mathbb{G}_\psi$ fulfills Eq. (6) when $\mathbb{U}(0, \alpha) = \alpha$, or Eq. (7) when $\mathbb{U}(0, \alpha) = 0$.

($\Leftarrow$) On the contrary, if $\mathbb{G}$ has the form in Eq. (10), since the commutativity of $\mathbb{U}$ and $\mathbb{G}$, without loss of generality, let us assume that $x \leq y \leq z$ for any $x, y, z \in [0, 1]$.

- If $0 \leq x \leq y \leq z \leq \alpha$, then it is evident from $\mathbb{G}|_{[0,\alpha]^2} = \max$ and Lemma 2.17.
- If $0 \leq x \leq y \leq \alpha < z$, then it follows that $\mathbb{U}(x, z) = \max(x, z) = z$ and $\mathbb{G}(x, z) = \mathbb{G}(y, z) = \mathbb{G}(\alpha, z) \geq \alpha$. Thus, we have
$$\mathbb{U}(x, \mathbb{G}(y, z)) = \mathbb{G}(y, z) = \mathbb{G}(\alpha, z) = \mathbb{G}(\mathbb{U}(x, y), \mathbb{U}(x, z)).$$
- If $0 \leq x \leq \alpha < y \leq z \leq 1$, then $\mathbb{U}(x, y) = y$, $\mathbb{U}(x, z) = z$ and
$$\mathbb{U}(x, \mathbb{G}(y, z)) = \mathbb{G}(y, z) = \mathbb{G}(\mathbb{U}(x, y), \mathbb{U}(x, z)).$$
- If $\alpha \leq x \leq y \leq z \leq 1$, then it is clear from Proposition 4.2 and Corollary 4.3.

□

Example 4.11. *Let us consider the following idempotent uninorm* $\mathbb{U}$ *with* $e = \frac{1}{2}$

$$\mathbb{U}(x, y) = \begin{cases} \max(x, y) & \text{if } y \geq \mathcal{A}(x), \\ \min(x, y) & \text{otherwise}, \end{cases} \tag{12}$$

where the associated function $\mathcal{A}(x)$ *with* $\alpha = \mathcal{A}(0) = \frac{2}{3}$ *has the form*

$$\mathcal{A}(x) = \begin{cases} \frac{2}{3} - \frac{1}{3}x & \text{if } x \in [0, \frac{1}{2}), \\ 2 - 3x & \text{if } x \in [\frac{1}{2}, \frac{2}{3}), \\ 0 & \text{if } x \in [\frac{2}{3}, 1]. \end{cases}$$

Let $\mathbb{G}$ *be a grouping function given by*

$$\mathbb{G}(x, y) = \begin{cases} \max(x, y) & \text{if } (x, y) \in [0, \frac{2}{3}]^2, \\ \frac{2}{3} + \frac{1}{3}\mathbb{G}_\psi(3x - 2, 3y - 2) & \text{if } (x, y) \in [\frac{2}{3}, 1]^2, \\ \frac{2}{3} + \frac{1}{3}\mathbb{G}_\psi(3\max(x, y) - 2, 0) & \text{otherwise}, \end{cases} \tag{13}$$

where $\mathbb{G}_\psi$ *is a weak grouping function with the weak coefficient* $w = \frac{1}{2}$ *and has the form*

$$\mathbb{G}_\psi(x,y) = \begin{cases} 1-\sqrt{2-2x} & \text{if } (x,y) \in [\frac{1}{2},1] \times [0, 1-\sqrt{2-2x}], \\ 1-\sqrt{2-2y} & \text{if } (y,x) \in [\frac{1}{2},1] \times [0, 1-\sqrt{2-2y}, 1], \\ \min(x,y) & \text{otherwise.} \end{cases}$$

Then from Proposition 4.2 and Lemma 4.10 we have that $\mathbb{U}$ *is distributive over* $\mathbb{G}$.

Theorem 4.12. *Let* $\mathbb{F}$ *be an idempotent null-uninorm with the associated function* $\beta = \mathcal{A}(a) < 1$ *and* $\mathbb{G}$ *be a grouping function. Then* $\mathbb{F}$ *is distributive over* $\mathbb{G}$ *iff* $\mathbb{G} = (\langle 0, a, \mathbb{G}_a\rangle, \langle a, 1, \mathbb{G}_b\rangle)$, *i.e.,* $\mathbb{G}$ *has the formula*

$$\mathbb{G}(x,y) = \begin{cases} a\mathbb{G}_a(\frac{x}{a}, \frac{y}{a}) & \text{if } (x,y) \in [0,a]^2, \\ a + (1-a)\mathbb{G}_b(\frac{x-a}{1-a}, \frac{y-a}{1-a}) & \text{if } (x,y) \in [a,1]^2, \\ a + (1-a)\mathbb{G}_b(\frac{\max(x,y)-a}{1-a}, 0) & \text{otherwise,} \end{cases} \tag{14}$$

where $\mathbb{G}_a$ *and* $\mathbb{G}_b$ *are grouping functions given by Eqs. (7) and (10), respectively.*

Proof. The verification is similar to Theorem 3.4. Thus, we also omit the specific proof and only present the corresponding figures in the following. □

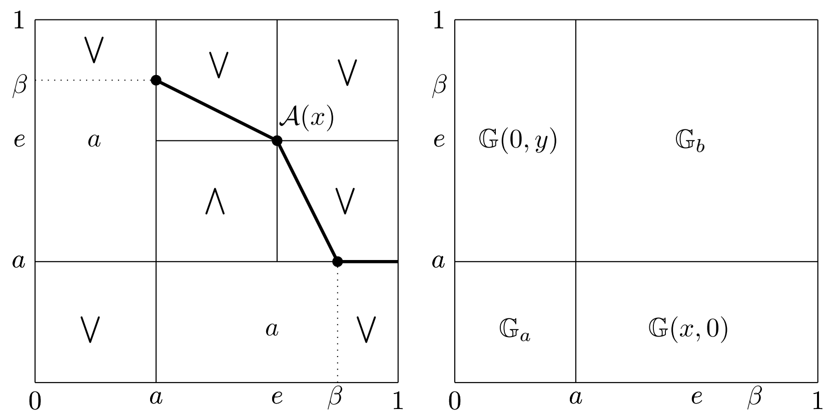

Figure 2: Structures of $\mathbb{F}$ (left) and $\mathbb{G}$ (right) in Theorem 4.12.

Example 4.13. *Consider the following idempotent null-uninorm* $\mathbb{F}$ *with* $a = \frac{1}{3}$, $e = \frac{2}{3}$:

$$\mathbb{F}(x,y) = \begin{cases} \min(x,y) & \text{if } (x,y) \in [\frac{1}{3}, \frac{2}{3}]^2, \text{ or} \\ & \text{if } (x,y) \in B(\frac{2}{3}) \text{ and } y < \mathcal{A}(x), \text{ or} \\ & \text{if } (x,y) \in B(\frac{2}{3}),\ y = \mathcal{A}(x) \text{ and } x < \mathcal{A}(\mathcal{A}(x)), \\ \frac{1}{3} & \text{if } (x,y) \in [0,\frac{1}{3}] \times [\frac{1}{3},1] \cup [\frac{1}{3},1] \times [0,\frac{1}{3}], \\ \max(x,y) & \text{otherwise,} \end{cases}$$

where $B(\frac{2}{3}) = [\frac{1}{3}, \frac{2}{3}] \times [\frac{2}{3}, 1] \cup [\frac{2}{3}, 1] \times [\frac{1}{3}, \frac{2}{3}]$ *and the associated function with* $\mathcal{A}(\frac{1}{3}) = \frac{3}{4}$ *has the form*

$$\mathcal{A}(x) = \begin{cases} -\frac{1}{4}x + \frac{5}{6} & \text{if } x \in [\frac{1}{3}, \frac{2}{3}), \\ -4x + \frac{10}{3} & \text{if } x \in [\frac{2}{3}, \frac{3}{4}), \\ \frac{1}{3} & \text{if } x \in [\frac{3}{4}, 1]. \end{cases}$$

Let $\mathbb{G}$ *be a grouping function satisfying*

$$\mathbb{G}(x,y) = \begin{cases} \frac{1}{3}\mathbb{G}_a(3x, 3y) & \text{if } (x,y) \in [0,\frac{1}{3}]^2, \\ \frac{1}{3} + \frac{2}{3}\mathbb{G}_b(\frac{3}{2}x - 1, \frac{3}{2}y - 1) & \text{if } (x,y) \in [\frac{1}{3},1]^2, \\ \frac{1}{3} + \frac{2}{3}\mathbb{G}_b(\frac{3}{2}\max(x,y) - 1, 0) & \text{otherwise,} \end{cases}$$

where $\mathbb{G}_a$ *and* $\mathbb{G}_b$ *are grouping functions respectively with the form*

$$\mathbb{G}_a(x,y) = \begin{cases} 1-\sqrt{1-x} & \text{if } (x,y) \in (0,1] \times [0, 1-\sqrt{1-x}], \\ 1-\sqrt{1-y} & \text{if } (x,y) \in [0, 1-\sqrt{1-y}] \times (0,1], \\ \min(x,y) & \text{otherwise,} \end{cases}$$

and the form in Eq. (13). Then from Corollary 4.3 and Theorem 4.12 we have that $\mathbb{F}$ *is distributive over* $\mathbb{G}$.

5 Conclusion and future works

In this article, we studied the distributivity of idempotent null-uninorms over overlap or grouping functions through introducing a class of weak overlap or grouping functions and studied the related results on them. Furthermore, by discussing the different values of the underlying uninorm's associated function on the interval endpoints and comparing them with the weak coefficient of the weak grouping functions we fully characterized the distributivity and presented some positive examples.

Although we have also studied the distributivity conditions of null-uninorms over overlap or grouping functions which have been discussed in [55]. What we need to point out is that the specific objects and results of our research are completely different from the existing ones, which are embodied in the following aspects.

(1) The specific research objects are completely different. The underlying operators of null-uninorms in [55] are continuous and Archimedean, i.e., strict or nilpotent t-norms (t-conorms). But the underlying operators of null-uninorms in our paper are $\mathbb{T}_M$ and $\mathbb{S}_M$ which are non-Archimedean. They are completely different aggregation operators. Therefore, the intersection of these two types of null-uninorms is empty.

(2) The obtained results are completely different. The equivalent distributivity conditions of null-uninorms with continuous Archimedean underlying operators over overlap or grouping functions are obtained under the conditions with overlap (grouping) functions satisfying 1-section deflation (1-section inflation) in the Reference [55]. But without any additional conditions, we have obtained the necessary and sufficient distributivity conditions of idempotent null-uninorms over these two functions and the corresponding structures by analyzing the values of underlying uninorms' associated functions at the interval endpoints. Thus, the obtained results are completely different. And more importantly, our achieved results generalize the problem raised for the future work in [34]. Besides, it is worth noting that, in application, we observe that the authors in [25] have proved that the pair of conjunction and fuzzy implications constituted by the idempotent uninorm obtained with the classical negation as a generator and its residual implication is optimal in the edge detection applications of image processing. And it is obvious that the idempotent null-uninorms discussed in the paper generalize idempotent uninorms, which indicates that the obtained distributivity results may also have possible applications in image processing.

For the future work, we will investigate the distributivity between other types of aggregation operators such as uninorms with continuous underlying operators, uninorms locally internal on the boundary and overlap (grouping) functions, even more general binary aggregation operators and those two functions.

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