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Original paper

Some aggregation operators for IVI-octahedron sets and their application to MCDGM

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Abstract

In this paper, in order to apply the concept of IVI-octahedron sets to MCDGM problems, we define some aggregation operators via IVI-octahedron sets and obtain some their properties. We define some aggregation operators via IVI-octahedron sets and obtain some their properties. We present a MCGDM method with linguistic variables in IVI-octahedron set environment. Finally, we give a numerical examples for MCGDM problems.

Keywords: IVI-Octahedron set, score function, accuracy function, IVI-octahedron Bonferroni mean operator, IVI-octahedron averaging operator, IVI-octahedron geometric operator, generalized IVI-octahedron averaging operator, generalized IVI-octahedron geometric operator.

1 Introduction

To express the real world as it is, Zadeh [27] initially introduced the concept of fuzzy sets as the generalization of classical sets in 1965. After then, many researchers (See [1, 9, 17, 28]) have been trying to find a mathematical expression of uncertainties and ambiguities which can be applied not only to Mathematics but also to engineering, medicine and social sciences, etc.

Decision making theory is very important in foreign policy, national defense policy, economic policy, medical diagnosis, etc. Decision making problems have used mainly three tools of correlation coefficients, similarity measures and aggregation operators. In the real world, we frequently encounter with decision making problems with uncertainty and vagueness that can be difficult to solve with the classical methods. A number of techniques have been developed to solve these and it can be seen from the literature that numerous researchers have discussed decision-making problems based on various types of fuzzy sets (See [3, 18, 23, 25] for the method of similarity measures and [4, 8, 16, 24] for the method of correlation coefficients). Recently, Şenel et al. [19] defined some similarity measures between octahedron sets proposed by Lee et al. [12] and applied them to multicriteria group decision making (MCGDM) problems.

An aggregation operator is one of tools which solve decision making problems. First of all, we would like to examine the research trends in aggregation operators. Xu and Chen [22] defined an interval-valued intuitionistic fuzzy Bonferroni mean operator and applied it to a multi-criteria decision making (MCDM). Liu et al. [13] proposed a Bonferroni mean operator based on intuitionistic fuzzy sets and dealt with Multiple attribute group decision making (MAGDM) problems. Garg and Arora [7] introduced a Bonferroni mean aggregation operator under intuitionistic fuzzy soft sets and applied it to decision-making problems. Kaur and Garg [10] introduced a Bonferroni mean operator on cubic intuitionistic fuzzy and applied it to MAGDM. On the other hand, Zhang [30] defined some aggregation operators based on interval-valued intuitionistic hesitant fuzzy setsand applied them to group decision making (GDM).

The purpose of our study is to propose several aggregation operators based on IVI-octahedron sets by Kim et al. [11] and apply them to MCGDM. To do this, this paper is organized as follows. In Section 2, we recall some basic concepts

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that are needed in next sections. In Section 3, we introduce several aggregation operators based on IVI-octahedron sets, and study some of their properties and some examples. In Section 4, in order to apply some IVI-octahedron aggregation operators to MCGDM, we propose an algorithm and give a numerical example for MCGDM problems to demonstrate the usefulness and applicability of our proposed method.

2 Preliminaries

In this section, we list some basic definitions needed in the next sections.

For a nonempty set X, let I^X denotes the set of all fuzzy sets in X and members of I^X will write λ, μ, ν, etc., where $I = [0, 1]$ (See [27]).

Each member of a set $I \oplus I = \{(a^{\in}, a^{\notin}) : (a^{\in}, a^{\notin}) \in I \times I \text{ and } a^{\in} + a^{\notin} \leq 1\}$ is called an *intuitionistic fuzzy number* (briefly, IFN). We denote intuitionistic fuzzy numbers $(a^{\in}, a^{\notin})$, $(b^{\in}, b^{\notin})$, $(c^{\in}, c^{\notin})$, etc. as $\bar{a}$, $\bar{b}$, $\bar{c}$, etc. It is well-known (Theorem 2.1 in [5]) that $(I \oplus I, \leq)$ is a complete distributive lattice with the greatest element $\bar{1}$ and the least element $\bar{0}$ satisfying DeMorgan's laws (See [5]).

For a nonempty set X, a mapping $\bar{A} = (A^{\in}, A^{\notin}) : X \to I \oplus I$ is called an *intuitionistic fuzzy set* (briefly, IFS) in X (See [1]).

The set of all closed subintervals of I is denoted by $[I]$, and members of $[I]$ are called *interval-valued numbers* and denoted by $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, etc., where $\widetilde{a} = [a^-, a^+]$ and $0 \leq a^- \leq a^+ \leq 1$. For a nonempty set X, a mapping $\widetilde{A} = [A^-, A^+] : X \to [I]$ is called an *interval-valued fuzzy set* (briefly, IVFS) in X (See [9, 15, 28]).

Let $[I] \oplus [I] = \{(\widetilde{a}^{\in}, \widetilde{a}^{\notin}) : (\widetilde{a}^{\in}, \widetilde{a}^{\notin}) \in [I] \times [I] \text{ and } a^{\in,+} + a^{\notin,+} \leq 1\}$, where

$$\widetilde{a}^{\in} = [a^{\in,-}, a^{\in,+}],\ \widetilde{a}^{\notin} = [a^{\notin,-}, a^{\notin,+}] \in [I].$$

Each member of $[I] \oplus [I]$ is called an *interval-valued intuitionistic fuzzy number* (briefly, IVIN). We denote IVINs $(\widetilde{a}^{\in}, \widetilde{a}^{\notin})$, $(\widetilde{b}^{\in}, \widetilde{b}^{\notin})$, $(\widetilde{c}^{\in}, \widetilde{c}^{\notin})$, etc. as $\widetilde{\widetilde{a}}$, $\widetilde{\widetilde{b}}$, $\widetilde{\widetilde{c}}$, etc.

For a nonempty set X, a mapping $\widetilde{\widetilde{A}} = (\widetilde{A}^{\in}, \widetilde{A}^{\notin}) : X \to [I] \oplus [I]$ is called an *interval-valued intuitionistic fuzzy set* (briefly, IVIS) in X, where for each $x \in X$, $\widetilde{A}^{\in} = [A^{\in,-}(x), A^{\in,+}(x)]$, $\widetilde{A}^{\notin} = [A^{\notin,-}(x), A^{\notin,+}(x)]$ and $A^{\in,+}(x) + A^{\notin,+}(x) \leq 1$ (See [2]).

Members of $([I] \oplus [I]) \times (I \oplus I) \times I$, denoted by Ω, are called *interval-valued intuitionistic fuzzy octahedron numbers* (briefly, IVI-octahedron numbers) and we write them as

$$\widetilde{\widetilde{\widetilde{a}}} = \left\langle \widetilde{\widetilde{a}}, \bar{a}, a \right\rangle,\ \widetilde{\widetilde{\widetilde{b}}} = \left\langle \widetilde{\widetilde{b}}, \bar{b}, b \right\rangle,\ \text{etc},$$

where $\widetilde{\widetilde{a}} = (\widetilde{a}^{\in}, \widetilde{a}^{\notin}) = ([a^{\in,-}, a^{\in,+}], [a^{\notin,-}, a^{\notin,+}])$, $\bar{a} = (a^{\in}, a^{\notin})$ (See [11]).

We define relations $\leq$ and $=$ on Ω as follows: for any $\widetilde{\widetilde{\widetilde{a}}}$, $\widetilde{\widetilde{\widetilde{b}}} \in \Omega$,

$$\widetilde{\widetilde{\widetilde{a}}} \leq \widetilde{\widetilde{\widetilde{b}}} \Longleftrightarrow \widetilde{\widetilde{a}} \leq \widetilde{\widetilde{b}},\ \bar{a} \leq \bar{b},\ a \leq b, \quad \widetilde{\widetilde{\widetilde{a}}} = \widetilde{\widetilde{\widetilde{b}}} \Longleftrightarrow \widetilde{\widetilde{\widetilde{a}}} \leq \widetilde{\widetilde{\widetilde{b}}},\ \widetilde{\widetilde{\widetilde{b}}} \leq \widetilde{\widetilde{\widetilde{a}}}.$$

For any $(\widetilde{\widetilde{\widetilde{a}}}_j)_{j \in J} \subset \Omega$, its inf $\bigwedge_{j \in J} \widetilde{\widetilde{\widetilde{a}}}_j$ and sup $\bigvee_{j \in J} \widetilde{\widetilde{\widetilde{a}}}_j$ are defined as follows:

$$\bigwedge_{j \in J} \widetilde{\widetilde{\widetilde{a}}}_j = \left\langle \bigwedge_{j \in J} \widetilde{\widetilde{a}}_j, \bigwedge_{j \in J} \bar{a}_j, \bigwedge_{j \in J} a_j \right\rangle, \quad \bigvee_{j \in J} \widetilde{\widetilde{\widetilde{a}}}_j = \left\langle \bigvee_{j \in J} \widetilde{\widetilde{a}}_j, \bigvee_{j \in J} \bar{a}_j, \bigvee_{j \in J} a_j \right\rangle.$$

For a nonempty set X, a mapping $\mathcal{A} = \left\langle \widetilde{\widetilde{A}}, \bar{A}, A \right\rangle : X \to ([I] \oplus [I]) \times (I \oplus I) \times I$ is called an *interval-valued intuitionistic fuzzy octahedron set* (briefly, IVI-octahedron set) in X (See [11]).

Throughout this paper, let $X = \{x_1, x_2, x_3, \cdots, x_n\}$ be a universal set.

3 IVI-octahedron Bonferroni mean value

In this section, we define some basic operational laws between IVI-octahedron numbers. By using them, we define operators $IVIOBM$ [resp. $IVIOWBM$], $IVIOA$ [resp. $IVIOWA$], $IVIOG$ [resp. $IVIOWG$] and $GIVIOA$ [resp. $GIVIOWA$] on Ω, and study some properties and give some examples respectively.

Definition 3.1. *Let* $\tilde{\tilde{\bar{a}}} = \left\langle \tilde{\bar{a}}, \bar{a}, a \right\rangle \in \Omega$. *Then*

(*i*) *the score function* $S(\tilde{\tilde{\bar{a}}})$ *of* $\tilde{\tilde{\bar{a}}}$ *is defined as follows:*

$$S(\tilde{\tilde{\bar{a}}}) = \frac{1}{3}(a^{\in,-} + a^{\in,+} - a^{\not\in,-} - a^{\not\in,+} + a) + (a^{\in} - a^{\not\in}), \tag{3.1}$$

(*ii*) *the accuracy function* $H(\tilde{\tilde{\bar{a}}})$ *of* $\tilde{\tilde{\bar{a}}}$ *is defined as follows:*

$$H(\tilde{\tilde{\bar{a}}}) = \frac{1}{3}(a^{\in,-} + a^{\in,+} + a^{\not\in,-} + a^{\not\in,+} + a) + (a^{\not\in} + a^{\in}). \tag{3.2}$$

It is obvious that $-2 \leq S(\tilde{\tilde{\bar{a}}}) \leq 2$ *and* $0 \leq H(\tilde{\tilde{\bar{a}}}) \leq 2$.

Definition 3.2. *Let* $\tilde{\tilde{\bar{a}}},\ \tilde{\tilde{\bar{b}}} \in \Omega$ *and let* $k > 0$ *be a real number. Then the operations* $\tilde{\tilde{\bar{a}}} \oplus \tilde{\tilde{\bar{b}}}$, $\tilde{\tilde{\bar{a}}} \otimes \tilde{\tilde{\bar{b}}}$, $k\tilde{\tilde{\bar{a}}}$ *and* $\tilde{\tilde{\bar{a}}}^k$ *are defined as follows:*

(*i*) $\tilde{\tilde{\bar{a}}} \oplus \tilde{\tilde{\bar{b}}}$
$= < ([a^{\in,-} + b^{\in,-} - a^{\in,-}b^{\in,-}, a^{\in,+} + b^{\in,+} - a^{\in,+}b^{\in,+}], [a^{\not\in,-}b^{\not\in,-}, a^{\not\in,+}b^{\not\in,+}]), (a^{\in} + b^{\in} - a^{\in}b^{\in}, a^{\not\in}b^{\not\in}), a + b - ab >$,

(*ii*) $\tilde{\tilde{\bar{a}}} \otimes \tilde{\tilde{\bar{b}}}$
$= < ([a^{\in,-}b^{\in,-}, a^{\in,+}b^{\in,+}], [a^{\not\in,-} + b^{\not\in,-} - a^{\not\in,-}b^{\not\in,-}, a^{\not\in,+} + b^{\not\in,+} - a^{\not\in,+}b^{\not\in,+}]), (a^{\in}b^{\in}, a^{\not\in} + b^{\not\in} - a^{\not\in}b^{\not\in}), ab >$,

(*iii*) $k\tilde{\tilde{\bar{a}}}$
$= < ([1 - (1 - a^{\in,-})^k, 1 - (1 - a^{\in,+})^k], [(a^{\not\in,-})^k, (a^{\not\in,+})^k]), (1 - (1 - a^{\not\in})^k, (a^{\in})^k), 1 - (1 - a)^k >$,

(*iv*) $\tilde{\tilde{\bar{a}}}^k$
$= < ([(a^{\in,-})^k, (a^{\in,+})^k], [1 - (1 - a^{\not\in,-})^k, 1 - (1 - a^{\not\in,+})^k]), ((a^{\not\in})^k, 1 - (1 - a^{\in})^k), a^k >$.

Proposition 3.3. *For any* $\tilde{\tilde{\bar{a}}},\ \tilde{\tilde{\bar{b}}} \in \Omega$, $\tilde{\tilde{\bar{a}}} \oplus \tilde{\tilde{\bar{b}}},\ \tilde{\tilde{\bar{a}}} \otimes \tilde{\tilde{\bar{b}}},\ k\tilde{\tilde{\bar{a}}},\ \tilde{\tilde{\bar{a}}}^k \in \Omega$.

Proof. From Definition 3.2 and the proof of Theorem 8 in [30], the proofs are easy. □

Definition 3.4. *A mapping* $IVIOBM^{p,q} : \Omega^n \to \Omega$ *is called an IVI-octahedron Bonferroni mean (briefly, IVIOBM) operator on* Ω*, if for each* $(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \tilde{\tilde{\bar{a}}}_n) \in \Omega^n$,

$$IVIOBM^{p,q}(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \tilde{\tilde{\bar{a}}}_n) = \left[\frac{1}{n(n-1)} \bigoplus_{i,\ j=1,\ i\neq j}^{n} (\tilde{\tilde{\bar{a}}}_i^{\,p} \otimes \tilde{\tilde{\bar{a}}}_j^{\,q})\right]^{\frac{1}{p+q}}, \tag{3.3}$$

where $p,\ q > 0$ *are real numbers.*

Proposition 3.5. *An IVIOBM operator is well-defined, i.e., for each* $(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \tilde{\tilde{\bar{a}}}_n) \in \Omega^n$,

$$IVIOBM^{p,q}(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \tilde{\tilde{\bar{a}}}_n) \in \Omega.$$

In fact, $IVIOBM^{p,q}(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \tilde{\tilde{\bar{a}}}_n)$ *is given by:*

$$IVIOBM^{p,q}(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \tilde{\tilde{\bar{a}}}_n) = \left\langle ([a^{\in,-}, a^{\in,+}], [a^{\not\in,-}, a^{\not\in,+}]), (a^{\in}, a^{\not\in}), a \right\rangle, \tag{3.4}$$

where

$$a^{\in,-} = \left[1 - \Pi_{i,\ j=1,\ i\neq j}^{n}(1 - (a_i^{\in,-})^p (a_j^{\in,-})^q)^{\frac{1}{n(n-1)}}\right]^{\frac{1}{p+q}},$$

$$a^{\in,+} = \left[1 - \Pi_{i,\ j=1,\ i\neq j}^{n}(1-(a_i^{\in,+})^p(a_j^{\in,+})^q)^{\frac{1}{n(n-1)}}\right]^{\frac{1}{p+q}},$$

$$a^{\notin,-} = 1 - \left[1 - \Pi_{i,\ j=1,\ i\neq j}^{n}(1-(1-a_i^{\notin,-})^p(1-a_j^{\notin,-})^q)^{\frac{1}{n(n-1)}}\right]^{\frac{1}{p+q}},$$

$$a^{\notin,+} = 1 - \left[1 - \Pi_{i,\ j=1,\ i\neq j}^{n}(1-(1-a_i^{\notin,+})^p(1-a_j^{\notin,+})^q)^{\frac{1}{n(n-1)}}\right]^{\frac{1}{p+q}},$$

$$a^{\in} = \left[1 - \Pi_{i,\ j=1,\ i\neq j}^{n}(1-(a_i^{\notin})^p(a_j^{\notin})^q)^{\frac{1}{n(n-1)}}\right]^{\frac{1}{p+q}},$$

$$a^{\notin} = 1 - \left[1 - \Pi_{i,\ j=1,\ i\neq j}^{n}(1-(1-a_i^{\in})^p(1-a_j^{\in})^q)^{\frac{1}{n(n-1)}}\right]^{\frac{1}{p+q}},$$

$$a = \left[1 - \Pi_{i,\ j=1,\ i\neq j}^{n}(1-a_i^p a_j^q)^{\frac{1}{n(n-1)}}\right]^{\frac{1}{p+q}}.$$

Proof. Let $\tilde{\tilde{\bar{a}}}_i,\ \tilde{\tilde{\bar{a}}}_j \in \Omega$ and let $p,\ q$ be any two positive real numbers. Then by Definition 3.2 (iv), we have

$$\tilde{\tilde{\bar{a}}}_i^{\,p} = \tilde{\tilde{\bar{\delta}}}_i = \left\langle \tilde{\tilde{\delta}}_i, \bar{\delta}_i, \delta_i \right\rangle,\ \tilde{\tilde{\bar{a}}}_j^{\,q} = \tilde{\tilde{\bar{\delta}}}_j = \left\langle \tilde{\tilde{\delta}}_j, \bar{\delta}_j, \delta_j \right\rangle. \tag{3.5}$$

Thus by Definition 3.2 (i), we get

$$\tilde{\tilde{\bar{a}}}_i^{\,p} \otimes \tilde{\tilde{\bar{a}}}_j^{\,q} = \tilde{\tilde{\bar{\delta}}} = \left\langle \tilde{\tilde{\delta}}, \bar{\delta}, \delta \right\rangle. \tag{3.6}$$

Now by induction on n, we can prove similarly to the proof of Theorem 3 in [10] that (3.7) holds:

$$\bigoplus_{i,\ j=1,\ i\neq j}^{n} (\tilde{\tilde{\bar{a}}}_i^{\,p} \otimes \tilde{\tilde{\bar{a}}}_j^{\,q}) = \tilde{\tilde{\bar{\eta}}} = \left\langle \tilde{\tilde{\eta}}, \bar{\eta}, \eta \right\rangle. \tag{3.7}$$

So by Definition 3.2 (iii), we have

$$\frac{1}{n(n-1)}\left[\bigoplus_{i,\ j=1,\ i\neq j}^{n} (\tilde{\tilde{\bar{a}}}_i^{\,p} \otimes \tilde{\tilde{\bar{a}}}_j^{\,q})\right] = \tilde{\tilde{\bar{\zeta}}} = \left\langle \tilde{\tilde{\zeta}}, \bar{\zeta}, \zeta \right\rangle. \tag{3.8}$$

Hence by the definition of IVIOBM, we can see that (3.4) holds.

Now let $\tilde{\tilde{\bar{a}}}_i = \left\langle \tilde{\tilde{a}}_i, \bar{a}_i, a_i \right\rangle \in \Omega$, $(i = 1,\ 2,\ \cdots,\ n)$. Then by the procedure of the proof of Theorem 3 in [10], we can easily see that (3.9) holds:

$$0 \leq a^{\in,+} + a^{\notin,+} \leq 1,\ 0 \leq a^{\in} + a^{\notin} \leq 1,\ 0 \leq a \leq 1. \tag{3.9}$$

Thus $IVIOBM^{p,q}(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \tilde{\tilde{\bar{a}}}_n) \in \Omega$. This completes the proof. □

Proposition 3.6. *Let $\tilde{\tilde{\bar{a}}}_i,\ \tilde{\tilde{\bar{b}}}_i \in \Omega$ $(i = 1,\ 2,\ \cdots,\ n)$. Then $IVIOBM^{p,q}$ has the following properties:*

(1) *$IVIOBM^{p,q}$ is idempotent, i.e., if $\tilde{\tilde{\bar{a}}}_i = \tilde{\tilde{\bar{a}}}$ for all i, then*

$$IVIOBM^{p,q}(\tilde{\tilde{\bar{a}}}, \tilde{\tilde{\bar{a}}}, \cdots, \tilde{\tilde{\bar{a}}}) = \tilde{\tilde{\bar{a}}},$$

(2) *$IVIOBM^{p,q}$ is monotone, i.e., if $\tilde{\tilde{\bar{a}}}_i \leq_1 \tilde{\tilde{\bar{b}}}_i$ for all i, then*

$$IVIOBM^{p,q}(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \tilde{\tilde{\bar{a}}}_n) \leq IVIOBM^{p,q}(\tilde{\tilde{\bar{b}}}_1, \tilde{\tilde{\bar{b}}}_2, \cdots, \tilde{\tilde{\bar{b}}}_n),$$

(3) *$IVIOBM^{p,q}$ is commutative, i.e., if $(\tilde{\tilde{\bar{b}}}_1, \tilde{\tilde{\bar{b}}}_2, \cdots, \tilde{\tilde{\bar{b}}}_n)$ is any permutation of $(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \tilde{\tilde{\bar{a}}}_n)$, then*

$$IVIOBM^{p,q}(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \tilde{\tilde{\bar{a}}}_n) = IVIOBM^{p,q}(\tilde{\tilde{\bar{b}}}_1, \tilde{\tilde{\bar{b}}}_2, \cdots, \tilde{\tilde{\bar{b}}}_n),$$

(4) *$IVIOBM^{p,q}$ is bounded, i.e., if a^- and a^+ are the lower and upper bound of $\tilde{\tilde{\bar{a}}}_i$, then*

$$a^- \leq IVIOBM^{p,q}(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \tilde{\tilde{\bar{a}}}_n) \leq a^+,$$

where

$$a^- = \left\langle ([a^{\in,-}_{\min}, a^{\in,+}_{\min}], [a^{\notin,-}_{\max}, a^{\notin,+}_{\mathrm{mmax}}]), (a^{\in}_{\max}, a^{\notin}_{\min}), a_{\min} \right\rangle, \qquad a^+ = \left\langle ([a^{\in,-}_{\max}, a^{\in,+}_{\max}], [a^{\notin,-}_{\min}, a^{\notin,+}_{\min}]), (a^{\in}_{\min}, a^{\notin}_{\max}), a_{\max} \right\rangle.$$

Proof. Since the proofs are almost similar to that of properties 1, 2, 3, and 4 in [10], they are omitted. □

Definition 3.7. *Let $\tilde{\tilde{\bar{a}}}_i \in \Omega$ and let $w = (w_1, w_2, \cdots, w_n)^T$ be a weighted vector such that $w_i > 0$ and $\Sigma_{i=1}^n w_i = 1$ ($i = 1,\ 2,\ \cdots,\ n$). Then a weighted IVIOBM over Ω, denoted by $IVIOWBM^{p,q}_w$, is the mapping $IVIOWBM^{p,q}_w : \Omega^n \to \Omega$ is given by:*

$$IVIOWBM^{p,q}_w(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \tilde{\tilde{\bar{a}}}_n) = \left[\frac{1}{n(n-1)} \bigoplus_{i,\ j=1,\ i\neq j}^{n} \left((w_i\tilde{\tilde{\bar{a}}}_i)^p \otimes (w_j\tilde{\tilde{\bar{a}}}_j)^q\right)\right]^{\frac{1}{p+q}}, \tag{3.10}$$

where $p,\ q > 0$ are real numbers.

Proposition 3.8. *An IVIOWBM operator is well-defined, i.e., for each $(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \tilde{\tilde{\bar{a}}}_n) \in \Omega^n$,*

$$IVIOWBM^{p,q}_w(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \tilde{\tilde{\bar{a}}}_n) \in \Omega,$$

and $IVIOWBM^{p,q}_w(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \tilde{\tilde{\bar{a}}}_n)$ is given by:

$$IVIOWBM^{p,q}_w(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \tilde{\tilde{\bar{a}}}_n) = \left\langle ([a^{\in,-}, a^{\in,+}], [a^{\notin,-}, a^{\notin,+}]), (a^{\in}, a^{\notin}), a \right\rangle, \tag{3.11}$$

where

$$a^{\in,-} = \left[1 - \Pi^n_{i,\ j=1,\ i\neq j}\left(1 - \left(1 - \left(1 - a^{\in,-}_i\right)^{w_i}\right)^p \left(1 - \left(1 - a^{\in,-}_j\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right]^{\frac{1}{p+q}},$$

$$a^{\in,+} = \left[1 - \Pi^n_{i,\ j=1,\ i\neq j}\left(1 - \left(1 - \left(1 - a^{\in,+}_i\right)^{w_i}\right)^p \left(1 - \left(1 - a^{\in,+}_j\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right]^{\frac{1}{p+q}},$$

$$a^{\notin,-} = 1 - \left[1 - \Pi^n_{i,\ j=1,\ i\neq j}\left(1 - \left(1 - \left(a^{\notin,-}_i\right)^{w_i}\right)^p \left(1 - \left(a^{\notin,-}_j\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right]^{\frac{1}{p+q}},$$

$$a^{\notin,+} = 1 - \left[1 - \Pi^n_{i,\ j=1,\ i\neq j}\left(1 - \left(1 - \left(a^{\notin,+}_i\right)^{w_i}\right)^p \left(1 - \left(a^{\notin,+}_j\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right]^{\frac{1}{p+q}},$$

$$a^{\in} = \left[1 - \Pi^n_{i,\ j=1,\ i\neq j}\left(1 - \left(1 - \left(1 - a^{\notin}_i\right)^{w_i}\right)^p \left(1 - \left(1 - a^{\notin}_j\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right]^{\frac{1}{p+q}},$$

$$a^{\notin} = 1 - \left[1 - \Pi^n_{i,\ j=1,\ i\neq j}\left(1 - (1 - (a^{\in}_i)^{w_i})^p \left(1 - \left(a^{\in}_j\right)^{w_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right]^{\frac{1}{p+q}},$$

$$a = \left[1 - \Pi^n_{i,\ j=1,\ i\neq j}(1 - (1 - (1 - a_i)^{w_i})^p (1 - (1 - a_j)^{w_j})^q)^{\frac{1}{n(n-1)}}\right]^{\frac{1}{p+q}}$$

and $\kappa = (w_1, w_2, \cdots, w_n)^T$ is the associated weight vector such that each $w_i > 0$ and $\Sigma^n_{i=-} w_i = 1$.

Proof. The proof is similar to that of Proposition 3.5. □

Definition 3.9. *Let $\tilde{\tilde{a}}_i \in \Omega$ $(i = 1, 2, \cdots, n)$ and let $w = (w_1, w_2, \cdots, w_n)^T$ be the weighted vector of $\tilde{\tilde{a}}_1$ $(i = 1, 2, \cdots, n)$ such that $w_i \in [0,1]$ and $\Sigma_{i=1}^n = 1$. Then an IVI-octahedron weighted averaging (briefly, IVIOWA) operator is a mapping $IVIOWA : \Omega^n \to \Omega$ defined by:*

$$IVIOWA(\tilde{\tilde{a}}_1, \tilde{\tilde{a}}_2, \cdots, \tilde{\tilde{a}}_n) = \bigoplus_{i=1}^n (w_i \tilde{\tilde{a}}_i). \tag{3.12}$$

If $w = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})^T$, then the IVIOWA operator reduced to an IVI-octahedron averaging (briefly, IVIOA) operator given by:

$$IVIOA(\tilde{\tilde{a}}_1, \tilde{\tilde{a}}_2, \cdots, \tilde{\tilde{a}}_n) = \bigoplus_{i=1}^n (\frac{1}{n} \tilde{\tilde{a}}_i). \tag{3.13}$$

Proposition 3.10. *An IVIOWA operator is well-defined, i.e., for each $(\tilde{\tilde{a}}_1, \tilde{\tilde{a}}_2, \cdots, \tilde{\tilde{a}}_n) \in \Omega^n$,*

$$IVIOWA(\tilde{\tilde{a}}_1, \tilde{\tilde{a}}_2, \cdots, \tilde{\tilde{a}}_n) \in \Omega,$$

and $IVIOWA(\tilde{\tilde{a}}_1, \tilde{\tilde{a}}_2, \cdots, \tilde{\tilde{a}}_n)$ is given by:

$$IVIOWA(\tilde{\tilde{a}}_1, \tilde{\tilde{a}}_2, \cdots, \tilde{\tilde{a}}_n) = \tilde{\tilde{a}} = \left\langle ([a^{\in,-}, a^{\in,+}], [a^{\notin,-}, a^{\notin,+}]), (a^{\in}, a^{\notin}), a \right\rangle, \tag{3.14}$$

where

$$\tilde{\tilde{a}}^{\in} = \left[1 - \Pi_{i=1}^n (1 - a_i^{\in,-})^{w_i}, 1 - \Pi_{i=1}^n (1 - a_i^{\in,+})^{w_i}\right], \qquad \tilde{\tilde{a}}^{\notin} = \left[\Pi_{i=1}^n (a_i^{\notin,-})^{w_i}, \Pi_{i=1}^n (a_i^{\notin,+})^{w_i}\right],$$
$$\bar{a} = \left(1 - \Pi_{i=1}^n (1 - a_i^{\in})^{w_i}, \Pi_{i=1}^n (a_i^{\notin})^{w_i}\right), \ a = 1 - \Pi_{i=1}^n (1 - a_i)^{w_i}.$$

Proof. The proof of the first part is obvious from Definition 3.9 and Proposition 3.3. By using the mathematical induction on n and Definition 3.2, we can prove the second part.

□

Definition 3.11. *Let $\tilde{\tilde{a}}_i \in \Omega$ $(i = 1, 2, \cdots, n)$ and let $w = (w_1, w_2, \cdots, w_n)^T$ be the weighted vector of $\tilde{\tilde{a}}_1$ $(i = 1, 2, \cdots, n)$ such that $w_i \in [0,1]$ and $\Sigma_{i=1}^n w_i = 1$. Then an IVI-octahedron weighted geometric (briefly, IVIOWG) operator is a mapping $IVIOWG : \Omega^n \to \Omega$ defined by:*

$$IVIOWG(\tilde{\tilde{a}}_1, \tilde{\tilde{a}}_2, \cdots, \tilde{\tilde{a}}_n) = \bigotimes_{i=1}^n (\tilde{\tilde{a}}_i^{\,w_i}). \tag{3.15}$$

If $w = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})^T$, then the IVIOWG operator reduced to an IVI-octahedron geometric (briefly, IVIOG) operator given by:

$$IVIOG(\tilde{\tilde{a}}_1, \tilde{\tilde{a}}_2, \cdots, \tilde{\tilde{a}}_n) = \bigotimes_{i=1}^n (\tilde{\tilde{a}}_i^{\,\frac{1}{n}}). \tag{3.16}$$

Proposition 3.12. *An IVIOWG operator is well-defined, i.e., for each $(\tilde{\tilde{a}}_1, \tilde{\tilde{a}}_2, \cdots, \tilde{\tilde{a}}_n) \in \Omega^n$,*

$$IVIOWG(\tilde{\tilde{a}}_1, \tilde{\tilde{a}}_2, \cdots, \tilde{\tilde{a}}_n) \in \Omega,$$

and $IVIOWG(\tilde{\tilde{a}}_1, \tilde{\tilde{a}}_2, \cdots, \tilde{\tilde{a}}_n)$ is given by:

$$IVIOWG(\tilde{\tilde{a}}_1, \tilde{\tilde{a}}_2, \cdots, \tilde{\tilde{a}}_n) = \tilde{\tilde{a}} = \left\langle ([a^{\in,-}, a^{\in,+}], [a^{\notin,-}, a^{\notin,+}]), (a^{\in}, a^{\notin}), a \right\rangle, \tag{3.17}$$

where

$$\tilde{\tilde{a}}^{\in} = \left[\Pi_{i=1}^n (a_i^{\in,-})^{w_i}, \Pi_{i=1}^n (a_i^{\in,+})^{w_i}\right], \qquad \tilde{\tilde{a}}^{\notin} = \left[1 - \Pi_{i=1}^n (1 - a_i^{\notin,-})^{w_i}, 1 - \Pi_{i=1}^n (1 - a_i^{\notin,+})^{w_i}\right],$$
$$\bar{a} = \left(\Pi_{i=1}^n (a_i^{\in})^{w_i}, 1 - \Pi_{i=1}^n (1 - a_i^{\notin})^{w_i}\right), \ a = \Pi_{i=1}^n (a_i)^{w_i}.$$

Proof. The proof is similar to that of Proposition 3.10. □

Now by combining generators IVIOWA and IVIOWG with the generalizes mean proposed by Dyckhoff and Pedrycz in [6], we define the generalized IVI-octahedron weighted averaging operator and the generalized IVI-octahedron weighted geometric operator, respectively.

Definition 3.13. *Let $\tilde{\tilde{\bar{a}}}_i \in \Omega$ $(i = 1,\ 2,\ \cdots,\ n)$ and let $w = (w_1, w_2, \cdots, w_n)^T$ be the weighted vector of $\tilde{\tilde{\bar{a}}}_i$ $(i = 1,\ 2,\ \cdots,\ n)$ such that $w_i \in [0, 1]$ and $\Sigma_{i=1}^n w_i = 1$. Let $\lambda > 0$ be a parameter.*

(i) A generalized IVI-octahedron weighted averaging (briefly, GIVIOWA) operator with a parameter λ, denoted by $GIVIOWA_\lambda$, is a mapping $GIVIOWA_\lambda : \Omega^n \to \Omega$ defined by:

$$GIVIOWA_\lambda(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \widetilde{\widetilde{\bar{a}_n}}) = \left(\bigoplus_{i=1}^n (w_i \tilde{\tilde{\bar{a}}}_i^{\lambda})\right)^{\frac{1}{\lambda}}. \tag{3.18}$$

(ii) A generalized IVI-octahedron weighted geometric (briefly, GIVIOWG) operator with a parameter λ, denoted by $GIVIOWG_\lambda$, is a mapping $GIVIOWG_\lambda : \Omega^n \to \Omega$ defined by:

$$GIVIOWG_\lambda(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \widetilde{\widetilde{\bar{a}_n}}) = \frac{1}{\lambda}\left(\bigotimes_{i=1}^n (\lambda \tilde{\tilde{\bar{a}}}_i)^{w_i}\right). \tag{3.19}$$

Remark 3.1. (1) *If $\lambda = 1$, then GIVIOWA [resp. GIVIOWG] operator reduces to the IVIOWA [resp. IVIOWG] operator.*

(2) *When $w_i = \frac{1}{n}$, a generalized IVI-octahedron averaging (briefly, GIVIOA) operator and a generalized IVI-octahedron geometric (briefly, GIVIOG) operator with a parameter λ, denoted by $GIVIOA_\lambda$ and $GIVIOG_\lambda$, are defined as follows, respectively:*

$$GIVIOWA_\lambda(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \widetilde{\widetilde{\bar{a}_n}}) = \left(\bigoplus_{i=1}^n (\frac{1}{n} \tilde{\tilde{\bar{a}}}_i^{\lambda})\right)^{\frac{1}{\lambda}}, \tag{3.20}$$

$$GIVIOWG_\lambda(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \widetilde{\widetilde{\bar{a}_n}}) = \frac{1}{\lambda}\left(\bigotimes_{i=1}^n (\lambda \tilde{\tilde{\bar{a}}}_i)^{\frac{1}{n}}\right). \tag{3.21}$$

Proposition 3.14. *Let $\lambda > 0$ be a parameter.*

(1) *A GIVIOWA operator is well-defined, i.e., for each $(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \widetilde{\widetilde{\bar{a}_n}}) \in \Omega^n$,*

$$GIVIOWA_\lambda(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \widetilde{\widetilde{\bar{a}_n}}) \in \Omega,$$

and $GIVIOWA_\lambda(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \widetilde{\widetilde{\bar{a}_n}})$ is given by:

$$GIVIOWA_\lambda(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \widetilde{\widetilde{\bar{a}_n}}) = \tilde{\tilde{\bar{a}}}, \tag{3.22}$$

where

$$\tilde{\tilde{a}}^{\in} = \left[\left(1 - \Pi_{i=1}^n \left(1 - (a_i^{\in,-})^\lambda\right)^{w_i}\right)^{\frac{1}{\lambda}}, \left(1 - \Pi_{i=1}^n \left(1 - (a_i^{\in,+})^\lambda\right)^{w_i}\right)^{\frac{1}{\lambda}}\right], \qquad a^{\notin,-} = 1 - \left(1 - \Pi_{i=1}^n \left(1 - (1 - a_i^{\notin,-})^\lambda\right)^{w_i}\right)^{\frac{1}{\lambda}},$$

$$a^{\notin,+} = 1 - \left(1 - \Pi_{i=1}^n \left(1 - (1 - a_i^{\notin,+})^\lambda\right)^{w_i}\right)^{\frac{1}{\lambda}}, \qquad \bar{a} = \left(\left(1 - \Pi_{i=1}^n \left(1 - (a_i^{\in})^\lambda\right)^{w_i}\right)^{\frac{1}{\lambda}}, 1 - \left(1 - \Pi_{i=1}^n \left(1 - (1 - a_i^{\notin})^\lambda\right)^{w_i}\right)^{\frac{1}{\lambda}}\right),$$

$$a = \left(1 - \Pi_{i=1}^n \left(1 - (a_i)^\lambda\right)^{w_i}\right)^{\frac{1}{\lambda}}.$$

(2) *A GIVIOWG operator is well-defined, i.e., for each $(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \widetilde{\widetilde{\bar{a}_n}}) \in \Omega^n$,*

$$GIVIOWG_\lambda(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \widetilde{\widetilde{\bar{a}_n}}) \in \Omega,$$

and $GIVIOWG_\lambda(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \widetilde{\widetilde{\bar{a}_n}})$ is given by:

$$GIVIOWG_\lambda(\tilde{\tilde{\bar{a}}}_1, \tilde{\tilde{\bar{a}}}_2, \cdots, \widetilde{\widetilde{\bar{a}_n}}) = \tilde{\tilde{\bar{a}}}, \tag{3.23}$$

where

$$a^{\in,-} = 1 - \left(1 - \Pi_{i=1}^{n}\left(1 - (1 - a_i^{\in,-})^{\lambda}\right)^{w_i}\right)^{\frac{1}{\lambda}}, \qquad a^{\in,+} = 1 - \left(1 - \Pi_{i=1}^{n}\left(1 - (1 - a_i^{\in,+})^{\lambda}\right)^{w_i}\right)^{\frac{1}{\lambda}},$$

$$\widetilde{\widetilde{a}}^{\not\in} = \left[\left(1 - \Pi_{i=1}^{n}\left(1 - (a_i^{\not\in,-})^{\lambda}\right)^{w_i}\right)^{\frac{1}{\lambda}}, \left(1 - \Pi_{i=1}^{n}\left(1 - (a_i^{\not\in,+})^{\lambda}\right)^{w_i}\right)^{\frac{1}{\lambda}}\right],$$

$$\bar{a} = \left(1 - \left(1 - \Pi_{i=1}^{n}\left(1 - (1 - a_i^{\in})^{\lambda}\right)^{w_i}\right)^{\frac{1}{\lambda}}, \left(1 - \Pi_{i=1}^{n}\left(1 - (a_i^{\not\in})^{\lambda}\right)^{w_i}\right)^{\frac{1}{\lambda}}\right), \qquad a = 1 - \left(1 - \Pi_{i=1}^{n}\left(1 - (1 - a_i)^{\lambda}\right)^{w_i}\right)^{\frac{1}{\lambda}}.$$

Proof. The proof is similar to that of Proposition 3.10. □

Proposition 3.15. *Let $\widetilde{\widetilde{a}}_i \in \Omega$ $(i = 1,\ 2,\ \cdots,\ n)$ and let $w = (w_1, w_2, \cdots, w_n)^T$ be the weighted vector of $\widetilde{\widetilde{a}}_i$ $(i = 1,\ 2,\ \cdots,\ n)$ such that $w_i \in [0, 1]$ and $\Sigma_{i=1}^{n} w_i = 1$. Let $\lambda > 0$ be a parameter.*

(1) *A GIVIOWA operator $GIVIOWA_{\lambda}(\widetilde{\widetilde{a}}_1, \widetilde{\widetilde{a}}_2, \cdots, \widetilde{\widetilde{a}}_n)$ is monotonically increasing with respect to λ, i.e., for any $\lambda_1,\ \lambda_2 > 0$ such that $\lambda_1 \leq \lambda_2$,*

$$GIVIOWA_{\lambda_1}(\widetilde{\widetilde{a}}_1, \widetilde{\widetilde{a}}_2, \cdots, \widetilde{\widetilde{a}}_n) \leq GIVIOWA_{\lambda_2}(\widetilde{\widetilde{a}}_1, \widetilde{\widetilde{a}}_2, \cdots, \widetilde{\widetilde{a}}_n).$$

(2) *A GIVIOWG operator $GIVIOWG_{\lambda}(\widetilde{\widetilde{a}}_1, \widetilde{\widetilde{a}}_2, \cdots, \widetilde{\widetilde{a}}_n)$ is monotonically decreasing with respect to λ, i.e., for any $\lambda_1,\ \lambda_2 > 0$ such that $\lambda_1 \leq \lambda_2$,*

$$GIVIOWG_{\lambda_1}(\widetilde{\widetilde{a}}_1, \widetilde{\widetilde{a}}_2, \cdots, \widetilde{\widetilde{a}}_n) \leq GIVIOWG_{\lambda_2}(\widetilde{\widetilde{a}}_1, \widetilde{\widetilde{a}}_2, \cdots, \widetilde{\widetilde{a}}_n).$$

Proof. The proofs are similar to that of Theorem 3.8 in [29]. □

Lemma 3.16 (See [20,21])**.** *Let $x_i > 0$, $\lambda_i > 0$ $(i = 1,\ 2,\ \cdots,\ n)$ and let $w\Sigma_{i=1}^{n}\lambda_i = 1$. Then*

$$\Pi_{i=1}^{n}(x_i)_i^{\lambda} \leq \Sigma_{i=1}^{n}\lambda_i x_i. \tag{3.24}$$

Proposition 3.17. *Let $\widetilde{\widetilde{a}}_i \in \Omega$ $(i = 1,\ 2,\ \cdots,\ n)$ and let $w = (w_1, w_2, \cdots, w_n)^T$ be the weighted vector of $\widetilde{\widetilde{a}}_i$ $(i = 1,\ 2,\ \cdots,\ n)$ such that $w_i \in [0, 1]$ and $\Sigma_{i=1}^{n} w_i = 1$. Let $\lambda > 0$ be a parameter. Then*

$$IVIOWG(\widetilde{\widetilde{a}}_1, \widetilde{\widetilde{a}}_2, \cdots, \widetilde{\widetilde{a}}_n) \leq GIVIOWA_{\lambda}(\widetilde{\widetilde{a}}_1, \widetilde{\widetilde{a}}_2, \cdots, \widetilde{\widetilde{a}}_n). \tag{3.25}$$

Proof. By using Lemma 3.16 and the proof of Theorem 3.20 in [30], we can prove that (3.29) holds. □

From Proposition 3.17, we can easily see that the IVI-octahedron numbers obtained by IVIOWG operator are not bigger than the ones obtained by GIVIOWA operator for any λ. If $\lambda = 1$, then we have easily the following.

Corollary 3.18. *Let $\widetilde{\widetilde{a}}_i \in \Omega$ $(i = 1,\ 2,\ \cdots,\ n)$ and let $w = (w_1, w_2, \cdots, w_n)^T$ be the weighted vector of $\widetilde{\widetilde{a}}_i$ $(i = 1,\ 2,\ \cdots,\ n)$ such that $w_i \in [0, 1]$ and $\Sigma_{i=1}^{n} w_i = 1$. Then*

$$IVIOWG(\widetilde{\widetilde{a}}_1, \widetilde{\widetilde{a}}_2, \cdots, \widetilde{\widetilde{a}}_n) \leq IVIOWA(\widetilde{\widetilde{a}}_1, \widetilde{\widetilde{a}}_2, \cdots, \widetilde{\widetilde{a}}_n). \tag{3.26}$$

Proposition 3.19. *Let $\widetilde{\widetilde{a}}_i \in \Omega$ $(i = 1,\ 2,\ \cdots,\ n)$ and let $w = (w_1, w_2, \cdots, w_n)^T$ be the weighted vector of $\widetilde{\widetilde{a}}_i$ $(i = 1,\ 2,\ \cdots,\ n)$ such that $w_i \in [0, 1]$ and $\Sigma_{i=1}^{n} w_i = 1$. Let $\lambda > 0$ be a parameter. Then*

$$GIVIOWG_{\lambda}(\widetilde{\widetilde{a}}_1, \widetilde{\widetilde{a}}_2, \cdots, \widetilde{\widetilde{a}}_n) \leq IVIOWA(\widetilde{\widetilde{a}}_1, \widetilde{\widetilde{a}}_2, \cdots, \widetilde{\widetilde{a}}_n). \tag{3.27}$$

Proof. The proof follows from Lemma 3.16 and the proof of Theorem 3.22 in [30]. □

Proposition 3.20. *Let $\widetilde{\widetilde{a}}_i \in \Omega$ $(i = 1,\ 2,\ \cdots,\ n)$ and let $w = (w_1, w_2, \cdots, w_n)^T$ be the weighted vector of $\widetilde{\widetilde{a}}_i$ $(i = 1,\ 2,\ \cdots,\ n)$ such that $w_i \in [0, 1]$ and $\Sigma_{i=1}^{n} w_i = 1$. Let $\lambda > 0$ be a parameter. Then*

$$GIVIOWG_{\lambda}(\widetilde{\widetilde{a}}_1, \widetilde{\widetilde{a}}_2, \cdots, \widetilde{\widetilde{a}}_n) \leq GIVIOWA_{\lambda}(\widetilde{\widetilde{a}}_1, \widetilde{\widetilde{a}}_2, \cdots, \widetilde{\widetilde{a}}_n). \tag{3.28}$$

Proof. From Lemma 3.16 and the proof of Theorem 3.23 in [30], we can show that (3.32) holds. □

4 MCGDM based on some IVI-octahedron aggregation operators

In this section, by using some IVI-octahedron aggregation operators proposed in Section 3, we solve the MCGDM under the IVI-octahedron sets environment. For it, the following assumption or notations are used to present the MCGDM problems for evaluating these with an IVI-octahedron set environment. Let $A = \{A_1, A_2, \cdots, A_m\}$ be the set of m alternatives and let $C = \{C_1, C_2, \cdots, C_n\}$ be a collection of n attributes with the weight vector $w = (w_1, w_2, \cdots, w_n)^T$ such that $w_i \in [0, 1]$ $(i = 1,\ 2,\ \cdots,\ N)$ and $\Sigma_{i=1}^n w_i = 1$. Assume that these alternatives are evaluated by an expert which give their preferences related to each alternative A_i $(i = 1,\ 2,\ \cdots,\ n)$ under the IVI-octahedron sets environment, and these values can be considered as IVI-octahedron numbers $D = (\tilde{\tilde{a}}_{ij})_{m\times n}$, where $\tilde{\tilde{a}}_{ij} = \left\langle ([a_{ij}^{\in,-}, a_{ij}^{\in,+}], [a_{ij}^{\notin,-}, a_{ij}^{\notin,+}]), (a_{ij}^{\in}, a_{ij}^{\notin}), a_{ij} \right\rangle \in \Omega$ $(i = 1,\ 2,\ \cdots,\ n;\ j = 1,\ 2,\ \cdots,\ m)$ represents the priority values of alternative A_i given by decision maker.

In general, attributes can be classified into two types: benefit attributes (i.e., the bigger the attribute values, the better) and cost attributes (i.e., the smaller the attribute values, the better) in an MCGDM problem. In other words, the attribute set C can be divided into two subsets: the subset of benefit attributes and cost attributes, respectively. If all the attributes are of the same type, then the rating values do not need normalization. However there are benefit attributes and cost attributes in an MCGDM problem. In such cases, we transform the rating values of the cost type into the rating values of the benefit type. In order to find the best alternative(s), we propose some steps.

Step 1: We collect the information rating of alternatives corresponding to criteria and summarize in the form of an IVI-octahedron number $\tilde{\tilde{a}}_{ij} \in \Omega$ $(i = 1,\ 2,\ \cdots,\ n;\ j = 1,\ 2,\ \cdots,\ m)$. These rating values are expressed as a decision matrix D as follows:

$$M = \begin{bmatrix} & C_1 & C_2 & \cdots & C_n \\ A_1 & \tilde{\tilde{a}}_{11} & \tilde{\tilde{a}}_{12} & \cdots & \tilde{\tilde{a}}_{1n} \\ A_2 & \tilde{\tilde{a}}_{21} & \tilde{\tilde{a}}_{22} & \cdots & \tilde{\tilde{a}}_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_m & \tilde{\tilde{a}}_{m1} & \tilde{\tilde{a}}_{m1} & \cdots & \tilde{\tilde{a}}_{mn} \end{bmatrix}. \tag{4.1}$$

Step 2: We normalize these collective information decision making matrix by transforming the rating values of a cost type into a benefit type, if necessary, by using the following normalized formula:

$$\tilde{\tilde{r}}_{ij} = \begin{cases} \tilde{\tilde{a}}_{ij} \text{ for a benifit type criterian} \\ \tilde{\tilde{a}}_{ij}^{c} \text{ for a cost type criterian,} \end{cases} \tag{4.2}$$

where $\tilde{\tilde{a}}_{ij}^{c} = \left\langle ([a_{ij}^{\notin,-}, a_{ij}^{\notin,+}], [a_{ij}^{\in,-}, a_{ij}^{\in,+}]), (a_{ij}^{\notin}, a_{ij}^{\in}), 1 - a_{ij} \right\rangle$. Then we obtain the normalized decision matrix $R = (\tilde{\tilde{r}}_{ij})_{m\times n}$.

Step 3: We aggregate the different preference values $\tilde{\tilde{r}}_{ij}$ $(j = 1\ ,2\ ,\cdots,\ n)$ of the alternatives A_i into the collective one $\tilde{\tilde{r}}_i$ $(i = 1\ ,2\ ,\cdots,\ m)$ by using an IVIOWBM aggregation operator for a real positive number $p,\ q$, or an IVIOWA [resp. IVIOWG, GIVIOWA or GIVIOWG] aggregation operator as follows:

$$\tilde{\tilde{r}}_{ij} = \left\langle ([r_{ij}^{\in,-}, r_{ij}^{\in,+}], [r_{ij}^{\notin,-}, r_{ij}^{\notin,+}]), (r_{ij}^{\in}, r_{ij}^{\notin}), r_{ij} \right\rangle, \tag{4.3}$$

where

$$\tilde{\tilde{a}}_i = IVIOWBM_w^{p,q}(\tilde{\tilde{r}}_{i1}, \tilde{\tilde{r}}_{i2}, \cdots, \tilde{\tilde{r}}_{in}),\ \text{(See (3.11))}, \qquad \tilde{\tilde{b}}_i = IVIOWA(\tilde{\tilde{r}}_{i1}, \tilde{\tilde{r}}_{i2}, \cdots, \tilde{\tilde{r}}_{in}),\ \text{(See (3.14))},$$

$$\tilde{\tilde{c}}_i = IVIOWG(\tilde{\tilde{r}}_{i1}, \tilde{\tilde{r}}_{i2}, \cdots, \tilde{\tilde{r}}_{in}),\ \text{(See (3.17))}, \qquad \tilde{\tilde{d}}_i = GIVIOWA(\tilde{\tilde{r}}_{i1}, \tilde{\tilde{r}}_{i2}, \cdots, \tilde{\tilde{r}}_{in})\ \text{(See (3.22))},$$

$$\tilde{\tilde{e}}_i = GIVIOWG(\tilde{\tilde{r}}_{i1}, \tilde{\tilde{r}}_{i2}, \cdots, \tilde{\tilde{r}}_{in}),\ \text{(See (3.23))}.$$

Step 4: We calculate the score value of IVI-octahedron number $\tilde{\tilde{r}}_i$ by using the Equation (3.1) as follows:

$$S(\tilde{\tilde{r}}_i) = \frac{1}{3}(r_i^{\in,-} + r_i^{\in,+} - r_i^{\notin,-} - r_i^{\notin,+} + r_i) + (r_i^{\in} - r_i^{\notin}). \tag{4.4}$$

Step 5: We rank the alternative A_i $(i = 1,\ 2,\ \cdots,\ m)$ with the order of their score value $S(\tilde{\tilde{\overline{r}}}_i)$.

Now we give a practical example to illustrate the application of some aggregation operators proposed in Section 3.

Example 4.1 (See 5.1 Case Study [10]). *Proper inventory management is the first step of the ladder of good production levels. Any shortage of raw material in inventory may disrupt the whole manufacturing cycle which in-turn can incur a huge loss to the company. Suppose that a Food company wants to keep track of various inventory items. The company produces mainly four kinds of foods, say A_1=Beverages, A_2=Edible oils, A_3=Pickles and A_4=Bakery items. For manufacturing these food items, the stock re-ordering decisions for ingredients in inventory are to be taken in account on three factors, say C_1=Cost price, C_2=Storage facilities and C_3=Staleness level. It is obvious that $\{C_1, C_3\}$ and $\{C_2\}$ are the subsets of benefit attributes and cost attributes, respectively. Let $w = (0.20, 0.38, 0.42)^T$ be the weight vector of these factors.*

Step 1: *Let the decision matrix D composed of the information rating values of alternatives corresponding to each attribute be expressed as follows:*

$$M = \begin{bmatrix} & C_1 & C_2 & C_3 \\ A_1 & \tilde{\tilde{\overline{a}}}_{11} & \tilde{\tilde{\overline{a}}}_{12} & \tilde{\tilde{\overline{a}}}_{13} \\ A_2 & \tilde{\tilde{\overline{a}}}_{21} & \tilde{\tilde{\overline{a}}}_{22} & \tilde{\tilde{\overline{a}}}_{23} \\ A_3 & \tilde{\tilde{\overline{a}}}_{31} & \tilde{\tilde{\overline{a}}}_{32} & \tilde{\tilde{\overline{a}}}_{23} \\ A_4 & \tilde{\tilde{\overline{a}}}_{41} & \tilde{\tilde{\overline{a}}}_{41} & \tilde{\tilde{\overline{a}}}_{43} \end{bmatrix}, \tag{4.5}$$

where $\tilde{\tilde{\overline{a}}}_{11} = \langle([0.50, 0.60], [0.10, 0.20]), (0.40, 0.20), 0.60\rangle$, $\tilde{\tilde{\overline{a}}}_{12} = \langle([0.20, 0.30], [0.40, 0.50]), (0.30, 0.20), 0.40\rangle$,
$\tilde{\tilde{\overline{a}}}_{13} = \langle([0.40, 0.60], [0.20, 0.30]), (0.20, 0.35), 0.25\rangle$, $\tilde{\tilde{\overline{a}}}_{21} = \langle([0.20, 0.30], [0.40, 0.50]), (0.40, 0.60), 0.45\rangle$,
$\tilde{\tilde{\overline{a}}}_{22} = \langle([0.15, 0.25], [0.30, 0.35]), (0.20, 0.30), 0.20\rangle$, $\tilde{\tilde{\overline{a}}}_{23} = \langle([0.20, 0.40], [0.10, 0.20]), (0.40, 0.50), 0.35\rangle$,
$\tilde{\tilde{\overline{a}}}_{31} = \langle([0.50, 0.60], [0.20, 0.30]), (0.20, 0.40), 0.30\rangle$, $\tilde{\tilde{\overline{a}}}_{32} = \langle([0.40, 0.60], [0.25, 0.35]), (0.30, 0.40), 0.45\rangle$,
$\tilde{\tilde{\overline{a}}}_{33} = \langle([0.50, 0.70], [0.10, 0.15]), (0.30, 0.50), 0.55\rangle$, $\tilde{\tilde{\overline{a}}}_{41} = \langle([0.30, 0.50], [0.10, 0.30]), (0.10, 0.30), 0.15\rangle$,
$\tilde{\tilde{\overline{a}}}_{42} = \langle([0.40, 0.55], [0.15, 0.20]), (0.30, 0.40), 0.25\rangle$, $\tilde{\tilde{\overline{a}}}_{43} = \langle([0.40, 0.50], [0.20, 0.30]), (0.45, 0.35), 0.40\rangle$.

Step 2: *From (4.2), we have the normalized decision making matrix R:*

$$R = \begin{bmatrix} & C_1 & C_2 & C_3 \\ A_1 & \tilde{\tilde{\overline{r}}}_{11} & \tilde{\tilde{\overline{r}}}_{12} & \tilde{\tilde{\overline{r}}}_{13} \\ A_2 & \tilde{\tilde{\overline{r}}}_{21} & \tilde{\tilde{\overline{r}}}_{22} & \tilde{\tilde{\overline{r}}}_{23} \\ A_3 & \tilde{\tilde{\overline{r}}}_{31} & \tilde{\tilde{\overline{r}}}_{32} & \tilde{\tilde{\overline{r}}}_{23} \\ A_4 & \tilde{\tilde{\overline{r}}}_{41} & \tilde{\tilde{\overline{r}}}_{41} & \tilde{\tilde{\overline{r}}}_{43} \end{bmatrix}, \tag{4.6}$$

where
$\tilde{\tilde{\overline{r}}}_{11} = \langle([0.10, 0.20], [0.50, 0.60]), (0.20, 0.40), 0.40\rangle$, $\tilde{\tilde{\overline{r}}}_{12} = \langle([0.20, 0.30], [0.40, 0.50]), (0.30, 0.20), 0.40\rangle$,
$\tilde{\tilde{\overline{r}}}_{13} = \langle([0.20, 0.30], [0.40, 0.60]), (0.35, 0.20), 0.75\rangle$, $\tilde{\tilde{\overline{r}}}_{21} = \langle([0.40, 0.50], [0.20, 0.30]), (0.60, 0.40), 0.55\rangle$,
$\tilde{\tilde{\overline{r}}}_{22} = \langle([0.15, 0.25], [0.30, 0.35]), (0.20, 0.30), 0.20\rangle$, $\tilde{\tilde{\overline{r}}}_{23} = \langle([0.10, 0.20], [0.20, 0.40]), (0.50, 0.40), 0.65\rangle$,
$\tilde{\tilde{\overline{r}}}_{31} = \langle([0.20, 0.30], [0.50, 0.60]), (0.40, 0.20), 0.70\rangle$, $\tilde{\tilde{\overline{r}}}_{32} = \langle([0.40, 0.60], [0.25, 0.35]), (0.30, 0.40), 0.45\rangle$,
$\tilde{\tilde{\overline{r}}}_{33} = \langle([0.10, 0.15], [0.50, 0.70]), (0.50, 0.30), 0.45\rangle$, $\tilde{\tilde{\overline{r}}}_{41} = \langle([0.10, 0.30], [0.30, 0.50]), (0.30, 0.10), 0.85\rangle$,
$\tilde{\tilde{\overline{r}}}_{42} = \langle([0.40, 0.55], [0.15, 0.20]), (0.30, 0.40), 0.25\rangle$, $\tilde{\tilde{\overline{r}}}_{43} = \langle([0.20, 0.30], [0.40, 0.50]), (0.35, 0.45), 0.60\rangle$.

Step 3: *Let us calculate IVIOWBM aggregation operator for a real positive number $p = q = 1$, or an IVIOWA [resp. IVIOWG, GIVIOWA or GIVIOWG] aggregation operator for $\tilde{\tilde{\overline{r}}}_i = (\tilde{\tilde{\overline{r}}}_{i1}, \tilde{\tilde{\overline{r}}}_{i2}, \tilde{\tilde{\overline{r}}}_{i3})$ $(i = 1,\ 2,\ 3,\ 4)$. Then from (3.12), (3.15), (3.21), (3.27) and (3.28), we get the followings respectively:*

$$IVIOWBM_w^{1,1}(\tilde{\tilde{\overline{r}}}_i) = \tilde{\tilde{\overline{a}}}_i, \tag{4.7}$$

where

$\widetilde{\widetilde{a_1}} = \langle([0.06, 0.10], [0.76, 0.83]), (0.11, 0.64), 0.22\rangle$, $\widetilde{\widetilde{a_2}} = \langle([0.06, 0.11], [0.63, 0.71]), (0.16, 0.72), 0.18\rangle$,
$\widetilde{\widetilde{a_3}} = \langle([0.08, 0.12], [0.75, 0.83]), (0.15, 0.68), 0.21\rangle$, $\widetilde{\widetilde{a_4}} = \langle([0.08, 0.15], [0.66, 0.74]), (0.12, 0.69), 0.24\rangle$.

$$IVIOWA(\tilde{\tilde{\overline{r}}}_i) = \tilde{\tilde{\overline{b}}}_i, \tag{4.8}$$

where

$$\tilde{\tilde{\tilde{b}}}_1 = \langle([0.18, 0.28], [0.42, 0.56]), (0.30, 0.23), 0.58\rangle, \tilde{\tilde{\tilde{b}}}_2 = \langle([0.19, 0.29], [0.23, 0.36]), (0.43, 0.36), 0.50\rangle,$$
$$\tilde{\tilde{\tilde{b}}}_3 = \langle([0.25, 0.39], [0.38, 0.52]), (0.41, 0.31), 0.51\rangle, \tilde{\tilde{\tilde{b}}}_4 = \langle([0.27, 0.41], [0.26, 0.35]), (0.32, 0.32), 0.58\rangle.$$

$$IVIOWG(\tilde{\tilde{\tilde{r}}}_i) = \tilde{\tilde{\tilde{c}}}_i, \tag{4.9}$$

where

$$\tilde{\tilde{\tilde{c}}}_1 = \langle([0.17, 0.28], [0.42, 0.56]), (0.30, 0.24), 0.52\rangle, \tilde{\tilde{\tilde{c}}}_2 = \langle([0.15, 0.26], [0.24, 0.36]), (0.37, 0.36), 0.40\rangle,$$
$$\tilde{\tilde{\tilde{c}}}_3 = \langle([0.19, 0.29], [0.42, 0.57]), (0.39, 0.32), 0.49\rangle, \tilde{\tilde{\tilde{c}}}_4 = \langle([0.23, 0.38], [0.29, 0.40]), (0.32, 0.37), 0.46\rangle.$$

$$GIVIOWA_2(\tilde{\tilde{\tilde{r}}}_i) = \tilde{\tilde{\tilde{d}}}_i, \tag{4.10}$$

where

$$\widetilde{\widetilde{\widetilde{d}}}_1 = \langle([0.18, 0.28], [0.42, 0.56]), (0.31, 0.23), 0.60\rangle, \widetilde{\widetilde{\widetilde{d}}}_2 = \langle([0.22, 0.31], [0.23, 0.36]), (0.45, 0.36), 0.52\rangle,$$
$$\widetilde{\widetilde{\widetilde{d}}}_3 = \langle([0.28, 0.42], [0.38, 0.50]), (0.42, 0.31), 0.52\rangle, \widetilde{\widetilde{\widetilde{d}}}_4 = \langle([0.29, 0.42], [0.26, 0.34]), (0.32, 0.31), 0.61\rangle.$$

$$GIVIOWG_2(\tilde{\tilde{\tilde{r}}}_i) = \tilde{\tilde{\tilde{e}}}_i, \tag{4.11}$$

where

$$\tilde{\tilde{\tilde{e}}}_1 = \langle([0.17, 0.28], [0.42, 0.57]), (0.29, 0.26), 0.50\rangle, \tilde{\tilde{\tilde{e}}}_2 = \langle([0.15, 0.26], [0.24, 0.36]), (0.35, 0.37), 0.38\rangle,$$
$$\tilde{\tilde{\tilde{e}}}_3 = \langle([0.19, 0.28], [0.43, 0.59]), (0.39, 0.33), 0.49\rangle, \tilde{\tilde{\tilde{e}}}_4 = \langle([0.22, 0.37], [0.31, 0.42]), (0.32, 0.39), 0.43\rangle.$$

Step 4: *Let the score value of each alternative* A_i $(i = 1,\ 2,\ 3,\ 4)$ $S(\widetilde{\widetilde{\widetilde{a}}}_i)$, $S(\tilde{\tilde{\tilde{b}}}_i)$, $S(\tilde{\tilde{\tilde{c}}}_i)$, $S(\tilde{\tilde{\tilde{d}}}_i)$ *and* $S(\tilde{\tilde{\tilde{e}}}_i)$. *Then from (4.4), (4.7), (4.8), (4.9), (4.10) and (4.11), we have the followings:*

$$S(\widetilde{\widetilde{\widetilde{a}}}_1) = -0.937450,\ S(\widetilde{\widetilde{\widetilde{a}}}_2) = -0.891893,\ S(\widetilde{\widetilde{\widetilde{a}}}_3) = -0.916037,\ S(\widetilde{\widetilde{\widetilde{a}}}_4) = -0.877611, \tag{4.12}$$
$$S(\tilde{\tilde{\tilde{b}}}_1) = 0.096178,\ S(\tilde{\tilde{\tilde{b}}}_2) = 0.196810,\ S(\tilde{\tilde{\tilde{b}}}_3) = 0.181943,\ S(\tilde{\tilde{\tilde{b}}}_4) = 0.217392, \tag{4.13}$$
$$S(\tilde{\tilde{\tilde{c}}}_1) = 0.045577,\ S(\tilde{\tilde{\tilde{c}}}_2) = 0.074093,\ S(\tilde{\tilde{\tilde{c}}}_3) = 0.067680,\ S(\tilde{\tilde{\tilde{c}}}_4) = 0.070614, \tag{4.14}$$
$$S(\widetilde{\widetilde{\widetilde{d}}}_1) = 0.110087,\ S(\widetilde{\widetilde{\widetilde{d}}}_2) = 0.243477,\ S(\widetilde{\widetilde{\widetilde{d}}}_3) = 0.224479,\ S(\widetilde{\widetilde{\widetilde{d}}}_4) = 0.255142, \tag{4.15}$$
$$S(\tilde{\tilde{\tilde{e}}}_1) = 0.025478,\ S(\tilde{\tilde{\tilde{e}}}_2) = 0.046259,\ S(\tilde{\tilde{\tilde{e}}}_3) = 0.040407,\ S(\tilde{\tilde{\tilde{e}}}_4) = 0.029634. \tag{4.16}$$

Thus from (4.12), (4.13), (4.14), (4.15) and (4.16), we obtain Table 1.

$$S(\widetilde{\widetilde{\widetilde{a}}}_4) \geq S(\widetilde{\widetilde{\widetilde{a}}}_2) \geq S(\widetilde{\widetilde{\widetilde{a}}}_3) \geq S(\widetilde{\widetilde{\widetilde{a}}}_1), \quad S(\tilde{\tilde{\tilde{b}}}_4) \geq S(\tilde{\tilde{\tilde{b}}}_2) \geq S(\tilde{\tilde{\tilde{b}}}_3) \geq S(\tilde{\tilde{\tilde{b}}}_1),$$
$$S(\tilde{\tilde{\tilde{c}}}_2) \geq S(\tilde{\tilde{\tilde{c}}}_4) \geq S(\tilde{\tilde{\tilde{c}}}_3) \geq S(\tilde{\tilde{\tilde{c}}}_1), \quad S(\widetilde{\widetilde{\widetilde{d}}}_4) \geq S(\widetilde{\widetilde{\widetilde{d}}}_2) \geq S(\widetilde{\widetilde{\widetilde{d}}}_3) \geq S(\widetilde{\widetilde{\widetilde{d}}}_1),$$
$$S(\tilde{\tilde{\tilde{e}}}_2) \geq S(\tilde{\tilde{\tilde{e}}}_3) \geq S(\tilde{\tilde{\tilde{e}}}_4) \geq S(\tilde{\tilde{\tilde{e}}}_1).$$

Table 1: Values and ranking orders of $S(\tilde{\tilde{\tilde{b}}}_i)$, $S(\tilde{\tilde{\tilde{c}}}_i)$, $S(\tilde{\tilde{\tilde{d}}}_i)$, and $S(\tilde{\tilde{\tilde{e}}}_i)$, $i = 1, 2, 3, 4$.

Score values				Ranking order
$S(\tilde{\tilde{\tilde{b}}}_1)$	$S(\tilde{\tilde{\tilde{b}}}_2)$	$S(\tilde{\tilde{\tilde{b}}}_3)$	$S(\tilde{\tilde{\tilde{b}}}_4)$	
0.0962	0.1968	0.1819	0.2174	.0 $S(\tilde{\tilde{\tilde{b}}}_4) \geq S(\tilde{\tilde{\tilde{b}}}_2) \geq S(\tilde{\tilde{\tilde{b}}}_3) \geq S(\tilde{\tilde{\tilde{b}}}_1)$
$S(\tilde{\tilde{\tilde{c}}}_1)$	$S(\tilde{\tilde{\tilde{c}}}_2)$	$S(\tilde{\tilde{\tilde{c}}}_3)$	$S(\tilde{\tilde{\tilde{c}}}_4)$	
0.0456	0.0741	0.0677	0.0706	.0 $S(\tilde{\tilde{\tilde{c}}}_2) \geq S(\tilde{\tilde{\tilde{c}}}_4) \geq S(\tilde{\tilde{\tilde{c}}}_3) \geq S(\tilde{\tilde{\tilde{c}}}_1)$
$S(\widetilde{\widetilde{\widetilde{d}}}_1)$	$S(\widetilde{\widetilde{\widetilde{d}}}_2)$	$S(\widetilde{\widetilde{\widetilde{d}}}_3)$	$S(\widetilde{\widetilde{\widetilde{d}}}_4)$	
0.1101	0.2435	0.2245	0.2551	.0 $S(\widetilde{\widetilde{\widetilde{d}}}_4) \geq S(\widetilde{\widetilde{\widetilde{d}}}_2) \geq S(\widetilde{\widetilde{\widetilde{d}}}_3) \geq S(\widetilde{\widetilde{\widetilde{d}}}_1)$
$S(\tilde{\tilde{\tilde{e}}}_1)$	$S(\tilde{\tilde{\tilde{e}}}_2)$	$S(\tilde{\tilde{\tilde{e}}}_3)$	$S(\tilde{\tilde{\tilde{e}}}_4)$	
0.0255	0.0463	0.0404	0.0296	.0 $S(\tilde{\tilde{\tilde{e}}}_2) \geq S(\tilde{\tilde{\tilde{e}}}_3) \geq S(\tilde{\tilde{\tilde{e}}}_4) \geq S(\tilde{\tilde{\tilde{e}}}_1)$

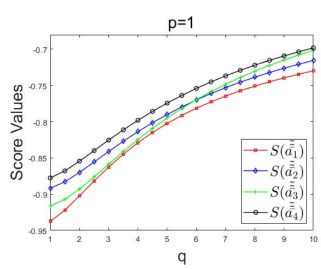
In Table 1, the values and ranking orders of $S(\tilde{\tilde{b}}_i)$, $S(\tilde{\tilde{c}}_i)$, $S(\tilde{\tilde{d}}_i)$, $S(\tilde{\tilde{e}}_i)$, $i = 1, 2, 3, 4$ are summarized.

In Table 2, the values and ranking orders of $S(\tilde{\tilde{a}}_i)$, $i = 1, 2, 3, 4$ with the parameters $p = 1, 2, 3, 4$ and $q = 1, 3, 5, 7, 9$ are summarized.

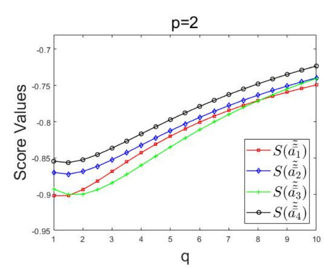
Table 2: Values and ranking orders of $S(\tilde{\tilde{a}}_i)$, $i = 1, 2, 3, 4$ with $p = 1, 2, 3, 4$ and $q = 1, 3, 5, 7, 9$.

p	q	$S(\widetilde{\tilde{a}_1})$	$S(\widetilde{\tilde{a}_2})$	$S(\widetilde{\tilde{a}_3})$	$S(\widetilde{\tilde{a}_4})$	Ranking order
$p=1$	$q=1$	-0.9374	-0.8919	-0.9160	-0.8776	$S(\widetilde{\tilde{a}_4}) \geq S(\widetilde{\tilde{a}_2}) \geq S(\widetilde{\tilde{a}_3}) \geq S(\widetilde{\tilde{a}_1})$
	$q=3$	-0.8624	-0.8410	-0.8585	-0.8253	$S(\widetilde{\tilde{a}_4}) \geq S(\widetilde{\tilde{a}_2}) \geq S(\widetilde{\tilde{a}_3}) \geq S(\widetilde{\tilde{a}_1})$
	$q=5$	-0.8023	-0.7900	-0.7948	-0.7743	$S(\widetilde{\tilde{a}_4}) \geq S(\widetilde{\tilde{a}_2}) \geq S(\widetilde{\tilde{a}_3}) \geq S(\widetilde{\tilde{a}_1})$
	$q=7$	-0.7646	-0.7529	-0.7483	-0.7367	$S(\widetilde{\tilde{a}_4}) \geq S(\widetilde{\tilde{a}_3}) \geq S(\widetilde{\tilde{a}_2}) \geq S(\widetilde{\tilde{a}_1})$
	$q=9$	-0.7394	-0.7262	-0.7148	-0.7092	$S(\widetilde{\tilde{a}_4}) \geq S(\widetilde{\tilde{a}_3}) \geq S(\widetilde{\tilde{a}_2}) \geq S(\widetilde{\tilde{a}_1})$
$p=2$	$q=1$	-0.9022	-0.8699	-0.8931	-0.8544	$S(\widetilde{\tilde{a}_4}) \geq S(\widetilde{\tilde{a}_2}) \geq S(\widetilde{\tilde{a}_3}) \geq S(\widetilde{\tilde{a}_1})$
	$q=3$	-0.8687	-0.8525	-0.8844	-0.8365	$S(\widetilde{\tilde{a}_4}) \geq S(\widetilde{\tilde{a}_2}) \geq S(\widetilde{\tilde{a}_1}) \geq S(\widetilde{\tilde{a}_3})$
	$q=5$	-0.8199	-0.8124	-0.8348	-0.7971	$S(\widetilde{\tilde{a}_4}) \geq S(\widetilde{\tilde{a}_2}) \geq S(\widetilde{\tilde{a}_1}) \geq S(\widetilde{\tilde{a}_3})$
	$q=7$	-0.7845	-0.7778	-0.7900	-0.7624	$S(\widetilde{\tilde{a}_4}) \geq S(\widetilde{\tilde{a}_2}) \geq S(\widetilde{\tilde{a}_1}) \geq S(\widetilde{\tilde{a}_3})$
	$q=9$	-0.7592	-0.7508	-0.7548	-0.7347	$S(\widetilde{\tilde{a}_4}) \geq S(\widetilde{\tilde{a}_2}) \geq S(\widetilde{\tilde{a}_3}) \geq S(\widetilde{\tilde{a}_1})$
$p=3$	$q=1$	-0.8624	-0.8410	-0.8585	-0.8253	$S(\widetilde{\tilde{a}_4}) \geq S(\widetilde{\tilde{a}_2}) \geq S(\widetilde{\tilde{a}_3}) \geq S(\widetilde{\tilde{a}_1})$
	$q=3$	-0.8560	-0.8474	-0.8875	-0.8316	$S(\widetilde{\tilde{a}_4}) \geq S(\widetilde{\tilde{a}_2}) \geq S(\widetilde{\tilde{a}_1}) \geq S(\widetilde{\tilde{a}_3})$
	$q=5$	-0.8207	-0.8203	-0.8580	-0.8059	$S(\widetilde{\tilde{a}_4}) \geq S(\widetilde{\tilde{a}_2}) \geq S(\widetilde{\tilde{a}_1}) \geq S(\widetilde{\tilde{a}_3})$
	$q=7$	-0.7904	-0.7909	-0.8188	-0.7769	$S(\widetilde{\tilde{a}_4}) \geq S(\widetilde{\tilde{a}_1}) \geq S(\widetilde{\tilde{a}_2}) \geq S(\widetilde{\tilde{a}_3})$
	$q=9$	-0.7673	-0.7657	-0.7846	-0.7511	$S(\widetilde{\tilde{a}_4}) \geq S(\widetilde{\tilde{a}_2}) \geq S(\widetilde{\tilde{a}_1}) \geq S(\widetilde{\tilde{a}_3})$
$p=4$	$q=1$	-0.8289	-0.8137	-0.8245	-0.7980	$S(\widetilde{\tilde{a}_4}) \geq S(\widetilde{\tilde{a}_2}) \geq S(\widetilde{\tilde{a}_3}) \geq S(\widetilde{\tilde{a}_1})$
	$q=3$	-0.8384	-0.8351	-0.8762	-0.8201	$S(\widetilde{\tilde{a}_4}) \geq S(\widetilde{\tilde{a}_2}) \geq S(\widetilde{\tilde{a}_1}) \geq S(\widetilde{\tilde{a}_3})$
	$q=5$	-0.8153	-0.8203	-0.8695	-0.8072	$S(\widetilde{\tilde{a}_4}) \geq S(\widetilde{\tilde{a}_1}) \geq S(\widetilde{\tilde{a}_2}) \geq S(\widetilde{\tilde{a}_3})$
	$q=7$	-0.7908	-0.7972	-0.8389	-0.7849	$S(\widetilde{\tilde{a}_4}) \geq S(\widetilde{\tilde{a}_1}) \geq S(\widetilde{\tilde{a}_2}) \geq S(\widetilde{\tilde{a}_3})$
	$q=9$	-0.7703	-0.7749	-0.8073	-0.7622	$S(\widetilde{\tilde{a}_4}) \geq S(\widetilde{\tilde{a}_1}) \geq S(\widetilde{\tilde{a}_2}) \geq S(\widetilde{\tilde{a}_3})$

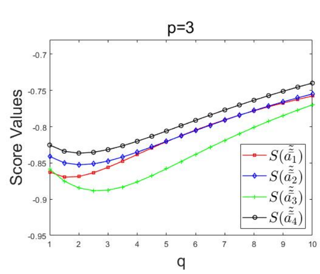
Figure 1 is the graphs of score values of $\tilde{\tilde{a}}_i$, $i = 1, 2, 3, 4$, when the parameter q vary from 1 to 10 by fixing the parameter $p = 1, 2, 3, 4$, respectively.



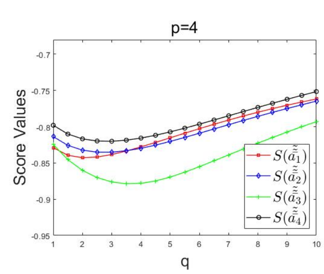
(a) Case 1: $p = 1$, $q = 1 \sim 10$



(b) Case 2: $p = 2$, $q = 1 \sim 10$



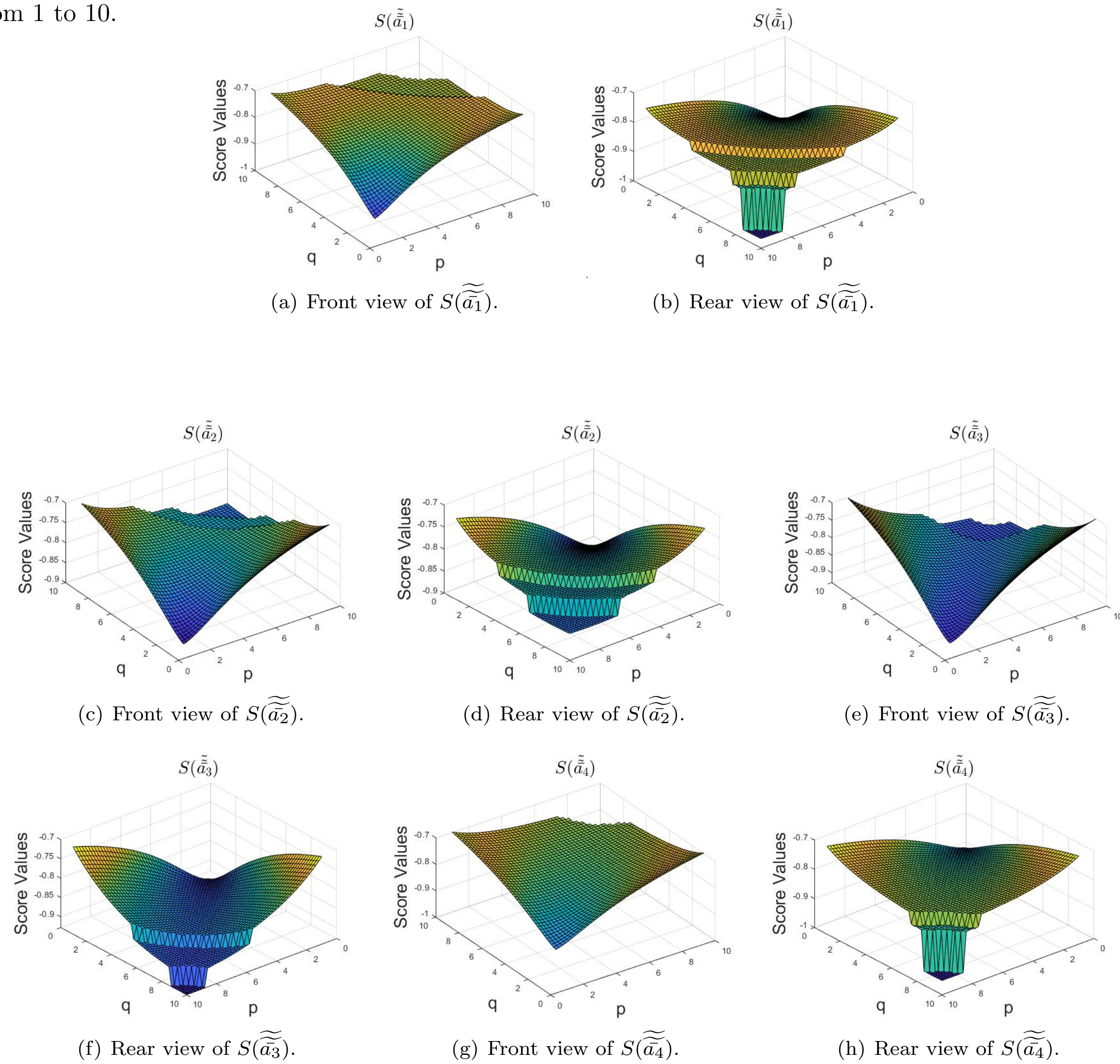
(c) Case 3: $p = 3$, $q = 1 \sim 10$



(d) Case 4: $p = 4$, $q = 1 \sim 10$

Figure 1: Score values of $\tilde{\tilde{a}}_i$, $i = 1, 2, 3, 4$ for different values of parameter q, when the parameter p is fixed.

Figure 2 is the graphs of score values of $\tilde{\tilde{\tilde{a}}}_i$, $i = 1, 2, 3, 4$, when the parameter p vary 1 to 10 and the parameter q vary from 1 to 10.



(a) Front view of $S(\widetilde{\widetilde{\widetilde{a_1}}})$. (b) Rear view of $S(\widetilde{\widetilde{\widetilde{a_1}}})$.

(c) Front view of $S(\widetilde{\widetilde{\widetilde{a_2}}})$. (d) Rear view of $S(\widetilde{\widetilde{\widetilde{a_2}}})$. (e) Front view of $S(\widetilde{\widetilde{\widetilde{a_3}}})$.

(f) Rear view of $S(\widetilde{\widetilde{\widetilde{a_3}}})$. (g) Front view of $S(\widetilde{\widetilde{\widetilde{a_4}}})$. (h) Rear view of $S(\widetilde{\widetilde{\widetilde{a_4}}})$.

Figure 2: Score values of $\tilde{\tilde{\tilde{a}}}_i$, $i = 1, 2, 3, 4$ for the variation of the parameters p and q.

5 Conclusions

We defined some aggregation operators based on IVI-octahedron sets, i.e., IVIOBM, IVIOA, IVIOG, GIVIOA and GIVIOG and proposed an algorithm applying them to MCGDM problems. Moreover, we gave a practical example to illustrate the application of some aggregation operators. In the future, we expect that one can apply IVI-octahedron sets to group and ring theories, BCI/BCK-algebras, topologies, category theory and decision-making problems by correlation coefficients or similarity measures, etc.

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