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Original paper

Restricted equivalence functions induced from fuzzy implication functions

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Abstract

Restricted equivalence function, as an effective tool for the theoretical research and practical applications of fuzzy sets and systems along with fuzzy logic, has been continuously considered by scholars since it was proposed. In particular, recently, Bustince, Campión, De Miguel et al. (H. Bustince, M.J. Campión, L. De Miguel, E. Induráin, Strong negations and restricted equivalence functions revisited: An analytical and topological approach, Fuzzy Sets and Systems (2021), https://doi.org/10.1016/j.fss.2021.10.013.) investigated it using analytical and topological approach and proposed an open problem to ask whether the binary function $F(x,y) = T(I(x,y), I(y,x))$ obtained from a t-norm T and a fuzzy implication function I is a restricted equivalence function or not. In this paper, we pay attention to this problem and give positive answer of it. Specifically, first, we consider the binary functions obtained from overlap functions and fuzzy implication functions by following the construction way of F and get the necessary and sufficient condition that makes such obtained F to be a restricted equivalence function. Second, we introduce the so-called ♡-functions, which are binary functions on unit closed interval with few additional axioms and obtain the necessary and sufficient condition that ensures the binary function constructed via any non-decreasing ♡-function and fuzzy implication function as the way of F to be a restricted equivalence function. Finally, we give the necessary and sufficient condition that makes F to be a restricted equivalence function.

Keywords: Restricted equivalence functions, fuzzy implication functions, triangular norms, overlap functions.

1 Introduction

Restricted equivalence functions, as the extension of equivalence functions proposed by Fodor and Roubens in [13], were introduced by Bustince, Barrenechea, and Pagola in [3] in 2006. After that time, they were often used to build similarity measures involved in fuzzy sets and interval-valued fuzzy sets when handling with the problems related to fuzzy sets and systems along with fuzzy logic, especially for their applications. For example, in 2007, Bustince, Barrenechea and Pagola [4] applied restricted equivalence functions to calculate the threshold of an image. In 2008, Bustince, Barrenechea and Pagola [5] investigated the relationship among restricted dissimilarity functions, restricted equivalence functions and normal E_N-functions. In 2009, Jurio, Pagola, Paternain et al. [15] introduced interval-valued restricted equivalence functions to measure the equivalence between the intervals associated to different pixels. In 2017, Palmeira and Bedregal [17] gave a characterization of interval-valued restricted equivalence functions on the lattice $L([0,1])$. In 2018, Palmeira, Bedregal, Bustince et al. [18] showed two different methods for extending lattice-valued restricted equivalence functions used for constructing similarity measures on L-fuzzy sets. In 2021, Miguel, Santiago, Wagner et al. [8] extended the concept of restricted equivalence function for type-2 fuzzy sets and used them to generate a similarity measure for type-2 fuzzy sets.

In particular, recently, for the sake of better enable researchers to understand the structure and construction of restricted equivalence functions, Bustince, Campión, De Miguel et al. [6] proposed an analytical and topological

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approach to study restricted equivalence functions. In addition, they raised an open problem as follows:

≪ ***If we are given a fuzzy implication function*** I ***and a t-norm*** T***, we may wonder whether the binary function*** $F : [0,1]^2 \longrightarrow [0,1]$ ***given by***

$$F(x,y) = T(I(x,y), I(y,x)) \ (x,y \in [0,1]),$$

is a restricted equivalence function on the unit closed interval $[0,1]$ ***or not.***≫

In this paper, we focus on this problem and seek its answer. To be more specific, the main general objectives of this paper are as follows.

- To consider the binary functions obtained from overlap functions and fuzzy implication functions by following the construction way of F whether are restricted equivalence functions or not, which supply us the manner to construct restricted equivalence functions on the basis of overlap functions and fuzzy implication functions.
- To investigate if F is a restricted equivalence function or not when the t-norms degenerate into some weaker cases.
- To search the conditions under which F can be a restricted equivalence function for arbitrary t-norm and fuzzy implication function.

The rest of the paper is organized as follows. In Section 2, we recall some needed concepts and existing results. In Section 3, we study the restricted equivalence functions derived from overlap functions and fuzzy implication functions. In Section 4, we investigate the restricted equivalence functions induced from t-norms and fuzzy implication functions. In the last section, this study is summarized.

2 Preliminaries

In this section, we list several essential known concepts and existing results, which include restricted equivalence functions, fuzzy implication functions, t-norms and overlap functions.

Definition 2.1. [3, 6] *A binary function* $RE : [0,1]^2 \longrightarrow [0,1]$ *is said to be a restricted equivalence function if, for any* $x, y, z \in [0,1]$*, it holds that:*

$(RE1)$ $RE(x,y) = RE(y,x)$;

$(RE2)$ $RE(x,y) = 1$ *iff* $x = y$;

$(RE3)$ $RE(x,y) = 0$ *iff* $(x,y) = (0,1)$ *or* $(x,y) = (1,0)$;

$(RE4)$ $RE(x,z) \leq RE(x,y)$ *whenever* $x \leq y \leq z$.

Definition 2.2. [1, 24] *A binary function* $I : [0,1]^2 \longrightarrow [0,1]$ *is called a fuzzy implication function if, for all* $x, y, z \in [0,1]$*, it holds that:*

$(I1)$ $I(x,z) \geq I(y,z)$ *whenever* $x \leq y$;

$(I2)$ $I(x,y) \leq I(x,z)$ *whenever* $y \leq z$;

$(I3)$ $I(0,0) = I(1,1) = 1$;

$(I4)$ $I(1,0) = 0$.

Definition 2.3. [1, 10, 19] *Let* $I : [0,1]^2 \longrightarrow [0,1]$ *be a fuzzy implication function. Then it is said to satisfy:*

(IP) *The identity principle, if* $I(x,x) = 1$ *for any* $x \in [0,1]$;

(OP) *The ordering property, if* $I(x,y) = 1 \Leftrightarrow x \leq y$ *for any* $x, y \in [0,1]$;

(SCC0) *The strong corner condition for 0, if* $I(x,y) = 0$ *implies* $x = 1$ *and* $y = 0$ *for any* $x, y \in [0,1]$.

Definition 2.4. [14, 16] *A binary function* $T : [0,1]^2 \longrightarrow [0,1]$ *is said to be a t-norm if, for all* $x, y, z \in [0,1]$*, it holds that:*

$(T1)$ $T(x,y) = T(y,x)$;

$(T2)$ $T(x, T(y, z)) = T(T(x, y), z)$;

$(T3)$ $T(x, y) \leq T(x, z)$ *whenever* $y \leq z$;

$(T4)$ $T(x, 1) = x$.

Moreover, a t-norm T is said to be

(T5) continuous if it is continuous in both arguments at the same time;

(T6) positive if $T(x, y) = 0$ implies either $x = 0$ or $y = 0$.

It is well known that, for any t-norm, there naturally exists a fuzzy implication function defined as follows.

Definition 2.5. [1, 16] *For any t-norm* $T : [0, 1]^2 \longrightarrow [0, 1]$, *the binary function* $I_T : [0, 1]^2 \longrightarrow [0, 1]$ *defined, for all* $x, y \in [0, 1]$, *by*

$$I_T(x, y) = \sup\{t \in [0, 1] \mid T(x, t) \leq y\},$$

is a fuzzy implication function and say that I_T *is the residual implication derived from* T.

In the following example, we show some certain t-norms and the residual implications derived from them, which are from [16, 22].

Example 2.6. (1) *The minimum t-norm* $T_M : [0, 1]^2 \longrightarrow [0, 1]$ *and its residual implication* $I_{T_M} : [0, 1]^2 \longrightarrow [0, 1]$ *are given, respectively, by*

$$T_M(x, y) = \min\{x, y\},$$

and

$$I_{T_M}(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{if } x > y. \end{cases}$$

(2) *The product t-norm* $T_P : [0, 1]^2 \longrightarrow [0, 1]$ *and its residual implication* $I_{T_P} : [0, 1]^2 \longrightarrow [0, 1]$ *are given, respectively, by*

$$T_P(x, y) = xy,$$

and

$$I_{T_P}(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ \frac{y}{x} & \text{if } x > y. \end{cases}$$

(3) *The Łukasiewicz t-norm* $T_L : [0, 1]^2 \longrightarrow [0, 1]$ *and its residual implication* $I_{T_L} : [0, 1]^2 \longrightarrow [0, 1]$ *are given, respectively, by*

$$T_L(x, y) = \max\{x + y - 1, 0\},$$

and

$$I_{T_L}(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ \min\{1 - x + y, 1\} & \text{if } x > y. \end{cases}$$

(4) *The nilpotent minimum t-norm* $T_{NM} : [0, 1]^2 \longrightarrow [0, 1]$ *and its residual implication* $I_{T_{NM}} : [0, 1]^2 \longrightarrow [0, 1]$ *are given, respectively, by*

$$T_{NM}(x, y) = \begin{cases} 0 & \text{if } x + y \leq 1, \\ \min\{x, y\} & \text{if } x + y > 1 \end{cases}$$

and

$$I_{T_{NM}}(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ \max\{1 - x, y\} & \text{if } x > y. \end{cases}$$

Definition 2.7. [7] *A binary function* $O : [0,1]^2 \longrightarrow [0,1]$ *is said to be an overlap function if, for any* $x, y, z \in [0,1]$, *it holds that:*

(O1) *O is commutative;*

(O2) $O(x,y) = 0$ *iff* $xy = 0$;

(O3) $O(x,y) = 1$ *iff* $xy = 1$;

(O4) $O(x,y) \leq O(x,z)$ *whenever* $y \leq z$;

(O5) *O is continuous.*

In the following example, we show some common overlap functions, which are from [2, 9, 20].

Example 2.8. (1) *Any positive continuous t-norm is an overlap function.*

(2) *The function* $O_{mM} : [0,1]^2 \longrightarrow [0,1]$ *given by*

$$O_{mM}(x,y) = \min\{x,y\}\max\{x^2,y^2\},$$

is an overlap function.

(3) *For any* $p > 0$, *the function* $O_p : [0,1]^2 \longrightarrow [0,1]$ *given by*

$$O_p(x,y) = x^p y^p,$$

is an overlap function.

(4) *The function* $O_{Mid} : [0,1]^2 \longrightarrow [0,1]$ *given by*

$$O_{Mid}(x,y) = xy\frac{x+y}{2},$$

is an overlap function.

(5) *The function* $O_{DB} : [0,1]^2 \longrightarrow [0,1]$ *given by*

$$O_{DB}(x,y) = \begin{cases} \frac{2xy}{x+y} & \text{if } x+y \neq 0, \\ 0 & \text{if } x+y = 0, \end{cases}$$

is an overlap function.

3 Restricted equivalence functions derived from overlap functions and fuzzy implication functions

In this section, we focus on the construction method of restricted equivalence functions on the basis of overlap functions and fuzzy implication functions. The major conclusion of this section is that we get the necessary and sufficient condition that makes the binary function constructed via any overlap function and fuzzy implication function as the way of the open problem proposed by Bustince, Campión, De Miguel et al. in [6] to be a restricted equivalence function.

Theorem 3.1. *Let* $O : [0,1]^2 \longrightarrow [0,1]$ *be an overlap function and* $I : [0,1]^2 \longrightarrow [0,1]$ *be a fuzzy implication function. Then the following statements are equivalent.*

(i) *I satisfies* ***(OP)*** *and* ***(SCC0)***.

(ii) *The binary function* $RE_{O,I} : [0,1]^2 \longrightarrow [0,1]$ *given, for each* $x, y \in [0,1]$, *by*

$$RE_{O,I}(x,y) = O(I(x,y), I(y,x)), \tag{1}$$

is a restricted equivalence function.

Proof. **(i) implies (ii):** Notice that the proof method of this conclusion is similar with the way taken in the proof of Proposition 6.3 in [6] and, for the sake of completeness, here, we verify that $RE_{O,I}$ satisfies conditions (RE1)–(RE4) of Definition 2.1 as follows.

(RE1) It follows immediately from item (O1) of Definition 2.7 and the definition of $RE_{O,I}$.

(RE2) For $x, y \in [0,1]$, it follows from item (O3) of Definition 2.7 and **(OP)** that

$$\begin{aligned} RE_{O,I}(x,y) = 1 &\Leftrightarrow O(I(x,y), I(y,x)) = 1 \\ &\Leftrightarrow I(x,y) = 1 \text{ and } I(y,x) = 1 \\ &\Leftrightarrow x \le y \text{ and } y \le x \\ &\Leftrightarrow x = y. \end{aligned}$$

(RE3) For $x, y \in [0,1]$, it follows from item (O2) of Definition 2.7 and **(SCC0)** that

$$\begin{aligned} RE_{O,I}(x,y) = 0 &\Leftrightarrow O(I(x,y), I(y,x)) = 0 \\ &\Leftrightarrow I(x,y) = 0 \text{ or } I(y,x) = 0 \\ &\Leftrightarrow (x,y) = (1,0) \text{ or } (y,x) = (1,0) \\ &\Leftrightarrow (x,y) = (1,0) \text{ or } (x,y) = (0,1). \end{aligned}$$

(RE4) For any $x, y, z \in [0,1]$ with $x \le y \le z$, it follows from item (O4) of Definition 2.7 and **(OP)** that

$$\begin{aligned} RE_{O,I}(x,z) &= O(I(x,z), I(z,x)) \\ &= O(1, I(z,x)) \\ &\le O(1, I(y,x)) \\ &= O(I(x,y), I(y,x)) \\ &= RE_{O,I}(x,y). \end{aligned}$$

(ii) implies (i): First, for any $x \in [0,1]$, it follows from item (RE2) of Definition 2.1 and item (O3) of Definition 2.7 that

$$\begin{aligned} RE_{O,I}(x,x) = 1 &\Rightarrow O(I(x,x), I(x,x)) = 1 \\ &\Rightarrow I(x,x) = 1. \end{aligned}$$

Thus, one concludes that I satisfies **(IP)**.

Furthermore, for any $x, y \in [0,1]$ with $x \le y$, it follows from item (I1) of Definition 2.2 and **(IP)** that

$$I(x,y) \ge I(y,y) = 1,$$

that is, $I(x,y) = 1$.

Conversely, for any $x, y \in [0,1]$, if $I(x,y) = 1$, then it must be $x \le y$. Otherwise, if $y < x$, then, by a similar way as that of above second paragraph, one has that $I(y,x) = 1$. And thus, using item (RE2) of Definition 2.1 and item (O3) of Definition 2.7, it follows from

$$RE_{O,I}(x,y) = O(I(x,y), I(y,x)) = 1,$$

that $x = y$, which is a contradiction with $y < x$.

Therefore, to sum up, one concludes that I satisfies **(OP)**.

On the other hand, for any $x, y \in [0,1]$, if $I(x,y) = 0$, then, from item (O2) of Definition 2.7, it holds that

$$RE_{O,I}(x,y) = O(I(x,y), I(y,x)) = 0.$$

Moreover, using item (RE3) of Definition 2.1, one gets that

$$(x,y) = (0,1) \text{ or } (x,y) = (1,0).$$

And thus, one asserts that $x = 1$ and $y = 0$, that is, I satisfies **(SCC0)**. Otherwise, if $x = 0$ and $y = 1$, then, from items (I2) and (I3) of Definition 2.2, it holds that

$$I(x, y) = I(0, 1) = 1,$$

which is a contradiction.

Based on the above discussion, one obtains that I satisfies **(OP)** and **(SCC0)**.

□

Remark 3.2. *Theorem 3.1 shows a common approach to obtain restricted equivalence function on the basis of overlap functions and fuzzy implication functions. For example, the binary function* $RE_{O_p,I_{T_{NM}}} : [0,1]^2 \longrightarrow [0,1]$ *given, for each* $x, y \in [0,1]$*, by*

$$RE_{O_p,I_{T_{NM}}}(x,y) = \begin{cases} 1 & \text{if } x = y, \\ (\max\{x, 1-y\})^p & \text{if } x < y, \\ (\max\{y, 1-x\})^p & \text{if } x > y \end{cases}$$

is the restricted equivalence function obtained from the overlap function O_p *given in item (3) of Example 2.8 and fuzzy implication function* $I_{T_{NM}}$ *given in item (4) of Example 2.6.*

4 Restricted equivalence functions derived from t-norms and fuzzy implication functions

In this section, we consider the conditions under which the binary function constructed as the way of the open problem proposed by Bustince, Campión, De Miguel et al. in [6] is a restricted equivalence function for any t-norm and fuzzy implication function. To deal the open problem, first, we introduce the concept of ♡-functions, which are binary functions on unit closed interval with few additional axioms. Second, we obtain the necessary and sufficient condition that makes the binary function constructed via any non-decreasing ♡-function and fuzzy implication function as the way of the open problem proposed by Bustince, Campión, De Miguel et al. in [6] to be a restricted equivalence function. Third, we give the necessary and sufficient condition that ensures the binary function as shown in the open problem proposed by Bustince, Campión, De Miguel et al. in [6] to be a restricted equivalence function.

Definition 4.1. *A binary function* $CN : [0,1]^2 \longrightarrow [0,1]$ *is said to be a ♡-function if, for any* $x \in [0,1]$*, it holds that:*

$(CN1)$ $CN(x, 1) = x$;

$(CN2)$ CN *is commutative.*

Remark 4.2. *Notice that the concept of ♡-functions defined in Definition 4.1 has a close relationship with the concept of t-seminorms*[1] *(t-seminorm is also named as semicopula in the literature such as [12, 11] etc.) proposed by Suárez Garcia and Gil Álvarez in [21] and the concept of pseudo-t-norms*[2] *proposed by Wang and Yu in [23]. Actually, any non-decreasing ♡-function is a commutative t-seminorm and a commutative pseudo-t-norm simultaneously and vice versa.*

Proposition 4.3. *Let* $CN : [0,1]^2 \longrightarrow [0,1]$ *be a ♡-function and* $I : [0,1]^2 \longrightarrow [0,1]$ *be a fuzzy implication function. Consider the following statements.*

(i) I *satisfies* ***(OP)*** *and* ***(SCC0)****.*

(ii) *The binary function* $RE_{CN,I} : [0,1]^2 \longrightarrow [0,1]$ *given, for each* $x, y \in [0,1]$*, by*

$$RE_{CN,I}(x, y) = CN(I(x, y), I(y, x)), \tag{2}$$

is a restricted equivalence function.

[1] A binary function $S : [0,1]^2 \longrightarrow [0,1]$ is said to be a t-seminorm if it is non-decreasing in each variable and $S(x, 1) = S(1, x) = x$ for any $x \in [0,1]$ [21].

[2] A binary function $R : [0,1]^2 \longrightarrow [0,1]$ is said to be a pseudo-t-norm if it satisfies that $R(1, x) = x, R(0, x) = 0$ and $y \le z$ implies that $R(x, y) \le R(x, z)$ for any $x, y, z \in [0,1]$ [23].

Then (i) $\Rightarrow$ (ii). Moreover, if CN is non-decreasing, then (i) $\Leftrightarrow$ (ii).

Proof. **(i) implies (ii):** We prove that $RE_{CN,I}$ satisfies conditions (RE1)–(RE4) of Definition 2.1 as follows.

(RE1) It follows immediately from item (CN2) of Definition 4.1 and the definition of $RE_{CN,I}$.

(RE2) For $x, y \in [0,1]$ with $x = y$, using item (CN1) of Definition 4.1 and **(OP)**, it can be checked in a similar way as the proof of (RE2) part in "(i) implies (ii) side of Theorem 3.1 that $RE_{CN,I}(x,y) = 1$."

Conversely, for $x, y \in [0,1]$, if $RE_{CN,I}(x,y) = 1$, then, one can draw the following verifications.

Case 1. If $x \le y$, then, by **(OP)**, it holds that $I(x,y) = 1$. Furthermore, by item (CN1) of Definition 4.1 and **(OP)**, one has that

$$\begin{aligned} RE_{CN,I}(x,y) = 1 &\Rightarrow CN(I(x,y), I(y,x)) = 1 \\ &\Rightarrow CN(1, I(y,x)) = 1 \\ &\Rightarrow I(y,x) = 1 \\ &\Rightarrow y \le x. \end{aligned}$$

Thus, one concludes that $x = y$.

Case 2. If $y < x$, then, using item (CN1) of Definition 4.1 and **(OP)**, it can be checked in a similar way as above Case 1 that $x \le y$, which is a contradiction.

Therefore, one concludes that $RE_{CN,I}(x,y) = 1$ iff $x = y$.

(RE3) For $x, y \in [0,1]$, it follows from item (I4) of Definition 2.2, **(OP)** and item (CN1) of Definition 4.1 that

$$\begin{aligned} (x,y) = (0,1) \text{ or } (x,y) = (1,0) &\Rightarrow (x,y) = (0,1) \text{ or } (y,x) = (0,1) \\ &\Rightarrow I(x,y) = 1, I(y,x) = 0 \text{ or } I(x,y) = 0, I(y,x) = 1 \\ &\Rightarrow CN(I(x,y), I(y,x)) = 0 \\ &\Rightarrow RE_{CN,I}(x,y) = 0. \end{aligned}$$

Conversely, for $x, y \in [0,1]$, if $RE_{CN,I}(x,y) = 0$, then, using item (CN1) of Definition 4.1, **(OP)** and **(SCC0)**, it can be checked in a similar way as above Case 1 that $(x,y) = (0,1)$ when $x \le y$ and $(x,y) = (1,0)$ when $y < x$. And thus, one concludes that $RE_{CN,I}(x,y) = 0$ implies that $(x,y) = (0,1)$ or $(x,y) = (1,0)$.

Therefore, one gets that $RE_{CN,I}(x,y) = 0$ iff $(x,y) = (0,1)$ or $(x,y) = (1,0)$.

(RE4) For any $x, y, z \in [0,1]$ with $x \le y \le z$, using item (CN1) of Definition 4.1 and **(OP)**, it can be proven in a similar way as the proof of (RE4) part in "(i) implies (ii)" side of Theorem 3.1 that $RE_{CN,I}(x,z) \le RE_{CN,I}(x,y)$.

(ii) implies (i): It can be proved in a similar way as the proof of "(ii) implies (i)" side of Theorem 3.1 under the condition of that CN is non-decreasing.

□

Theorem 4.4. *Let $T : [0,1]^2 \longrightarrow [0,1]$ be a commutative t-seminorm or a commutative pseudo-t-norm or a t-norm and $I : [0,1]^2 \longrightarrow [0,1]$ be a fuzzy implication function. Then the following statements are equivalent.*

*(i) I satisfies **(OP)** and **(SCC0)**.*

(ii) The binary function $RE_{T,I} : [0,1]^2 \longrightarrow [0,1]$ given, for each $x, y \in [0,1]$, by

$$RE_{T,I}(x,y) = T(I(x,y), I(y,x)), \tag{3}$$

is a restricted equivalence function.

Proof. Based on Remark 4.2 and the fact that any t-norm is a non-decreasing ♡-function, it follows immediately from Proposition 4.3. □

Remark 4.5. *(i) Eq. (3) becomes the restricted equivalence function given in Proposition 6.3 of literature [6] when T is taken as T_M in Theorem 4.4.*

(*ii*) *Proposition 4.3 gives a common approach to obtain restricted equivalence function on the basis of ♡-functions and fuzzy implication functions. For example, the binary function* $RE_{CN,I_{T_L}} : [0,1]^2 \longrightarrow [0,1]$ *given, for each* $x, y \in [0,1]$, *by*

$$RE_{CN,I_{T_L}}(x,y) = \begin{cases} 1 & \text{if } x = y, \\ \min\{1-y+x, 1\} & \text{if } x < y, \\ \min\{1-x+y, 1\} & \text{if } x > y \end{cases}$$

is the restricted equivalence function obtained from the ♡-function $CN : [0,1]^2 \longrightarrow [0,1]$ *given, for each* $x, y \in [0,1]$, *by*

$$CN(x,y) = \begin{cases} y & \text{if } x = 1, \\ x & \text{if } y = 1, \\ 1 & \text{otherwise} \end{cases}$$

and fuzzy implication function I_{T_L} *given in item (3) of Example 2.6.*

(*iii*) *Theorem 4.4 gives a common approach to obtain restricted equivalence function on the basis of t-norms and fuzzy implication functions. For example, the binary function* $RE_{T_{NM},I_{T_{NM}}} : [0,1]^2 \longrightarrow [0,1]$ *given, for each* $x, y \in [0,1]$, *by*

$$RE_{T_{NM},I_{T_{NM}}}(x,y) = \begin{cases} 1 & \text{if } x = y, \\ \max\{x, 1-y\} & \text{if } x < y, \\ \max\{y, 1-x\} & \text{if } x > y \end{cases}$$

is the restricted equivalence function obtained from the t-norm T_{NM} *and fuzzy implication function* $I_{T_{NM}}$ *given in item (4) of Example 2.6. Actually,* $RE_{T_{NM},I_{T_{NM}}}$ *equals to* $RE_{O_1,I_{T_{NM}}}$ *given in Remark 3.2.*

5 Concluding remarks

This work mainly considers the open problem proposed by Bustince, Campión, De Miguel et al. in [6] for the constructions of restricted equivalence functions by means of t-norms and fuzzy implication functions. The main results of this work are as follows.

- We obtain the necessary and sufficient condition under which the binary functions derived from overlap functions and fuzzy implication functions following the construction way as in [6] for t-norms and fuzzy implication functions are the restricted equivalence functions.
- After introducing the concept of ♡-functions, we get the necessary and sufficient condition under which the binary function $RE_{T,I}$ given, for any non-decreasing ♡-function T and fuzzy implication function I, as $RE_{T,I}(x,y) = T(I(x,y), I(y,x))$ is a restricted equivalence function.
- We obtain the necessary and sufficient condition under which the binary function $RE_{T,I}$ is a restricted equivalence function for any t-norm T and fuzzy implication function I.

For further work, we plan to search the properties of the restricted equivalence functions derived from overlap functions (resp. ♡-functions) and fuzzy implication functions. In addition, the applications of the restricted equivalence functions obtained in this work in practical application problems also is a further research topic.

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References

[1] M. Baczyński, B. Jayaram, *Fuzzy implications*, Springer, Berlin, 2008.

[2] B. Bedregal, G. P. Dimuro, H. Bustince, E. Barrenechea, *New results on overlap and grouping functions*, Information Sciences, **249** (2013), 148-170.

[3] H. Bustince, E. Barrenechea, M. Pagola, *Restricted equivalence functions*, Fuzzy Sets and Systems, **157**(17) (2006), 2333-2346.

[4] H. Bustince, E. Barrenechea, M. Pagola, *Image thresholding using restricted equivalence functions and maximizing the measures of similarity*, Fuzzy Sets and Systems, **158**(5) (2007), 495-51.

[5] H. Bustince, E. Barrenechea, M. Pagola, *Relationship between restricted dissimilarity functions, restricted equivalence functions and normal E_N-functions: Image thresholding invariant*, Pattern Recognition Letters, **29**(4) (2008), 525-536.

[6] H. Bustince, M. J. Campión, L. De Miguel, E. Induráin, *Strong negations and restricted equivalence functions revisited: An analytical and topological approach*, Fuzzy Sets and Systems, **441** (2022), 110-129.

[7] H. Bustince, J. Fernández, R. Mesiar, J. Montero, R. Orduna, *Overlap functions*, Nonlinear Analysis, **72**(3) (2010), 1488-1499.

[8] L. De Miguel, R. Santiago, C. Wagner, J. M. Garibaldi, Z. Takáč, A. F. R. L. de Hierro, H. Bustince, *Extension of restricted equivalence functions and similarity measures for type-2 fuzzy sets*, IEEE Transactions on Fuzzy Systems, **30**(9) (2022), 4005-4016.

[9] G. P. Dimuro, B. Bedregal, *Archimedean overlap functions: The ordinal sum and the cancellation, idempotency and limiting properties*, Fuzzy Sets and Systems, **252** (2014), 39-54.

[10] G. P. Dimuro, B. Bedregal, R. H. N. Santiago, *On (G, N)-implications derived from grouping functions*, Information Sciences, **279** (2014), 1-17.

[11] F. Durante, J. Quesada-Molina, C. Sempi, *Semicopulas: Characterizations and applicability*, Kybernetika, **42**(8) (2006), 287-302.

[12] F. Durante, C. Sempi, *Semicopulæ*, Kybernetika, **41**(3) (2005), 315-328.

[13] J. Fodor, M. Roubens, *Fuzzy preference modelling and multicriteria decision support*, Springer Science and Business Media, 1994.

[14] P. Helbin, *On some categories of triangular norms on the real unit interval*, Iranian Journal of Fuzzy Systems, **19**(5) (2022), 183-198.

[15] A. Jurio, M. Pagola, D. Paternain, C. Lopez-Molina, P. Melo-Pinto, *Interval-valued restricted equivalence functions applied on clustering techniques*, Proceedings of the Joint 2009, International Fuzzy Systems Association World Congress and 2009, European Society of Fuzzy Logic and Technology Conference, Lisbon, Portugal, July 20-24, 2009 DBLP, (2009).

[16] E. P. Klement, R. Mesiar, E. Pap, *Triangular norms*, Kluwer Academic Publisher, Dordrecht, 2000.

[17] E. S. Palmeira, B. Bedregal, *Restricted equivalence function on $L([0, 1])$*, in: P. Melin, O. Castillo, J. Kacprzyk, M. Reformat, W. Melek (eds.), Fuzzy Logic in Intelligent System Design, NAFIPS 2017, Advances in Intelligent Systems and Computing, Springer, Cham, **648** (2018), 410-420.

[18] E. S. Palmeira, B. Bedregal, H. Bustince, D. Paternain, L. D. Miguel, *Application of two different methods for extending latticevalued restricted equivalence functions used for constructing similarity measures on L-fuzzy sets*, Information Sciences, **441** (2018), 95-112.

[19] J. Qiao, *$R_{\mathscr{O}}$-implications induced from C_L-overlap functions on complete lattices*, Soft Computing, **26**(17) (2022), 8229-8243.

[20] J. Qiao, B. Q. Hu, *On interval additive generators of interval overlap functions and interval grouping functions*, Fuzzy Sets and Systems, **323** (2017), 19-55.

[21] F. Suárez Garcia, P. Gil Álvarez, *Two families of fuzzy integrals*, Fuzzy Sets and Systems, **18**(1) (1986), 67-81.

[22] J. T. Wang, X. L. Xin, *Monadic algebras of an involutive monoidal t-norm based logic*, Iranian Journal of Fuzzy Systems, **19**(3) (2022), 187-202.

[23] Z. Wang, Y. Yu, *Pseudo-t-norms and implication operators on a complete Brouwerian lattice*, Fuzzy Sets and Systems, **132**(1) (2002), 113-124.

[24] Y. Zhang, D. C. Li, *Fuzzy implications satisfying the generalized hypothetical syllogism based on the Bandler-Kohout subproduct with semicopulas*, Iranian Journal of Fuzzy Systems, (2022). DOI: 10.22111/IJFS.2022.7280.