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Original paper

On linearly ordered index sets for ordinal sums in the sense of A. H. Clifford yielding uninorms

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Abstract

This paper focuses on the topic of ordinal sums of semigroups in the sense of A. H. Clifford – a method for constructing a new semigroup from a given system of semigroups indexed by a linearly ordered index set. We completely describe the linearly ordered index set for an ordinal sum of semigroups yielding a uninorm.

Keywords: Uninorm, ordinal sum, order.

1 Introduction, basic notions and results

Equipped with a uninorm as binary operation, the unit interval is linearly ordered Abelian semigroup with neutral element $e \in [0,1]$. Uninorms were introduced by Yager and Rybalov in [12] with the idea of allowing certain kind of aggregation functions combining the maximum and the minimum, depending on an element $e \in [0,1]$. This operation U is known as a *triangular norm* (t-norm for short) whenever $e = 1$, while as a *triangular conorm* (t-conorm for short) whenever $e = 0$. For any other value $e \in]0,1[$, the structure of U on $[0,e]^2$ and $[e,1]^2$ is closely related to t-norms and t-conorms. Uninorms have proved to be useful not only in aggregation process but in fields such as decision making, information fusion, subjective evaluations, optimization and control, expert systems, neural networks, fuzzy systems modelling, pseudo-analysis and measure/integral theory, image processing and approximate reasoning, etc (see the recent survey [7] for details).

The ordinal sum of two disjoint posets $(X_1, \leq_1)$ and $(X_2, \leq_2)$ is defined by taking the following order relation on $X_1 \cup X_2$: $x \leq y$ if and only if ($x, y \in X_1$ and $x \leq_1 y$, or $x, y \in X_2$ and $x \leq_2 y$, or $x \in X_1$ and $y \in X_2$). It was introduced by Birkhoff [1] in 1940, and the original source of ordinal sums.

Following Climescu's idea [3], Clifford [2] introduced ordinal sums of semigroups as a method for constructing a new semigroup from a given system of semigroups indexed by a linearly ordered index set. Here, we recall the following fundamental result [5, Theorem 3.42] that generalizes Clifford's result.

Theorem 1.1. *Let $(A, \preceq)$ be a linearly ordered set with $A \neq \emptyset$ and $(G_\alpha)_{\alpha \in A}$ with $G_\alpha = (X_\alpha, *_\alpha)$ a family of semigroups. Assume that for all $\alpha, \beta \in A$ with $\alpha \prec \beta$ the sets X_α and X_β are either disjoint or that $X_\alpha \cap X_\beta = \{x_{\alpha\beta}\}$, where $x_{\alpha\beta}$ is both the neutral element of $*_\alpha$ and the annihilator of $*_\beta$, and where for each $\gamma \in A$ with $\alpha \prec \gamma \prec \beta$ we have $X_\gamma = \{x_{\alpha\beta}\}$. Put $X = \bigcup_{\alpha \in A} X_\alpha$ and define the binary operation $*$ on X by*

$$x * y = \begin{cases} x *_\alpha y & \text{if } (x,y) \in X_\alpha^2, \\ x & \text{if } (x,y) \in X_\alpha \times X_\beta \text{ and } \alpha \prec \beta, \\ y & \text{if } (x,y) \in X_\alpha \times X_\beta \text{ and } \beta \prec \alpha. \end{cases} \tag{1}$$

Then $(X, *)$ is a semigroup, and it is called the ordinal sum of semigroups $((X_\alpha, *_\alpha))_{\alpha \in A}$. Moreover, $(X, *)$ is a commutative semigroup if and only if, for each $\alpha \in A$, the semigroup $(X_\alpha, *_\alpha)$ is commutative.

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A t-subnorm [4] is a binary operation $T : [0,1]^2 \to [0,1]$ such that $([0,1],T)$ is an abelian, linearly ordered semigroup which satisfies $T(x,y) \le \min(x,y)$ for all $x,y \in [0,1]$. Dually, a t-supconorm is a binary operation $S : [0,1]^2 \to [0,1]$ such that $([0,1],S)$ is an abelian, linearly ordered semigroup which satisfies $S(x,y) \ge \max(x,y)$ for all $x,y \in [0,1]$.

We can apply here the result of Clifford to uninorms for they are special semigroups. Jenei [4, Corollary 2] introduced the *ordinal sum of t-subnorms* as follows:

Proposition 1.2. [4, Corollary 2] *Let A be a finite or countably infinite index set and $(]a_\alpha,b_\alpha[)_{\alpha\in A}$ be a system of disjoint open subintervals of $[0,1]$. Let $(V_\alpha)_{\alpha\in A}$ be a family of t-subnorms such that if $b_{\alpha_0}=1$ for some $\alpha_0 \in A$ then V_{α_0} is a t-norm, and if $b_{\alpha_1}=a_{\alpha_2}$ for some $\alpha_1,\alpha_2 \in A$, then either V_{α_1} is a t-norm or V_{α_2} has no zero divisors. Then the following function $T : [0,1]^2 \to [0,1]$ is a t-norm:*

$$T(x,y)=\begin{cases} a_\alpha+(b_\alpha-a_\alpha)V_\alpha\left(\frac{x-a_\alpha}{b_\alpha-a_\alpha},\frac{y-a_\alpha}{b_\alpha-a_\alpha}\right) & \text{if } (x,y)\in]a_\alpha,b_\alpha]^2,\\ \min(x,y) & \text{otherwise.}\end{cases}$$

The resulting t-norm T will be referred to as an ordinal sum of t-subnorms.

Klement, Mesiar, and Pap [6, Theorem 3.1] pointed out that the construction above is the most general way to obtain a t-norm as an ordinal sum of semigroups. For a uninorm U with a neutral element e, several necessary conditions to obtain $([0,1],U)$ as an ordinal sum of semigroups were presented in [8, Proposition 7]:

Proposition 1.3. *Let $(A,\preceq)$ be a linearly ordered set with $A\ne\emptyset$ and $((X_\alpha,*_\alpha))_{\alpha\in A}$ be a family of semigroups such that $(X_\alpha)_{\alpha\in A}$ is a partition of the closed unit interval $[0,1]$. If operation $* : [0,1]^2\to[0,1]$ given by (1) is a uninorm with the neutral element $e\in[0,1]$ then we have:*

(i) *Each X_α has a form $X_\alpha = I_\alpha \cup J_\alpha$, where I_α is a subinterval of $[0,e]$ and J_α is a subinterval of $[e,1]$.*

(ii) *Each semigroup $(X_\alpha,*_\alpha)$ is a linearly ordered Abelian semigroup, where for the operation $*_\alpha$ we have $x *_\alpha y \le \min(x,y)$ for all $x,y\in I_\alpha$; $x *_\alpha y \ge \max(x,y)$ for all $x,y\in J_\alpha$; and $x *_\alpha y \in [\min(x,y),\max(x,y)]$ for all $x\in I_\alpha$, $y\in J_\alpha$ and all $x\in J_\alpha$, $y\in I_\alpha$.*

(iii) *The order $\preceq$ on A is compatible with the usual order $\le$ on $[0,e]$ and reversed to the usual order $\le$ on $[e,1]$, i.e., for $\alpha,\beta\in A$ we have $\alpha\prec\beta$ if and only if $x<y$ for all $x\in I_\alpha$ and $y\in I_\beta$ and $u>v$ for all $u\in J_\alpha$ and $v\in J_\beta$.*

Several special cases for an ordinal sum construction yielding uninorms were studied. For example, Mesiarová-Zemánková [8] discussed the cases when X_α is either $[a_\alpha,b_\alpha[\cup]c_\alpha,d_\alpha]$ or a singleton; Su, Zong and Mesiarová-Zemánková [11] investigated the cases when X_α has one of the five forms (singleton, closed interval, open interval, left-open interval, right-open interval). This paper is a continuation of [8, 11] and focuses on the most general cases, that is, each X_α has the form as (i) in Proposition 1.3. We specify the linear order on the index set A.

2 Main results

In [11], the cases when each X_α is either a singleton or an interval were discussed, and the index set A is divided into three parts: $\alpha_\top$ (an element of A such that $e\in X_{\alpha_\top}$), $\underline{A}=\{\alpha\in A \mid X_\alpha\subseteq[0,e[\}$ and $\overline{A}=\{\alpha\in A\mid X_\alpha\subseteq]e,1]\}$, that is to say, $A=\underline{A}\cup\{\alpha_\top\}\cup\overline{A}$. However, for the most general cases (see Proposition 1.3), we do not always have $A=\underline{A}\cup\{\alpha_\top\}\cup\overline{A}$ (see Example 1 below) because each X_α has a form $X_\alpha=I_\alpha\cup J_\alpha$, where I_α is a subinterval of $[0,e]$ and J_α is a subinterval of $[e,1]$. To investigate the most general cases, we introduce the following symbols:

$$\begin{aligned}&\alpha_\top : \text{an element of } A \text{ such that } e\in X_{\alpha_\top},\\ &\underset{\sim}{A}=\{\alpha\in A\mid X_\alpha\cap[0,e[\ne\emptyset\}\setminus\{\alpha_\top\},\\ &\tilde{A}=\{\alpha\in A\mid X_\alpha\subseteq[e,1]\}\setminus\{\alpha_\top\}.\end{aligned}$$

Hence $A=\underset{\sim}{A}\cup\{\alpha_\top\}\cup\tilde{A}$.

Proposition 2.1. *Under the same hypotheses of Proposition 1.3 (i) and (ii), if $* : [0,1]^2\to[0,1]$, defined by (1), is a uninorm with neutral element e, then*

(i) *$\alpha\preceq\alpha_\top$ for all $\alpha\in A$.*

(ii) *If $\alpha, \beta \in \underset{\sim}{A}$, then $\alpha \prec \beta$ if and only if there exist $x_0 \in I_\alpha$ and $y_0 \in I_\beta$ such that $x_0 < y_0$. In particular, if $J_\alpha \neq \emptyset$ and $J_\beta \neq \emptyset$, then $\alpha \prec \beta$ if and only if there exist $x_0 \in J_\alpha$ and $y_0 \in J_\beta$ such that $x_0 > y_0$.*

(iii) *If $\alpha, \beta \in \tilde{A}$, then $\alpha \prec \beta$ if and only if there exist $x_0 \in J_\alpha$ and $y_0 \in J_\beta$ such that $x_0 > y_0$.*

(iv) *If $\alpha \in \underset{\sim}{A}$, $\beta \in \tilde{A}$ and $J_\alpha \neq \emptyset$, then $\alpha \prec \beta$ if and only if there exist $x_0 \in J_\alpha$ and $y_0 \in J_\beta$ such that $x_0 > y_0$.*

Observe that the condition that there exist $x_0 \in I_\alpha$ and $y_0 \in I_\beta$ such that $x_0 < y_0$ is equivalent to that $x < y$ for all $x \in I_\alpha$ and $y \in I_\beta$ because I_α and I_β are subintervals of $[0, e]$.

Proof. (i) If $\alpha \in A$ and $\alpha \neq \alpha_\top$, then $e * x = x$ for any $x \in X_\alpha$ and further $\alpha \prec \alpha_\top$ by (1).

(ii) Consider $\alpha, \beta \in \underset{\sim}{A}$. If there exist $x_0 \in I_\alpha$ and $y_0 \in I_\beta$ such that $x_0 < y_0$, then $x_0 < y_0 < e$ and $x_0 * y_0 \leq \min(x_0, y_0) = x_0$ because $*$ is a uninorm with neutral element e. By (1), we then have $x_0 * y_0 = x_0$ and further $\alpha \prec \beta$. The reverse implication is directly derived from Theorem 1.1 and the fact that $*$ is a uninorm with neutral element e. Assume that $J_\alpha \neq \emptyset$ and $J_\beta \neq \emptyset$, then there exist $x_0 \in J_\alpha$ and $y_0 \in J_\beta$ with $x_0, y_0 \geq e$. If $\alpha \prec \beta$, then $U(x_0, y_0) = x_0$ by Theorem 1.1, which, together with the fact that $*$ is a uninorm with neutral element e, gives $x_0 > y_0$. If $x_0 > y_0$, then $U(x_0, y_0) \geq \max(x_0, y_0) = x_0$ and further $U(x_0, y_0) = x_0$ by Theorem 1.1. Thus, $\alpha \prec \beta$.

(iii) The proof is similar to that of (ii).

(iv) Consider $\alpha \in \underset{\sim}{A}$, $\beta \in \tilde{A}$ and $J_\alpha \neq \emptyset$. If there exist $x_0 \in J_\alpha$ and $y_0 \in J_\beta$ such that $x_0 > y_0$ then $x_0 > y_0 > e$ and further $x_0 * y_0 \geq \max(x_0, y_0) = x_0$. Thus, $x_0 * y_0 = x_0$ by (1), implying $\alpha \prec \beta$. The reverse implication is directly derived from Theorem 1.1. □

We will denote $\mathcal{L}_A$ the set of all linear orders on A that fulfill the conditions (i)–(iv) in Proposition 2.1.

Theorem 2.2. *Under the same hypotheses of Proposition 1.3 (i) and (ii), suppose that $* : [0,1]^2 \to [0,1]$ is defined by (1). Then $*$ is a uninorm with neutral element e if and only if $\preceq \in \mathcal{L}_A$ and $(X_{\alpha_\top}, *_{\alpha_\top})$ is a linearly ordered abelian semigroup with neutral element $e \in]0,1[$.*

Proof. The necessity follows from Proposition 2.1.

Conversely, suppose that $\preceq \in \mathcal{L}_A$. Plainly, $*$ is an operation of an abelian semigroup. Since $\alpha \preceq \alpha_\top$ for all $\alpha \in A$ and $(X_{\alpha_\top}, *_{\alpha_\top})$ is a linearly ordered abelian semigroup with neutral element e, we then have that e is the neutral element of $*$. Suppose that $x \in X_\alpha$, $y \in X_\beta$ and $z \in X_\gamma$ with $y < z$. To prove the monotonicity of $*$, we consider the following cases:

- If $\alpha = \beta = \gamma$, then $x * y = x *_\alpha y \leq x *_\alpha z = x * z$ for $(X_\alpha, *_\alpha)$ is a linearly ordered Abelian semigroup.
- If $\alpha \neq \beta$ and $\beta = \gamma$, then either $x * y = x = x * z$ (in this case, $\alpha \prec \beta$) or $x * y = y < z = x * z$ (in this case, $\beta \prec \alpha$).
- If $\beta \prec \gamma$, then $y < e$ (the case $y = e$ implies $\beta \succeq \gamma$ by Proposition 2.1(i) and the case $z > y > e$ implies $\beta \succeq \gamma$ by Proposition 2.1(ii)–(iv)).
  - If $\alpha \prec \beta \prec \gamma$, then $x * y = x = x * z$.
  - If $\alpha = \beta \prec \gamma$, then $x * z = x$, $x * y \leq \min(x, y)$ whenever $x \in I_\alpha$ and $x * y \in [y, x]$ whenever $x \in J_\alpha$. Thus, $x * y \leq x * z$.
  - If $\beta \prec \alpha \prec \gamma$, then $x * y = y$ and $x * z = x$. If $x \geq e$, then $x * y < x * z$. If $x < e$, then, by $\beta \prec \alpha$ and (ii) in Proposition 2.1, $y < x$ and further $x * y < x * z$.
  - If $\beta \prec \gamma \prec \alpha$, then $x * y = y < z = x * z$.
  - If $\beta \prec \alpha = \gamma$, then $x * y = y$. Since $(X_\alpha, *_\alpha)$ is a linearly ordered Abelian semigroup, we obtain $x * z \in I_\alpha \cup J_\alpha$. From $\beta \prec \alpha$, we have $y < u$ for all $u \in I_\alpha$. Thus, $x * z > y = x * y$.
- If $\beta \succ \gamma$, then a proof similar to that of the case $\beta \prec \gamma$ can show that $x * y \leq x * z$.

Summarizing the above cases, we know that $*$ is increasing in each variable. □

Let $\underset{\sim}{A}^* := \{\alpha \in \underset{\sim}{A} \mid J_\alpha = \emptyset\}$. In virtue of Proposition 2.1, we only need to specify the linear order $\preceq$ on $\underset{\sim}{A}^* \times \tilde{A} \cup \tilde{A} \times \underset{\sim}{A}^*$. Before doing this, several special cases should be illustrated:

- If $\widetilde{A}=\emptyset$ (implying $A=\underset{\sim}{A}\cup\{\alpha_\top\}$), then $*$ is a uninorm with neutral element e if and only if conditions (i) and (ii) in Proposition 2.1 hold. In this case, the order structure on the index set A is clear.
- If $\widetilde{A}\neq\emptyset$ and $\underset{\sim}{A}^*=\emptyset$, then $*$ is a uninorm with neutral element e if and only if conditions (i)– (iv) in Proposition 2.1 hold. In this case, the order structure on the index set A is also clear.

Next, we investigate the case $\widetilde{A}\neq\emptyset$ and $\underset{\sim}{A}^*\neq\emptyset$.

Proposition 2.3. *Under the same hypotheses of Proposition 1.3 (i) and (ii), suppose that* $*:[0,1]^2\to[0,1]$*, defined by* (1)*, is a uninorm with neutral element* e*. Consider the mapping* $\mathbf{g}:\underset{\sim}{A}^*\to\widetilde{A}\cup\{\alpha_\top\}$ *defined by*

$$\mathbf{g}(\alpha)=\inf\{\beta\in\widetilde{A}\cup\{\alpha_\top\}\mid *=\min \ \textit{on}\ X_\alpha\times X_\beta\}. \tag{2}$$

Then

(i) $\mathbf{g}$ *is an increasing mapping.*

(ii) *If* $\alpha\in\underset{\sim}{A}$, $\beta\in\widetilde{A}$ *and* $\mathbf{g}(\alpha)\prec\beta$, *then* $\alpha\prec\beta$.

(iii) *If* $\alpha\in\underset{\sim}{A}$, $\beta\in\widetilde{A}$ *and* $\beta\prec\mathbf{g}(\alpha)$, *then* $\beta\prec\alpha$.

Proof. (i) Suppose that $\alpha_1,\alpha_2\in\underset{\sim}{A}^*$ with $\alpha_1\prec\alpha_2$. Then, by Proposition 2.1(ii), we have $x_1<x_2$ for any $x_1\in X_{\alpha_1}$ and $x_2\in X_{\alpha_2}$. The monotonicity of $*$ yields

$$\{\beta\in\widetilde{A}\cup\{\alpha_\top\}\mid *=\min\ \text{on}\ X_{\alpha_2}\times X_\beta\}\subseteq\{\beta\in\widetilde{A}\cup\{\alpha_\top\}\mid *=\min\ \text{on}\ X_{\alpha_1}\times X_\beta\},$$

implying that $\mathbf{g}$ is an increasing mapping.

(ii) If $\beta\prec\alpha$, then $*=\max$ on $X_\alpha\times X_\beta$, which, together with the monotonicity of $*$ and Proposition 2.1(iii), gives $*=\max$ on $X_\alpha\times X_\gamma$ for all $\gamma\in\widetilde{A}$ with $\gamma\prec\beta$. As a consequence, $\beta\preceq\mathbf{g}(\alpha)$, contradicting $\mathbf{g}(\alpha)\prec\beta$.

(iii) The proof is similar to that of (ii). □

For any increasing mapping $\mathbf{g}:\underset{\sim}{A}^*\to\widetilde{A}\cup\{\alpha_\top\}$, we can define a linear order $\preceq$ on A as follows:

$$\alpha\prec\beta \text{ if } \begin{cases} \text{either } \alpha\neq\alpha_\top \text{ and } \beta=\alpha_\top,\\ \text{or } \alpha,\beta\in\underset{\sim}{A} \text{ and there exist } x\in I_\alpha \text{ and } y\in I_\beta \text{ such that } x<y,\\ \text{or } \alpha,\beta\in\widetilde{A} \text{ and there exist } x\in J_\alpha \text{ and } y\in J_\beta \text{ such that } x>y,\\ \text{or } \alpha\in\underset{\sim}{A}\backslash\underset{\sim}{A}^*,\beta\in\widetilde{A} \text{ and there exist } x\in J_\alpha \text{ and } y\in J_\beta \text{ such that } x>y,\\ \text{or } \alpha\in\widetilde{A},\beta\in\underset{\sim}{A}\backslash\underset{\sim}{A}^* \text{ and there exist } x\in J_\alpha \text{ and } y\in J_\beta \text{ such that } x<y,\\ \text{or } \alpha\in\underset{\sim}{A}^*,\beta\in\widetilde{A} \text{ and } \mathbf{g}(\alpha)\prec\beta,\\ \text{or } \alpha\in\widetilde{A},\beta\in\underset{\sim}{A}^* \text{ and } \alpha\prec\mathbf{g}(\beta), \end{cases} \tag{3}$$

and in the cases ($\alpha\in\underset{\sim}{A}^*,\beta\in\widetilde{A}$ and $\beta=\mathbf{g}(\alpha)$) and ($\alpha\in\widetilde{A},\beta\in\underset{\sim}{A}^*$ and $\alpha=\mathbf{g}(\beta)$), either $\alpha\prec\beta$ or $\beta\prec\alpha$ is defined.

From Theorem 2.2 and Proposition 2.3, the following corollary follows:

Corollary 2.4. *Under the same hypotheses of Proposition 1.3 (i) and (ii), suppose that* $*:[0,1]^2\to[0,1]$ *is defined by* (1)*. Then* $*$ *is a uninorm with neutral element* e *if and only if there exist an increasing mapping* $\mathbf{g}$ *such that the linear order* $\preceq$ *on* A *is defined by* (3) *and* $(X_{\alpha_\top},*_{\alpha_\top})$ *is a linearly ordered abelian semigroup with neutral element* $e\in]0,1[$.

Finally, we illustrate our results by an example from [8, Example 1].

Example 2.5. *Let* $A = \{1,2,3,4,5\}$ *be an index set, and let* $G_1 = ([0, \frac{1}{4}] \cup \{\frac{3}{4}\}, *_1)$, $G_2 = (\{\frac{1}{2}\}, \min)$, $G_3 = (]\frac{1}{4}, \frac{1}{2}[, \min)$, $G_4 = (]\frac{1}{2}, \frac{3}{4}[, \max)$, $G_5 = (]\frac{3}{4}, 1], \max)$ *be commutative semigroups, where* $*_1$ *is given by where* $*_\alpha$ *is for* $x \le y$ *given by*

$$x *_1 y = \begin{cases} \frac{3}{4} & \text{if } x = y = \frac{3}{4}, \\ x & \text{if } x \in \left[0, \frac{1}{8}\right], y = \frac{3}{4}, \\ \frac{1}{4} & \text{if } x \in \left]\frac{1}{8}, \frac{1}{4}\right], y = \frac{3}{4}, \\ \min\left(x, \frac{1}{8}\right) \cdot \min\left(y, \frac{1}{8}\right) & \text{if } (x,y) \in \left[0, \frac{1}{4}\right]. \end{cases}$$

Next, we show how to obtain a uninorm with neutral element $\frac{1}{2}$ *by the ordinal sum of* $\{G_i\}_{i=1,2,3,4,5}$.

(a) *Plainly,* $\underset{\sim}{A} = \{1,3\}$, $\underset{\sim}{A}^* = \{3\}$, $\widetilde{A} = \{4,5\}$ *and* $\alpha_\top = 2$.

(b) *From Proposition 2.1, we must have that 2 is the largest element of* A, $1 \prec 3$ *as* $\underset{\sim}{A} = \{1,3\}$, $5 \prec 1 \prec 4$ *because of* $J_1 \neq \emptyset$, $\widetilde{A} = \{4,5\}$.

(c) *Consider the operation* $\mathbf{g} : \{3\} \to \{2,4,5\}$ *is given by* $\mathbf{g}(3) = 5$. *By Eq.* (3), *we then obtain a linear order* $5 \prec 1 \prec 3 \prec 4 \prec 2$, *and the resulting uninorm* U *is given for* $x \le y$ *by*

$$U(x,y) = \begin{cases} \min\left(x, \frac{1}{8}\right) \cdot \min\left(y, \frac{1}{8}\right) & \text{if } (x,y) \in \left[0, \frac{1}{4}\right]^2, \\ \min(x,y) & \text{if } (x,y) \in \left[0, \frac{1}{2}\right]^2 \setminus \left[0, \frac{1}{4}\right]^2, \\ \max(x,y) & \text{if } (x,y) \in \left[\frac{1}{2}, 1\right], \\ \min(x,y) & \text{if } x \in \left[0, \frac{1}{2}\right], y \in \left[\frac{1}{2}, \frac{3}{4}\right[, \\ \max(x,y) & \text{if } x \in \left[0, \frac{1}{2}\right], y \in \left]\frac{3}{4}, 1\right], \\ x & \text{if } x \in \left[0, \frac{1}{8}\right], y = \frac{3}{4}, \\ \frac{1}{4} & \text{if } x \in \left]\frac{1}{8}, \frac{1}{4}\right], y = \frac{3}{4}, \\ y & \text{if } x \in \left]\frac{1}{4}, \frac{1}{2}\right], y = \frac{3}{4}. \end{cases}$$

Moreover, we have $\underline{A} = \{3\}$, $\overline{A} = \{4,5\}$ *and* $\alpha_\top = 2$. *However,* $\bigcup_{i=2}^{5} X_i \neq [0,1]$.

3 Conclusions

Clifford [2] introduced ordinal sums of semigroups as a method for constructing a new semigroup from a given system of semigroups indexed by a linearly ordered index set. Several special cases for an ordinal sum construction yielding uninorms were studied [8, 11]. This paper is a continuation of [8, 11]. We focused on the most general cases and specified the linear order on the index set A.

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