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Interval-valued q-rung orthopair fuzzy integrals and their application in multi-criteria group decision making

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Abstract

The generalized interval-valued orthopair fuzzy sets provide an extension of Yager's generalized orthopair fuzzy sets, where membership and non-membership degrees are subsets of closed interval $[0,1]$. Due to the uncertainty and ambiguity of real life, it is more superior for decision makers to provide their judgments by intervals rather than crisp numbers. Moreover, in the era of huge scale and rapid updating of information, individual weights have been quietly diluted, and the integration of information one by one is time-consuming and complicated. In recent years, some scholars have conducted research on the calculus of generalized orthopair fuzzy sets, but no research has further revealed the intrinsic connection between the integrals of generalized interval-valued orthopair fuzzy sets and traditional aggregation operators, which is very important in applications such as large group decision making. In order to fill this theoretical gap, this paper aims to study the integrals of generalized interval-valued orthopair fuzzy functions. In detail, we define the indefinite integral starting from the inverse operations of the interval-valued q-rung orthopair fuzzy functions (IVq-ROFFs)' derivatives, and some fundamental properties with rigorous mathematical proofs are also discussed. To be more practical, we continue to develop definite integrals for both simplified and generalized IVq-ROFFs. Besides, we give the corresponding Newton-Leibniz formula through limit procedure, which shows the calculation relationship between the indefinite and definite integrals of the IVq-ROFFs. After obtaining the basic calculus results under generalized interval-valued orthopair fuzzy circumstance, we further reveal the inherent link between the integrals of generalized IVq-ROFFs and the traditional discrete aggregation operators. Finally, the practicability and feasibility of the proposed definite integral models are illustrated by an example of public health emergency group decision-making, and sensitivity analysis and comparison are also carried out.

Keywords: Fuzzy sets, decision making, aggregation operators, information fusion.

1 Introduction

Multi-attribute group decision making (MAGDM) is one of the most important branches of modern decision theory. A large number of decision-making problems in management science, operation research, economics and engineering are conducted by decision-making groups [5, 10, 12, 30, 31]. In general, MAGDM is an activity that evaluates alternatives by a group of decision makers (DMs) and determines the most appropriate alternatives accordingly. One of the key issues in MAGDM is how to represent information about the attributes given by the DMs. Another key issue is how to aggregate information about attributes and provide an alternative ranking. With regard to the first issue, orthopair fuzzy set (OFS) [3, 19] is a powerful tool to handle vagueness and uncertainty in real life. OFSs describe their membership grades as a pair of values from the unit interval, one of which represents support membership and the other represents against membership. The two most typical examples of orthopair fuzzy sets are intuitionistic fuzzy sets (IFSs) [2] and Pythagorean fuzzy sets (PFSs) [24].

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The main difference between these two types of fuzzy sets is that the sum of the support and against memberships required by IFS does not exceed one, while the PFS only requires that the square of the support and against memberships be limited to one. Therefore, when considering some practical problems, PFS can better handle fuzzy information and solve many decision-making problems that IFS cannot solve [22, 29]. Then, Yager [25, 26] introduced another general class of orthopair fuzzy set, called q-rung orthopair fuzzy set (q-ROFS), which requires that the sum of the qth power of support and against membership be less than or equal to one. A significant benefit of this relaxation is that it gives DMs more freedoms to provide information about the membership grads, which makes q-ROFSs a wider range of practical applications. The only restriction is that if the support for or against membership is one, the other must be zero. That is, the orthopair $\langle 1, 0\rangle$ can be considered a true or complete membership, and the orthopair $\langle 0, 1\rangle$ can be considered a false or completely non-membership. We can find that the q-ROFs are generalization of IFSs and PFSs.

Because q-ROFS has a wider constraint space and stronger modeling capabilities. Recently, many scholars have studied the aggregation of q-rung orthopair fuzzy numbers (q-ROFNs), which is also the second key problem in GDM. Firstly, some basic operations of q-ROFNs were presented [4, 7, 16, 18] to facilitate calculation. Then, various aggregation operators were also proposed, such as the q-rung orthopair fuzzy generalized Heronian mean and geometric Heronian mean operators [21]; the q-rung orthopair fuzzy Archimedean Bonferroni mean (BM) operator and the q-rung orthopair fuzzy weighted Archimedean BM operator [15]; the q-rung orthopair fuzzy power Maclaurin symmetric mean (MSM) operator and the q-rung orthopair fuzzy power weighted MSM operator [14]; the q-rung orthopair fuzzy $I_\delta, M_{\delta,v}$ and $K_{\delta,v}$ operators [1]; the q-rung orthopair fuzzy G-MSM (q-ROFGMSM) and q-rung orthopair fuzzy Geo-MSM (q-ROFGGMSM) operators [17]; the generalized q-rung orthopair fuzzy Einstein interactive weighted geometric operator [6]; the q-rung orthopair fuzzy N-soft weighted average (q-ROFNSWA) operator and q-rung orthopair fuzzy N-soft weighted geometric (q-ROFNSWG) operator [28] etc. However, the q-rung orthopair fuzzy information aggregation techniques mentioned above still have some shortcomings: (i) These surveys focus only on the aggregation of relatively fewer and discrete q-ROFNs; (ii) In these aggregation methods, the weight of each q-ROFN has a considerable influence on the aggregation result.

In addition, although q-ROFS has shown its applicability in many aspects, its limitations have gradually emerged. In some cases, due to insufficient information, it is difficult for DMs to accurately quantify their judgements with crisp numbers. In this situation, it is more convenient for the DMs to provide their judgements by a subset of the closed interval [0,1]. Therefore, Joshi et al. proposed the concept of interval-valued q-rung orthopair fuzzy set (IVq-ROFS) [11], whose membership and nonmembership degrees are all intervals rather than two real numbers, and some set operations were also studied, for example: negation, union and intersection. Then, Gao et al.[9] discussed four basic operations of interval-valued q-rung orthopair fuzzy values (IVq-ROFVs) and introduced the detailed definition of interval-valued q-rung orthopair fuzzy functions (IVq-ROFFs). Wan et al.[20] studied interval-valued q-rung orthopair fuzzy choquet integral operators and their application in group decision-making. Liang et al.[13] generalized interval-valued q-rung orthopair fuzzy integral (IVq-ROFI) to three-way decision. But there is no research has further revealed the intrinsic connection between the IVq-ROFI and the traditional aggregation operators, which is very important in large group decision making. In order to fill this theoretical gap, in this paper, we focus on the IVq-ROFIs and discuss their connections in detail. It is the most important and fundamental part of the interval-valued q-rung orthopair fuzzy calculus (IVq-ROFC) theory system, and has direct and powerful applications in reality.

To this end, the structure of this paper is as follows: Section 2 makes some preparations for the whole work. Section 3 focuses on the indefinite integrals of IVq-ROFFs. In addition, some of basic integral properties are discussed. In Section 4, we investigate the definite integrals of the IVq-ROFFs and discuss their desirable characteristics. Using the definition of q-ROFIL in [8], we give the corresponding Newton-Leibniz formula through limit procedure, which shows the calculation relationship between the indefinite and definite integrals of IVq-ROFFs. It is worth noting that the definite integrals have much more applications in real life. Later, we show that the properties of IVq-ROFIs can also be applied to the generalized IVq-ROFFs case, which allows us to use it accurately and immediately in the aggregation methods. In Section 5, a practical example concerning group decision making of public health emergency is given to illustrate the application of the IVq-ROFIs. Finally, we end up this paper with some concluding remarks in Section 6.

2 Preliminaries

As a preparation for further discussions, we first review and provide some relevant concepts, definitions and results for IVq-ROFS, IVq-ROFV and IVq-ROFF.

Definition 2.1. [9] *An IVq-ROFS $\mathbb{S}$ over the set X is defined as:*

$$\mathbb{S} = \{\langle x, \widetilde{\mu}_{\mathbb{S}}(x), \widetilde{v}_{\mathbb{S}}(x)\rangle \mid x \in X\},$$

where $\widetilde{\mu}_{\mathbb{S}}(x)$ *and* $\widetilde{v}_{\mathbb{S}}(x)$ *are subsets of* $[0,1]$, *and the following conditions are required:*

$$\sup_{x\in X}(\widetilde{\mu}_{\mathbb{S}}(x))^q+\sup_{x\in X}(\widetilde{v}_{\mathbb{S}}(x))^q\leq 1 \quad \text{for each} \quad q\geq 1.$$

Let

$$\underline{\mu}=\inf_{x\in X}\widetilde{\mu}_{\mathbb{S}}(x)\leq\sup_{x\in X}\widetilde{\mu}_{\mathbb{S}}(x)=\overline{\mu} \quad \text{and} \quad \underline{v}=\inf_{x\in X}\widetilde{v}_{\mathbb{S}}(x)\leq\sup_{x\in X}\widetilde{v}_{\mathbb{S}}(x)=\overline{v}.$$

We express the IVq-ROFS as the interval form

$$\mathbb{S}=\left\{\langle x,\,[\underline{\mu},\overline{\mu}],[\underline{v},\overline{v}]\rangle\,|\,x\in X\right\}.$$

Furthermore, the IVq-ROFV is denoted by

$$\alpha=\langle[\underline{\mu}_\alpha,\overline{\mu}_\alpha],[\underline{v}_\alpha,\overline{v}_\alpha]\rangle,$$

and let $\mathbb{S}$ *be the set consisting of all IVq-ROFVs.*

Remark 2.2. *In special case of* $\underline{\mu}_\alpha=\overline{\mu}_\alpha$ *and* $\underline{v}_\alpha=\overline{v}_\alpha$, *we arrive at the ordinary q-ROFN*

$$\alpha=\langle[\underline{\mu}_\alpha,\overline{\mu}_\alpha],[\underline{v}_\alpha,\overline{v}_\alpha]\rangle\triangleq\langle\mu_\alpha,v_\alpha\rangle.$$

In the following, we introduce some basic operations of IVq-ROFVs, including $\oplus$, $\ominus$, $\otimes$, $\oslash$, and so on. They are the basis for analysis and calculation throughout the study.

Definition 2.3. [9] *Let* $\alpha=\left\langle\left[\underline{\mu}_\alpha,\overline{\mu}_\alpha\right],[\underline{v}_\alpha,\overline{v}_\alpha]\right\rangle$ *and* $\beta=\left\langle\left[\underline{\mu}_\beta,\overline{\mu}_\beta\right],[\underline{v}_\beta,\overline{v}_\beta]\right\rangle$ *be two IVq-ROFVs. Then*

$$\alpha\oplus\beta=\left\langle\left[\left(1-(1-\underline{\mu}_\alpha^q)(1-\underline{\mu}_\beta^q)\right)^{\frac{1}{q}},\left(1-(1-\overline{\mu}_\alpha^q)(1-\overline{\mu}_\beta^q)\right)^{\frac{1}{q}}\right],[\underline{v}_\alpha\underline{v}_\beta,\overline{v}_\alpha\overline{v}_\beta]\right\rangle, \tag{1}$$

$$\alpha\otimes\beta=\left\langle\left[\underline{\mu}_\alpha\underline{\mu}_\beta,\overline{\mu}_\alpha\overline{\mu}_\beta\right],\left[\left(1-(1-\underline{v}_\alpha^q)(1-\underline{v}_\beta^q)\right)^{\frac{1}{q}},\left(1-(1-\overline{v}_\alpha^q)(1-\overline{v}_\beta^q)\right)^{\frac{1}{q}}\right]\right\rangle. \tag{2}$$

The inverse operations of (1)-(2) *are defined respectively as follows:*

$$\beta\ominus\alpha=\left\langle\left[\left(\frac{\underline{\mu}_\beta^q-\underline{\mu}_\alpha^q}{1-\underline{\mu}_\alpha^q}\right)^{\frac{1}{q}},\left(\frac{\overline{\mu}_\beta^q-\overline{\mu}_\alpha^q}{1-\overline{\mu}_\alpha^q}\right)^{\frac{1}{q}}\right],\left[\frac{\underline{v}_\beta}{\underline{v}_\alpha},\frac{\overline{v}_\beta}{\overline{v}_\alpha}\right]\right\rangle, \tag{3}$$

with the conditions

$$(\underline{\mu}_\beta^q-\underline{\mu}_\alpha^q)(1-\overline{\mu}_\alpha^q)\leq(1-\underline{\mu}_\alpha^q)(\overline{\mu}_\beta^q-\overline{\mu}_\alpha^q),\ \ \underline{v}_\beta\overline{v}_\alpha\leq\overline{v}_\beta\underline{v}_\alpha,\ \ \frac{\overline{v}_\beta}{\overline{v}_\alpha}\leq\frac{(1-\overline{\mu}_\beta^q)^{\frac{1}{q}}}{(1-\overline{\mu}_\alpha^q)^{\frac{1}{q}}}\leq 1;$$

$$\beta\oslash\alpha=\left\langle\left[\frac{\underline{\mu}_\beta}{\underline{\mu}_\alpha},\frac{\overline{\mu}_\beta}{\overline{\mu}_\alpha}\right],\left[\left(\frac{\underline{v}_\beta^q-\underline{v}_\alpha^q}{1-\underline{v}_\alpha^q}\right)^{\frac{1}{q}},\left(\frac{\overline{v}_\beta^q-\overline{v}_\alpha^q}{1-\overline{v}_\alpha^q}\right)^{\frac{1}{q}}\right]\right\rangle, \tag{4}$$

with the conditions

$$\underline{\mu}_\beta\overline{\mu}_\alpha\leq\overline{\mu}_\beta\underline{\mu}_\alpha,\ \ (\underline{v}_\beta^q-\underline{v}_\alpha^q)(1-\overline{v}_\alpha^q)\leq(1-\underline{v}_\alpha^q)(\overline{v}_\beta^q-\overline{v}_\alpha^q),\ \ \frac{\overline{\mu}_\beta}{\overline{\mu}_\alpha}\leq\frac{(1-\overline{v}_\beta^q)^{\frac{1}{q}}}{(1-\overline{v}_\alpha^q)^{\frac{1}{q}}}\leq 1.$$

Furthermore, we have the following scalar-multiplication and power-multiplication, that is,

$$\lambda\alpha=\left\langle\left[(1-(1-\underline{\mu}_\alpha^q)^\lambda)^{\frac{1}{q}},(1-(1-\overline{\mu}_\alpha^q)^\lambda)^{\frac{1}{q}}\right],\left[\underline{v}_\alpha^\lambda,\overline{v}_\alpha^\lambda\right]\right\rangle, \tag{5}$$

and

$$\alpha^\lambda=\left\langle\left[\underline{\mu}_\alpha^\lambda,\overline{\mu}_\alpha^\lambda\right],\left[(1-(1-\underline{v}_\alpha^q)^\lambda)^{\frac{1}{q}},(1-(1-\overline{v}_\alpha^q)^\lambda)^{\frac{1}{q}}\right]\right\rangle, \tag{6}$$

for all $\lambda\geq 0$.

Additionally, we introduce two common aggregation techniques for IVq-ROFVs:

Definition 2.4. [9] *Assume that* $\alpha_i = \left\langle \left[\underline{\mu}_{\alpha_i}, \overline{\mu}_{\alpha_i}\right], \left[\underline{v}_{\alpha_i}, \overline{v}_{\alpha_i}\right] \right\rangle$ $(i = 1, 2, \cdots, n)$ *are a collection of IVq-ROFVs, and assume that the weight function* w_i *satisfies* $w_i \in [0, 1]$ *and* $\sum_{j=1}^{n} w_i = 1$. *Then, we define the interval-valued q-rung orthopair fuzzy weighted averaging (IVq-ROFWA) operator as follows:*

$$IVq-ROFWA : (\alpha_1, \alpha_2, \cdots, \alpha_n) \triangleq \oplus_{i=1}^{n} w_i \alpha_i,$$

where the right-hand term has the specific expression as:

$$\oplus_{i=1}^{n} w_i \alpha_i = \left\langle \left[\left(1 - \prod_{i=1}^{n} (1 - \underline{\mu}_{\alpha_i}^{q})^{w_i}\right)^{\frac{1}{q}}, \left(1 - \prod_{i=1}^{n} (1 - \overline{\mu}_{\alpha_i}^{q})^{w_i}\right)^{\frac{1}{q}} \right], \left[\prod_{i=1}^{n} \underline{v}_{\alpha_i}^{w_i}, \prod_{i=1}^{n} \overline{v}_{\alpha_i}^{w_i} \right] \right\rangle, \tag{7}$$

with $\prod$ *being the product of the multi-elements.*

Similarly, the following definition can be obtained:

Definition 2.5. [9] *Assume that* α_i *and* w_i *are the same as in Definition 2.4. Then, we define the interval-valued q-rung orthopair fuzzy weighted geometric (IVq-ROFWG) operator as follows:*

$$IVq-ROFWG : (\alpha_1, \alpha_2, \cdots, \alpha_n) \triangleq \otimes_{i=1}^{n} \alpha_i^{w_i},$$

where the right-hand terms are explicitly expressed as:

$$\otimes_{i=1}^{n} \alpha_i^{w_i} = \left\langle \left[\prod_{i=1}^{n} \underline{\mu}_{\alpha_i}^{w_i}, \prod_{i=1}^{n} \overline{\mu}_{\alpha_i}^{w_i} \right], \left[\left(1 - \prod_{i=1}^{n} (1 - \underline{v}_{\alpha_i}^{q})^{w_i}\right)^{\frac{1}{q}}, \left(1 - \prod_{i=1}^{n} (1 - \overline{v}_{\alpha_i}^{q})^{w_i}\right)^{\frac{1}{q}} \right] \right\rangle.$$

After aggregation, we usually need to compare IVq-ROFVs in decision-making problems. Inspired by Xu[23], we propose the following score function and accuracy function to rank IVq-ROFVs:

Definition 2.6. *Let* $\alpha = \langle [\underline{\mu}_{\alpha}, \overline{\mu}_{\alpha}], [\underline{v}_{\alpha}, \overline{v}_{\alpha}] \rangle$ *be an IVq-ROFV. Then the score function is defined as:*

$$S(\alpha) = \frac{sgn\left(\underline{\mu}_{\alpha}^{q} - \underline{v}_{\alpha}^{q}\right) \left|\underline{\mu}_{\alpha}^{q} - \underline{v}_{\alpha}^{q}\right|^{\frac{1}{q}} + sgn\left(\overline{\mu}_{\alpha}^{q} - \overline{v}_{\alpha}^{q}\right) \left|\overline{\mu}_{\alpha}^{q} - \overline{v}_{\alpha}^{q}\right|^{\frac{1}{q}}}{2}, \tag{8}$$

where $sgn(x) = 1$ *if* $x \geq 0$ *and* $sgn(x) = -1$ *if* $x < 0$. *The accuracy function is defined as:*

$$T(\alpha) = \frac{\left(\underline{\mu}_{\alpha}^{q} + \underline{v}_{\alpha}^{q}\right)^{\frac{1}{q}} + \left(\overline{\mu}_{\alpha}^{q} + \overline{v}_{\alpha}^{q}\right)^{\frac{1}{q}}}{2}. \tag{9}$$

For given IVq-ROFVs $\alpha_1 = \langle [\underline{\mu}_{\alpha_1}, \overline{\mu}_{\alpha_1}], [\underline{v}_{\alpha_1}, \overline{v}_{\alpha_1}] \rangle$ *and* $\alpha_2 = \langle [\underline{\mu}_{\alpha_2}, \overline{\mu}_{\alpha_2}], [\underline{v}_{\alpha_2}, \overline{v}_{\alpha_2}] \rangle$, *we have*

- *If* $S(\alpha_1) < S(\alpha_2)$, *then* $\alpha_1 < \alpha_2$.
- *If* $S(\alpha_1) = S(\alpha_2)$ *and* $T(\alpha_1) < T(\alpha_2)$, *then* $\alpha_1 < \alpha_2$.
- *If* $S(\alpha_1) = S(\alpha_2)$ *and* $T(\alpha_1) = T(\alpha_2)$, *then* $\alpha_1 = \alpha_2$.

Remark 2.7. *Definition 2.6 generalizes definitions in the paper* [23] *to nonlinear cases. We give some comments on the score function. Instead of*

$$S(\alpha) = \frac{\left(\underline{\mu}_{\alpha}^{q} - \underline{v}_{\alpha}^{q}\right) + \left(\overline{\mu}_{\alpha}^{q} - \overline{v}_{\alpha}^{q}\right)}{2},$$

our formula (8) *ensures that much reasonable distribution of score values over the interval* $[-1, 1]$, *because*

$$sgn\left(\underline{\mu}_{\alpha}^{q} - \underline{v}_{\alpha}^{q}\right) \left|\underline{\mu}_{\alpha}^{q} - \underline{v}_{\alpha}^{q}\right|^{\frac{1}{q}} \sim \underline{\mu}_{\alpha} - \underline{v}_{\alpha}, \ \forall \ q \geq 1.$$

When q *is large enough, it eliminates the concentration of* $S(\alpha)$ *near at zero. Therefore, Formula* (8) *seems to be more accurate in practical applications. See* [27] *for comparison.*

Here, the sign function sgn is to guarantee that the value of $S(\alpha)$ *is from -1 to 1.*

Definition 2.8. [9] *Let the functions f, g of two variables have the form*

$$f(x,y),\ g(x,y)\ :[0,1]\times[0,1]\mapsto[0,1]. \tag{10}$$

Putting

$$\underline{f}_\alpha=\underline{f}(\underline{\mu}_\alpha,\underline{v}_\alpha)\le\overline{f}(\overline{\mu}_\alpha,\overline{v}_\alpha)=\overline{f}_\alpha\quad\text{and}\quad\underline{g}_\alpha=\underline{g}(\underline{\mu}_\alpha,\underline{v}_\alpha)\le\overline{g}(\overline{\mu}_\alpha,\overline{v}_\alpha)=\overline{g}_\alpha,$$

with the restriction $\overline{f}^q_\alpha+\overline{g}^q_\alpha\le1$. Then,

$$\varphi(\alpha)\triangleq\left\langle\left[\underline{f}_\alpha,\overline{f}_\alpha\right],\ \left[\underline{g}_\alpha,\overline{g}_\alpha\right]\right\rangle, \tag{11}$$

is called an IVq-ROFF with respect to $\alpha\in\mathbb{S}$.

Remark 2.9. *Definition 2.8 generalizes the definition of IVq-ROFFs in our previous work* [9].

We also need some subsets of $\mathbb{S}$ to ensure that the basic operations of IVq-ROFFs are meaningful and closed under q-ROFS.

Proposition 2.10. *Assume that $\varphi(\alpha)$ is defined in Definition 2.8.*
(i) *For fixed $\alpha\in\mathbb{S}$, we define*

$$\mathbb{S}^\varphi_\oplus(\alpha)\triangleq\left\{\varphi(\beta)|\beta\in\mathbb{S}_\oplus(\alpha)\ :\ \begin{array}{c}(\underline{f}^q_\beta-\underline{f}^q_\alpha)(1-\overline{f}^q_\alpha)\le(1-\underline{f}^q_\alpha)(\overline{f}^q_\beta-\overline{f}^q_\alpha),\\ \underline{g}_\beta\overline{g}_\alpha\le\overline{g}_\beta\underline{g}_\alpha,\ \dfrac{\overline{g}_\beta}{\overline{g}_\alpha}\le\left(\dfrac{1-\overline{f}^q_\beta}{1-\overline{f}^q_\alpha}\right)^{\frac{1}{q}}\le1\end{array}\right\},$$

with $\mathbb{S}_\oplus(\alpha)\triangleq\{\beta\in\mathbb{S}\,|\,\beta\oplus\alpha\in\mathbb{S}\}$. This implies that $\varphi(\beta)\ominus\varphi(\alpha)$ is still an IVq-ROFF, as long as $\varphi(\beta)\in\mathbb{S}^\varphi_\oplus(\alpha)$.
(ii) *For fixed $\alpha\in\mathbb{S}$,*

$$\mathbb{S}^\varphi_\otimes(\alpha)\triangleq\left\{\varphi(\beta)|\beta\in\mathbb{S}_\otimes(\alpha)\ :\ \begin{array}{c}(\underline{g}^q_\beta-\underline{g}^q_\alpha)(1-\overline{g}^q_\alpha)\le(1-\underline{g}^q_\alpha)(\overline{g}^q_\beta-\overline{g}^q_\alpha),\\ \underline{f}_\beta\overline{f}_\alpha\le\overline{f}_\beta\underline{f}_\alpha,\ \dfrac{\overline{f}_\beta}{\overline{f}_\alpha}\le\left(\dfrac{1-\overline{g}^q_\beta}{1-\overline{g}^q_\alpha}\right)^{\frac{1}{q}}\le1\end{array}\right\},$$

with $\mathbb{S}_\otimes(\alpha)\triangleq\{\beta\in\mathbb{S}\,|\,\beta\otimes\alpha\in\mathbb{S}\}$. Similarly, $\varphi(\beta)\oslash\varphi(\alpha)$ is still an IVq-ROFF for any $\varphi(\beta)\in\mathbb{S}^\varphi_\otimes(\alpha)$.

Limit procedure is a powerful tool in mathematical analysis. In the light of the limit of the elementary functions, we introduce the definition of IVq-ROFF limit as follows:

Definition 2.11. [9] [Limit of IVq-ROFF] *For given $\alpha\in\mathbb{S}$ and $\varphi\in\mathbb{S}^\varphi_\oplus(\alpha)$, we define the limit of IVq-ROFF at the point α*

$$\begin{aligned}\lim_{\beta\to\alpha}\varphi(\beta)&=\lim_{\beta\to\alpha}\left\langle\left[\underline{f}_\beta,\overline{f}_\beta\right],\ \left[\underline{g}_\beta,\overline{g}_\beta\right]\right\rangle\\ &\triangleq\left\langle\left[\lim_{(\underline{\mu}_\beta,\underline{v}_\beta)\to(\underline{\mu}_\alpha,\underline{v}_\alpha)}\underline{f}_\beta,\ \lim_{(\underline{\mu}_\beta,\overline{v}_\beta)\to(\overline{\mu}_\alpha,\underline{v}_\alpha)}\overline{f}_\beta\right],\ \left[\lim_{(\underline{\mu}_\beta,\underline{v}_\beta)\to(\underline{\mu}_\alpha,\underline{v}_\alpha)}\underline{g}_\beta,\ \lim_{(\overline{\mu}_\beta,\overline{v}_\beta)\to(\underline{\mu}_\alpha,\overline{v}_\alpha)}\overline{g}_\beta\right]\right\rangle,\end{aligned}$$

where, and in what follows, the $\beta\to\alpha$ means the IVq-ROFV β approaches α in q-ROFS, while in the second line, the limit signs are the same as in elementary multi-calculus.

Based on Definition 2.11, we can define the continuity of IVq-ROFF via limit procedure.

Definition 2.12. [9] [Continuity of IVq-ROFF] *For given $\alpha\in\mathbb{S}$ and $\varphi\in\mathbb{S}^\varphi_\oplus(\alpha)$, we say that IVq-ROFF $\varphi(\alpha)$ is continuous at α, provided that*

$$\lim_{\beta\to\alpha}\varphi(\beta)=\varphi(\alpha).$$

Moreover, if φ is continuous at every point $\alpha\in\mathbb{S}$, then we say that φ is continuous IVq-ROFF.

After getting Definitions 2.11 and 2.12, we can derive the following basic operational rules:

Theorem 2.13. *Assume that the IVq-ROFFs φ and ϕ are defined in Definition 2.8. The following assertions hold true:*

(i) If both the components f and g are continuous in elementary calculus, then $\varphi(\alpha)$ is continuous in IVq-ROFS.

(ii) If φ and ϕ are continuous, then $\varphi(\alpha)\oplus\varphi(\beta)$, $\varphi(\alpha)\ominus\varphi(\beta)$, $\varphi(\alpha)\otimes\varphi(\beta)$, $\varphi(\alpha)\oslash\varphi(\beta)$, as well as the compound $\varphi(\phi)$ are still continuous, so long as their operations make sense in IVq-ROFS.

The derivative formula as well as the operational laws of IVq-ROFFs have already been established in our previous work [9]. We display them in Theorem 2.14-Theorem 2.16 below for the usage of this paper:

Theorem 2.14. [9] *Let the IVq-ROFF $\varphi(\alpha)$ be as in Definition 2.8, $\alpha\in\mathbb{S}$ and $\beta\in\mathbb{S}^{\varphi}_{\oplus}(\alpha)$. Let the partial derivative of functions f and g, i.e., $\frac{\partial f_\alpha}{\partial\mu_\alpha}$, $\frac{\partial f_\alpha}{\partial v_\alpha}$, $\frac{\partial g_\alpha}{\partial\mu_\alpha}$, $\frac{\partial g_\alpha}{\partial v_\alpha}$, be continuous functions. Suppose*

$$\frac{\partial \underline{f}^q_\alpha}{\partial \underline{v}^q_\alpha}=0=\frac{\partial \overline{f}^q_\alpha}{\partial \overline{v}^q_\alpha},\quad \frac{\partial \underline{g}_\alpha}{\partial \underline{\mu}_\alpha}=0=\frac{\partial \overline{g}_\alpha}{\partial \overline{\mu}_\alpha}, \tag{12}$$

and

$$0\le\frac{(1-\underline{\mu}^q_\alpha)}{(1-\underline{f}^q_\alpha)}\frac{\partial \underline{f}^q_\alpha}{\partial \underline{\mu}^q_\alpha}\le\frac{(1-\underline{\mu}^q_\alpha)}{(1-\underline{f}^q_\alpha)}\frac{\partial \underline{f}^q_\alpha}{\partial \underline{\mu}^q_\alpha}\le 1,\quad 0\le\frac{(1-\overline{\mu}^q_\alpha)}{(1-\overline{f}^q_\alpha)}\frac{\partial \overline{f}^q_\alpha}{\partial \overline{\mu}^q_\alpha}\le\frac{\overline{v}_\alpha}{\overline{g}_\alpha}\frac{\partial \overline{g}_\alpha}{\partial \overline{v}_\alpha}\le\frac{\underline{v}_\alpha}{\underline{g}_\alpha}\frac{\partial \underline{g}_\alpha}{\partial \underline{v}_\alpha}\le 1. \tag{13}$$

Then, $\varphi(\alpha)$ admits its derivative at α, i.e.,

$$\frac{\mathcal{T}\varphi(\alpha)}{\mathcal{T}\alpha}=\left\langle\left[\left(\frac{(1-\underline{\mu}^q_\alpha)}{(1-\underline{f}^q_\alpha)}\frac{\partial \underline{f}^q_\alpha}{\partial \underline{\mu}^q_\alpha}\right)^{\frac{1}{q}},\left(\frac{(1-\overline{\mu}^q_\alpha)}{(1-\overline{f}^q_\alpha)}\frac{\partial \overline{f}^q_\alpha}{\partial \overline{\mu}^q_\alpha}\right)^{\frac{1}{q}}\right],\left[\left(1-\frac{\underline{v}_\alpha}{\underline{g}_\alpha}\frac{\partial \underline{g}_\alpha}{\partial \underline{v}_\alpha}\right)^{\frac{1}{q}},\left(1-\frac{\overline{v}_\alpha}{\overline{g}_\alpha}\frac{\partial \overline{g}_\alpha}{\partial \overline{v}_\alpha}\right)^{\frac{1}{q}}\right]\right\rangle. \tag{14}$$

Theorem 2.15. [9] *For given constant IVq-ROFV*

$$C=\langle[\underline{c}_1,\overline{c}_1],\,[\underline{c}_2,\overline{c}_2]\rangle,$$

with $0\le\underline{c}_i\le\overline{c}_i\le 1$ $(i=1,2)$ being the constants, one has

$$\frac{\mathcal{T}}{\mathcal{T}\alpha}(\varphi(\alpha)\oplus C)=\frac{\mathcal{T}}{\mathcal{T}\alpha}(\varphi(\alpha)\ominus C)=\frac{\mathcal{T}\varphi(\alpha)}{\mathcal{T}\alpha}.$$

Theorem 2.16. [9] *Let the IVq-ROFFs $\varphi(\alpha)$ and $\phi(\alpha)$ satisfy Definition 2.8. Under the same hypotheses in Theorem 2.14, we have*

$$\frac{\mathcal{T}\varphi(\phi(\alpha))}{\mathcal{T}\alpha}=\frac{\mathcal{T}\varphi(\phi)}{\mathcal{T}\phi}\otimes\frac{\mathcal{T}\phi(\alpha)}{\mathcal{T}\alpha},$$

so long as the derivatives exist.

Proposition 2.17. [8] *Consider the sequence $\{a_n\}_{n=1}^{\infty}$ with $a_n\in(-1,1)$. Then the series $\prod_{n=1}^{\infty}(1+a_n)$ is convergent if and only if $\sum_{n=1}^{\infty}\ln(1+a_n)$ is summable. In particular, if*

$$\sum_{n=1}^{\infty}\ln(1+a_n)=L,\ \prod_{n=1}^{\infty}(1+a_n)=P.$$

Then $P=e^L$. Furthermore, the necessary and sufficient condition of convergence of series $\prod_{n=1}^{\infty}(1+a_n)$ is that $\sum_{n=1}^{\infty}a_n$ is summable.

3 Indefinite integral of IVq-ROFFs

Integral provides a platform for processing global information through microscopic methods. It is an important part of calculus. In the framework of IVq-ROFS, the integral of IVq-ROFF is a physical motivation, closely related to the aggregation methods, and thus has a wide range of applications in our life. Therefore, it is necessary to survey the IVq-ROFF integral theory. This section focuses on the indefinite integral of IVq-ROFF, starting with the definition of primitive.

Definition 3.1 (Primitive)**.** *Suppose that the IVq-ROFFs $\phi(\alpha)$ and $\Phi(\alpha)$ are well defined in $\mathbb{S}$. If*

$$\frac{\mathcal{T}\Phi(\alpha)}{\mathcal{T}\alpha}=\phi(\alpha),\quad \alpha\in\mathbb{S},$$

then Φ is called a primitive of ϕ in $\mathbb{S}$.

Recalling Theorem 2.15, it satisfies that, for any constant IVq-ROFF C,

$$\frac{\mathcal{T}}{\mathcal{T}\alpha}(\Phi(\alpha)\oplus C)=\frac{\mathcal{T}}{\mathcal{T}\alpha}\Phi(\alpha).$$

This implies that if Φ is a primitive of ϕ, so does $\Phi(\alpha)\oplus C$, in other words, the primitive ϕ is not unique. This motivates the definition of indefinite integral.

Definition 3.2. *We define all the primitives of ϕ as its indefinite integral, given by*

$$\int\phi(\alpha)\mathcal{T}\alpha.$$

Clearly,

$$\int\phi(\alpha)\mathcal{T}\alpha=\Phi(\alpha)\oplus C, \tag{15}$$

where $\Phi(\alpha)$ is one of any primitives of $\phi(\alpha)$, and the constant IVq-ROFF C is arbitrarily given.

By Definition 3.1 and Definition 3.2, we see that integral operation of IVq-ROFF is in fact the inverse of its derivative. Remember that in Theorem 2.14, simple calculation provides us the following examples:

$$\int\langle[0,1],[1,1]\rangle d\alpha=\langle[1,1],[0,0]\rangle\oplus C,$$

$$\int\langle[1,1],[0,0]\rangle d\alpha=\langle[\underline{\mu}_\alpha,\overline{\mu}_\alpha],[\underline{v}_\alpha,\overline{v}_\alpha]\rangle\oplus C,$$

$$\int\left\langle\left[\left(\frac{3(\underline{\mu}_\alpha^{2q}-\underline{\mu}_\alpha^{3q})}{1-\underline{\mu}_\alpha^{3q}}\right)^{\frac{1}{q}},\left(\frac{3(\overline{\mu}_\alpha^{2q}-\overline{\mu}_\alpha^{3q})}{1-\overline{\mu}_\alpha^{3q}}\right)^{\frac{1}{q}}\right],\left[0,\left(\frac{1}{2}\right)^{\frac{1}{q}}\right]\right\rangle\mathcal{T}\alpha=\left\langle\left[\underline{\mu}_\alpha^{3},\overline{\mu}_\alpha^{3}\right],\left[\underline{v}_\alpha,\overline{v}_\alpha^{\frac{1}{2}}\right]\right\rangle\oplus C.$$

Theorem 3.3. *Let the IVq-ROFF*

$$\varphi(\alpha)=\left\langle\left[\underline{f}_\alpha,\overline{f}_\alpha\right],\left[\underline{g}_\alpha,\overline{g}_\alpha\right]\right\rangle, \tag{16}$$

be continuous. Assume that

$$\frac{\partial\underline{f}_\alpha}{\partial\underline{v}_\alpha}=\frac{\partial\overline{f}_\alpha}{\partial\overline{v}_\alpha}=0=\frac{\partial\underline{g}_\alpha}{\partial\underline{\mu}_\alpha}=\frac{\partial\overline{g}_\alpha}{\partial\overline{\mu}_\alpha}. \tag{17}$$

Then

$$\begin{aligned}\int\phi(\alpha)\mathcal{T}\alpha=\Bigg\langle&\left[\left(1-c_1^1\exp\left\{-\int\frac{q\underline{\mu}_\alpha^{q-1}\underline{f}_\alpha^q}{1-\underline{\mu}_\alpha^q}d\underline{\mu}_\alpha\right\}\right)^{\frac{1}{q}},\left(1-c_1^2\exp\left\{-\int\frac{q\overline{\mu}_\alpha^{q-1}\overline{f}_\alpha^q}{1-\overline{\mu}_\alpha^q}d\overline{\mu}_\alpha\right\}\right)^{\frac{1}{q}}\right],\\&\left[c_2^1\exp\left\{\int\frac{1-\underline{g}_\alpha^q}{\underline{v}_\alpha}d\underline{v}_\alpha\right\},c_2^2\exp\left\{\int\frac{1-\overline{g}_\alpha^q}{\overline{v}_\alpha}d\overline{v}_\alpha\right\}\right]\Bigg\rangle,\end{aligned} \tag{18}$$

where the constants $c_i^j\ (i,j=1,2)$ are given.

Remark 3.4. *The integrals in* (18) *are the same as in elementary calculus. The continuity condition is for the existence of their meaningfulness.*

Proof of Theorem 3.3. Express

$$\int \phi(\alpha)\mathcal{T}\alpha = \left\langle \left[\underline{F}_\alpha, \overline{F}_\alpha\right], \left[\underline{G}_\alpha, \overline{G}_\alpha\right] \right\rangle.$$

Then, it suffices to prove

$$\underline{F}_\alpha^q = 1 - c_1^1 \exp\left\{-\int \frac{q\underline{\mu}_\alpha^{q-1}\underline{f}_\alpha^q}{1-\underline{\mu}_\alpha^q} d\underline{\mu}_\alpha\right\}, \quad \overline{F}_\alpha^q = 1 - c_1^2 \exp\left\{-\int \frac{q\overline{\mu}_\alpha^{q-1}\overline{f}_\alpha^q}{1-\underline{\mu}_\alpha^q} d\overline{\mu}_\alpha\right\}, \tag{19}$$

and

$$\underline{G}_\alpha = c_2 \exp\left\{\int \frac{1-\underline{g}_\alpha^q}{\underline{v}_\alpha} d\underline{v}_\alpha\right\}, \quad \overline{G}_\alpha = c_2 \exp\left\{\int \frac{1-\overline{g}_\alpha^q}{\overline{v}_\alpha} d\overline{v}_\alpha\right\}. \tag{20}$$

Form one hand, Definition 3.1 tells that

$$\frac{\mathcal{T}}{\mathcal{T}\alpha}\left\langle \left[\underline{F}_\alpha, \overline{F}_\alpha\right], \left[\underline{G}_\alpha, \overline{G}_\alpha\right] \right\rangle = \frac{\mathcal{T}}{\mathcal{T}\alpha}\int \phi(\alpha)\mathcal{T}\alpha = \phi(\alpha) = \left\langle \left[\underline{f}_\alpha, \overline{f}_\alpha\right], \left[\underline{g}_\alpha, \overline{g}_\alpha\right] \right\rangle. \tag{21}$$

Form another hand, Theorem 2.14 implies that

$$\frac{\mathcal{T}}{\mathcal{T}\alpha}\left\langle \left[\underline{F}_\alpha, \overline{F}_\alpha\right], \left[\underline{G}_\alpha, \overline{G}_\alpha\right] \right\rangle = \left\langle \left[\left(\frac{(1-\underline{\mu}_\alpha^q)}{(1-\underline{F}_\alpha^q)}\frac{\partial \underline{F}_\alpha^q}{\partial \underline{\mu}_\alpha^q}\right)^{\frac{1}{q}}, \left(\frac{(1-\overline{\mu}_\alpha^q)}{(1-\overline{F}_\alpha^q)}\frac{\partial \overline{F}_\alpha^q}{\partial \overline{\mu}_\alpha^q}\right)^{\frac{1}{q}}\right], \left[\left(1-\frac{\underline{v}_\alpha}{\underline{G}_\alpha}\frac{\partial \underline{G}_\alpha}{\partial \underline{v}_\alpha}\right)^{\frac{1}{q}}, \left(1-\frac{\overline{v}_\alpha}{\overline{G}_\alpha}\frac{\partial \overline{G}_\alpha}{\partial \overline{v}_\alpha}\right)^{\frac{1}{q}}\right] \right\rangle. \tag{22}$$

In accordance with (21)-(22), we need to prove

$$\frac{(1-\underline{\mu}_\alpha^q)}{(1-\underline{F}_\alpha^q)}\frac{\partial \underline{F}_\alpha^q}{\partial \underline{\mu}_\alpha^q} = \underline{f}_\alpha^q, \quad \frac{(1-\overline{\mu}_\alpha^q)}{(1-\overline{F}_\alpha^q)}\frac{\partial \overline{F}_\alpha^q}{\partial \overline{\mu}_\alpha^q} = \overline{f}_\alpha^q, \tag{23}$$

and

$$1-\frac{\underline{v}_\alpha}{\underline{G}_\alpha}\frac{\partial \underline{G}_\alpha}{\partial \underline{v}_\alpha} = \underline{g}_\alpha^q, \quad 1-\frac{\overline{v}_\alpha}{\overline{G}_\alpha}\frac{\partial \overline{G}_\alpha}{\partial \overline{v}_\alpha} = \overline{g}_\alpha^q. \tag{24}$$

In fact, using (17), we can re-write the former one in (23) as

$$\frac{d\underline{F}_\alpha^q}{1-\underline{F}_\alpha^q} = \frac{\underline{f}^q}{1-\underline{\mu}_\alpha^q} d\underline{\mu}_\alpha^q.$$

Solving directly the differential equation gives birth to

$$\underline{F}_\alpha^q = 1 - c_1^1 \exp\left\{-\int \frac{q\underline{\mu}_\alpha^{q-1}\underline{f}_\alpha^q}{1-\underline{\mu}_\alpha^q} d\underline{\mu}_\alpha\right\}.$$

We justify the other terms in a similar argument, and complete the proof of Theorem 3.3.

As a direct application of Theorem 3.3, we have

Example 3.5. *Consider*

$$\phi(\alpha) = \langle [\mu_\alpha^2, \mu_\alpha], [v_\alpha^3, v_\alpha] \rangle.$$

Then, by Formula (18) *in Theorem 3.3, we find*

$$\int \phi(\alpha)\mathcal{T}\alpha = \left\langle \left[c_1^1\left(1-e^{\frac{\mu_\alpha^{2q}}{2}+\mu_\alpha^q}(1-\mu_\alpha^q)\right)^{\frac{1}{q}}, c_1^2\left(1-e^{\mu_\alpha^q}(1-\mu_\alpha^q)\right)^{\frac{1}{q}}\right], \left[c_2^1\frac{v_\alpha}{e^{\frac{1}{3q}v_\alpha^{3q}}}, c_2^2\frac{v_\alpha}{e^{\frac{1}{q}v_\alpha^q}}\right] \right\rangle.$$

Corollary 3.6. *Suppose that the IVq-ROFFs* $\Phi_1(\alpha)$ *and* $\Phi_2(\alpha)$ *are two primitives of* ϕ*. Then*

$$\Phi_1 \ominus \Phi_2 = C,$$

where C *denotes a constant IVq-ROFF.*

Proof. Using Definition 2.3, we easily deduce the commutative law

$$(\alpha_1 \oplus C_1) \ominus (\alpha_2 \oplus C_2) = (\alpha_1 \ominus \alpha_2) \oplus (C_1 \ominus C_2).$$

It follows from (15) and (18) that

$$\begin{aligned}
\Phi_1 \ominus \Phi_2 = &\left\langle \left[\left(1 - c_1^1 \exp\left\{ -\int \frac{q\underline{\mu}_\alpha^{q-1}\underline{f}_\alpha^q}{1-\underline{\mu}_\alpha^q} d\underline{\mu}_\alpha \right\}\right)^{\frac{1}{q}}, \left(1 - c_1^2 \exp\left\{ -\int \frac{q\overline{\mu}_\alpha^{q-1}\overline{f}_\alpha^q}{1-\overline{\mu}_\alpha^q} d\overline{\mu}_\alpha \right\}\right)^{\frac{1}{q}} \right], \right.\\
&\left. \left[c_2^1 \exp\left\{ \int \frac{1-\underline{g}_\alpha^q}{\underline{v}_\alpha} d\underline{v}_\alpha \right\}, c_2^2 \exp\left\{ \int \frac{1-\overline{g}_\alpha^q}{\overline{v}_\alpha} d\overline{v}_\alpha \right\} \right] \right\rangle \\
&\ominus \left\langle \left[\left(1 - d_1^1 \exp\left\{ -\int \frac{q\underline{\mu}_\alpha^{q-1}\underline{f}_\alpha^q}{1-\underline{\mu}_\alpha^q} d\underline{\mu}_\alpha \right\}\right)^{\frac{1}{q}}, \left(1 - d_1^2 \exp\left\{ -\int \frac{q\overline{\mu}_\alpha^{q-1}\overline{f}_\alpha^q}{1-\overline{\mu}_\alpha^q} d\overline{\mu}_\alpha \right\}\right)^{\frac{1}{q}} \right], \right.\\
&\left. \left[d_2^1 \exp\left\{ \int \frac{1-\underline{g}_\alpha^q}{\underline{v}_\alpha} d\underline{v}_\alpha \right\}, d_2^2 \exp\left\{ \int \frac{1-\overline{g}_\alpha^q}{\overline{v}_\alpha} d\overline{v}_\alpha \right\} \right] \right\rangle \\
&\oplus (C_1 \ominus C_2)\\
= &\left\langle \left[\left(\frac{d_1^1 - c_1^1}{d_1^1}\right)^{\frac{1}{q}}, \left(\frac{d_1^2 - c_1^2}{d_1^2}\right)^{\frac{1}{q}} \right], \left[\frac{c_2^1}{d_2^1}, \frac{c_2^2}{d_2^2} \right] \right\rangle \oplus (C_1 \ominus C_2).
\end{aligned}$$

This completes the proof.

Theorem 3.7 (Algebraic manipulations)**.** *Let two continuous IVq-ROFFs of the form*

$$\varphi(\alpha) = \left\langle \left[\underline{f}_1, \overline{f}_1\right], \left[\underline{g}_1, \overline{g}_1\right] \right\rangle \quad \text{and} \quad \phi(\alpha) = \left\langle \left[\underline{f}_2, \overline{f}_2\right], \left[\underline{g}_2, \overline{g}_2\right] \right\rangle. \tag{25}$$

Then,

$$\int \varphi(\alpha)\mathcal{T}\alpha \oplus \int \phi(\alpha)\mathcal{T}\alpha = \int \left\langle \left[\left(\underline{f}_1^q + \underline{f}_2^q\right)^{\frac{1}{q}}, \left(\overline{f}_1^q + \overline{f}_2^q\right)^{\frac{1}{q}} \right], \left[\left(\underline{g}_1^q - (1-\underline{g}_2^q)\right)^{\frac{1}{q}}, (\overline{g}_1^q - (1-\overline{g}_2^q))^{\frac{1}{q}} \right] \right\rangle \mathcal{T}\alpha, \tag{26}$$

and

$$\int \varphi(\alpha)\mathcal{T}\alpha \ominus \int \phi(\alpha)\mathcal{T}\alpha = \int \left\langle \left[\left(\underline{f}_1^q - \underline{f}_2^q\right)^{\frac{1}{q}}, \left(\overline{f}_1^q - \overline{f}_2^q\right)^{\frac{1}{q}} \right], \left[\left(\underline{g}_1^q + (1-\underline{g}_2^q)\right)^{\frac{1}{q}}, (\overline{g}_1^q + (1-\overline{g}_2^q))^{\frac{1}{q}} \right] \right\rangle \mathcal{T}\alpha, \tag{27}$$

provided that the right-hand quantities of them are meaningful in IVq-ROFS.

Proof. We first prove (26). Observe from Definition 2.3, Formula (18), properties of elementary integrals, we infer

$$\begin{aligned}
&\int \left\langle \left[\underline{f}_1, \overline{f}_1\right], \left[\underline{g}_1, \overline{g}_1\right] \right\rangle \mathcal{T}\alpha \oplus \int \left\langle \left[\underline{f}_2, \overline{f}_2\right], \left[\underline{g}_2, \overline{g}_2\right] \right\rangle \mathcal{T}\alpha\\
&= \left\langle \left[\left(1 - c_1^1 \exp\left\{ -\int \frac{q\underline{\mu}_\alpha^{q-1}\underline{f}_1^q}{1-\underline{\mu}_\alpha^q} d\underline{\mu}_\alpha \right\}\right)^{\frac{1}{q}}, \left(1 - c_1^2 \exp\left\{ -\int \frac{q\overline{\mu}_\alpha^{q-1}\overline{f}_1^q}{1-\overline{\mu}_\alpha^q} d\overline{\mu}_\alpha \right\}\right)^{\frac{1}{q}} \right], \right.\\
&\quad \left. \left[c_2^1 \exp\left\{ \int \frac{1-\underline{g}_1^q}{\underline{v}_\alpha} d\underline{v}_\alpha \right\}, c_2^2 \exp\left\{ \int \frac{1-\overline{g}_1^q}{\overline{v}_\alpha} d\overline{v}_\alpha \right\} \right] \right\rangle \\
&\quad \oplus \left\langle \left[\left(1 - d_1^1 \exp\left\{ -\int \frac{q\underline{\mu}_\alpha^{q-1}\underline{f}_2^q}{1-\underline{\mu}_\alpha^q} d\underline{\mu}_\alpha \right\}\right)^{\frac{1}{q}}, \left(1 - d_1^2 \exp\left\{ -\int \frac{q\overline{\mu}_\alpha^{q-1}\overline{f}_2^q}{1-\overline{\mu}_\alpha^q} d\overline{\mu}_\alpha \right\}\right)^{\frac{1}{q}} \right], \right.\\
&\quad \left. \left[d_2^1 \exp\left\{ \int \frac{1-\underline{g}_2^q}{\underline{v}_\alpha} d\underline{v}_\alpha \right\}, d_2^2 \exp\left\{ \int \frac{1-\overline{g}_2^q}{\overline{v}_\alpha} d\overline{v}_\alpha \right\} \right] \right\rangle \\
&= \left\langle \left[\underline{A}, \overline{A}\right], \left[c_2^1 d_2^1 \exp\left\{ \int \frac{(1-\underline{g}_1^q) + (1-\underline{g}_2^q)}{\underline{v}_\alpha} d\underline{v}_\alpha \right\}, c_2^2 d_2^2 \exp\left\{ \int \frac{(1-\overline{g}_1^q) + (1-\overline{g}_2^q)}{\overline{v}_\alpha} d\overline{v}_\alpha \right\} \right] \right\rangle,
\end{aligned} \tag{28}$$

where

$$\underline{A}^q = 1 - c_1^1 d_1^1 \exp\left\{-\int \frac{q\underline{\mu}_\alpha^{q-1}(\underline{f}_1^q + \underline{f}_2^q)}{1-\underline{\mu}_\alpha^q} d\underline{\mu}_\alpha\right\},$$

and

$$\overline{A}^q = 1 - c_1^2 d_1^2 \exp\left\{-\int \frac{q\overline{\mu}_\alpha^{q-1}(\overline{f}_1^q + \overline{f}_2^q)}{1-\overline{\mu}_\alpha^q} d\overline{\mu}_\alpha\right\}.$$

For simplicity, only the deduction of $\underline{A}$ will be demonstrated. Owing to the addition arithmetic,

$$\begin{aligned}\underline{A}^q =& \left(1 - c_1^1 \exp\left\{-\int \frac{q\underline{\mu}_\alpha^{q-1}\underline{f}_1^q}{1-\underline{\mu}_\alpha^q}\right\}\right) + \left(1 - d_1^1 \exp\left\{-\int \frac{q\underline{\mu}_\alpha^{q-1}\underline{f}_2^q}{1-\underline{\mu}_\alpha^q}\right\}\right)\\ &- \left(1 - c_1^1 \exp\left\{-\int \frac{q\underline{\mu}_\alpha^{q-1}\underline{f}_1^q}{1-\underline{\mu}_\alpha^q}\right\}\right) \times \left(1 - d_1^1 \exp\left\{-\int \frac{q\underline{\mu}_\alpha^{q-1}\underline{f}_2^q}{1-\underline{\mu}_\alpha^q}\right\}\right)\\ =& 1 - c_1^1 d_1^1 \exp\left\{-\int \frac{q\underline{\mu}_\alpha^{q-1}\underline{f}_1^q}{1-\underline{\mu}_\alpha^q}\right\} \exp\left\{-\int \frac{q\underline{\mu}_\alpha^{q-1}\underline{f}_2^q}{1-\underline{\mu}_\alpha^q}\right\}\\ =& 1 - c_1^1 d_1^1 \exp\left\{-\int \frac{q\underline{\mu}_\alpha^{q-1}(\underline{f}_1^q + \underline{f}_2^q)}{1-\underline{\mu}_\alpha^q}\right\}.\end{aligned}$$

Again using (18), we insert the above equality into (28) and obtain

$$\begin{aligned}&\left\langle \left[\underline{A}, \overline{A}\right], \left[c_2^1 \exp\left\{\int \frac{(1-\underline{g}_1^q) + (1-\underline{g}_2^q)}{\underline{v}_\alpha} d\underline{v}_\alpha\right\}, c_2^1 \exp\left\{\int \frac{(1-\overline{g}_1^q) + (1-\overline{g}_2^q)}{\overline{v}_\alpha} d\overline{v}_\alpha\right\}\right]\right\rangle\\ &= \int \left\langle \left[\left(\underline{f}_1^q + \underline{f}_2^q\right)^{\frac{1}{q}}, \left(\overline{f}_1^q + \overline{f}_2^q\right)^{\frac{1}{q}}\right], \left[\left(\underline{g}_1^q - (1-\underline{g}_2^q)\right)^{\frac{1}{q}}, (\overline{g}_1^q - (1-\overline{g}_2^q))^{\frac{1}{q}}\right]\right\rangle \mathcal{T}\alpha.\end{aligned} \tag{29}$$

This, along with (25) and (28), gives birth to the desired (26).

By (18) and the difference arithmetic of IVq-ROFFs, we easily check (27). The proof is finished.

Corollary 3.8 (Scalar-multiplication)**.** *For given constant $\lambda \in [0,1]$, it satisfies*

$$\lambda \int \langle [\underline{f}, \overline{f}], [\underline{g}, \overline{g}]\rangle \mathcal{T}\alpha = \int \langle \left[\lambda^{\frac{1}{q}}\underline{f}, \lambda^{\frac{1}{q}}\overline{f}\right], \left[(1-\lambda(1-\underline{g}^q))^{\frac{1}{q}}, (1-\lambda(1-\underline{g}^q))^{\frac{1}{q}}\right]\rangle \mathcal{T}\alpha. \tag{30}$$

Proof. It yields from Theorem 2.16 that

$$\begin{aligned}\frac{\mathcal{T}}{\mathcal{T}\alpha}\left(\lambda \int \langle [\underline{f}, \overline{f}], [\underline{g}, \overline{g}]\rangle \mathcal{T}\alpha\right) &= \frac{\mathcal{T}}{\mathcal{T}(\int \langle [\underline{f}, \overline{f}], [\underline{g}, \overline{g}]\rangle \mathcal{T}\alpha)}\left(\lambda \int \langle [\underline{f}, \overline{f}], [\underline{g}, \overline{g}]\rangle \mathcal{T}\alpha\right) \otimes \frac{\mathcal{T}}{\mathcal{T}\alpha}(\int \langle [\underline{f}, \overline{f}], [\underline{g}, \overline{g}]\rangle \mathcal{T}\alpha)\\ &= \left\langle \left[\lambda^{\frac{1}{q}}, \lambda^{\frac{1}{q}}\right], \left[(1-\lambda)^{\frac{1}{q}}, (1-\lambda)^{\frac{1}{q}}\right]\right\rangle \otimes \langle [\underline{f}, \overline{f}], [\underline{g}, \overline{g}]\rangle.\end{aligned}$$

This combines with the properties of primitive yields

$$\lambda \int \langle [\underline{f}, \overline{f}], [\underline{g}, \overline{g}]\rangle \mathcal{T}\alpha \ominus \int \langle \left[\lambda^{\frac{1}{q}}\underline{f}, \lambda^{\frac{1}{q}}\overline{f}\right], \left[(1-\lambda(1-\underline{g}^q))^{\frac{1}{q}}, (1-\lambda(1-\underline{g}^q))^{\frac{1}{q}}\right]\rangle \mathcal{T}\alpha = C. \tag{31}$$

Selecting $\lambda = 1$ in (31) yields $C = \langle [0,0], [1,1]\rangle$. We conclude (30).

In the light of (26), one should expect that (30) is also valid for $\lambda > 1$. To see this, we consider the case $\lambda = 2$ and give a heuristic proof. Select in (26)

$$\varphi = \langle [\underline{f}, \overline{f}], [\underline{g}, \overline{g}]\rangle = \phi,$$

and deduce

$$\lambda \int \langle f, g\rangle \mathcal{T}\alpha = 2\int [\underline{f}, \overline{f}], [\underline{g}, \overline{g}]\rangle \mathcal{T}\alpha = \int \langle \left[2^{\frac{1}{q}}\underline{f}, 2^{\frac{1}{q}}\overline{f}\right], \left[(1-2(1-\underline{g}^q))^{\frac{1}{q}}, (1-2(1-\underline{g}^q))^{\frac{1}{q}}\right]\rangle \mathcal{T}\alpha.$$

This justifies the validity of $\lambda = 2$.

4 Definite integral of IVq-ROFFs

In order to understand the definite integral of IVq-ROFF, we first explain some concepts and parameters, as shown in Table 1. Then, some definitons and main theorems are introduced.

Table 1: The definitions of notations

Notation	Definition
$\mathbb{S} = \left\{\langle x,\, [\underline{\mu}, \overline{\mu}], [\underline{v}, \overline{v}]\rangle \mid x \in X\right\}$	Interval-valued q-rung orthopair fuzzy set (IVq-ROFS)
$\underline{\mu}, \overline{\mu}$	$\underline{\mu} = \inf_{x\in X} \widetilde{\mu}_{\mathbb{S}}(x) \leq \sup_{x\in X} \widetilde{\mu}_{\mathbb{S}}(x) = \overline{\mu}$
$\underline{v}, \overline{v}$	$\underline{v} = \inf_{x\in X} \widetilde{v}_{\mathbb{S}}(x) \leq \sup_{x\in X} \widetilde{v}_{\mathbb{S}}(x) = \overline{v}$
$\alpha = \langle[\underline{\mu}_\alpha, \overline{\mu}_\alpha], [\underline{v}_\alpha, \overline{v}_\alpha]\rangle$	Interval-valued q-rung orthopair fuzzy value (IVq-ROFV)
$f(x, y),\ g(x, y)\ : [0, 1] \times [0, 1] \mapsto [0, 1]$	weight function
$\underline{f}_\alpha, \overline{f}_\alpha$	$\underline{f}_\alpha = \underline{f}(\underline{\mu}_\alpha, \underline{v}_\alpha) \leq \overline{f}(\overline{\mu}_\alpha, \overline{v}_\alpha) = \overline{f}_\alpha$
$\underline{g}_\alpha, \overline{g}_\alpha$	$\underline{g}_\alpha = \underline{g}(\underline{\mu}_\alpha, \underline{v}_\alpha) \leq \overline{g}(\overline{\mu}_\alpha, \overline{v}_\alpha) = \overline{g}_\alpha$
$\varphi(\alpha) \triangleq \left\langle \left[\underline{f}_\alpha, \overline{f}_\alpha\right],\, \left[\underline{g}_\alpha, \overline{g}_\alpha\right] \right\rangle$	Interval-valued q-rung orthopair fuzzy function (IVq-ROFF)
$\mathcal{T}$	Derivative symbol
$[\underline{F}_\alpha, \overline{F}_\alpha]$	Integral $[\underline{\mu}, \overline{\mu}]$ results of IVq-ROFIs
$[\underline{G}_\alpha, \overline{G}_\alpha]$	Integral $[\underline{v}, \overline{v}]$ results of IVq-ROFIs
$L(\alpha, \beta)$	q-rung orthopair fuzzy integral line (q-ROFIL)

4.1 Definitions and main theorems

The definite integral of IVq-ROFF has the same sprit as the indefinite integral in computation, but has a more specific application. In this section, we analyze under addition operation the study of integral theory in it field of IVq-ROFS. For the sake of narrative convenience, we first introduce the concepts of the *simplified IVq-ROFF* and the *generalized IVq-ROFF* in Table1.

Definition 4.1. *Assume in addition that all the function components of IVq-ROFF in Definition 2.8 have the common variables, that is,*

$$\underline{f}_\alpha = \underline{f}(\mu_\alpha, v_\alpha) \leq \overline{f}(\mu_\alpha, v_\alpha) = \overline{f}_\alpha \quad \text{and} \quad \underline{g}_\alpha = \underline{g}(\mu_\alpha, v_\alpha) \leq \overline{g}(\mu_\alpha, v_\alpha) = \overline{g}_\alpha. \tag{32}$$

Then, the IVq-ROFF having the form

$$\varphi(\alpha) = \left\langle \left[\underline{f}_\alpha(\mu_\alpha, v_\alpha), \overline{f}_\alpha(\mu_\alpha, v_\alpha)\right],\, \left[\underline{g}_\alpha(\mu_\alpha, v_\alpha), \overline{g}_\alpha(\mu_\alpha, v_\alpha)\right] \right\rangle \triangleq \left\langle \left[\underline{f}_\alpha, \overline{f}_\alpha\right],\, \left[\underline{g}_\alpha, \overline{g}_\alpha\right] \right\rangle, \tag{33}$$

is called simplified IVq-ROFF. In comparison, the IVq-ROFF in Definition 2.8 is called generalized IVq-ROFF if the condition (32) *is not required.*

We remark that the *simplified IVq-ROFF* (with the condition (32)) enables us to explore the ordinary q-rung orthopair fuzzy integral line (q-ROFIL) defined in [8] without causing confusion, and therefore, simplify the mathematical proof in the IVq-ROFF circumstance.

Definition 4.2. [8] *Let $L(\alpha, \beta) \subset \mathbb{S}$ be a line from α to β, with two q-ROFNs in $\mathbb{S}$*

$$\alpha = \langle \mu_\alpha, v_\alpha \rangle \quad \text{and} \quad \beta = \langle \mu_\beta, v_\beta \rangle, \tag{34}$$

are given. Then, the line $L(\alpha, \beta)$ is called a q-ROFIL going from α to β, provided that

$$t_1,\, t_2 \in L(\alpha, \beta) \quad \text{and} \quad t_2 \in \mathbb{S}_{\oplus}(t_1).$$

This q-ROFIL is clearly present and easy to construct. In addition, q-ROFIL is much more flexible than the linear case $q = 1$, for example, allowing concave curves.

According to Definition 4.2, we have the following definition:

Definition 4.3. *Let $\alpha, \beta \in \mathbb{S}$ be as in (34), and let ϕ be the simplified IVq-ROFF and well defined in $L(\alpha,\beta)$. Insert finitely many $\alpha_i = \langle \mu_i, v_i\rangle$ $(i = 0, 1, \cdots, n)$ such that*

$$\alpha = \alpha_0 \lhd \alpha_1 \lhd \cdots \lhd \alpha_{n-1} \lhd \alpha_n = \beta.$$

Choose arbitrarily $\xi_i \in L(\alpha_{i-1}, \alpha_i)$ and consider

$$\begin{aligned}\oplus_{i=1}^{n}\phi(\xi_i)\otimes(\alpha_i\ominus\alpha_{i-1}) &= \oplus_{i=1}^{n}\left\langle\left[\underline{f}_{\xi_i},\overline{f}_{\xi_i}\right],\left[\underline{g}_{\xi_i},\overline{g}_{\xi_i}\right]\right\rangle\otimes\left\langle\left(\frac{\mu_{\alpha_i}^q-\mu_{\alpha_{i-1}}^q}{1-\mu_{\alpha_{i-1}}^q}\right)^{\frac{1}{q}},\frac{v_{\alpha_i}}{v_{\alpha_{i-1}}}\right\rangle\\ &= \oplus_{i=1}^{n}\left\langle\left[\underline{f}_{\xi_i}\left(\frac{\mu_{\alpha_i}^q-\mu_{\alpha_{i-1}}^q}{1-\mu_{\alpha_{i-1}}^q}\right)^{\frac{1}{q}},\overline{f}_{\xi_i}\left(\frac{\mu_{\alpha_i}^q-\mu_{\alpha_{i-1}}^q}{1-\mu_{\alpha_{i-1}}^q}\right)^{\frac{1}{q}}\right],\right.\\ &\qquad\left.\left[\left(\underline{g}_{\xi_i}^q+\frac{v_{\alpha_i}^q}{v_{\alpha_{i-1}}^q}-\underline{g}_{\xi_i}^q\frac{v_{\alpha_i}^q}{v_{\alpha_{i-1}}^q}\right)^{\frac{1}{q}},\left(\overline{g}_{\xi_i}^q+\frac{v_{\alpha_i}^q}{v_{\alpha_{i-1}}^q}-\overline{g}_{\xi_i}^q\frac{v_{\alpha_i}^q}{v_{\alpha_{i-1}}^q}\right)^{\frac{1}{q}}\right]\right\rangle\\ &\triangleq\left\langle\left[\underline{F}_n,\overline{F}_n\right],\left[\underline{G}_n,\underline{G}_n\right]\right\rangle.\end{aligned} \tag{35}$$

Define

$$\lambda_1 = \max_i\{1\le i\le n \mid \mu_i-\mu_{i-1}\} \quad\text{and}\quad \lambda_2=\min_i\left\{1\le i\le n \mid \frac{v_i}{v_{i-1}}\right\}. \tag{36}$$

If there is a constant IVq-ROFF $\langle[\underline{F},\overline{F}],[\underline{G},\overline{G}]\rangle$ which relies only on α, β and ϕ, such that

$$\lim\oplus_{i=1}^{n}\phi(\xi_i)\otimes(\alpha_i\ominus\alpha_{i-1})=\left\langle\left[\lim_{\lambda_1\to0}\underline{F}_n,\lim_{\lambda_1\to0}\overline{F}_n\right],\left[\lim_{\lambda_2\to1}\underline{G}_n,\lim_{\lambda_2\to1}\overline{G}_n\right]\right\rangle=\langle[\underline{F},\overline{F}],[\underline{G},\overline{G}]\rangle.$$

Then, the simplified IVq-ROFF ϕ is summable over $L(\alpha,\beta)$, in particular,

$$\int_{L(\alpha,\beta)}\phi(\xi)\mathcal{T}\xi=\int_{L(\alpha,\beta)}\left\langle\left[\underline{f}_{\xi},\overline{f}_{\xi}\right],\left[\underline{g}_{\xi},\overline{g}_{\xi}\right]\right\rangle\mathcal{T}\xi=\langle[\underline{F},\overline{F}],[\underline{G},\overline{G}]\rangle. \tag{37}$$

Theorem 4.4 (Newton-Leibniz)**.** *Let the hypotheses in Definition 4.3 hold true, and the simplified IVq-ROFF satisfying*

$$\frac{\partial\underline{f}_{\alpha}}{\partial v_{\alpha}}=\frac{\partial\overline{f}_{\alpha}}{\partial v_{\alpha}}=0=\frac{\partial\underline{g}_{\alpha}}{\partial\mu_{\alpha}}=\frac{\partial\overline{g}_{\alpha}}{\partial\mu_{\alpha}}. \tag{38}$$

Then,

$$\begin{aligned}\int_{L(\alpha,\beta)}\left\langle\left[\underline{f}_{\xi},\overline{f}_{\xi}\right],\left[\underline{g}_{\xi},\overline{g}_{\xi}\right]\right\rangle\mathcal{T}\xi=&\left\langle\left[\left(1-\exp\left\{-\int_{\mu_\alpha}^{\mu_\beta}\frac{q\mu_{\xi}^{q-1}\underline{f}_{\xi}^q}{1-\mu_{\xi}^q}d\mu_{\xi}\right\}\right)^{\frac{1}{q}},\left(1-\exp\left\{-\int_{\mu_\alpha}^{\mu_\beta}\frac{q\mu_{\xi}^{q-1}\overline{f}_{\xi}^q}{1-\mu_{\xi}^q}d\mu_{\xi}\right\}\right)^{\frac{1}{q}}\right],\right.\\ &\left.\left[\exp\left\{\int_{v_\alpha}^{v_\beta}\frac{1-\underline{g}_{\xi}^q}{v_{\xi}}dv_{\xi}\right\},\exp\left\{\int_{v_\alpha}^{v_\beta}\frac{1-\overline{g}_{\xi}^q}{v_{\xi}}dv_{\xi}\right\}\right]\right\rangle.\end{aligned} \tag{39}$$

Remark 4.5. *Actually,* (39) *can be considered as the sum of IVq-ROFFs. Specifically, if we choose* (35) *in Definition 4.3 as:*

$$\underline{f}_{\xi_i}=\overline{f}_{\xi_i}=\left(\frac{(1-\mu_{i-1})^{1+w_i}}{(\mu_i^q-\mu_{i-1}^q)}\right)^{\frac{1}{q}}\quad\text{and}\quad\underline{g}_{\xi_i}=\overline{g}_{\xi_i}=\left(\frac{v_{i-1}^{q(1+w_i)}-v_i^q}{(v_{i-1}^q-v_i^q)}\right)^{\frac{1}{q}},$$

we approximate the weighted averaging operator (q-ROFWA), namely,

$$\begin{aligned}\int_{L(\alpha,\beta)}\left\langle\left[\underline{f}_{\xi},\overline{f}_{\xi}\right],\left[\underline{g}_{\xi},\overline{g}_{\xi}\right]\right\rangle\mathcal{T}\xi&\approx\oplus_{i=1}^{n}\phi(\xi_i)\otimes(\alpha_i\ominus\alpha_{i-1})\\&=\oplus_{i=1}^{n}w_i\alpha_i\\&=q-ROFWA:(\alpha_1,\alpha_2,\cdots,\alpha_n),\end{aligned}$$

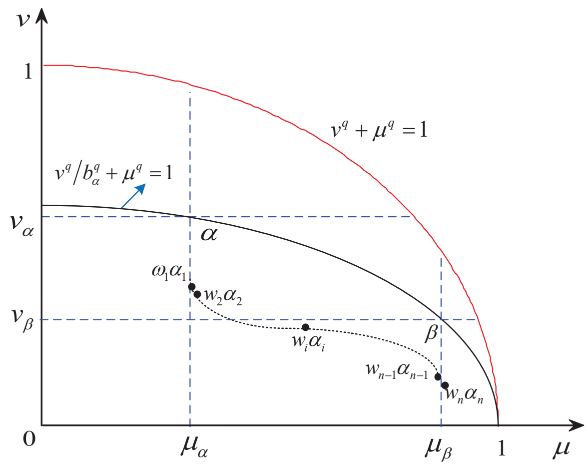

Figure 1: The relationship between integral and weighted averaging operator

where w_i $(i = 1, 2, \cdots, n)$ *is the weight of* α_i*. As shown in Figure 1 below:*

More generally, we can get an in-depth understanding of the approximation of Definition 2.4 from Theorem 4.4 in the current interval case. It is interesting to discuss the relationship between these discrete and continuous expressions. We will report them in detail in the forthcoming paper.

Proof of Theorem 4.4. By (37), it suffices to prove

$$\underline{F}^q = \lim_{\lambda_1 \to 0} \underline{F}_n^q = 1 - \exp\left\{ - \int_{\mu_\alpha}^{\mu_\beta} \frac{q\mu_\xi^{q-1} \underline{f}_\xi^q}{1 - \mu_\xi^q} d\mu_\xi \right\}, \tag{40}$$

$$\overline{F}^q = \lim_{\lambda_1 \to 0} \overline{F}_n^q = 1 - \exp\left\{ - \int_{\mu_\alpha}^{\mu_\beta} \frac{q\mu_\xi^{q-1} \overline{f}_\xi^q}{1 - \mu_\xi^q} d\mu_\xi \right\}, \tag{41}$$

and

$$\underline{G} = \lim_{\lambda_2 \to 1} \underline{G}_n = \exp\left\{ \int_{v_\alpha}^{v_\beta} \frac{1 - \underline{g}_\xi}{v_\xi} dv_\xi \right\}, \quad \overline{G} = \lim_{\lambda_2 \to 1} \overline{G}_n = \exp\left\{ \int_{v_\alpha}^{v_\beta} \frac{1 - \overline{g}_\xi}{v_\xi} dv_\xi \right\}. \tag{42}$$

We first consider $\underline{F}$. From (35) and Definition 2.3, we receive

$$\underline{F}_n^q = 1 - \prod_i \left(1 - \underline{f}_i^q \frac{\mu_i^q - \mu_{i-1}^q}{1 - \mu_{i-1}^q} \right). \tag{43}$$

Notice that

$$\prod_i \left(1 - \underline{f}_i^q \frac{\mu_i^q - \mu_{i-1}^q}{1 - \mu_{i-1}^q} \right) = \exp\left\{ \ln \prod_i \left(1 - \underline{f}_i^q \frac{\mu_i^q - \mu_{i-1}^q}{1 - \mu_{i-1}^q} \right) \right\} = \exp\left\{ \sum_i \ln \left(1 - \underline{f}_i^q \frac{\mu_i^q - \mu_{i-1}^q}{1 - \mu_{i-1}^q} \right) \right\}. \tag{44}$$

Making use of (44) and Proposition 2.17, we send $\lambda_1 \to 0$ in (43) and receive

$$\begin{aligned} \lim_{\lambda_1 \to 0} \underline{F}_n^q &= 1 - \lim_{\lambda_1 \to 0} \exp\left\{ \sum_i \ln \left(1 - \underline{f}_i^q \frac{\mu_i^q - \mu_{i-1}^q}{1 - \mu_{i-1}^q} \right) \right\} \\ &= 1 - \exp\left\{ \lim_{\lambda_1 \to 0} \sum_i \ln \left(1 - \underline{f}_i^q \frac{\mu_i^q - \mu_{i-1}^q}{1 - \mu_{i-1}^q} \right) \right\} \\ &= 1 - \exp\left\{ - \int_{\mu_\alpha}^{\mu_\beta} \underline{f}_\xi^q \frac{q\xi^{q-1}}{1 - \xi^q} d\xi \right\}, \end{aligned}$$

where in the equalities we have used the properties of integral in elementary calculus, the continuity of exponential function, $\underline{f}$ and μ. This proves (40).

The proofs of (41)-(42) can be dealt similarly. So, we complete the proof of Theorem 4.4.

By Theorem 4.4, it is clear to see that

$$\int_{L(\alpha,\alpha)} \phi(\xi)\mathcal{T}\xi = \langle [0,0],[1,1]\rangle\,, \tag{45}$$

$$\int_{L(\alpha,\beta)} \langle [0,0],[1,1]\rangle\,\mathcal{T}\xi = \beta \ominus \alpha, \tag{46}$$

$$\int_{L(\alpha,\beta)} \langle [0,0],[1,1]\rangle\,\mathcal{T}\xi = \langle [0,0],[1,1]\rangle\,. \tag{47}$$

It is worthy to mention that, like the additive q-ROFF integrals, the value of $\int_{L(\alpha,\beta)} \phi(\xi)\mathcal{T}\xi$ depends only on where the end-points α and β are located, but not on how the route from α to β goes. We present an example to illustrate the phenomenon:

Example 4.6. *Consider*

$$\phi(\alpha) = \langle [\underline{f}_\alpha, \overline{f}_\alpha], [\underline{g}_\alpha, \overline{g}_\alpha]\rangle = \left\langle [0.5,1], \left[v_\alpha^{\frac{1}{q}}, v_\alpha^{\frac{1}{q}}\right]\right\rangle.$$

Select the integral lines L_1 and L_2 as follows:

$$L_1 := \left\{\langle \mu_\xi, v_\xi\rangle \mid \frac{v_\xi - v_\alpha}{\mu_\xi - \mu_\alpha} = \frac{v_\beta - v_\alpha}{\mu_\beta - \mu_\alpha}, \quad \mu_\xi \in [\mu_\alpha,\, \mu_\beta]\right\},$$

and

$$L_2 := \left\{\langle \mu_\xi, v_\xi\rangle \mid \begin{matrix} v_\xi = \varphi(\mu_\xi) > 0, & \mu_\xi \in [\mu_\alpha,\, \mu_\beta], \\ v_\alpha = \varphi(\mu_\alpha), & v_\beta = \varphi(\mu_\beta) \quad \varphi' < 0,\ \varphi'' > 0\end{matrix}\right\},$$

in which the end-points

$$\alpha = \langle 0.2, 0.5\rangle \quad \text{and} \quad \beta = \langle 0.8, 0.3\rangle. \tag{48}$$

Then, by (39)*, direct calculation gives*

$$\begin{aligned}\int_{L_1(\alpha,\beta)} \phi(\xi)\mathcal{T}\xi &= \left\langle \left[\left(1 - \left(\frac{1-\mu_\beta}{1-\mu_\alpha}\right)^{(0.5)^q}\right)^{\frac{1}{q}}, \left(\frac{\mu_\beta - \mu_\alpha}{1-\mu_\alpha}\right)^{\frac{1}{q}}\right], \left[\frac{v_\beta}{v_\alpha} e^{0.5(v_\alpha^2 - v_\beta^2)}, \frac{v_\beta}{v_\alpha} e^{v_\alpha - v_\beta}\right]\right\rangle \\ &= \left\langle \left[\left(1 - (0.25)^{(0.5)^q}\right)^{\frac{1}{q}}, (0.75)^{\frac{1}{q}}\right], [0.6e^{0.08}, 0.6e^{0.2}]\right\rangle.\end{aligned}$$

However,

$$\int_{L_2(\alpha,\beta)} \phi(\xi)\mathcal{T}\xi = \left\langle \left[\left(1 - e^{-(0.5)^q} 0.25\right)^{\frac{1}{q}}, (0.75)^{\frac{1}{q}}\right], \left[0.6e^{0.08}, 0.6e^{0.2}\right]\right\rangle.$$

This implies

$$\int_{L_2(\alpha,\beta)} \phi(\xi)\mathcal{T}\xi = \int_{L_1(\alpha,\beta)} \phi(\xi)\mathcal{T}\xi,$$

so long as the end-points α and β of L_1 and L_2 are the same.

4.2 Some basic properties

In what follows, we assume that the *simplified IVq-ROFFs* ϕ and

$$\phi_i(\alpha) = \left\langle \left[\underline{f}_i, \overline{f}_i\right], \left[\underline{g}_i, \overline{g}_i\right]\right\rangle, \quad i = 1,2,3\cdots. \tag{49}$$

Additionally,

$$\phi_1(\alpha) \preceq \phi_2(\alpha), \tag{50}$$

if and only if

$$\underline{f}_1 \leq \underline{f}_2,\ \overline{f}_1 \leq \overline{f}_2 \quad \text{and } \underline{g}_1 \geq \underline{g}_2,\ \overline{g}_1 \geq \overline{g}_2.$$

The contribution in this subsection is devoted to proving the following several theorems, which are responsible to some basic operational laws of IVq-ROFIs of *simplified IVq-ROFFs*.

Theorem 4.7 (Comparison theorem)**.** *(i) If $\phi_1 \preceq \phi_2$, then*

$$\int_{L(\alpha,\beta)} \phi_1(\xi)\mathcal{T}\xi \preceq \int_{L(\alpha,\beta)} \phi_2(\xi)\mathcal{T}\xi.$$

(ii) If $\gamma \in \mathbb{S}_{\oplus}(\beta)$ and $\beta \in \mathbb{S}_{\oplus}(\alpha)$, then

$$\int_{L(\alpha,\beta)} \phi(\xi)\mathcal{T}\xi \preceq \int_{L(\alpha,\gamma)} \phi(\xi)\mathcal{T}\xi.$$

Proof. The proof follows directly from Formulas (39) and (49). And it also can be shown in the following Figure 2

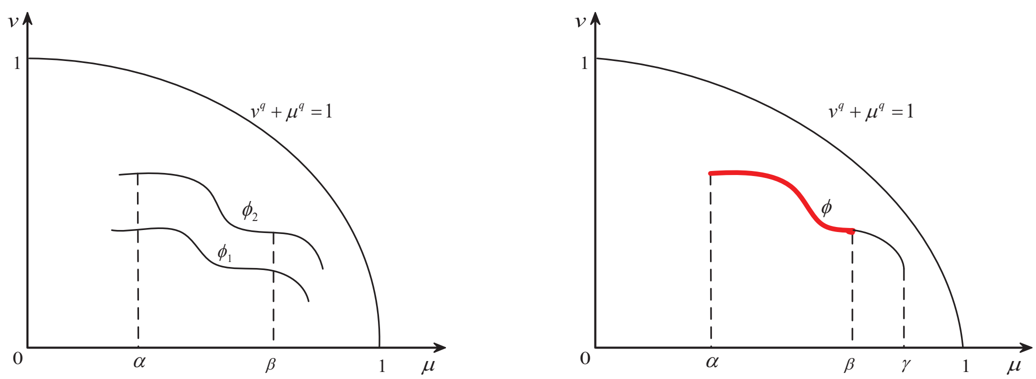

Figure 2: Geometric representation of the comparison theorem

Theorem 4.8 (Algebraic operations of integrand)**.**

$$\begin{aligned}\int_{L(\alpha,\beta)} \phi_1(\xi)\mathcal{T}\xi \oplus \int_{L(\alpha,\beta)} \phi_2(\xi)\mathcal{T}\xi = \int_{L(\alpha,\beta)} \Big\langle \Big[\left(\underline{f}_1^q + \underline{f}_2^q\right)^{\frac{1}{q}}, \left(\overline{f}_1^q + \overline{f}_2^q\right)^{\frac{1}{q}} \Big], \\ \Big[\left(1-(1-\underline{g}_1^q)-(1-\underline{g}_2^q)\right)^{\frac{1}{q}}, (1-(1-\overline{g}_1^q)-(1-\overline{g}_2^q))^{\frac{1}{q}} \Big] \Big\rangle \mathcal{T}\xi.\end{aligned} \tag{51}$$

Proof. By Theorem 4.4 and (49), it has

$$\begin{aligned}\int_{L(\alpha,\beta)} \phi_1(\xi)\mathcal{T}\xi \oplus \int_{L(\alpha,\beta)} \phi_2(\xi)\mathcal{T}\xi = &\Bigg\langle \Bigg[\left(1-\exp\left\{-\int_{\mu_\alpha}^{\mu_\beta} \frac{q\mu_\xi^{q-1}\underline{f}_1^q}{1-\mu_\xi^q}\right\}\right)^{\frac{1}{q}}, \left(1-\exp\left\{-\int_{\mu_\alpha}^{\mu_\beta} \frac{q\mu_\xi^{q-1}\overline{f}_1^q}{1-\mu_\xi^q}\right\}\right)^{\frac{1}{q}} \Bigg], \\ &\left[\exp\left\{\int_{v_\alpha}^{v_\beta} \frac{1-\underline{g}_1^q}{v_\xi}\right\}, \exp\left\{\int_{v_\alpha}^{v_\beta} \frac{1-\overline{g}_1^q}{v_\xi}\right\}\right]\Bigg\rangle \\ &\oplus \Bigg\langle \Bigg[\left(1-\exp\left\{-\int_{\mu_\alpha}^{\mu_\beta} \frac{q\mu_\xi^{q-1}\underline{f}_2^q}{1-\mu_\xi^q}\right\}\right)^{\frac{1}{q}}, \left(1-\exp\left\{-\int_{\mu_\alpha}^{\mu_\beta} \frac{q\mu_\xi^{q-1}\overline{f}_2^q}{1-\mu_\xi^q}\right\}\right)^{\frac{1}{q}} \Bigg], \\ &\left[\exp\left\{\int_{v_\alpha}^{v_\beta} \frac{1-\underline{g}_2^q}{v_\xi}\right\}, \exp\left\{\int_{v_\alpha}^{v_\beta} \frac{1-\overline{g}_2^q}{v_\xi}\right\}\right]\Bigg\rangle.\end{aligned} \tag{52}$$

Using the basic operation of IVq-ROFFs, we continue to compute the right-hand side of (52) and receive

$$\begin{aligned}&\int_{L(\alpha,\beta)}\phi_1(\xi)\mathcal{T}\xi\oplus\int_{L(\alpha,\beta)}\phi_2(\xi)\mathcal{T}\xi\\&=\left\langle\left[\left(1-\exp\left\{-\int_{\mu_\alpha}^{\mu_\beta}\frac{q\mu_\xi^{q-1}(\underline{f}_1^q+\underline{f}_2^q)}{1-\mu_\xi^q}\right\}\right)^{\frac{1}{q}},\left(1-\exp\left\{-\int_{\mu_\alpha}^{\mu_\beta}\frac{q\mu_\xi^{q-1}(\overline{f}_1^q+\overline{f}_2^q)}{1-\mu_\xi^q}\right\}\right)^{\frac{1}{q}}\right],\right.\\&\quad\left.\left[\exp\left\{\int_{v_\alpha}^{v_\beta}\frac{(1-\underline{g}_1^q)+(1-\underline{g}_2^q)}{v_\xi}\right\},\exp\left\{\int_{v_\alpha}^{v_\beta}\frac{(1-\overline{g}_1^q)+(1-\overline{g}_2^q)}{v_\xi}\right\}\right]\right\rangle.\end{aligned}\tag{53}$$

Combining (53) with (39), we conclude (51). The proof of Theorem 4.8 is done.

Theorem 4.9 (Scalar-multiplication)**.** *It satisfies that*

$$\lambda\int_{L(\alpha,\beta)}\phi(\xi)\mathcal{T}\xi=\int_{L(\alpha,\beta)}\left\langle\left[\lambda^{\frac{1}{q}}\underline{f},\lambda^{\frac{1}{q}}\overline{f}\right],\left[(1-\lambda(1-\underline{g}^q))^{\frac{1}{q}},(1-\lambda(1-\overline{g}^q))^{\frac{1}{q}}\right]\right\rangle\mathcal{T}\xi,\tag{54}$$

where the constant $\lambda\in[0,1]$ is given.

Proof. Again using (39), one deduces

$$\begin{aligned}\lambda\int_{L(\alpha,\beta)}\phi(\xi)\mathcal{T}\xi&=\lambda\left\langle\left[\left(1-\exp\left\{-\int_{\mu_\alpha}^{\mu_\beta}\frac{q\mu_\xi^{q-1}\underline{f}_\xi^q}{1-\mu_\xi^q}\right\}\right)^{\frac{1}{q}},\left(1-\exp\left\{-\int_{\mu_\alpha}^{\mu_\beta}\frac{q\mu_\xi^{q-1}\overline{f}_\xi^q}{1-\mu_\xi^q}\right\}\right)^{\frac{1}{q}}\right],\right.\\&\qquad\left.\left[\exp\left\{\int_{v_\alpha}^{v_\beta}\frac{1-\underline{g}_\xi^q}{v_\xi}\right\},\exp\left\{\int_{v_\alpha}^{v_\beta}\frac{1-\overline{g}_\xi^q}{v_\xi}\right\}\right]\right\rangle\\&=\left\langle[\underline{A}^{\frac{1}{q}},\overline{A}^{\frac{1}{q}}],\left[\exp\left\{\int_{v_\alpha}^{v_\beta}\frac{1-\underline{g}_\xi^q}{v_\xi}\right\},\exp\left\{\int_{v_\alpha}^{v_\beta}\frac{1-\overline{g}_\xi^q}{v_\xi}\right\}\right]\right\rangle,\end{aligned}$$

where

$$\underline{A}=1-\left(1-\left(1-\exp\left\{-\int_{\mu_\alpha}^{\mu_\beta}\frac{q\mu_\xi^{q-1}\underline{f}_\xi^q}{1-\mu_\xi^q}\right\}\right)\right)^\lambda=1-\exp\left\{-\int_{\mu_\alpha}^{\mu_\beta}\frac{q\mu_\xi^{q-1}\lambda\underline{f}_\xi^q}{1-\mu_\xi^q}\right\},$$

and similarly,

$$\overline{A}=1-\exp\left\{-\int_{\mu_\alpha}^{\mu_\beta}\frac{q\mu_\xi^{q-1}\lambda\overline{f}_\xi^q}{1-\mu_\xi^q}\right\}.$$

Remark 4.10. *In fact, Theorem 4.9 is still valid for $\lambda>1$.*

Theorem 4.11 (Algebraic operation of integral line)**.** *Assume that*

$$\gamma\in\mathbb{S}_\oplus(\beta),\quad\beta\in\mathbb{S}_\oplus(\alpha),\quad\alpha\in\mathbb{S},$$

and the IVq-ROFF ϕ is continuous. Then

$$\int_{L(\alpha,\beta)}\phi(\xi)\mathcal{T}\xi\oplus\int_{L(\beta,\gamma)}\phi(\xi)\mathcal{T}\xi=\int_{L(\alpha,\gamma)}\phi(\xi)\mathcal{T}\xi.\tag{55}$$

Proof. By Formula (1) in Definition 2.3, the classical properties in elementary calculus, we compute

$$\begin{aligned}
&\left\langle\left[\left(1-\exp\left\{-\int_{\mu_\alpha}^{\mu_\beta}\frac{q\mu_\xi^{q-1}\underline{f}^q}{1-\mu_\xi^q}\right\}\right)^{\frac{1}{q}},\left(1-\exp\left\{-\int_{\mu_\alpha}^{\mu_\beta}\frac{q\mu_\xi^{q-1}\overline{f}^q}{1-\mu_\xi^q}\right\}\right)^{\frac{1}{q}}\right],\right.\\
&\quad\left.\left[\exp\left\{\int_{v_\alpha}^{v_\beta}\frac{1-\underline{g}^q}{v_\xi}\right\},\exp\left\{\int_{v_\alpha}^{v_\beta}\frac{1-\overline{g}^q}{v_\xi}\right\}\right]\right\rangle\\
&\oplus\left\langle\left[\left(1-\exp\left\{-\int_{\mu_\beta}^{\mu_\gamma}\frac{q\mu_\xi^{q-1}\underline{f}^q}{1-\mu_\xi^q}\right\}\right)^{\frac{1}{q}},\left(1-\exp\left\{-\int_{\mu_\beta}^{\mu_\gamma}\frac{q\mu_\xi^{q-1}\overline{f}^q}{1-\mu_\xi^q}\right\}\right)^{\frac{1}{q}}\right],\right.\\
&\quad\left.\left[\exp\left\{\int_{v_\beta}^{v_\gamma}\frac{1-\underline{g}^q}{v_\xi}\right\},\exp\left\{\int_{v_\beta}^{v_\gamma}\frac{1-\overline{g}^q}{v_\xi}\right\}\right]\right\rangle\\
&=\left\langle\left[\left(1-\exp\left\{-\int_{\mu_\alpha}^{\mu_\gamma}\frac{q\mu_\xi^{q-1}\underline{f}^q}{1-\mu_\xi^q}\right\}\right)^{\frac{1}{q}},\left(1-\exp\left\{-\int_{\mu_\alpha}^{\mu_\gamma}\frac{q\mu_\xi^{q-1}\overline{f}^q}{1-\mu_\xi^q}\right\}\right)^{\frac{1}{q}}\right],\right.\\
&\quad\left.\left[\exp\left\{\int_{v_\alpha}^{v_\gamma}\frac{1-\underline{g}^q}{v_\xi}\right\},\exp\left\{\int_{v_\alpha}^{v_\gamma}\frac{1-\overline{g}^q}{v_\xi}\right\}\right]\right\rangle.
\end{aligned}\tag{56}$$

The combination of (39) with (56) yields (55), and thus completes the proof of Theorem 4.11.

Theorem 4.12 (Estimate value theorem)**.** *Assume in addition that the simplified IVq-ROFF ϕ is continuous. Then there are two constants $\underline{M}$, $\overline{M}$ such that*

$$\underline{M}\otimes(\beta\ominus\alpha)\preceq\int_{L(\alpha,\beta)}\phi(\xi)\mathcal{T}\xi\preceq\overline{M}\otimes(\beta\ominus\alpha).\tag{57}$$

Proof. We only check the former inequality " $\preceq$ " in (57) for illustration. Recalling (39), we have

$$\begin{aligned}
\int_{L(\alpha,\beta)}\phi(\xi)\mathcal{T}\xi=&\left\langle\left[\left(1-\exp\left\{-\int_{\mu_\alpha}^{\mu_\beta}\frac{q\mu_\xi^{q-1}\underline{f}^q}{1-\mu_\xi^q}\right\}\right)^{\frac{1}{q}},\left(1-\exp\left\{-\int_{\mu_\alpha}^{\mu_\beta}\frac{q\mu_\xi^{q-1}\overline{f}^q}{1-\mu_\xi^q}\right\}\right)^{\frac{1}{q}}\right],\right.\\
&\left.\left[\exp\left\{\int_{v_\alpha}^{v_\beta}\frac{1-\underline{g}^q}{v_\xi}\right\},\exp\left\{\int_{v_\alpha}^{v_\beta}\frac{1-\overline{g}^q}{v_\xi}\right\}\right]\right\rangle.
\end{aligned}\tag{58}$$

By Theorem 2.13, we see that $\underline{f}$, $\overline{f}$ and $\underline{g}$, $\overline{g}$ are continuous. Putting

$$\underline{m}_{11}=\min_{\mu\in[\mu_\alpha,\mu_\beta]}\underline{f},\quad \underline{m}_{12}=\min_{\mu\in[\mu_\alpha,\mu_\beta]}\overline{f}.$$

Thus, direct computation shows

$$1-\exp\left\{-\int_{\mu_\alpha}^{\mu_\beta}\frac{q\mu_\xi^{q-1}\underline{f}^q}{1-\mu_\xi^q}\right\}\geq 1-\exp\left\{-\underline{m_{11}}^q\int_{\mu_\alpha}^{\mu_\beta}\frac{q\mu_\xi^{q-1}}{1-\mu_\xi^q}\right\}=1-\left(1-\frac{\mu_\beta^q-\mu_\alpha^q}{1-\mu_\alpha^q}\right)^{\underline{m_{11}}^q}$$

and

$$1-\exp\left\{-\int_{\mu_\alpha}^{\mu_\beta}\frac{q\mu_\xi^{q-1}\overline{f}^q}{1-\mu_\xi^q}d\mu_\xi\right\}\geq 1-\left(1-\frac{\mu_\beta^q-\mu_\alpha^q}{1-\mu_\alpha^q}\right)^{\underline{m_{12}}^q}.$$

Putting

$$\underline{m}_{21}=\min_{v\in[v_\alpha,v_\beta]}\underline{g},\quad \underline{m}_{22}=\min_{v\in[v_\alpha,v_\beta]}\overline{g}.$$

Then, it has

$$\exp\left\{\int_{v_\alpha}^{v_\beta}\frac{1-\underline{g}^q}{v_\xi}dv_\xi\right\}\leq\exp\left\{(1-\underline{m_2}^q)\int_{v_\alpha}^{v_\beta}\frac{1}{v_\xi}dv_\xi\right\}=\left(\frac{v_\beta}{v_\alpha}\right)^{1-\underline{m_2}^q}.$$

We insert the last two inequalities back into (58), to discover

$$\begin{aligned}\int_{L(\alpha,\beta)}\phi(\xi)\mathcal{T}\xi \succeq & \Bigg\langle\Bigg[\left(1-\left(1-\frac{\mu_\beta^q-\mu_\alpha^q}{1-\mu_\alpha^q}\right)^{\underline{m_{11}}^q}\right)^{\frac{1}{q}}, \left(1-\left(1-\frac{\mu_\beta^q-\mu_\alpha^q}{1-\mu_\alpha^q}\right)^{\underline{m_{12}}^q}\right)^{\frac{1}{q}}\Bigg], \\ & \left[\left(\frac{v_\beta}{v_\alpha}\right)^{1-\underline{m_{21}}^q}, \left(\frac{v_\beta}{v_\alpha}\right)^{1-\underline{m_{22}}^q}\right]\Bigg\rangle \\ \succeq & \Bigg\langle\Bigg[\left(1-\left(1-\frac{\mu_\beta^q-\mu_\alpha^q}{1-\mu_\alpha^q}\right)^{\underline{m_{11}}^q}\right)^{\frac{1}{q}}, \left(1-\left(1-\frac{\mu_\beta^q-\mu_\alpha^q}{1-\mu_\alpha^q}\right)^{\underline{m_{11}}^q}\right)^{\frac{1}{q}}\Bigg], \\ & \left[\left(\frac{v_\beta}{v_\alpha}\right)^{1-\underline{m_{22}}^q}, \left(\frac{v_\beta}{v_\alpha}\right)^{1-\underline{m_{22}}^q}\right]\Bigg\rangle,\end{aligned} \tag{59}$$

where in the last $\succeq$ we have used the facts $\underline{m}_{11} \leq \underline{m}_{12}$ and $\underline{m}_{21} \leq \underline{m}_{22}$.

Let $\underline{M} = \min\{\underline{m_{11}}^q, 1-\underline{m_{22}}^q\}$. We continue to estimate (59) and obtain

$$\begin{aligned}\int_{L(\alpha,\beta)}\phi(\xi)d\xi \succeq & \Bigg\langle\Bigg[\left(1-\left(1-\frac{\mu_\beta^q-\mu_\alpha^q}{1-\mu_\alpha^q}\right)^{\underline{m_{11}}^q}\right)^{\frac{1}{q}}, \left(1-\left(1-\frac{\mu_\beta^q-\mu_\alpha^q}{1-\mu_\alpha^q}\right)^{\underline{m_{11}}^q}\right)^{\frac{1}{q}}\Bigg], \\ & \left[\left(\frac{v_\beta}{v_\alpha}\right)^{1-\underline{m_{22}}^q}, \left(\frac{v_\beta}{v_\alpha}\right)^{1-\underline{m_{22}}^q}\right]\Bigg\rangle \\ = & \left\langle\left(1-\left(1-\frac{\mu_\beta^q-\mu_\alpha^q}{1-\mu_\alpha^q}\right)^{\underline{M}}\right)^{\frac{1}{q}}, \left(\frac{v_\beta}{v_\alpha}\right)^{\underline{M}}\right\rangle \\ = & \underline{M}(\beta\ominus\alpha).\end{aligned} \tag{60}$$

In Theorem 4.13 below, we refine the proof of Theorem 4.12 and deduce the mean value theorem of IVq-ROFF.

Theorem 4.13 (Mean value theorem). *Under the same hypotheses made in Theorem 4.12. There exists asimplified IVq-ROFF* $\phi(\eta) = \left\langle\left[\underline{f}_\eta, \overline{f}_\eta\right], \left[\underline{g}_\eta, \overline{g}_\eta\right]\right\rangle$ *with some fixed* $\eta \in L(\alpha,\beta)$*, such that*

$$\phi(\eta)\otimes(\beta\ominus\alpha) = \int_{L(\alpha,\beta)}\phi(\xi)\mathcal{T}\xi. \tag{61}$$

Proof. By the continuity of $\underline{f}(\mu_\xi, v_\xi)$, and the non-negativity of $\frac{\mu_\xi}{1-\mu_\xi}$, one has

$$\int_{\mu_\alpha}^{\mu_\beta}\frac{q\mu_\xi^{q-1}\underline{f}_\xi^q}{1-\mu_\xi^q}d\mu_\xi = f_{\eta_{\underline{f}}}^q\int_{\mu_\alpha}^{\mu_\beta}\frac{q\mu_\xi^{q-1}}{1-\mu_\xi^q}d\mu_\xi, \quad \text{for some} \quad \eta_{\underline{f}} \in L(\alpha,\beta),$$

and

$$\int_{\mu_\alpha}^{\mu_\beta}\frac{q\mu_\xi^{q-1}\underline{f}_\xi^q}{1-\mu_\xi^q}d\mu_\xi = f_{\eta_{\overline{f}}}^q\int_{\mu_\alpha}^{\mu_\beta}\frac{q\mu_\xi^{q-1}}{1-\mu_\xi^q}d\mu_\xi, \quad \text{for some} \quad \eta_{\overline{f}} \in L(\alpha,\beta).$$

The same argument runs that

$$\int_{v_\alpha}^{v_\beta}\frac{1-\underline{g}_\xi^q}{v_\xi}dv_\xi = (1-g_{\eta_{\underline{g}}}^q)\int_{v_\alpha}^{v_\beta}\frac{1}{v_\xi}dv_\xi, \quad \text{for some} \quad \eta_{\underline{g}} \in L(\alpha,\beta),$$

and

$$\int_{v_\alpha}^{v_\beta}\frac{1-\overline{g}_\xi^q}{v_\xi}dv_\xi = (1-g_{\eta_{\overline{g}}}^q)\int_{v_\alpha}^{v_\beta}\frac{1}{v_\xi}dv_\xi, \quad \text{for some} \quad \eta_{\underline{g}} \in L(\alpha,\beta).$$

Therefore,

$$1-\exp\left\{-\int_{\mu_\alpha}^{\mu_\beta}\frac{q\mu_\xi^{q-1}\underline{f}_\xi^q}{1-\mu_\xi^q}d\mu_\xi\right\}=1-\exp\left\{-f_{\eta_{\underline{f}}}^q\int_{\mu_\alpha}^{\mu_\beta}\frac{q\mu_\xi^{q-1}\underline{f}_\xi^q}{1-\mu_\xi^q}d\mu_\xi\right\}=1-\left(1-\frac{\mu_\beta^q-\mu_\alpha^q}{1-\mu_\alpha^q}\right)^{f_{\eta_{\underline{f}}}^q},$$

and

$$1-\exp\left\{-\int_{\mu_\alpha}^{\mu_\beta}\frac{q\mu_\xi^{q-1}\overline{f}_\xi^q}{1-\mu_\xi^q}d\mu_\xi\right\}=1-\left(1-\frac{\mu_\beta^q-\mu_\alpha^q}{1-\mu_\alpha^q}\right)^{f_{\eta_{\overline{f}}}^q}.$$

In a similar way,

$$\exp\left\{\int_{v_\alpha}^{v_\beta}\frac{1-\underline{g}_\xi^q}{v_\xi}dv_\xi\right\}=\exp\left\{(1-g_{\eta_{\underline{g}}}^q)\int_{v_\alpha}^{v_\beta}\frac{1}{v_\xi}dv_\xi\right\}=\left(\frac{v_\beta}{v_\alpha}\right)^{(1-g_{\eta_{\underline{g}}}^q)},$$

and

$$\exp\left\{\int_{v_\alpha}^{v_\beta}\frac{1-\overline{g}_\xi^q}{v_\xi}dv_\xi\right\}=\left(\frac{v_\beta}{v_\alpha}\right)^{(1-g_{\eta_{\overline{g}}}^q)}.$$

Insert the last four inequalities back into (58), to discover

$$\begin{aligned}\int_{L(\alpha,\beta)}\phi(\xi)\mathcal{T}\xi=\Bigg\langle&\left[\left(1-\left(1-\frac{\mu_\beta^q-\mu_\alpha^q}{1-\mu_\alpha^q}\right)^{f_{\eta_{\underline{f}}}^q}\right)^{\frac{1}{q}},\left(1-\left(1-\frac{\mu_\beta^q-\mu_\alpha^q}{1-\mu_\alpha^q}\right)^{f_{\eta_{\overline{f}}}^q}\right)^{\frac{1}{q}}\right],\\&\left[\left(\frac{v_\beta}{v_\alpha}\right)^{(1-g_{\eta_{\underline{g}}}^q)},\left(\frac{v_\beta}{v_\alpha}\right)^{(1-g_{\eta_{\overline{g}}}^q)}\right]\Bigg\rangle.\end{aligned}\tag{62}$$

On the other hand, if we define

$$\underline{f}_\eta^q=\frac{1-\left(1-\frac{\mu_\beta^q-\mu_\alpha^q}{1-\mu_\alpha^q}\right)^{f_{\eta_{\underline{f}}}^q}}{\frac{\mu_\beta-\mu_\alpha^q}{1-\mu_\alpha^q}},\quad \overline{f}_\eta^q=\frac{1-\left(1-\frac{\mu_\beta^q-\mu_\alpha^q}{1-\mu_\alpha^q}\right)^{f_{\eta_{\overline{f}}}^q}}{\frac{\mu_\beta-\mu_\alpha^q}{1-\mu_\alpha^q}},$$

and

$$\underline{g}_\eta^q=1-\frac{1-\left(\frac{v_\beta}{v_\alpha}\right)^{q(1-g_{\eta_{\underline{g}}}^q)}}{1-\frac{v_\beta^q}{v_\alpha^q}},\quad \overline{g}_\eta^q=1-\frac{1-\left(\frac{v_\beta}{v_\alpha}\right)^{q(1-g_{\eta_{\overline{g}}}^q)}}{1-\frac{v_\beta^q}{v_\alpha^q}},$$

we infer

$$\begin{aligned}&\Bigg\langle\left[\left(1-\left(1-\frac{\mu_\beta^q-\mu_\alpha^q}{1-\mu_\alpha^q}\right)^{f_{\eta_{\underline{f}}}^q}\right)^{\frac{1}{q}},\left(1-\left(1-\frac{\mu_\beta^q-\mu_\alpha^q}{1-\mu_\alpha^q}\right)^{f_{\eta_{\overline{f}}}^q}\right)^{\frac{1}{q}}\right],\ \left[\left(\frac{v_\beta}{v_\alpha}\right)^{(1-g_{\eta_{\underline{g}}}^q)},\left(\frac{v_\beta}{v_\alpha}\right)^{(1-g_{\eta_{\overline{g}}}^q)}\right]\Bigg\rangle\\&=\Bigg\langle\left[\underline{f}_\eta\left(\frac{\mu_\beta-\mu_\alpha^q}{1-\mu_\alpha^q}\right)^{\frac{1}{q}},\overline{f}_\eta\left(\frac{\mu_\beta-\mu_\alpha^q}{1-\mu_\alpha^q}\right)^{\frac{1}{q}}\right],\ \left[\left(1-(1-\underline{g}_\eta^q)\left(1-\frac{v_\beta^q}{v_\alpha^q}\right)\right)^{\frac{1}{q}},\left(1-(1-\overline{g}_\eta^q)\left(1-\frac{v_\beta^q}{v_\alpha^q}\right)\right)^{\frac{1}{q}}\right]\Bigg\rangle.\end{aligned}\tag{63}$$

Making use of (62) and (63), we calculate

$$\begin{aligned}\int_{L(\alpha,\beta)}\phi(\xi)\mathcal{T}\xi&=\left\langle\left[\underline{f}_\eta,\overline{f}_\eta\right],\left[\underline{g}_\eta,\overline{g}_\eta\right]\right\rangle\otimes\left\langle\left[\left(\frac{\mu_\beta-\mu_\alpha^q}{1-\mu_\alpha^q}\right)^{\frac{1}{q}},\left(\frac{\mu_\beta-\mu_\alpha^q}{1-\mu_\alpha^q}\right)^{\frac{1}{q}}\right],\left[\frac{v_\beta}{v_\alpha},\frac{v_\beta}{v_\alpha}\right]\right\rangle\\&=\left\langle\left[\underline{f}_\eta,\overline{f}_\eta\right],\left[\underline{g}_\eta,\overline{g}_\eta\right]\right\rangle\otimes\left\langle\left(\frac{\mu_\beta-\mu_\alpha^q}{1-\mu_\alpha^q}\right)^{\frac{1}{q}},\frac{v_\beta}{v_\alpha}\right\rangle\\&=\phi(\eta)\otimes(\beta\ominus\alpha).\end{aligned}$$

This is (61), and the proof is completed.

Remark 4.14. *The continuity hypothesis on IVq-ROFF seems essential in Theorem 4.12 and Theorem 4.13. However, in proving Theorems 4.7-4.11, we only require that IVq-ROFFs be continuous piecewise.*

Although the integral theories established in Section 4 are focused on the *simplified IVq-ROFFs*, we can establish the counterparts for the *generalized IVq-ROFFs* by the similar argument used in Section 4.

Theorem 4.4' [Newton-Leibniz for generalized IVq-ROFFs] *Let the IVq-ROFF defined in Definition 2.8 satisfy*

$$\frac{\partial \underline{f}_\alpha}{\partial \underline{v}_\alpha} = \frac{\partial \overline{f}_\alpha}{\partial \overline{v}_\alpha} = 0 = \frac{\partial \underline{g}_\alpha}{\partial \underline{\mu}_\alpha} = \frac{\partial \overline{g}_\alpha}{\partial \overline{\mu}_\alpha}. \tag{64}$$

Then,

$$\int_{L(\alpha,\beta)} \left\langle \left[\underline{f}_\xi, \overline{f}_\xi\right], \left[\underline{g}_\xi, \overline{g}_\xi\right] \right\rangle \mathcal{T}\xi =$$
$$\left\langle \left[\left(1 - \exp\left\{ -\int_{\underline{\mu}_\alpha}^{\underline{\mu}_\beta} \frac{q\mu_\xi^{q-1} \underline{f}_\xi^q}{1-\mu_\xi^q} d\mu_\xi \right\}\right)^{\frac{1}{q}}, \left(1 - \exp\left\{ -\int_{\overline{\mu}_\alpha}^{\overline{\mu}\beta} \frac{q\mu_\xi^{q-1} \overline{f}_\xi^q}{1-\mu_\xi^q} d\mu_\xi \right\}\right)^{\frac{1}{q}} \right], \right. \tag{65}$$
$$\left. \left[\exp\left\{ \int_{\underline{v}_\alpha}^{\underline{v}_\beta} \frac{1-\underline{g}_\xi^q}{v_\xi} dv_\xi \right\}, \exp\left\{ \int_{\overline{v}_\alpha}^{\overline{v}_\beta} \frac{1-\overline{g}_\xi^q}{v_\xi} dv_\xi \right\} \right] \right\rangle.$$

Proof. The proof is the same sprit of that in Theorem 4.4, but needs more delicate computation and slight modification of interval q-rung orthopair fuzzy integral line.

Remark 4.15. *Following the ideas used in Subsections 4.2, we can deduce the operational laws for generalized IVq-ROFFs, which are parallel to those in Theorems 4.7 -4.13.*

5 The application of the integrals

In this paper, we give a practical example to illustrate the application of the integrals of IVq-ROFFs in solving group decision-making problem.

5.1 A practical application

Example 5.1. *In order to effectively prevent, timely control and eliminate the hazards of public health emergencies. Better protect the public's physical health and life safety, and maintain social stability. The government evaluates the disaster-bearing capability of public health emergencies in the three cities $A_i (i = 1, 2, 3)$ under its jurisdiction. The four evaluation attributes considered include: Disaster monitoring and forecasting capabilities (C_1) , disaster prevention capabilities (C_2), disaster emergency rescue capabilities (C_3), post-disaster recovery capabilities (C_4). A panel composed of three experts $E_k (k = 1, 2, 3)$ is invited to assess these three cities with respect to these four attributes. The kth expert provides his/her assessment value over the city A_i with respect to the attributes $C_j (j = 1, 2, \cdots, 4)$ using the IVq-ROFVs $r_{ij}^k = \langle [\underline{\mu}_{ij}^k, \overline{\mu}_{ij}^k], [\underline{v}_{ij}^k, \overline{v}_{ij}^k] \rangle$. All $r_{ij}^k (i, k = 1, 2, 3; j = 1, 2, \cdots, 4)$ are contained in the interval-valued q-rung orthopair fuzzy decision matrices $R^k = (r_{ij}^k)_{3\times4} (k = 1, 2, 3)$, as shown in Tables 2, 3 and 4.*

The first step is to aggregate the assessment values given by the experts. Here, we take the first aggregated value for example. r_{11} is aggregated by three IVq-ROFVs: $r_{11}^1 = \langle [0.2, 0.7], [0.6, 0.8] \rangle$, $r_{11}^2 = \langle [0.2, 0.6], [0.6, 0.8] \rangle$,$r_{11}^3 = \langle [0.1, 0.3], [0.4, 0.5] \rangle$.

Next, we define the end-points $\mathcal{O} = \langle [0, 0], [1, 1] \rangle$ and $\beta = \langle [1, 1], [0, 0] \rangle$. Build the IVq-ROFF

$$\varphi \triangleq \left\langle \left[\frac{\underline{f}(\mu)}{n}, \frac{\overline{f}(\mu)}{n} \right], \left[\frac{\underline{g}(v)}{n}, \frac{\overline{g}(v)}{n} \right] \right\rangle,$$

with explicit form as follows $(n = 3)$:

$$\frac{\underline{f}(\mu)}{3} = \begin{cases} 1, & 0 \le \mu < 0.1, \\ 2/3, & 0.1 \le \mu < 0.2, \\ 0, & 0.2 \le \mu \le 1, \end{cases} \quad \frac{\overline{f}(\mu)}{3} = \begin{cases} 1, & 0 \le \mu < 0.3, \\ 2/3, & 0.3 \le \mu < 0.6, \\ 1/3, & 0.6 \le \mu < 0.7, \\ 0, & 0.7 \le \mu \le 1, \end{cases}$$

and

$$\frac{\underline{g}(v)}{3}=\begin{cases}1, & 0\le v\le 0.4,\\ 2/3, & 0.4<v\le 0.6,\\ 0, & 0.6\le v\le 1,\end{cases}\quad \frac{\overline{g}(v)}{3}=\begin{cases}1, & 0<v\le 0.5.\\ 2/3, & 0.5<v\le 0.8.\\ 0, & 0.8\le v\le 1.\end{cases}$$

With the above preparations, taking $q=5$, *using Formula* (65) *in Theorem 4.4', we deduce*

$$\begin{aligned}&\int_{L(\mathcal{O},\beta)}\left\langle\left[\frac{\underline{f}(\mu)}{3},\frac{\overline{f}(\mu)}{3}\right],\left[\frac{\underline{g}(v)}{3},\frac{\overline{g}(v)}{3}\right]\right\rangle\mathcal{T}\xi=\\ &\left\langle\left[\left(1-\exp\left\{-\int_0^{0.2}\frac{5\mu^4\left(\frac{\underline{f}(\mu)}{3}\right)^5}{1-\mu^5}d\mu\right\}\right)^{\frac{1}{5}},\left(1-\exp\left\{-\int_0^{0.7}\frac{5\mu^4\left(\frac{\overline{f}(\mu)}{3}\right)^5}{1-\mu^5}d\mu\right\}\right)^{\frac{1}{5}}\right],\right.\\ &\left.\left[\exp\left\{\int_1^{0.4}\frac{1-\left(\frac{\underline{g}(v)}{3}\right)^5}{v}dv\right\},\exp\left\{\int_1^{0.5}\frac{1-\left(\frac{\overline{g}(v)}{3}\right)^5}{v}dv\right\}\right]\right\rangle\\ &\approx\langle[0.14,0.42],[0.42,0.53]\rangle.\end{aligned}\tag{66}$$

In the same way, we can get the other aggregated values. Thus, the aggregative IVq-rung orthopair fuzzy decision matrix $R=(r_{ij})_{3\times 4}$ *is obtained as listed in Table 5:*

The next step is to aggregate the assessment values for every city. The calculating process is similar to the first step, thus we obtain the overall assessment $Val[A_i]$ *of the cities* $A_i(i=1,2,3)$ *as follows:*

$$Val[A_1]=\langle[0.22,0.48],[0.23,0.34]\rangle,$$

$$Val[A_2]=\langle[0.58,0.73],[0.26,0.48]\rangle,$$

$$Val[A_3]=\langle[0.32,0.55],[0.44,0.64]\rangle.$$

Table 2: The decision matrix R^1

	C_1	C_2	C_3	C_4
A_1	$\langle[0.2,0.7],[0.6,0.8]\rangle$	$\langle[0.3,0.5],[0.5,0.7]\rangle$	$\langle[0.1,0.3],[0.7,0.9]\rangle$	$\langle[0.4,0.7),[0.6,0.9]\rangle$
A_2	$\langle[0.4,0.8],[0.6,0.8]\rangle$	$\langle[0.2,0.7],[0.3,0.6]\rangle$	$\langle[0.2,0.3],[0.5,0.6]\rangle$	$\langle[0.5,0.7),[0.8,0.9]\rangle$
A_3	$\langle[0.5,0.7],[0.4,0.6]\rangle$	$\langle[0.6,0.7],[0.3,0.4]\rangle$	$\langle[0.3,0.4],[0.5,0.8]\rangle$	$\langle[0.3,0.5],[0.4,0.8]\rangle$

Table 3: The decision matrix R^2

	C_1	C_2	C_3	C_4
A_1	$\langle[0.2,0.6],[0.6,0.8]\rangle$	$\langle[0.3,0.7),[0.3,0.6]\rangle$	$\langle[0.1,0.3],[0.4,0.7]\rangle$	$\langle[0.5,0.6],[0.8,0.9]\rangle$
A_2	$\langle[0.3,0.4],[0.6,0.7]\rangle$	$\langle[0.4,0.5],[0.5,0.8]\rangle$	$\langle[0.6,0.7],[0.5,0.8]\rangle$	$\langle[0.6,0.8],[0.7,0.9]\rangle$
A_3	$\langle[0.3,0.5],[0.4,0.6]\rangle$	$\langle[0.2,0.4],[0.7,0.8]\rangle$	$\langle[0.1,0.3],[0.4,0.5]\rangle$	$\langle[0.3,0.7],[0.4,0.6]\rangle$

According to Definition 2.6, we get the scores of the three cities as follows:

$$S[A_1]=\frac{(0.22^5-0.23^5)^{\frac{1}{5}}+(0.48^5-0.34^5)^{\frac{1}{5}}}{2}=0.17,$$

$$S[A_2]=\frac{(0.58^5-0.26^5)^{\frac{1}{5}}+(0.73^5-0.48^5)^{\frac{1}{5}}}{2}=0.64,$$

Table 4: The decision matrix R^3

	C_1	C_2	C_3	C_4
A_1	$\langle[0.1, 0.3], [0.4, 0.5]\rangle$	$\langle[0.2, 0.7], [0.4, 0.8]\rangle$	$\langle[0.7, 0.9], [0.2, 0.3]\rangle$	$\langle[0.4, 0.6], [0.5, 0.9]\rangle$
A_2	$\langle[0.5, 0.7], [0.8, 0.9]\rangle$	$\langle[0.7, 0.8], [0.2, 0.4]\rangle$	$\langle[0.5, 0.8], [0.6, 0.9]\rangle$	$\langle[0.5, 0.7], [0.8, 0.9]\rangle$
A_3	$\langle[0.2, 0.3], [0.4, 0.6]\rangle$	$\langle[0.5, 0.7], [0.6, 0.9]\rangle$	$\langle[0.5, 0.7], [0.4, 0.6]\rangle$	$\langle[0.4, 0.8], [0.7, 0.8]\rangle$

Table 5: The integrated decision matrix R

	C_1	C_2	C_3	C_4
A_1	$\langle[0.14, 0.42], [0.42, 0.53]\rangle$	$\langle[0.23, 0.55], [0.31, 0.61]\rangle$	$\langle[0.47, 0.65], [0.22, 0.34]\rangle$	$\langle[0.42, 0.62], [0.51, 0.90]\rangle$
A_2	$\langle[0.32, 0.51], [0.62, 0.71]\rangle$	$\langle[0.29, 0.55], [0.21, 0.42]\rangle$	$\langle[0.34, 0.49], [0.51, 0.62]\rangle$	$\langle[0.52, 0.72], [0.71, 0.90]\rangle$
A_3	$\langle[0.24, 0.37], [0.40, 0.60]\rangle$	$\langle[0.34, 0.51], [0.33, 0.44]\rangle$	$\langle[0.21, 0.33], [0.41, 0.51]\rangle$	$\langle[0.32, 0.55], [0.43, 0.62]\rangle$

$$S[A_3] = \frac{(0.32^5 - 0.44^5)^{\frac{1}{5}} + (0.55^5 - 0.64^5)^{\frac{1}{5}}}{2} = -0.49,$$

and then we get the ranking of the cities: $A_2 \succ A_1 \succ A_3$. Therefore, A_2 has the highest capacity to support public health emergencies.

In summary, the application of the integrals of IVq-ROFFs in handling the decision-making problems with interval-valued q-rung orthopair fuzzy information have some advantages as follows:

1. In the integral process of IVq-ROFFs, DMs have more freedom to provide their own evaluation information, and the range of evaluation values that can be considered is broader, which is beneficial for consensus arrival in group decision-making.

2. In the multi-attribute group decision-making problems under interval-valued q-rung orthopair fuzzy environments, the weights of attributes are usually unknown. The integral method is very suitable for dealing with such group decision-making problems, because they do not need the weight of each IVq-ROFV, and avoid the influence of the the subjective judgement of the DMs on the final aggregation result, so it is convenient and more accurate.

5.2 Sensitivity analysis

In order to study the stability of the integral of IVq-ROFFs, on the basis of Example 5.1, we conduct sensitivity analysis. We will calculate the integration results when $q = 5, 6, 7$ and 8 respectively. Through comparing the integration results under different q values, we analyze the sensitivity of the integral of IVq-ROFFs.

Example 5.2. *Here are 20 citizens $T_i (i = 1, 2, \cdots, 20)$ who belong to the same city evaluates the disaster-bearing capability of public health emergencies in their city. We still consider the four evaluation attributes in Example 5.1: Disaster monitoring and forecasting capabilities (C_1) , disaster prevention capabilities (C_2), disaster emergency rescue capabilities (C_3), post-disaster recovery capabilities (C_4). The ith citizens provides his/her assessment value the city with respect to the attributes $C_j (j = 1, 2, \cdots, 4)$ using the IVq-ROFVs $r_{ij} = \langle[\underline{\mu}_{ij}, \overline{\mu}_{ij}], [\underline{v}_{ij}, \overline{v}_{ij}]\rangle$. All $r_{ij} (i = 1, 2, \cdots, 20; j = 1, 2, \cdots, 4)$ are contained in the interval-valued q-rung orthopair fuzzy decision matrices $R = (r_{ij})_{20\times4}$, as shown in Tables 6.*

First, when $q = 5$. Similarly, we take the first aggregated value for example. r_1 is aggregated by twenty IVq-ROFVs: $r_{11} = \langle[0.5, 0.6], [0.4, 0.8]\rangle$, $r_{21} = \langle[0.25, 0.7], [0.4, 0.5]\rangle, \cdots, r_{20,1} = \langle[0.5, 0.8], [0.6, 0.8]\rangle$.

Again, we define the end-points $\mathcal{O} = \langle[0, 0], [1, 1]\rangle$ and $\beta = \langle[1, 1], [0, 0]\rangle$. Build the IVq-ROFF

$$\varphi \triangleq \left\langle \left[\frac{\underline{f}(\mu)}{n}, \frac{\overline{f}(\mu)}{n}\right], \left[\frac{\underline{g}(v)}{n}, \frac{\overline{g}(v)}{n}\right] \right\rangle,$$

with explicit form as follows $(n = 20)$*:*

$$\frac{\underline{f}(\mu)}{20} = \begin{cases} 1, & 0 \le \mu < 0.1, \\ 19/20, & 0.1 \le \mu < 0.2, \\ 13/20, & 0.2 \le \mu < 0.3, \\ 8/20, & 0.3 \le \mu < 0.4, \\ 5/20, & 0.4 \le \mu < 0.5, \\ 0, & 0.5 \le \mu \le 1, \end{cases} \quad \frac{\overline{f}(\mu)}{20} = \begin{cases} 1, & 0 \le \mu < 0.3, \\ 19/20, & 0.3 \le \mu < 0.4, \\ 15/20, & 0.4 \le \mu < 0.5, \\ 9/20, & 0.5 \le \mu < 0.6, \\ 4/20, & 0.6 \le \mu < 0.7, \\ 2/20, & 0.7 \le \mu < 0.8, \\ 0, & 0.8 \le \mu \le 1, \end{cases}$$

and

$$\frac{\underline{g}(v)}{20} = \begin{cases} 1, & 0 \le v \le 0.1, \\ 18/20, & 0.1 < v \le 0.2, \\ 17/20, & 0.2 < v \le 0.3, \\ 13/20, & 0.3 < v \le 0.4, \\ 6/20, & 0.4 < v \le 0.5, \\ 4/20, & 0.5 < v \le 0.6, \\ 0, & 0.6 \le v \le 1, \end{cases} \quad \frac{\overline{g}(v)}{20} = \begin{cases} 1, & 0 < v \le 0.3. \\ 19/20, & 0.3 < v \le 0.4. \\ 17/20, & 0.4 < v \le 0.5. \\ 12/20, & 0.5 < v \le 0.6. \\ 9/20, & 0.6 < v \le 0.7. \\ 6/20, & 0.7 < v \le 0.8. \\ 1/20, & 0.8 < v \le 0.9. \\ 0, & 0.9 \le v \le 1. \end{cases}$$

With the above preparations, when $q = 5$*, using Formula* (65) *in Theorem 4.4', we deduce*

$$\begin{aligned} &\int_{L(\mathcal{O},\beta)} \left\langle \left[\frac{\underline{f}(\mu)}{20}, \frac{\overline{f}(\mu)}{20}\right], \left[\frac{\underline{g}(v)}{20}, \frac{\overline{g}(v)}{20}\right] \right\rangle \mathcal{T}\xi = \\ &\left\langle \left[\left(1 - \exp\left\{ - \int_0^{0.5} \frac{5\mu^4 \left(\frac{\underline{f}(\mu)}{20}\right)^5}{1-\mu^5} d\mu \right\}\right)^{\frac{1}{5}}, \left(1 - \exp\left\{ - \int_0^{0.8} \frac{5\mu^4 \left(\frac{\overline{f}(\mu)}{3}\right)^5}{1-\mu^5} d\mu \right\}\right)^{\frac{1}{5}} \right], \right. \\ &\left. \left[\exp\left\{ \int_1^{0.1} \frac{1 - \left(\frac{\underline{g}(v)}{20}\right)^5}{v} dv \right\}, \exp\left\{ \int_1^{0.3} \frac{1 - \left(\frac{\overline{g}(v)}{20}\right)^5}{v} dv \right\} \right] \right\rangle \\ &\approx \langle [0.23, 0.43], [0.19, 0.42] \rangle. \end{aligned} \tag{67}$$

In the same way, we can get the other aggregated values under four different q values. Thus, the aggregative IVq-ROFFs decision matrix $R = (r_{qj})_{4\times 4}$ *is obtained as listed in Table 7. By observing Table 7, it can be found that the integration results calculated under different q values are very similar. And with the increase of q value, the u value and the v value in the integration results have different degrees of decline. In comparison, v value decreases faster. But overall, the integration results are quite close, so it can be said that the IVq-ROFFs integration operator proposed in this paper is insensitive to the q value and has a high fault tolerance.*

For a more intuitive observation, we still integrate the evaluation values of twenty citizens on four aspects of the city. The calculating process is the same as above, thus we obtain the overall assessment under four different q values as follows:

$$\begin{aligned} Val_{q=5} &= \langle [0.18, 0.35], [0.19, 0.43] \rangle, \\ Val_{q=6} &= \langle [0.17, 0.33], [0.17, 0.42] \rangle, \\ Val_{q=7} &= \langle [0.17, 0.32], [0.16, 0.41] \rangle, \\ Val_{q=8} &= \langle [0.16, 0.32], [0.15, 0.39] \rangle. \end{aligned}$$

By observing the above integration results, we find that the final integration results are very close under different q values. Therefore, we can think that the integral of IVq-ROFFs is not sensitive to the q value, that is, we can get the stable integration results.

Through this example, we can draw a conclusion. The application of the integrals of IVq-ROFFs in handling the decision-making problems with IVq-ROFF information have some advantages as follows:

1. The integrals of IVq-ROFFs is less sensitive to q value. The integrals of IVq-ROFFs in handling the decision-making problems under different q values can obtain similar integration results. Therefore, when we deal with practical problems, we don't have to worry too much about the choice of q value. However, with the increase of q value, the integration result will deviate from the true value, and the error will gradually increase. It is necessary to select an appropriate q value according to the actual decision data.

2. The integrals of IVq-ROFFs can be used to deal with large amount of data. In Example 5.2, we increase the amount of data, but the computational complexity does not increase too much. This is because we use the evaluation information represented by IVq-ROFS, which discretizes the continuous information, so we do not have too much computational complexity to calculate the weight density function.

Table 6: The decision matrix R

	C_1	C_2	C_3	C_4
T_1	$\langle[0.5, 0.6], [0.4, 0.8]\rangle$	$\langle[0.5, 0.7], [0.3, 0.5]\rangle$	$\langle[0.3, 0.5], [0.5, 0.8]\rangle$	$\langle[0.4, 0.6], [0.3, 0.5]\rangle$
T_2	$\langle[0.5, 0.7], [0.4, 0.5]\rangle$	$\langle[0.6, 0.8], [0.4, 0.6]\rangle$	$\langle[0.3, 0.5], [0.4, 0.5]\rangle$	$\langle[0.4, 0.7), [0.3, 0.5]\rangle$
T_3	$\langle[0.3, 0.5], [0.4, 0.5]\rangle$	$\langle[0.3, 0.6], [0.5, 0.6]\rangle$	$\langle[0.2, 0.4], [0.4, 0.5]\rangle$	$\langle[0.3, 0.6], [0.4, 0.6]\rangle$
T_4	$\langle[0.2, 0.4], [0.3, 0.6]\rangle$	$\langle[0.4, 0.6], [0.3, 0.6]\rangle$	$\langle[0.1, 0.3], [0.2, 0.5]\rangle$	$\langle[0.3, 0.4], [0.1, 0.5]\rangle$
T_5	$\langle[0.4, 0.6], [0.4, 0.6]\rangle$	$\langle[0.6, 0.7], [0.5, 0.7]\rangle$	$\langle[0.2, 0.6], [0.3, 0.5]\rangle$	$\langle[0.4, 0.5], [0.4, 0.5]\rangle$
T_6	$\langle[0.2, 0.3], [0.1, 0.3]\rangle$	$\langle[0.3, 0.4], [0.3, 0.5]\rangle$	$\langle[0.1, 0.2], [0.6, 0.8]\rangle$	$\langle[0.2, 0.3], [0.1, 0.4]\rangle$
T_7	$\langle[0.2, 0.5], [0.5, 0.8]\rangle$	$\langle[0.4, 0.6], [0.5, 0.9]\rangle$	$\langle[0.2, 0.3], [0.3, 0.6]\rangle$	$\langle[0.4, 0.5], [0.4, 0.9]\rangle$
T_8	$\langle[0.2, 0.5], [0.3, 0.5]\rangle$	$\langle[0.4, 0.7], [0.4, 0.6]\rangle$	$\langle[0.2, 0.4], [0.3, 0.4]\rangle$	$\langle[0.3, 0.7], [0.3, 0.4]\rangle$
T_9	$\langle[0.1, 0.4], [0.2, 0.4]\rangle$	$\langle[0.3, 0.4], [0.3, 0.6]\rangle$	$\langle[0.1, 0.2], [0.5, 0.7]\rangle$	$\langle[0.1, 0.3], [0.3, 0.5]\rangle$
T_{10}	$\langle[0.3, 0.5], [0.1, 0.5]\rangle$	$\langle[0.3, 0.5], [0.2, 0.5]\rangle$	$\langle[0.2, 0.4], [0.6, 0.8]\rangle$	$\langle[0.2, 0.4], [0.3, 0.5]\rangle$
T_{11}	$\langle[0.2, 0.4], [0.4, 0.7]\rangle$	$\langle[0.4, 0.6], [0.4, 0.9]\rangle$	$\langle[0.1, 0.3], [0.3, 0.7]\rangle$	$\langle[0.2, 0.5], [0.3, 0.7]\rangle$
T_{12}	$\langle[0.2, 0.4], [0.3, 0.5]\rangle$	$\langle[0.2, 0.5], [0.4, 0.6]\rangle$	$\langle[0.2, 0.3], [0.5, 0.8]\rangle$	$\langle[0.1, 0.4], [0.3, 0.6]\rangle$
T_{13}	$\langle[0.5, 0.7], [0.6, 0.7]\rangle$	$\langle[0.5, 0.7], [0.6, 0.7]\rangle$	$\langle[0.4, 0.7], [0.5, 0.7]\rangle$	$\langle[0.3, 0.5], [0.5, 0.7]\rangle$
T_{14}	$\langle[0.3, 0.5], [0.3, 0.4]\rangle$	$\langle[0.4, 0.6], [0.4, 0.6]\rangle$	$\langle[0.1, 0.4], [0.4, 0.6]\rangle$	$\langle[0.3, 0.5], [0.4, 0.7]\rangle$
T_{15}	$\langle[0.5, 0.8], [0.6, 0.8]\rangle$	$\langle[0.6, 0.9], [0.7, 0.8]\rangle$	$\langle[0.4, 0.7], [0.5, 0.7]\rangle$	$\langle[0.5, 0.8], [0.5, 0.7]\rangle$
T_{16}	$\langle[0.4, 0.6], [0.5, 0.8]\rangle$	$\langle[0.4, 0.8], [0.6, 0.8]\rangle$	$\langle[0.3, 0.4], [0.4, 0.6]\rangle$	$\langle[0.3, 0.7], [0.4, 0.8]\rangle$
T_{17}	$\langle[0.3, 0.6], [0.4, 0.6]\rangle$	$\langle[0.4, 0.5], [0.3, 0.5]\rangle$	$\langle[0.2, 0.3], [0.4, 0.7]\rangle$	$\langle[0.3, 0.5], [0.4, 0.6]\rangle$
T_{18}	$\langle[0.4, 0.6], [0.6, 0.9]\rangle$	$\langle[0.5, 0.7], [0.7, 0.9]\rangle$	$\langle[0.2, 0.6], [0.4, 0.9]\rangle$	$\langle[0.3, 0.7], [0.6, 0.8]\rangle$
T_{19}	$\langle[0.2, 0.5], [0.4, 0.7]\rangle$	$\langle[0.2, 0.5], [0.4, 0.7]\rangle$	$\langle[0.1, 0.4], [0.3, 0.6]\rangle$	$\langle[0.2, 0.4], [0.3, 0.6]\rangle$
T_{20}	$\langle[0.5, 0.8], [0.6, 0.8]\rangle$	$\langle[0.6, 0.8], [0.8, 0.9]\rangle$	$\langle[0.4, 0.7], [0.6, 0.7]\rangle$	$\langle[0.5, 0.8], [0.7, 0.8]\rangle$

Table 7: The integrated decision matrix in different q values

	C_1	C_2	C_3	C_4
$q = 5$	$\langle[0.23, 0.43], [0.19, 0.42]\rangle$	$\langle[0.31, 0.50], [0.29, 0.53]\rangle$	$\langle[0.15, 0.31], [0.29, 0.50]\rangle$	$\langle[0.22, 0.41], [0.19, 0.46]\rangle$
$q = 6$	$\langle[0.22, 0.42], [0.17, 0.41]\rangle$	$\langle[0.31, 0.49], [0.28, 0.53]\rangle$	$\langle[0.14, 0.30], [0.28, 0.49]\rangle$	$\langle[0.22, 0.40], [0.18, 0.45]\rangle$
$q = 7$	$\langle[0.22, 0.41], [0.16, 0.40]\rangle$	$\langle[0.30, 0.49], [0.27, 0.52]\rangle$	$\langle[0.14, 0.29], [0.27, 0.48]\rangle$	$\langle[0.22, 0.39], [0.17, 0.45]\rangle$
$q = 8$	$\langle[0.21, 0.41], [0.15, 0.39]\rangle$	$\langle[0.30, 0.48], [0.27, 0.52]\rangle$	$\langle[0.14, 0.29], [0.27, 0.47]\rangle$	$\langle[0.22, 0.39], [0.16, 0.44]\rangle$

6 Conclusion

This paper is part of a series of work concerning the calculus theories under q-rung orthopair fuzzy environments. The main contribution of this paper is to establish the integral theories for IVq-ROFFs. Firstly, the concept of IVq-ROFF as well as some related definitions are given in Section 2. In this part, we present rigorous mathematical proofs of the IVq-ROFWA and IVq-ROFWG operators, are define the score and accuracy functions in a more reasonable way. Then, we start with the primitives of IVq-ROFF, which can be regarded as the inverse operations of derivative calculus in our previous work [9]. Next, we present the definition of indefinite IVq-ROFIs and their computation, and discuss some basic integral properties. Based on the indefinite integral theories, we study the definite IVq-ROFIs for both *simplified IVq-ROFFs* and *generalized IVq-ROFFs* under some suitable hypotheses. The explicit integral formula is derived in our theorems, and like the indefinite integral, we discuss the properties of IVq-ROFIs, such as comparison theorem, algebraic operations, mean value theorem, etc. Finally, we give an example to illustrate the advantages of the proposed q-ROFIs. In the future work, we can further study the applications of the IVq-ROFC theories in other fields, such as information retrieval, cluster analysis, and medical diagnosis, etc.

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