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Original paper

Fuzzy approximation of a fractional Lorenz system and a fractional financial crisis

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Abstract

Our aim in this paper is to study the fuzzy stability of a fractional Lorenz system in the sense of the Caputo-Fabrizio derivative and a fractional financial crisis in the sense of $\hbar-$Hilfer derivative. Defining a new type of fuzzy control function that has a dynamic situation helps us to investigate new stability results for these mathematical models.

Keywords: Fractional Lorenz system, stability, Caputo-Fabrizio derivative, fractional financial crisis, fuzzy sets.

1 Introduction

The butterfly effect, the possibility that a small change in initial conditions may have momentous effects, is a concept that was presented by an American mathematician and meteorologist Edward Lorenz. Nearly forty five years ago, he posed a question: does the flap of a butterfly's wings in Brazil set off a tornado in Texas trying to express why it is so difficult to earn a suitable weather forecast. In an experiment, he replaced the initial condition 0.506 with 0.506127. The effect was amazing. Lorenz was the first person who recognizes chaotic theory in the mathematical model of weather. His researches leads to a new field of study not only in mathematics but also in biology, physics, meteorology and so on. In fact, the Lorenz system is a model-driven Rayleigh-Bénard convection. For more details, we refer to [1, 4, 11]. In this paper, we consider the fractional version of the Lorenz system in the sense of the Caputo-Fabrizio derivative, and apply a fuzzy control function with the Mihet-Radu method, to investigate the fuzzy Ulam-Wright stability with the uniqueness of the solution. Also, we investigate the fuzzy Ulam-Wright stability of a financial crisis model presented by Korobeinikov [5], in the sense of $\hbar-$Hilfer derivative.

2 Preliminaries

Definition 2.1. [2] *Let $\varpi(\tau)$ be a continuous function and fractal differentiable on interval (σ,ρ) with order μ, therefore a Caputo fractal-fractional (FF) derivative of order η for $\varpi(\tau)$ with the following condition is given by*

Case 1: *A power-law type kernel:*

$$\mathsf{D}^{\eta,\mu}_{\sigma,\tau}\varpi(\tau)=\frac{1}{\Gamma(n-\eta)}\int_{\sigma}^{\tau}\frac{d\varpi(s)}{ds^{k}}(\tau-s)^{n-\eta-1}ds, \tag{1}$$

in which $n-1\in(0,\eta),\mu\in(-\infty,n]$ and

$$\frac{d\varpi(s)}{ds^{\mu}}=\lim_{\tau\longrightarrow s}\frac{\varpi(\tau)-\varpi(s)}{\tau^{\mu}-s^{\mu}}.$$

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Also,

$$\mathsf{D}^{\eta,\mu,\lambda}_{\sigma,\tau}\varpi(\tau) = \frac{1}{\Gamma(n-\eta)}\int_{\sigma}^{\tau}\frac{d^{\lambda}\varpi(s)}{ds^{k}}(\tau-s)^{n-\eta-1}ds, \tag{2}$$

and

$$\frac{d^{\lambda}\varpi(s)}{ds^{\mu}} = \lim_{\tau\longrightarrow s}\frac{\varpi^{\lambda}(\tau)-\varpi^{\lambda}(s)}{\tau^{\mu}-s^{\mu}},$$

in which $\lambda \in (-\infty, 1]$.

Case 2: *An exponential decay type kernel:*

$$\mathsf{D}^{\eta,\mu}_{\sigma,\tau}\varpi(\tau) = \frac{\Xi(\eta)}{1-\eta}\int_{\sigma}^{\tau}\frac{d\varpi(s)}{ds^{\mu}}e^{-\frac{\eta}{1-\eta}(\tau-s)}ds, \tag{3}$$

in which $\eta \in (0,\infty), \mu \in (-\infty, n], n \in \mathbb{N}$ *and* $\Xi(0) = \Xi(1) = 1$. *Also,*

$$\mathsf{D}^{\eta,\mu}_{\sigma,\tau,\lambda}\varpi(\tau) = \frac{\Xi(\eta)}{1-\eta}\int_{\sigma}^{\tau}\frac{d^{\lambda}\varpi(s)}{ds^{\mu}}e^{-\frac{\eta}{1-\eta}(\tau-s)}ds,$$

where $\mu, \eta, \lambda \in (0, 1]$.

Case 3: *The generalized Mittag-Leffler kernel:*

$$\mathsf{D}^{\eta,\mu}_{\sigma,\tau}\varpi(\tau) = \frac{\mho(\eta)}{1-\eta}\int_{\sigma}^{\tau}\frac{d\varpi(s)}{ds^{\mu}}\mathsf{E}_{\eta}(-\frac{\eta}{1-\eta}(\tau-s)^{\eta})ds. \tag{4}$$

Also,

$$\mathsf{D}^{\eta,\mu,\lambda}_{\sigma,\tau}\varpi(\tau) = \frac{\mho(\eta)}{1-\eta}\int_{\sigma}^{\tau}\frac{d^{\lambda}\varpi(s)}{ds^{\mu}}\mathsf{E}_{\eta}(-\frac{\eta}{1-\eta}(\tau-s)^{\eta})ds,$$

in which $\eta, \mu, \lambda \in (0, 1]$ *and* $\mho(\eta) = 1-\eta+\dfrac{\eta}{\Gamma(\eta)}$.

Definition 2.2. [2] *On* (σ, ρ), *consider the continuous mapping* $\varpi(\tau)$, *thus an* η *order FF integral of the map* $\varpi(\tau)$ *with the following conditions is given by*

Case 1: *A power-law type kernel:*

$$\mathsf{I}^{\eta,\mu}_{\sigma,\tau}(\varpi(\tau)) = \frac{\mu}{\Gamma(\eta)}\int_{\sigma}^{\tau}(\tau-s)^{\eta-1}s^{\mu-1}\varpi(s)ds. \tag{5}$$

Case 2: *An exponential decay type kernel:*

$$\mathsf{I}^{\eta,\mu}_{\sigma,\tau}(\varpi(\tau)) = \frac{\eta^{\mu}}{\Xi(\eta)}\int_{\sigma}^{\tau}s^{\eta-1}\varpi(s)ds + \frac{\mu(1-\eta)\tau^{\mu-1}\varpi(\tau)}{\Xi(\eta)}. \tag{6}$$

Case 3: *The generalized Mittag-Leffler kernel:*

$$\mathsf{I}^{\eta,\mu}_{\sigma,\tau}(\varpi(\tau)) = \frac{\eta^{\mu}}{\mho(\eta)}\int_{\sigma}^{\tau}s^{\eta-1}\varpi(s)(\tau-s)^{\eta-1}ds + \frac{\mu(1-\eta)\tau^{\mu-1}\varpi(\tau)}{\mho(\eta)}. \tag{7}$$

Definition 2.3. [8] *Let* (σ, ρ) *be a real interval and* $\alpha > 0$. *Suppose* $\hbar(\tau)$ *is a positive and increasing monotone function, with a continuous derivative* $\hbar'(\tau)$ *on* (σ, ρ). *Then the fractional integral of a function* $\varpi(\tau)$ *with respect to* $\hbar(\tau)$ *on* (σ, ρ) *is given by*

$$\mathsf{I}^{\alpha;\hbar}\varpi(\tau) = \frac{1}{\Gamma(\alpha)}\int_{\sigma}^{\tau}\hbar'(s)(\hbar(\tau)-\hbar(s))^{\alpha-1}\varpi(s)ds. \tag{8}$$

Suppose $\hbar'(\tau) \neq 0$ *and* $n \in \mathbb{N}$. *Then, the Riemann-Liouville derivative of a function* $\varpi(\tau)$ *with respect to* $\hbar(\tau)$ *of order* α *is given by*

$$\mathsf{D}^{\alpha;\hbar}\varpi(\tau) = \frac{1}{\hbar'(\tau)}\frac{d}{d\tau}I^{n-\alpha;\hbar}\varpi(\tau).$$

Now, let $\alpha \in (n-1,n)$ and $\varpi, \hbar \in \mathbb{C}^n([\sigma,\rho],\mathbb{R})$ such that $\hbar'(\tau) \neq 0$, and $\hbar$ is increasing on $[\sigma,\rho]$. Then, the $\hbar-$Hilfer fractional derivative of function $\varpi(\tau)$ of order α and type $\beta \in [0,1]$ is given by

$$
{}^{\mathcal{H}}\mathsf{D}_{\sigma}^{\alpha,\beta;\hbar}\varpi(\tau) = I_{\sigma}^{\beta(n-\alpha);\hbar}\left(\frac{1}{\hbar'(\tau)}\frac{d}{d\tau}\right)^n I_{\sigma}^{(1-\beta)(n-\alpha);\hbar}\varpi(\tau). \tag{9}
$$

Consider

$$
\mathrm{diag}\mathtt{M}_n([0,1]) = \left\{ \begin{bmatrix} \mathfrak{H}_1 & & \\ & \ddots & \\ & & \mathfrak{H}_n \end{bmatrix} = \mathrm{diag}[\mathfrak{H}_1, \cdots, \mathfrak{H}_n],\ \mathfrak{H}_1, \ldots, \mathfrak{H}_n \in [0,1] \right\},
$$

where $\mathrm{diag}\mathtt{M}_n([0,1])$ is equipped with the partial order relation:

$$
\boldsymbol{\mathfrak{H}} := \mathrm{diag}[\mathfrak{H}_1, \cdots, \mathfrak{H}_n],\ \boldsymbol{\mathfrak{K}} := \mathrm{diag}[\mathfrak{K}_1, \cdots, \mathfrak{K}_n] \in \mathrm{diag}\mathtt{M}_n([0,1]),
$$
$$
\boldsymbol{\mathfrak{H}} \preceq \boldsymbol{\mathfrak{K}} \Longleftrightarrow \mathfrak{H}_j \leq \mathfrak{K}_j.
$$

Some good reference for t-norms are [3], now we present a modified type of t-norms that can be apply for aggregation functions [10, 9]. In [8], the authors introduced MTN and CMTN as

Definition 2.4. *An MTN on $\mathrm{diag}\mathtt{M}_n([0,1])$ is an operation $\circledast : \mathrm{diag}\mathtt{M}_n([0,1]) \times \mathrm{diag}\mathtt{M}_n([0,1]) \to \mathrm{diag}\mathtt{M}_n([0,1])$ s.t.*
($\imath$) $(\forall\, \boldsymbol{\mathfrak{H}} \in \mathrm{diag}\mathtt{M}_n([0,1]))(\boldsymbol{\mathfrak{H}} \circledast \boldsymbol{1} = \boldsymbol{\mathfrak{H}})$ (boundary condition);
($\imath\imath$) $(\forall(\boldsymbol{\mathfrak{H}}, \boldsymbol{\mathfrak{K}}) \in (\mathrm{diag}\mathtt{M}_n([0,1]))^2)(\boldsymbol{\mathfrak{H}} \circledast \boldsymbol{\mathfrak{K}} = \boldsymbol{\mathfrak{K}} \circledast \boldsymbol{\mathfrak{H}})$ (commutativity);
($\imath\imath\imath$) $(\forall(\boldsymbol{\mathfrak{H}}, \boldsymbol{\mathfrak{K}}, \boldsymbol{\mathfrak{F}}) \in (\mathrm{diag}\mathtt{M}_n([0,1]))^3)(\boldsymbol{\mathfrak{H}} \circledast (\boldsymbol{\mathfrak{K}} \circledast \boldsymbol{\mathfrak{F}}) = (\boldsymbol{\mathfrak{H}} \circledast \boldsymbol{\mathfrak{K}}) \circledast \boldsymbol{\mathfrak{F}})$ (associativity);
($\imath v$) $(\forall(\boldsymbol{\mathfrak{H}}, \boldsymbol{\mathfrak{K}}, \boldsymbol{\mathfrak{F}}, \boldsymbol{\mathfrak{I}}) \in (\mathrm{diag}\mathtt{M}_n([0,1]))^4)(\boldsymbol{\mathfrak{H}} \preceq \boldsymbol{\mathfrak{K}};\ and\ \boldsymbol{\mathfrak{F}} \preceq \boldsymbol{\mathfrak{I}} \Longrightarrow \boldsymbol{\mathfrak{H}} \circledast \boldsymbol{\mathfrak{F}} \preceq \boldsymbol{\mathfrak{K}} \circledast \boldsymbol{\mathfrak{I}})$ (monotonicity).

For all $\boldsymbol{\mathfrak{H}}, \boldsymbol{\mathfrak{K}} \in \mathrm{diag}\mathtt{M}_n([0,1])$ and all sequences $\{\boldsymbol{\mathfrak{H}}_k\}$ and $\{\boldsymbol{\mathfrak{K}}_k\}$ converging to $\boldsymbol{\mathfrak{H}}$ and $\boldsymbol{\mathfrak{K}}$ assume

$$
\lim_k (\boldsymbol{\mathfrak{H}}_k \circledast \boldsymbol{\mathfrak{K}}_k) = \boldsymbol{\mathfrak{H}} \circledast \boldsymbol{\mathfrak{K}},
$$

then, $\circledast$ on $\mathrm{diag}\mathtt{M}_n([0,1])$ is a CMTN.
($\mathcal{E}1$) Define $\circledast_{\mathscr{M}} : \mathrm{diag}\mathtt{M}_n([0,1]) \times \mathrm{diag}\mathtt{M}_n([0,1]) \to \mathrm{diag}\mathtt{M}_n([0,1])$, such that,

$$
\boldsymbol{\mathfrak{H}} \circledast_{\mathscr{M}} \boldsymbol{\mathfrak{K}} = \mathrm{diag}[\mathfrak{H}_1, \cdots, \mathfrak{H}_n] \circledast_{\mathscr{M}} \mathrm{diag}[\mathfrak{K}_1, \cdots, \mathfrak{K}_n] = \mathrm{diag}[\min\{\mathfrak{H}_1, \mathfrak{K}_1\}, \cdots, \min\{\mathfrak{H}_n, \mathfrak{K}_n\}],
$$

therefore $\circledast_{\mathscr{M}}$ is CMTN (minimum CMTN).;
($\mathcal{E}2$) Define $\circledast_{\mathscr{P}} : \mathrm{diag}\mathtt{M}_n([0,1]) \times \mathrm{diag}\mathtt{M}_n([0,1]) \to \mathrm{diag}\mathtt{M}_n([0,1])$, such that,

$$
\boldsymbol{\mathfrak{H}} \circledast_{\mathscr{P}} \boldsymbol{\mathfrak{K}} = \mathrm{diag}[\mathfrak{H}_1, \cdots, \mathfrak{H}_n] \circledast_{\mathscr{P}} \mathrm{diag}[\mathfrak{K}_1, \cdots, \mathfrak{K}_n] = \mathrm{diag}[\mathfrak{H}_1.\mathfrak{K}_1, \cdots, \mathfrak{H}_n.\mathfrak{K}_n],
$$

then $\circledast_{\mathscr{P}}$ is CMTN (product CMTN).;
($\mathcal{E}3$) Define $\circledast_{\mathscr{L}} : \mathrm{diag}\mathtt{M}_n([0,1]) \times \mathrm{diag}\mathtt{M}_n([0,1]) \to \mathrm{diag}\mathtt{M}_n([0,1])$, such that,

$$
\boldsymbol{\mathfrak{H}} \circledast_{\mathscr{L}} \boldsymbol{\mathfrak{K}} = \mathrm{diag}[\mathfrak{H}_1, \cdots, \mathfrak{H}_n] \circledast_{\mathscr{L}} \mathrm{diag}[\mathfrak{K}_1, \cdots, \mathfrak{K}_n] = \mathrm{diag}[\max\{\mathfrak{H}_1 + \mathfrak{K}_1 - 1, 0\}, \cdots, \max\{\mathfrak{H}_n + \mathfrak{K}_n - 1, 0\}],
$$

therefore $\circledast_{\mathscr{L}}$ is CMTN (Lukasiewicz CMTN).

Note $\Pi_{j=1}^n \mathfrak{H}_j = \mathfrak{H}_1 \circledast ... \circledast \mathfrak{H}_n$, for $\mathfrak{H}_1, ..., \mathfrak{H}_n \in [0,1]$ and $\circledast_{\mathscr{M}} = \bigwedge$.

Also,

$$
\mathrm{diag}\left[0, \frac{1}{2}, \frac{3}{5}, 1\right] \circledast_{\mathscr{M}} \mathrm{diag}\left[\frac{2}{3}, \frac{3}{4}, \frac{7}{9}, \frac{1}{6}\right] = \mathrm{diag}\left[0, \frac{1}{2}, \frac{3}{5}, \frac{1}{6}\right],
$$

$$
\mathrm{diag}\left[0, \frac{1}{2}, \frac{3}{5}, 1\right] \circledast_{\mathscr{P}} \mathrm{diag}\left[\frac{2}{3}, \frac{3}{4}, \frac{7}{9}, \frac{1}{6}\right] = \mathrm{diag}\left[0, \frac{3}{8}, \frac{21}{45}, \frac{1}{6}\right],
$$

$$
\mathrm{diag}\left[0, \frac{1}{2}, \frac{3}{5}, 1\right] \circledast_{\mathscr{L}} \mathrm{diag}\left[\frac{2}{3}, \frac{3}{4}, \frac{7}{9}, \frac{1}{6}\right] = \mathrm{diag}\left[0, 0, \frac{11}{45}, \frac{1}{6}\right].
$$

In [8] MVF-set is defined by Θ^* and $\Theta \in \Theta^*$ i.e.,

$(\mathcal{C}1)$ $\Theta \in C[(\mathtt{W} \times ((0,+\infty))^n, \mathrm{diag}\mathtt{M}_n((0,1]))]$;
$(\mathcal{C}2)$ $\Theta(\mathtt{V}, .)$ is increasing, where $\mathtt{V} \in \mathtt{W}$;
$(\mathcal{C}3)$ $\lim_{\overrightarrow{\psi} \longrightarrow +\infty} \Theta(\mathtt{V}, \overrightarrow{\psi}) = \mathbf{1}$, where $\mathtt{V} \in \mathtt{W}$.

In Θ^* we introduce $\preceq$ by

$$\Theta \precsim \Theta_\circ \iff \Theta(\mathtt{V}, \overrightarrow{\psi}) \preceq \Theta_\circ(\mathtt{V}, \overrightarrow{\psi'}), \quad \forall\, \mathtt{V} \in \mathtt{W} \text{ and } \overrightarrow{\psi}, \overrightarrow{\psi'} \in ((0,+\infty))^n.$$

Now, we present an uncertain norm that has a dynamic situation since it depends on the time.

Definition 2.5. *Consider the CMTN $\circledast$, a linear-space $\mathtt{W}$ and MVF-set $\Psi : \mathtt{W} \times ((0,+\infty))^n \to diag\mathtt{M}_n((0,1])$. If*

$(\mathcal{S}1)$ $\Psi(\mathtt{V}, \overrightarrow{\psi}) = \mathbf{1}$, *for all* $\overrightarrow{\psi} \succ \overrightarrow{0}$ *if and only if* $\mathtt{V} = 0$;

$(\mathcal{S}2)$ $\Psi(\hbar\mathtt{V}, \overrightarrow{\psi}) = \Psi(\mathtt{V}, \frac{\overrightarrow{\psi}}{|\hbar|})$ *for all* $\mathtt{V} \in \mathtt{W}$, $\overrightarrow{\psi} \succ \overrightarrow{0}$ *and* $\hbar \in \mathbb{C}$ *with* $\hbar \neq 0$;

$(\mathcal{S}3)$ $\Psi(\mathtt{V} + \mathtt{V}', \overrightarrow{\psi} + \overrightarrow{\psi'}) \succeq \Psi(\mathtt{V}, \overrightarrow{\psi}) \circledast \Psi(\mathtt{V}', \overrightarrow{\psi'})$ *for all* $\mathtt{V}, \mathtt{V}' \in \mathtt{W}$ *and* $\overrightarrow{\psi}$, $\overrightarrow{\psi'} \succ \overrightarrow{0}$;

$(\mathcal{S}4)$ $\lim_{\overrightarrow{\psi} \longrightarrow +\infty} \Psi(\mathtt{V}, \overrightarrow{\psi}) = \mathbf{1}$, *for all* $\mathtt{V} \in \mathtt{W}$. *Then* $(\mathtt{W}, \Psi, \circledast)$ *is an MVFN-space.*

A complete MVFN-space is called an MVFB-space.

For instance the MVF-set Ψ, given by $\Psi(\mathtt{V}, \overrightarrow{\psi}) = \mathrm{diag}[\frac{\psi_1}{\psi_2+|\mathtt{V}|}, \mathbf{E}_\iota(-\frac{|\mathtt{V}|}{\psi_2}), \exp(-\frac{|\mathtt{V}|}{\psi_3})]$, is a matrix valued fuzzy norm, in which $\overrightarrow{\psi} \in ((0,+\infty))^3$, $\mathbf{E}_\iota, \iota \in (0,1]$, is the Mittag–Leffler function [7] and $(\mathtt{W}, \Psi, \circledast_{\mathscr{M}})$ is an MVFN-space.

Now, we present the Diaz-Margolis theorem from the literature:

Theorem 2.6. [8] *Let $\mathcal{B}$ be a set with a complete $[0,+\infty]$-valued metric Ω, and also let the self map ∇ on $\mathcal{B}$ satisfy $\Omega(\nabla q, \nabla u) \leq \kappa\Omega(u,q)$, $\kappa < 1$ is a Lipschitz constant. Assume $q \in \mathcal{B}$.*

If $\Omega(\nabla^m q, \nabla^{m+1} q) < \infty$, for all $m \geq m_0$; Then
$(\imath)$ the fixed–point t^* of ∇ is the convergence point of the sequence $\{\nabla^m q\}$;
$(\imath\imath)$ in the set $V = \{u \in \mathcal{B} \mid \Omega(\nabla^{m_0} q, u) < \infty\}$, u^* is the unique fixed point of ∇;
$(\imath\imath\imath)$ $(1-\kappa)\Omega(u,u^*) \leq \Omega(u, \nabla u)$ for every $u \in \mathcal{B}$.

Remark 2.7. *Consider the Wright function (Bessel Maitland function). The special function*

$$\mathsf{W}_{\alpha,\beta}(\varsigma) = \sum_{n=0}^{\infty} \frac{\varsigma^n}{n!\Gamma(\alpha n + \beta)},$$

is called Wright function, in which $\alpha > -1$, $\beta, \varsigma \in \mathbb{C}$ and Γ is the Gamma function.

Now, consider the following increasing Wright function

$$\mathsf{W}_{a,b}\left(-\frac{|\vartheta|}{\kappa}\right) = \sum_{k=0}^{\infty} \frac{\left(-\frac{|\vartheta|}{\kappa}\right)^k}{k!\Gamma(ak+b)},$$

in which $a, b \in [0,+\infty]$, $\vartheta \in \mathtt{W}$, $\kappa \in (0,+\infty)$.

We claim that $\left(\mathtt{W}, \mathsf{W}\left(-\frac{|\vartheta|}{\kappa}\right), *_{\mathscr{M}}\right)$ is an FN-space.

(1) *It is easy to show that $\mathsf{W}_{a,b}\left(-\frac{|\vartheta|}{\kappa}\right) = 1$ for every $\kappa \in (0,+\infty)$, if and only if $\vartheta = 0$.*

(2) *For any $\vartheta \in \mathtt{W}$ and $\kappa \in (0,+\infty)$, we have*

$$\mathsf{W}_{a,b}\left(-\frac{|\hbar\vartheta|}{\kappa}\right) = \sum_{k=0}^{\infty} \frac{-\frac{|\hbar\vartheta|^k}{\kappa}}{k!\Gamma(ak+b)} = \sum_{k=0}^{\infty} \frac{-\frac{|\vartheta|^k}{\frac{\kappa}{|\hbar|}}}{k!\Gamma(ak+b)} = \mathsf{W}_{a,b}\left(-\frac{|\vartheta|}{\frac{\kappa}{|\hbar|}}\right).$$

(3) *Let $\mathsf{W}_{a,b}\left(-\frac{|\vartheta|}{\kappa}\right) \leq \mathsf{W}_{a,b}\left(-\frac{|\vartheta'|}{\kappa'}\right)$. Then we have $\frac{|\vartheta'|}{\kappa'} \leq \frac{|\vartheta|}{\kappa}$, for any $\vartheta, \vartheta' \in \mathtt{W}$ and $\kappa, \kappa' \in (0,+\infty)$. Now, if $\vartheta = \vartheta'$, we have $\kappa \leq \kappa'$. Thus, otherwise, we have*

$$\frac{|\vartheta|}{\kappa} + \frac{|\vartheta|}{\kappa} \geq \frac{|\vartheta|}{\kappa} + \frac{|\vartheta'|}{\kappa'} \geq 2\frac{|\vartheta|}{\kappa+\kappa'} + 2\frac{|\vartheta'|}{\kappa+\kappa'} \geq 2\frac{|\vartheta+\vartheta'|}{\kappa+\kappa'},$$

therefore $\frac{|\vartheta|}{\kappa} \geq \frac{|\vartheta+\vartheta'|}{\kappa+\kappa'}$. *But* $-\frac{|\vartheta|}{\kappa} \leq -\frac{|\vartheta+\vartheta'|}{\kappa+\kappa'}$, *and also*

$$\sum_{k=0}^{\infty} \frac{\left(-\frac{|\vartheta|}{\kappa}\right)^k}{k!\Gamma(ak+b)} \leq \sum_{k=0}^{\infty} \frac{\left(-\frac{|\vartheta+\vartheta'|}{\kappa+\kappa'}\right)^k}{k!\Gamma(ak+b)}, \tag{10}$$

which implies that

$$\mathsf{W}_{a,b}\left(-\frac{|\vartheta|}{\kappa}\right) \leq \mathsf{W}_{a,b}\left(-\frac{|\vartheta+\vartheta'|}{\kappa+\kappa'}\right).$$

Hence we have,

$$\mathsf{W}_{a,b}\left(-\frac{|\vartheta+\vartheta'|}{\kappa+\kappa'}\right) \geq \min\left\{\mathsf{W}_{a,b}\left(-\frac{|\vartheta|}{\kappa}\right), \mathsf{W}_{a,b}\left(-\frac{|\vartheta'|}{\kappa'}\right)\right\},$$

for any $\vartheta, \vartheta' \in \mathtt{W}$ *and* $\kappa, \kappa' \in (0,+\infty)$. *Therefore,*

$$\Psi(\vartheta,\kappa) = \mathsf{W}_{a,b}\left(-\frac{|\vartheta|}{\kappa}\right),$$

denotes an FN and $(\mathtt{W}, \Psi, *_{\mathscr{M}})$ *is an FN-space, for all* $\vartheta \in \mathtt{W}$, $\kappa \in (0,+\infty)$ *and* $a, b \in [0,+\infty]$.

Now, we define a new model of uncertain control functions to investigate fuzzy Wright stability with the uniqueness of solution of two mathematical models, fractional Lorenz system in the sense of Caputo-Fabrizio derivative and a fractional financial crisis in the sense of $\hbar-$Hilfer derivative, our results extend some results of [6, 7, 12]. Throughout this paper, let $\circledast_{\mathscr{M}} = \circledast$.

3 Butterfly effect

Throughout this section, let $i = 1, 2, 3$. Consider the following Lorenz chaotic system

$$\begin{aligned} \Lambda_1'(\tau) &= \alpha(\Lambda_2 - \Lambda_1), \\ \Lambda_2'(\tau) &= \beta\Lambda_1 - \Lambda_2 - \Lambda_1\Lambda_3, \\ \Lambda_3'(\tau) &= \Lambda_1\Lambda_2 - \gamma\Lambda_3, \end{aligned} \tag{11}$$

in which $\Lambda_1 = \Lambda_1(\tau), \Lambda_2 = \Lambda_2(\tau)$ and $\Lambda_3 = \Lambda_3(\tau)$ are the dynamical variable of the system and $\alpha, \beta, \gamma \in \mathbb{R}$ are the related constants parameters.

Using (11), we get

$${}^{\mathcal{RL}}\mathsf{D}_{0,\tau}^{\eta,\mu,\lambda}\Lambda_i(\tau) = \Upsilon_i(\Lambda_1, \Lambda_2, \Lambda_3, \tau), \tag{12}$$

in which $\Upsilon_1(\Lambda_1, \Lambda_2, \Lambda_3, \tau) = \alpha(\Lambda_2 - \Lambda_1)$, $\Upsilon_2(\Lambda_1, \Lambda_2, \Lambda_3, \tau) = \beta\Lambda_1 - \Lambda_2 - \Lambda_1\Lambda_3$ and $\Upsilon_3(\Lambda_1, \Lambda_2, \Lambda_3, \tau) = \Lambda_1\Lambda_2 - \gamma\Lambda_3$. The equations in (12) can be expressed as follows

$${}^{\mathcal{RL}}\mathsf{D}_{0,\tau}^{\eta,\mu,\lambda}\Lambda_i(\tau) = \frac{\Xi(\eta)}{1-\eta}\frac{d}{d\tau^\mu}\int_0^\tau \mathrm{e}^{\frac{-\eta}{1-\eta}(\tau-s)}\Upsilon_i(\Lambda_1, \Lambda_2, \Lambda_3, \tau)ds. \tag{13}$$

Thus, equations in (13) can be written as follows

$${}^{\mathcal{RL}}\mathsf{D}_{0,\tau}^{\eta,\mu,\lambda}\Lambda_i(\tau) = \mu\tau^{\mu-1}\Upsilon_i(\Lambda_1, \Lambda_2, \Lambda_3, \tau). \tag{14}$$

Thus, equations in (14) can be given by

$${}^{\mathcal{CF}}\mathsf{D}_{0,\tau}^{\eta,\mu,\lambda}\Lambda_i(\tau) = \mu\tau^{\mu-1}\Upsilon_i(\Lambda_1, \Lambda_2, \Lambda_3, \tau). \tag{15}$$

Now, consider $\Lambda_i(0) = \Lambda_{i,0}$, as the initial conditions. Also, consider the following operators

$$\begin{aligned} &\phi_1(\tau, \Lambda_1) = \alpha(\Lambda_2 - \Lambda_1), \\ &\phi_2(\tau, \Lambda_2) = \beta\Lambda_1 - \Lambda_2 - \Lambda_1\Lambda_3, \\ &\phi_3(\tau, \Lambda_3) = \Lambda_1\Lambda_2 - \gamma\Lambda_3. \end{aligned} \tag{16}$$

Here, we let $\Delta_i = [\tau_0 - \sigma, \tau_0 + \sigma]$, and $\Delta'_i = [\Lambda_{i,0} - \rho_i, \Lambda_{i,0} + \rho_i]$.

Now, we show that ϕ_i satisfies the Lipschitz conditions with respect to Λ_1, Λ_2 and Λ_3, respectively. Let $\zeta_1, \zeta_2 \in \mathbb{C}(\Delta_1, \Delta'_1, \Delta'_2, \Delta'_3)$, therefore, we have

$$\begin{aligned} \Psi\bigg(\phi_1(\tau,\zeta_1) - \phi_1(\tau,\zeta_2), \overrightarrow{\psi}\bigg) &= \Psi\bigg(\alpha\Lambda_2(\tau) - \alpha\zeta_1 - \alpha\Lambda_2(\tau) - \alpha\zeta_2, \overrightarrow{\psi}\bigg) \\ &= \Psi\bigg(-\alpha\zeta_1 + \alpha\zeta_2, \overrightarrow{\psi}\bigg) = \Psi\bigg(\alpha(\zeta_1 - \zeta_2), \overrightarrow{\psi}\bigg) \\ &\succeq \Psi\bigg(\zeta_1 - \zeta_2, \frac{\overrightarrow{\psi}}{|\alpha|}\bigg), \end{aligned} \tag{17}$$

$$\begin{aligned} \Psi\bigg(\phi_2(\tau,\zeta_1) - \phi_2(\tau,\zeta_2), \overrightarrow{\psi}\bigg) &= \Psi\bigg((\beta\Lambda_1(\tau) - \zeta_1 - \Lambda_1(\tau)\Lambda_3(\tau)) - (\beta\Lambda_1(\tau) - \zeta_2 - \Lambda_1(\tau)\Lambda_3(\tau)), \overrightarrow{\psi}\bigg) \\ &= \Psi\bigg(-\zeta_1 + \zeta_2, \overrightarrow{\psi}\bigg) = \Psi\bigg(\zeta_1 - \zeta_2, \overrightarrow{\psi}\bigg), \end{aligned}$$

and

$$\begin{aligned} \Psi\bigg(\phi_3(\tau,\zeta_1) - \phi_3(\tau,\zeta_2), \overrightarrow{\psi}\bigg) &= \Psi\bigg((\Lambda_1(\tau)\Lambda_2 - \gamma\zeta_1) - (\Lambda_1(\tau)\Lambda_2 - \gamma\zeta_2), \overrightarrow{\psi}\bigg) \\ &= \Psi\bigg(-\gamma\zeta_1 + \gamma\zeta_2, \overrightarrow{\psi}\bigg) = \Psi\bigg(\gamma(t\zeta_1 - \zeta_2), \overrightarrow{\psi}\bigg) \\ &\succeq \Psi\bigg(\zeta_1 - \zeta_2, \frac{\overrightarrow{\psi}}{|\gamma|}\bigg). \end{aligned} \tag{18}$$

Considering (15), Assume the following inequality

$$\Psi\bigg({}^{\mathcal{CF}}\mathsf{D}^{\alpha,\beta,\hbar}\Lambda_i(\tau) - \Upsilon_i(\Lambda_1, \Lambda_2, \tau), \overrightarrow{\psi}\bigg) \succeq \varphi(\tau, \overrightarrow{\psi}); \quad i = 1, 2, 3, \tag{19}$$

in which $\overrightarrow{\psi} \in (0, +\infty)^n$.

We define a mapping $\Omega : \mathbb{C}(\Delta_1, \Delta'_1, \Delta'_2, \Delta'_3) \times \mathbb{C}(\Delta_1, \Delta'_1, \Delta'_2, \Delta'_3) \longrightarrow [0, +\infty]$ by

$$\Omega(\xi_1, \xi_2) = \inf\bigg\{\mathtt{C} \in (0, +\infty) : \Psi\bigg(\xi_1(\tau) - \xi_2(\tau), \overrightarrow{\psi}\bigg) \succeq \varphi(\tau, \frac{\overrightarrow{\psi}}{\mathtt{C}}); \forall \xi_1, \xi_2 \in \mathbb{C}(\Delta_1, \Delta'_1, \Delta'_2, \Delta'_3), \overrightarrow{\psi} \in (0, +\infty)^n\bigg\}.$$

Consider $\nabla : \mathbb{C}(\Delta_1, \Delta'_1, \Delta'_2, \Delta'_3) \to \mathbb{C}(\Delta_1, \Delta'_1, \Delta'_2, \Delta'_3)$ defined as follows:

$$\nabla\widehat{\Lambda}(\tau) = \widehat{\Lambda}_0 + \frac{\mu t^{\mu-1}(1-\eta)}{\Xi(\eta)}\widehat{\phi}(\tau, \widehat{\Lambda}(\tau)) + \frac{\eta^\mu}{\Xi(\eta)}\int_0^\tau \varsigma^{\tau-1}\widehat{\phi}(\varsigma, \widehat{\Lambda}(\varsigma))d\varsigma, \tag{20}$$

in which $\widehat{\Lambda}(\tau) = [\Lambda_1(\tau), \Lambda_2(\tau), \Lambda_3(\tau)]^T$, $\widehat{\Lambda_0}(\tau) = [\Lambda_{1,0}(\tau), \Lambda_{2,0}(\tau), \Lambda_{3,0}(\tau)]^T$ and also

$$\widehat{\phi}(\tau, \widehat{\Lambda}(\tau)) = [\phi_1(\tau, \Lambda_1), \phi_2(\tau, \Lambda_2), \phi_3(\tau, \Lambda_3)]^T.$$

Suppose $\Psi\bigg(\widehat{\Lambda}(\tau) - \widehat{\Lambda}'(\tau), \overrightarrow{\psi}\bigg) \succeq \varphi(\tau, \dfrac{\overrightarrow{\psi}}{\eth})$ in which $\eth \in (0, +\infty)$.

From (17)-(18) we imply that ∇ satisfies Lipschitz conditions with respect to $\widehat{\Lambda}(\tau)$. We have

$$\begin{aligned}
&\Psi\bigg(\nabla\widehat{\Lambda}(\tau)-\nabla\widehat{\Lambda}'(\tau),\overrightarrow{\psi}\bigg)\\
&=\Psi\bigg(\frac{\mu t^{\mu-1}(1-\eta)}{\Xi\eta}\widehat{\phi}(\tau,\widehat{\Lambda}(\tau))+\frac{\eta^{\mu}}{\Xi(\eta)}\int_0^{\tau}\varsigma^{\tau-1}\widehat{\phi}(\varsigma,\widehat{\Lambda}(\varsigma))d\varsigma-\frac{\mu t^{\mu-1}(1-\eta)}{\Xi(\eta)}\widehat{\phi}(\tau,\widehat{\Lambda}'(\tau))+\frac{\eta^{\mu}}{\Xi(\eta)}\int_0^{\tau}\varsigma^{\tau-1}\widehat{\phi}(\varsigma,\widehat{\Lambda}'(\varsigma))d\varsigma,\overrightarrow{\psi}\bigg)\\
&=\Psi\bigg(\frac{\mu t^{\mu-1}(1-\eta)}{\Xi(\eta)}\widehat{\phi}(\tau,\widehat{\Lambda}(\tau))-\widehat{\phi}(\tau,\widehat{\Lambda}'(\tau))+\frac{\eta^{\mu}}{\Xi(\eta)}\int_0^{\tau}\varsigma^{\tau-1}\widehat{\phi}(\varsigma,\widehat{\Lambda}(\varsigma))-\widehat{\phi}(\varsigma,\widehat{\Lambda}'(\varsigma))d\varsigma,\overrightarrow{\psi}\bigg)\\
&\succeq\Psi\bigg(\frac{\mu t^{\mu-1}(1-\eta)}{\Xi(\eta)}\wp\widehat{\Lambda}(\tau)-\widehat{\Lambda}'(\tau),\frac{1}{2}\overrightarrow{\psi}\bigg)\circledast\Psi\bigg(\frac{\eta^{\mu}}{\Xi(\eta)}\int_0^{\tau}\varsigma^{\tau-1}\widehat{\phi}(\varsigma,\widehat{\Lambda}(\varsigma))-\widehat{\phi}(\varsigma,\widehat{\Lambda}'(\varsigma))d\varsigma,\frac{1}{2}\overrightarrow{\psi}\bigg)\\
&\succeq\Psi\bigg(\mu\sigma^{\mu-1}(1-\eta)\wp\widehat{\Lambda}(\tau))-\widehat{\Lambda}'(\tau)),\frac{1}{2}\overrightarrow{\psi}\bigg)\circledast\Psi\bigg(\frac{\eta^{\mu}\sigma^{\mu}\wp}{\mu}\widehat{\Lambda}(\varsigma))-\widehat{\Lambda}'(\varsigma)),\frac{1}{2}\overrightarrow{\psi}\bigg)\\
&\succeq\varphi(\tau,\frac{\overrightarrow{\psi}}{2\eth\mu\sigma^{\mu-1}(1-\eta)\wp})\circledast\varphi(\tau,\frac{\overrightarrow{\psi}}{2\eth\eta^{\mu}\sigma^{\mu}\wp\mu^{-1}})\\
&\succeq\varphi(\tau,\frac{\overrightarrow{\psi}}{\max\{2\eth\mu\sigma^{\mu-1}(1-\eta)\wp,2\eth\eta^{\mu}\sigma^{\mu}\wp\mu^{-1},\}})
\end{aligned}$$

in which $\wp:=\max\{|\alpha|,|\gamma|,1\}$ and $\overrightarrow{\psi}\in(0,+\infty)^n$.

4 Economic model

Let $B_r:=\{(\chi_1,\chi_2):|(\chi_1,\chi_2)|_2\leq r\}$. Consider the following financial model

$$\begin{aligned}
\Lambda_1'(\tau)&=-\beta\Lambda_1\Lambda_2^{\alpha},\\
\Lambda_2'(\tau)&=\beta\Lambda_1\Lambda_2^{\alpha}-\frac{1}{\gamma}\Lambda_2,
\end{aligned}\tag{21}$$

in which γ ranges from a week to a few weeks, α is an average number of the activated units and β is a positive activation rate coefficient, and also, $\Lambda_1=\Lambda_1(\tau)$ and $\Lambda_2=\Lambda_2(\tau)$. Using (21), we get

$$^{\mathcal{H}}\mathsf{D}^{\alpha,\beta,\hbar}\Lambda_i(\tau)=\Upsilon_i(\Lambda_1,\Lambda_2,\tau);\qquad i=1,2,\tag{22}$$

where $\Upsilon_1(\Lambda_1,\Lambda_2,\tau)=-\beta\Lambda_1\Lambda_2^{\alpha}$ and $\Upsilon_2(\Lambda_1,\Lambda_2,\tau)=\beta\Lambda_1\Lambda_2^{\alpha}-\frac{1}{\gamma}\Lambda_2$. Suppose the initial conditions $\Lambda_1(0)=\Lambda_{1,0}$ and $\Lambda_2(0)=\Lambda_{2,0}$.

Assume that $\Delta_i=[\tau_0-\sigma,\tau_0+\sigma]$, and $\Delta_i'=B_r,\ i=1,2$.

Now, consider the following operators

$$\begin{aligned}
\phi_1(\tau,\Lambda_1)&=-\beta\Lambda_1\Lambda_2^{\alpha},\\
\phi_2(\tau,\Lambda_2)&=\beta\Lambda_1\Lambda_2^{\alpha}-\frac{1}{\gamma}\Lambda_2.
\end{aligned}\tag{23}$$

We show that ϕ_1 and ϕ_2 satisfy the Lipschitz conditions with respect to Λ_1 and Λ_2, respectively. Let $\zeta_1,\zeta_2\in\mathbb{C}(\Delta_1,\Delta_1',\Delta_2')$ therefore, we have

$$\begin{aligned}
\Psi\bigg(\phi_1(\tau,\zeta_1)-\phi_1(\tau,\zeta_2),\psi\bigg)&=\Psi\bigg(-\beta\zeta_1\Lambda_2^{\alpha}+\beta\zeta_2\Lambda_2^{\alpha},\psi\bigg)=\Psi\bigg(\beta\Lambda_2^{\alpha}(\zeta_2-\zeta_1),\psi\bigg)\\
&\succeq\Psi\bigg(\zeta_2-\zeta_1,\frac{1}{|\beta r^{\alpha}|}\psi\bigg),
\end{aligned}\tag{24}$$

and

$$\begin{aligned}
\Psi\bigg(\phi_2(\tau,\zeta_1)-\phi_2(\tau,\zeta_2),\psi\bigg)&=\Psi\bigg([\beta\Lambda_1\zeta_1^{\alpha}-\frac{1}{\gamma}\zeta_1]-[\beta\Lambda_1\zeta_2^{\alpha}-\frac{1}{\gamma}\zeta_2],\psi\bigg)=\Psi\bigg(\beta\Lambda_1(\zeta_1^{\alpha}-\zeta_2^{\alpha})+\frac{1}{\gamma}(\zeta_1-\zeta_2),\psi\bigg)\\
&\succeq\Psi\bigg(\beta|\Lambda_1||\zeta_1-\zeta_2||\zeta_1^{\alpha-1}+\zeta_1^{\alpha-2}\zeta_2+\ldots+\zeta_1\zeta_2^{\alpha-2}+\zeta_2^{\alpha-1}|+\frac{1}{\gamma}|\zeta_1-\zeta_2|,\psi\bigg)
\end{aligned}$$

$$
\begin{aligned}
&\succeq \Psi\bigg(\beta|\Lambda_1||\zeta_1-\zeta_2|[|\zeta_1^{\alpha-1}|+|\zeta_1^{\alpha-2}||\zeta_2|+\ldots+|\zeta_1||\zeta_2^{\alpha-2}|+|\zeta_2^{\alpha-1}|+\frac{1}{\gamma}|\zeta_1-\zeta_2|],\psi\bigg)\\
&\succeq \Psi\bigg(\Big[\beta|\Lambda_1|[|\zeta_1^{\alpha-1}|+|\zeta_1^{\alpha-2}||\zeta_2|+\ldots+|\zeta_1||\zeta_2^{\alpha-2}|+|\zeta_2^{\alpha-1}|]+\frac{1}{\gamma}\Big]|\zeta_1-\zeta_2|,\psi\bigg)\\
&\succeq \Psi\bigg([\beta r^{\alpha+1}+\frac{1}{\gamma}]|\zeta_1-\zeta_2|,\psi\bigg)\\
&\succeq \Psi\bigg(\zeta_1-\zeta_2,\frac{1}{|\beta r^{\alpha+1}+\frac{1}{\gamma}|}\psi\bigg). \qquad (25)
\end{aligned}
$$

Considering (22), Assume the following inequality

$$
\Psi\bigg({}^{\mathcal{H}}\mathsf{D}^{\alpha,\beta,\hbar}\Lambda_i(\tau)-\Upsilon_i(\Lambda_1,\Lambda_2,\tau),\psi\bigg)\succeq \mathbf{W}(\frac{-|\hbar(\zeta)-\hbar(0)|^{\alpha}}{\theta\psi});\ \ \theta\in(0,+\infty),\ i=1,2, \qquad (26)
$$

in which $\psi\in(0,+\infty)$ and

$$
\mathbf{W}(\frac{-|\hbar(\zeta)-\hbar(0)|^{\alpha}}{\theta\psi})=\operatorname{diag}\Big[\mathsf{W}_{a,b}(\frac{-|\hbar(\zeta)-\hbar(0)|^{\alpha}}{\theta\psi}),\mathsf{W}_{a,b}(\frac{-|\hbar(\zeta)-\hbar(0)|^{\alpha}}{\theta\psi}),\mathsf{W}_{a,b}(\frac{-|\hbar(\zeta)-\hbar(0)|^{\alpha}}{\theta\psi})\Big].
$$

Also suppose that

$$
\frac{1}{\Gamma(\alpha)}\int_0^{\tau}\hbar'(s)(\hbar(\tau)-\hbar(s))^{\alpha-1}\mathbf{W}\bigg(\frac{-|(\hbar(s)-\hbar(0)|^{\alpha}}{\theta\psi}\bigg)ds\succeq\mathbf{W}\bigg(\frac{-|(\hbar(s)-\hbar(0)|^{\alpha}}{\Pi\theta\psi}\bigg)ds, \qquad (27)
$$

in which $\Pi\in(0,1),\ \psi\in(0,\infty)$.

Definition 4.1. *Equation (22) has Wright stability with respect to $\mathbf{W}((\hbar(\tau)-\hbar(0))^{\alpha})$ if there exists $c>0$ such that, for all $\theta>0$ and any solution $\omega\in\mathbb{C}(\Delta_1,\Delta_1',\Delta_2')$ to (26) there exists a solution $\omega'\in\mathbb{C}(\Delta_1,\Delta_1',\Delta_2')$ to (22) with*

$$
\Psi(\omega(\tau)-\omega'(\tau),\psi)\succeq\mathbf{W}\bigg(\frac{-|(\hbar(\tau)-\hbar(0)|^{\alpha}}{c\theta\psi}\bigg),\qquad \psi\in(0,+\infty).
$$

Remark 4.2. *Assume that $\omega\in\mathbb{C}(\Delta_1,\Delta_1',\Delta_2')$ is a solution of inequality (26). Thus ω is a solution of the following inequality:*

$$
\begin{aligned}
&\Psi\bigg(\omega(\tau)-\omega_0(\tau)-\frac{1}{\Gamma(\alpha)}\int_0^{\zeta}\hbar'(s)(\hbar(\zeta)-\hbar(s))^{\alpha-1}\Upsilon(\omega_1,\omega_2,s)ds,\psi\bigg)\\
&\succeq\Psi\bigg(\frac{\theta}{\Gamma(\alpha)}\int_0^{\zeta}\hbar'(s)(\hbar(\zeta)-\hbar(s))^{\alpha-1}\mathsf{W}_{a,b}((\hbar(s)-\hbar(0))^{\alpha})ds,\psi\bigg)\\
&\succeq\Psi\bigg(\frac{\theta}{\Gamma(\alpha)}\int_0^{\zeta}\hbar'(s)(\hbar(\zeta)-\hbar(s))^{\alpha-1}\sum_{k=0}^{\infty}\frac{(\hbar(s)-\hbar(0))^{k\alpha}}{k!\Gamma(ak+b)}ds,\psi\bigg)\\
&=\Psi\bigg(\frac{\theta}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(ak+b)}\int_0^{\zeta}(\hbar(\zeta)-\hbar(s))^{\alpha-1}(\hbar(s)-\hbar(0))^{k\alpha}d\hbar(s),\psi\bigg)\\
&=\Psi\bigg(\frac{\theta}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(ak+b)}\int_0^{\hbar(\zeta)-\hbar(0)}(\hbar(\zeta)-\hbar(0)-u)^{\alpha-1}(u)^{k\alpha}du,\psi\bigg)\qquad u=\hbar(s)-\hbar(0))\\
&\succeq\Psi\bigg(\frac{\theta}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(ak+b)}(\hbar(\zeta)-\hbar(0))^{\alpha-1}\int_0^{\hbar(\zeta)-\hbar(0)}(1-\frac{u}{\hbar(\zeta)-\hbar(0)})^{\alpha-1}u^{k\alpha}du,\psi\bigg)\\
&=\Psi\bigg(\frac{\theta}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(ak+b)}(\hbar(\zeta)-\hbar(0))^{(k+1)\alpha}\int_0^{1}(1-v)^{\alpha-1}v^{k\alpha}dv,\psi\bigg)\qquad\bigg(v=\frac{u}{\hbar(\zeta)-\hbar(0)}\bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= \Psi\bigg(\frac{\theta}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(ak+b)}(\hbar(\zeta)-\hbar(0))^{(k+1)\alpha}\frac{\Gamma(k\alpha+1)\Gamma(\alpha)}{\Gamma((k+1)\alpha+1)},\psi\bigg)\\
&\succeq \Psi\bigg(\theta\sum_{k=0}^{\infty}\frac{(\hbar(\zeta)-\hbar(0))^{k\alpha}}{k!\Gamma(ak+b)},\psi\bigg)\\
&= \mathsf{W}_{a,b}\bigg(\frac{-|(\hbar(\zeta)-\hbar(0)|^{\alpha}}{\theta\psi}\bigg),
\end{aligned}
$$

in which $\psi \in (0,+\infty)$.

We define a mapping $\Omega : \mathbb{C}(\Delta_1,\Delta_1',\Delta_2',\Delta_3') \times \mathbb{C}(\Delta_1,\Delta_1',\Delta_2',\Delta_3') \longrightarrow [0,+\infty]$ by

$$
\Omega(\xi_1,\xi_2) = \inf\bigg\{\mathtt{C} \in (0,+\infty) : \Psi\bigg(\xi_1(\tau)-\xi_2(\tau),\psi\bigg) \succeq \mathbf{W}(\frac{-|\hbar(\zeta)-\hbar(0)|^{\alpha}}{\frac{\theta}{\mathtt{C}}\psi}); \forall \xi_1,\xi_2 \in \mathbb{C}(\Delta_1,\Delta_1',\Delta_2',\Delta_3'), \psi \in (0,+\infty)\bigg\}. \tag{28}
$$

Consider $\nabla : \mathbb{C}(\Delta_1,\Delta_1',\Delta_2',\Delta_3') \to \mathbb{C}(\Delta_1,\Delta_1',\Delta_2',\Delta_3')$ defined as follows:

$$
\nabla\widehat{\Lambda}(\tau) = \widehat{\Lambda}_0 + \frac{1}{\Gamma(\alpha)}\int_0^{\tau}\hbar'(s)(\hbar(\tau)-\hbar(s))^{\alpha-1}\widehat{\phi}(s,\widehat{\Lambda}(s))ds, \tag{29}
$$

in which $\widehat{\Lambda}(\tau) = [\Lambda_1(\tau),\Lambda_2(\tau),\Lambda_3(\tau)]^T$, $\widehat{\Lambda_0}(\tau) = [\Lambda_{1,0}(\tau),\Lambda_{2,0}(\tau),\Lambda_{3,0}(\tau)]^T$ and also

$$
\widehat{\phi}(\tau,\widehat{\Lambda}(\tau)) = [\phi_1(\tau,\Lambda_1),\phi_2(\tau,\Lambda_2),\phi_3(\tau,\Lambda_3)]^T.
$$

Let $\Psi\bigg(\widehat{\Lambda}(\tau)-\widehat{\Lambda}'(\tau),\psi\bigg) \succeq \mathbf{W}(\frac{-|\hbar(\zeta)-\hbar(0)|^{\alpha}}{\frac{\theta}{\eth}\psi})$, in which $\eth \in (0,+\infty)$.

From (24) and (25) we imply that ∇ satisfies Lipschipitz conditions with respect to $\widehat{\Lambda}(\tau)$. We have

$$
\begin{aligned}
& \Psi\bigg(\nabla\widehat{\Lambda}(\tau)-\nabla\widehat{\Lambda}'(\tau),\psi\bigg)\\
= \;& \Psi\bigg(\frac{1}{\Gamma(\alpha)}\int_0^{\tau}\hbar'(s)(\hbar(\tau)-\hbar(s))^{\alpha-1}\widehat{\phi}(s,\widehat{\Lambda}(s))ds - \frac{1}{\Gamma(\alpha)}\int_0^{\tau}\hbar'(s)(\hbar(\tau)-\hbar(s))^{\alpha-1}\widehat{\phi}(s,\widehat{\Lambda}'(s))ds,\psi\bigg)\\
= \;& \Psi\bigg(\frac{1}{\Gamma(\alpha)}\int_0^{\tau}\hbar'(s)(\hbar(\tau)-\hbar(s))^{\alpha-1}[\widehat{\phi}(s,\widehat{\Lambda}(s))-\widehat{\phi}(s,\widehat{\Lambda}'(s))]ds,\psi\bigg)\\
\succeq \;& \Psi\bigg(\frac{1}{\Gamma(\alpha)}\int_0^{\tau}\hbar'(s)(\hbar(\tau)-\hbar(s))^{\alpha-1}[\widehat{\Lambda}(s)-\widehat{\Lambda}'(s))]ds,\psi\bigg)\\
\succeq \;& \mathbf{W}(\frac{-|\hbar(\zeta)-\hbar(0)|^{\alpha}}{\frac{\Pi\theta}{\eth\nu}\psi}),
\end{aligned}
$$

in which $\nu = \max\{\frac{1}{\gamma} + 2\beta r^{\alpha+1}, \beta r^{\alpha}\}$, $\psi \in (0,+\infty)$ and $\Pi \in (0,1)$.

5 Concluding remarks

We considered mathematical models of Lorenz system and a financial crisis that via definitions of fractional calculus, represented them in the sense of Caputo-Fabrizio derivative and $\hbar-$Hilfer derivative, respectively. Afterward, we defined a new class of uncertain control function that had a dynamic case and used the Mihet-Radu method, to investigated the uncertain Wright stability and also uniqueness of solution.

Authors' contributions

SRA, writing–original draft preparation. RS, writing–original draft preparation and supervision and project administration. TA, methodology and supervision. SA, methodology and supervision, MC writing–original draft preparation. All authors read and approved the final manuscript.

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