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Original paper

New intuitionistic fuzzy operations, operators and topological structures

K. T. Atanassov[1] and R. Tsvetkov[2]

[1] *Department of Bioinformatics and Mathematical Modelling, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences, 105 Acad. G. Bonchev Str., 1113 Sofia, Bulgaria*

[1] *Intelligent Systems Laboratory, Prof. Asen Zlatarov University, Burgas-8010, Bulgaria*

[2] *Faculty of Applied Mathematics and Informatics, Technical University of Sofia, 1000 Sofia, Bulgaria*

krat@bas.bg, tzv@tu-sofia.bg

Abstract

Two new intuitionistic fuzzy operations (union and intersection) are defined. Based on them, two new topological operators (of a closure and of an interior types) are introduced. Some properties of these obects are studied. Based on them, four new intuitionistic fuzzy topological structures are introduced and some of their properties are discussed. Integral forms of both new intuitionistic fuzzy operations and of both intuitionistic fuzzy operators are given.

Keywords: Intuitionistic fuzzy operation, intuitionistic fuzzy operator, intuitionistic fuzzy set, intuitionistic fuzzy topological structure.

1 Introduction

The topology, as a new mathematical area appeared on the boundary of 19th and 20th century in the research of Jules Henri Poincaré and Felix Hausdorff and it obtains its foundations in the works of a pleiad of mathematicians. Among them are the names of Kazimierz Kuratowski, Pavel Alexandrov, Garrett Birkhoff, the group of French specialists working under the pseudonim Bourbaki and a lot of others.

In [19], the Polish mathematician Kuratowski gave the definition of a topological structure as an object that satisfies the conditions:

C1) $cl(A\Delta B) = cl(A)\Delta cl(B)$,

C2) $A \subseteq cl(A)$,

C3) $cl(cl(A)) = cl(A)$,

C4) $cl(O) = O$,

where $A, B \in X$, X is some fixed set of sets with a minimal element O, cl is the topological operator "closure" and $\Delta : X \times X \to X$ is the operation that generates cl; and

I1) $in(A\nabla B) = in(A)\nabla in(B)$,

I2) $in(A) \subseteq A$,

I3) $in(in(A)) = in(A)$,

I4) $in(I) = I$,

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where $A, B \in X$, X is the same set and I is its maximal element, *in* is the topological operator "interior" and $\nabla : X \times X \to X$ is the operation that generates *in*.

After the rise of Lotfi Zadeh's fuzzy sets, the next step was the introduction of the fuzzy topology (see. ,e.g., [11, 30, 37]) in the second half of XX century. The Intuitionistic Fuzzy Sets (IFSs, see [1, 2]), as extensions of fuzzy sets, opened new horizons for development of the topology. The research in this direction started with Dogan Čoker's results, published in the end of the last century (see [12, 13]) and of F. Gallego Lupiañez, from the beginning of the present century [21, 22]. As it is mentioned in [6], during the last 20 years, a lot of research in this area has been published, but all of them contain the intuitionistic fuzzy interpretations of existing objects in standard topology (see, e.g., [14, 16, 17, 20, 24, 26, 28, 29, 31, 33, 34, 35, 36, 38]).

The present paper is a continuation of [3, 4, 7] and other papers, where the ideas and definitions from the areas of (general) topology (see, e.g., [10, 19, 27]) are combined with these from modal logic (see, e.g., [9, 15, 25]) and from IFSs.

In [3], for a first time the concepts of a topological structure and modal logic were united, introducing Intuitionistic Fuzzy Modal Topological Structure (IFMTS). Unfortunately, by the moment there are only two pairs of intuitionistic fuzzy operations (from union and intersection types) defined over IFSs that generate such structure - the second one is described in [7]. All other pairs of operations generate structures that satisfy only a part of the C- or I-Kuratowski's conditions. In [4], these structures are called "feeble", because, as mentioned there, the term "weak" has another sense in topology.

In the present research, new pair of intuitionistic fuzzy operations from union and intersection type are introduced. Some of their basic properties are studied (Section 3). Based on them, new pair of topological operators (from closure and interior type) are defined and studied (Section 4) and four new Intuitionistic Fuzzy Feeble Modal Topological Structures (IFFMTSs) are described and illustrated in Section 5. The ideas are described for integral forms of the two new intuitionistic fuzzy operations and of the two new intuitionistic fuzzy operators (in Sections 4 and 5, respectively).

2 Preliminaries

Let everywhere below the set E, called a "universe", be fixed. Let its cardinality be $\varepsilon = |E|$ and let A be its subset. Then, the IFS in E has the form:

$$A^* = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\},$$

where functions $\mu_A : E \to [0, 1]$ and $\nu_A : E \to [0, 1]$ define degree of membership and of non-membership of element $x \in E$, respectively, to the set A, and for each $x \in E$:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

As usually, instead of A^* for brevity, we will use notation A.

Obviously, the IFSs are extensions of Lotfi Zadeh's fuzzy sets [39].

On Fig. 1, one of the geometrical interpretations of the IFSs (see, [1, 2]) is shown.

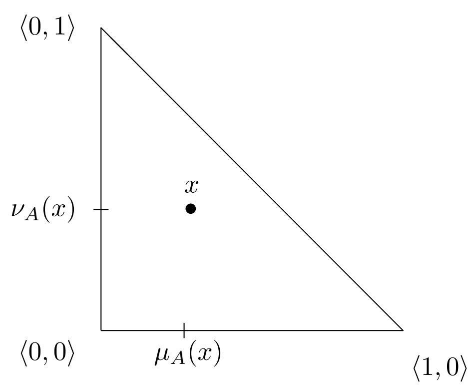

Figure 1: A geometrical interpretation of an element $x \in E$

Over IFSs a lot of intuitionistic fuzzy operations, relations and operators from modal, topological and level types, are defined (see, e.g., [2]). For the needs of the present research, below we give definitions of some of them.

$$\begin{array}{lcl}
A \subseteq B & \text{iff} & (\forall x \in E)(\mu_A(x) \leq \mu_B(x) \;\&\; \nu_A(x) \geq \nu_B(x));\\
A \supseteq B & \text{iff} & B \subseteq A;\\
A = B & \text{iff} & (\forall x \in E)(\mu_A(x) = \mu_B(x) \;\&\; \nu_A(x) = \nu_B(x));\\
\neg A & = & \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\};\\
A \cap_4 B & = & \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\};\\
A \cup_4 B & = & \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\};\\
A \cap_{13} B & = & \{\langle x, \mu_A(x)\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x)\nu_B(x))\rangle | x \in E\};\\
A \cup_{13} B & = & \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x), \nu_A(x)\nu_B(x)\rangle | x \in E\};\\
A@B & = & \{\langle x, \frac{\mu_A(x)+\mu_B(x)}{2}, \frac{\nu_A(x)+\nu_B(x)}{2}\rangle | x \in E\}.
\end{array}$$

The first two (simplest) intuitionistic fuzzy topological operators "closure" and "interior" defined over IFSs are (see, e.g., [2]):

$$\mathcal{C}(A) = \{\langle x, \sup_{y\in E} \mu_A(y), \inf_{y\in E} \nu_A(y)\rangle | x \in E\};$$

$$\mathcal{I}(A) = \{\langle x, \inf_{y\in E} \mu_A(y), \sup_{y\in E} \nu_A(y)\rangle | x \in E\}.$$

The first two (simplest) modal operators (see, e.g., [2]) are:

$$\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\};$$

$$\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle | x \in E\}.$$

They are intuitionistic fuzzy forms of the classical modal logic operators (see, e.g., [9, 15, 25]).

Following [1], the IFS A is called an "Intuitionistic Fuzzy Tautological set" (IFTS) if and only if for each $x \in E$:

$$\mu_A(x) \geq \nu_A(x),$$

and let:

$$O^* = \{\langle x, 0, 1\rangle | x \in E\},$$

$$U^* = \{\langle x, 0, 0\rangle | x \in E\},$$

$$E^* = \{\langle x, 1, 0\rangle | x \in E\}.$$

Therefore, for each IFS A:

$$O^* \subseteq A \subseteq E^*.$$

3 New intuitionistic fuzzy operations

Let the two IFSs A and B be given.

First, for them we define

$$A\#B = \{\langle x, \mu_A(x)\mu_B(x), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\},$$

$$A * B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \nu_A(x)\nu_B(x)\rangle | x \in E\}.$$

Immediately, we check that the definitions of these operations are correct. Really,

$$0 \leq \mu_A(x)\mu_B(x) \max(\mu_A(x), \mu_B(x)) \leq 1,$$

$$0 \leq \max(\nu_A(x), \nu_B(x)) \leq \nu_A(x)\nu_B(x) \leq 1,$$

$$\begin{aligned}0 &\leq \mu_A(x)\mu_B(x) + \max(\nu_A(x), \nu_B(x)) \\ &\leq \min(\mu_A(x), \mu_B(x)) + \max(\nu_A(x), \nu_B(x)) \\ &\leq \min(\mu_A(x), \mu_B(x)) + \max(1 - \mu_A(x), 1 - \mu_B(x)) \\ &= \min(\mu_A(x), \mu_B(x)) + 1 - \min(\mu_A(x), \mu_B(x)) \\ &= 1.\end{aligned}$$

and by analogy,

$$0 \leq \max(\mu_A(x), \mu_B(x)) + \nu_A(x)\nu_B(x) \leq 1.$$

In the present paper, we will use the notations "#" and "$*$" for both new operations, but in future they will obtain their own numbers in the sets of all operations "intersection" and "union", defined over IFSs.

Second, we see that these operations are dual, because for the IFSs A and B:

$$\begin{aligned}\neg(\neg A \# \neg B) &= \neg(\{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\} \# \{\langle x, \nu_B(x), \mu_B(x)\rangle | x \in E\}) \\ &= \neg\{\langle x, \nu_A(x)\nu_B(x), \max(\mu_A(x), \mu_B(x))\rangle | x \in E\} \\ &= \{\langle x, \max(\mu_A(x), \mu_B(x)), \nu_A(x)\nu_B(x)\rangle | x \in E\} \\ &= A * B,\end{aligned}$$

and by analogy,

$$A \# B = \neg(\neg A * \neg B).$$

Third, with respect to the values of the degrees of membership and of non-membership of an element $x \in E$ about two IFSs A and B the geometrical interpretation of operation $*$ and of operation # are shown on Fig. 2 and Fig. 3, respectively.

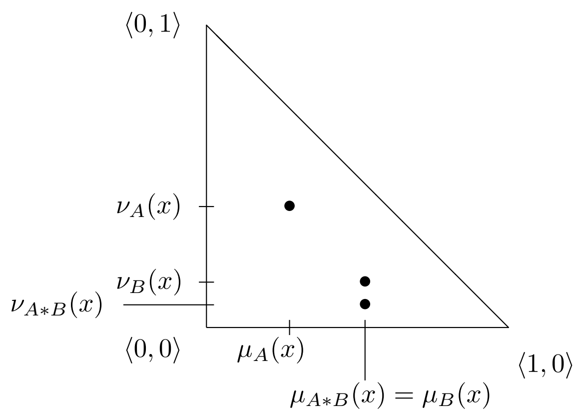

Figure 2: A geometrical interpretation of an element $x \in E$ about operation $*$

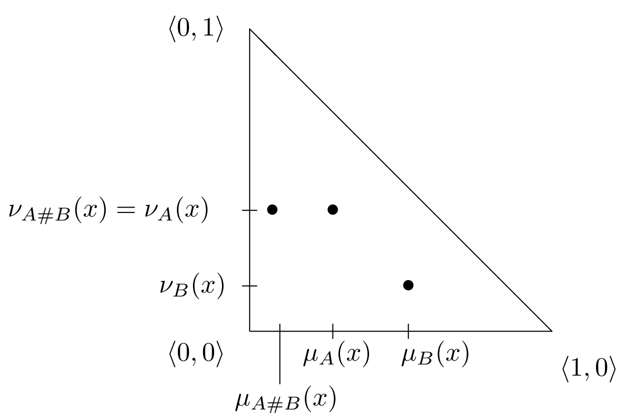

Figure 3: A geometrical interpretation of an element $x \in E$ about operation #

Using the above definitions, we can prove

Theorem 3.1. *For every two IFSs A and B,*

$$A\cap_{13} \subseteq A\#B \subseteq A \cap_4 B \subseteq A@B \subseteq A \cup_4 B \subseteq A * B \subseteq A \cap_{13} B.$$

Proof. The validity of Theorem 3.1 follows from the well-known equalities that are valid for arbitrary real numbers a and b and that in the present case are elements of the interval $[0, 1]$:

$$0 \leq ab \leq \min(a, b) \leq \frac{a+b}{2} \leq \max(a, b) \leq a + b - ab.$$

□

Now, following the well-known logical formula (see, e.g., [23]), but in the case of sets:

$$A \to B = \neg A \cup B,$$

we will construct a new intuitionistic fuzzy implication over IFSs that has the form

$$A \to B = \{\langle x, \max(\nu_A(x), \mu_B(x)), \mu_A(x)\nu_B(x)\rangle | x \in E\}.$$

For the new implication, we check that:

$$\begin{array}{lll} O^* \to O^* = E^*, & O^* \to U^* = E^*, & O^* \to E^* = E^*, \\ U^* \to O^* = U^*, & U^* \to U^* = U^*, & U^* \to E^* = E^*, \\ E^* \to O^* = O^*, & E^* \to U^* = U^*, & E^* \to E^* = E^*. \end{array}$$

In [18], George Klir and Bo Yuan discussed the following 9 axioms related to a fuzzy implication I.

Axiom 1: $(\forall x, y)(x \leq y \to (\forall z)(I(x, z) \geq I(y, z))$.
Axiom 2: $(\forall x, y)(x \leq y \to (\forall z)(I(z, x) \leq I(z, y))$.
Axiom 3: $(\forall y)(I(0, y) = 1)$.
Axiom 4: $(\forall y)(I(1, y) = y)$.
Axiom 5: $(\forall x)(I(x, x) = 1)$.
Axiom 6: $(\forall x, y, z)(I(x, I(y, z)) = I(y, I(x, z)))$.
Axiom 7: $(\forall x, y)(I(x, y) = 1$ iff $x \leq y)$.
Axiom 8: $(\forall x, y)(I(x, y) = I(N(y), N(x)))$, where N is an operation for negation.
Axiom 9: I is a continuous function.

As it is mentioned in [2], some of these axioms do not correspond to the case of intuitionistic fuzziness and by this reason, some modifications are needed.

If some axiom is valid only for IFTs about some intuitionistic fuzzy implication, its number is marked with an asterisk (*). These axioms in [2] are

Axiom 3*: $(\forall y)(I(0, y)$ is an IFT).
Axiom 5*: $(\forall x)(I(x, x)$ is an IFT).
Axiom 7*: $(\forall x, y)($if $I(x, y) = 1$, then $x \leq y$ and if $x \leq y$ then $I(x, y)$ is an IFT).

Of course, here x, y, z are IFSs, 0 and 1 are the special sets O^* and E^*, respectively and relations $\leq$ and $\geq$ are $\subseteq$ and $\supseteq$, respectively.

We must mention that, as it is discussed in [2], a lot of the intuitionistic fuzzy implications satisfy George Klir and Bo Yuan's axioms in the asterisk case. It is interesting to note that here Axiom 3 is valid in its original form too, as we show below.

Theorem 3.2. *For every three IFSs A, B, C, the new intuitionistic fuzzy implication $\to$ satisfies Axioms 1, 2, 3, 3*, 4, 5*, 6, 7*, 8, 9.*

Proof. We check the validity of the mentioned axioms consecutively.

For Axiom 1, let $A \subseteq B$, i.e. for each $x \in E$,

$$\mu_A(x) \leq \mu_B(x), \qquad \nu_A(x) \geq \nu_B(x).$$

Then from

$$A \to C = \{\langle x, \max(\nu_A(x), \mu_C(x)), \mu_A(x)\nu_C(x)\rangle | x \in E\},$$

and

$$B \to C = \{\langle x, \max(\nu_B(x), \mu_C(x)), \mu_B(x)\nu_C(x)\rangle | x \in E\},$$

it follows that

$$\max(\nu_A(x), \mu_C(x)) \geq \max(\nu_B(x), \mu_C(x)),$$
$$\mu_A(x)\nu_C(x) \leq \mu_B(x)\nu_C(x).$$

Therefore, $A \to C \supseteq B \to C$.

For Axiom 2, let again $A \subseteq B$. Then from

$$C \to A = \{\langle x, \max(\nu_C(x), \mu_A(x)), \mu_C(x)\nu_A(x)\rangle | x \in E\},$$

and

$$C \to B = \{\langle x, \max(\nu_C(x), \mu_B(x)), \mu_C(x)\nu_B(x)\rangle | x \in E\},$$

it follows that

$$\max(\nu_C(x), \mu_A(x)) \leq \max(\nu_C(x), \mu_B(x)),$$
$$\mu_C(x)\nu_A(x) \geq \mu_C(x)\nu_B(x).$$

Therefore $C \to A \subseteq C \to B$.

For Axioms 3, 3* and 4 we obtain directly that

$$O^* \to B = \{\langle x, \max(1, \mu_B(x)), 0.\nu_B(x)\rangle | x \in E\} = E^*,$$

i.e. $O^* \to B$ is IFT, too, and

$$E^* \to B = \{\langle x, \max(0, \mu_B(x)), 1.\nu_B(x)\rangle | x \in E\} = A.$$

Because from

$$A \to A = \{\langle x, \max(\nu_A(x), \mu_A(x)), \mu_A(x)\nu_A(x)\rangle | x \in E\},$$

it follows that for each $x \in E$:

$$\max(\mu_A(x), \nu_A(x)) \geq \mu_A(x)\nu_A(x),$$

then Axiom 5* is valid, but, obviously, there, elements $x \in E$ can exist so that

$$\max(\mu_A(x), \nu_A(x)) < 1, \qquad \mu_A(x)\nu_A(x) > 0,$$

i.e.,the original Axiom 5 is not valid.

For Axioms 6 we obtain that it is valid, because

$$\begin{aligned} A \to (B \to C) &= A \to \{\langle x, \max(\nu_B(x), \mu_C(x)), \mu_B(x)\nu_C(x)\rangle | x \in E\} \\ &= \{\langle x, \max(\nu_A(x), \max(\nu_B(x), \mu_C(x))), \mu_A(x)\mu_B(x)\nu_C(x)\rangle | x \in E\} \\ &= \{\langle x, \max(\nu_A(x), \nu_B(x), \mu_C(x)), \mu_A(x)\mu_B(x)\nu_C(x)\rangle | x \in E\} \\ &= \{\langle x, \max(\nu_B(x), \max(\nu_A(x), \mu_C(x))), \mu_B(x)\mu_A(x)\nu_C(x)\rangle | x \in E\} \\ &= B \to \{\langle x, \max(\nu_A(x), \mu_C(x)), \mu_A(x)\nu_C(x)\rangle | x \in E\} \\ &= B \to (A \to C). \end{aligned}$$

For Axiom 7*, first, we see that if

$$A \to B = \{\langle x, \max(\nu_A(x), \mu_B(x)), \mu_A(x)\nu_B(x)\rangle | x \in E\} = E^*,$$

then

$$\max(\nu_A(x), \mu_B(x)) = 1, \text{ and } \mu_A(x)\nu_B(x) = 0.$$

Therefore, $A = O^*$ and hence $A \subseteq B$, or $B = E^*$ and again $A \subseteq B$. Obviously, the cases $\nu_A(x) = 1$ and $\mu_A(x) > 0$, or $\mu_B(x) = 1$ and $\nu_B(x) > 0$ are impossible. On the other hand, if $A \subseteq B$, then $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$. Then

$$\max(\nu_A(x), \mu_B(x)) - \mu_A(x)\nu_B(x) \geq \mu_B(x) - \mu_A(x)\nu_B(x) \geq \mu_B(x) - \mu_A(x) \geq 0,$$

i.e., $A \rightarrow B$ is an IFT. Hence, Axiom 7* is checked. But, as we show above, if $A \subseteq B$, then $\max(\nu_A(x), \mu_B(x)) \geq \mu_A(x)\nu_B(x)$ and from here it does not follow that $\max(\nu_A(x), \mu_B(x)) = 1$ and $\mu_A(x)\nu_B(x) = 0$, i.e., in the general case Axiom 7 is not valid.

For Axioms 8 we obtain that it is valid, because

$$\begin{aligned} A \rightarrow B &= \{\langle x, \max(\nu_A(x), \mu_B(x)), \mu_A(x)\nu_B(x)\rangle | x \in E\} \\ &= \{\langle x, \max(\mu_B(x), \nu_A(x)), \nu_B(x)\mu_A(x)\rangle | x \in E\} \\ &= \{\langle x, \nu_B(x), \mu_B(x)\rangle | x \in E\} \rightarrow \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\} \\ &= \neg B \rightarrow \neg A. \end{aligned}$$

Axiom 9 is obviously valid, because functions "max" and "multiplication" preserve the continuity.

With this, Theorem 3.2 is proved. □

In [8], for a first time the idea for integral form of the degrees of membership and of non-membership of the IFS elements, was discussed. Here, we will show that the new intuitionistic fuzzy operations and operators also can have similar forms.

Let $E = [0, 1]$ be an universe, A and B be IFSs over it. Let μ_A, μ_B, ν_A and ν_B be measurable functions on $[0, 1]$. Then we define

$$A\#B = \{\langle x, \int_0^x \mu_A(t)\mu_B(t)\,dt, \int_0^x \max(\nu_A(t), \nu_B(t))\,dt\rangle | x \in E\},$$

$$A * B = \{\langle x, \int_0^x \max(\mu_A(t), \mu_B(t))\,dt, \int_0^x \nu_A(t)\nu_B(t)\,dt\rangle | x \in E\}.$$

where integrals are Lebesgue.

As above, we check that the definitions of these intuitionistic fuzzy operations are correct. Really, for operation # we see:

$$0 \leq \mu_A(t)\mu_B(t) \leq 1,$$

$$0 \leq \int_0^x \mu_A(t)\mu_B(t)\,dt \leq x \leq 1,$$

$$0 \leq \max(\nu_A(t), \nu_B(t)) \leq 1,$$

$$0 \leq \int_0^x \max(\nu_A(t), \nu_B(t))\,dt \leq x \leq 1.$$

$$\begin{aligned} 0 &\leq \int_0^x (\mu_A(t)\mu_B(t) + \max(\nu_A(t), \nu_B(t)))\,dt \\ &\leq \int_0^x (\min(\mu_A(t), \mu_B(t)) + \max(\nu_A(t), \nu_B(t)))\,dt \\ &\leq \int_0^x (\min(\mu_A(t), \mu_B(t)) + \max(1 - \mu_A(t), 1 - \mu_B(t)))\,dt \\ &\leq \int_0^x (\min(\mu_A(t), \mu_B(t)) + 1 - \min(\mu_A(t), \mu_B(t)))\,dt \\ &\leq x \leq 1, \end{aligned}$$

and by analogy,

$$0 \leq \int_0^x (\max(\mu_A(t), \mu_B(t)) + \nu_A(t)\nu_B(t))\,dt \leq 1.$$

4 New intuitionistic fuzzy topological operators

The new topological operators generated by operations "$*$" and "$\#$", respectively, have the forms:

$$\mathcal{Y}(A) = \{\langle x, \sup_{y\in E} \mu_A(y), \prod_{y\in E} \nu_A(y)\rangle | x \in E\},$$

$$\mathcal{Z}(A) = \{\langle x, \prod_{y\in E} \mu_A(y), \sup_{y\in E} \nu_A(y)\rangle | x \in E\}.$$

First, we check that the definitions of these operations are correct. Really,

$$0 \le \sup_{y\in E} \mu_A(y) \le 1,$$

$$0 \le \prod_{y\in E} \nu_A(y) \le 1,$$

$$0 \le \prod_{y\in E} \mu_A(y) \le 1,$$

$$0 \le \sup_{y\in E} \nu_A(y) \le 1,$$

$$0 \le \sup_{y\in E} \mu_A(y) + \prod_{y\in E} \nu_A(y) \le \sup_{y\in E} \mu_A(y) + \inf_{y\in E} \nu_A(y) \le 1,$$

$$0 \le \prod_{y\in E} \mu_A(y) + \sup_{y\in E} \nu_A(y) \le \inf_{y\in E} \mu_A(y) + \sup_{y\in E} \nu_A(y) \le 1.$$

Second, we see that these operators are dual, because for the IFS A:

$$\begin{aligned}\neg\mathcal{Z}(\neg A) &= \neg\mathcal{Z}\{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\}) \\ &= \neg\{\langle x, \prod_{y\in E} \nu_A(y), \sup_{y\in E} \mu_A(y)\rangle | x \in E\} \\ &= \{\langle x, \sup_{y\in E} \mu_A(y), \prod_{y\in E} \nu_A(y)\rangle | x \in E\} \\ &= \mathcal{Y}(A),\end{aligned}$$

and by analogy,

$$\neg\mathcal{Y}(\neg A) = \mathcal{Z}(A).$$

From the definitions, we see that for each IFS A:

$$\mathcal{Z}(A) \subseteq \mathcal{I}(A) \subseteq A \subseteq \mathcal{C}(A) \subseteq \mathcal{Y}(A).$$

The integral forms of the two new intuitionistic fuzzy topological operators generated by the respective intuitionistic fuzzy operations are:

$$\mathcal{Y}(A) = \{\langle x, \sup_{y\in E} \int_0^y \mu_A(t)\,dt, \prod_{y\in E} \int_0^y \nu_A(t)\,dt\rangle | x \in E\},$$

$$\mathcal{Z}(A) = \{\langle x, \prod_{y\in E} \int_0^y \mu_A(t)\,dt, \sup_{y\in E} \int_0^y \nu_A(t)\,dt\rangle | x \in E\}.$$

Immediately, we check, as above, that the definitions of these operators are correct.

Also, we see that both new operators are dual:

$$\begin{aligned}\neg\mathcal{Z}(\neg A) &= \neg\mathcal{Z}\{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\}) \\ &= \neg\{\langle x, \prod_{y\in E} \int_0^y \nu_A(t)\,dt, \sup_{y\in E} \int_0^y \mu_A(t)\,dt\rangle | x \in E\} \\ &= \{\langle x, \sup_{y\in E} \int_0^y \mu_A(t)\,dt, \prod_{y\in E} \int_0^y \nu_A(t)\,dt\rangle | x \in E\} \\ &= \mathcal{Y}(A).\end{aligned}$$

5 Four new intuitionistic fuzzy modal topological structures

If for each set X

$$\mathcal{P}(X) = \{Y|Y \subseteq X\},$$

then for each IFS A over the universe E:

$$\mathcal{P}(O^*) = \{O^*\}, \quad \mathcal{P}(E^*) = \{A|A \subseteq E^*\}.$$

Let $\mathcal{O}$ and $\mathcal{Q}$ be topological operators such that for each IFS $A \in \mathcal{P}(E^*)$:

$$\mathcal{O}(A) = \neg\mathcal{Q}(\neg A), \quad \mathcal{Q}(A) = \neg\mathcal{O}(\neg A).$$

Let $\Delta, \nabla : \mathcal{P}(E^*) \times \mathcal{P}(E^*) \to \mathcal{P}(E^*)$ be operations such that for every two IFSs $A, B \in \mathcal{P}(E^*)$:

$$A\nabla B = \neg(\neg A\Delta\neg B), \quad A\Delta B = \neg(\neg A\nabla\neg B).$$

Let $\circ$ and $\bullet$ be modal operators such that for each IFS $A \in \mathcal{P}(E^*)$:

$$\circ A = \neg \bullet \neg A, \quad \bullet A = \neg \circ \neg A.$$

In [3], the concept of an IFMTS was introduced and it was extended in [7] to *cl*-*cl*-IFMTS, *in*-*in*-IFMTS, *cl*-*in*-IFMTS and *in*-*cl*-IFMTS in respect of the type of the topological operator (from "closure" or from "interior" type) and of the modal operator, that at least conditionally can be from one of these two types. Unfortunately, all attempts of ours to find topological operators different from $\mathcal{C}$ and $\mathcal{I}$, operations different from $\cup_4$ and $\cap_4$ and modal operators different from $\Box$ and $\Diamond$, have failed. In all other cases, some of the conditions given in [3] must be changed with weak (feeble) ones. Because the term "weak topological structure" have another sense (see, e.g., [32]), in [4] we used the term "Intuitionistic Fuzzy Feeble Modal Topological Structure". For such structure, some conditions are changed with feeble ones. Below, we will mention these changes under line.

5.1 *cl*-*cl*-IFFMTS

The *cl*-*cl*-IFFMTS is the object $\langle \mathcal{P}(E^*), \mathcal{O}, \Delta, \circ \rangle$, where E is a fixed universe, $\mathcal{O} : \mathcal{P}(E^*) \to \mathcal{P}(E^*)$ is an operator of a closure type related to operation Δ; $\circ : \mathcal{P}(E^*) \to \mathcal{P}(E^*)$ is a modal operator and for every two IFSs $A, B \in \mathcal{P}(E^*)$ the following nine conditions hold:

CC1) $\mathcal{O}(A\Delta B) = \mathcal{O}(A)\Delta\mathcal{O}(B)$,

CC2) $A \subseteq \mathcal{O}(A)$,

CC3) $\mathcal{O}(O^*) = O^*$,

CC4) $\mathcal{O}(\mathcal{O}(A)) \supseteq \mathcal{O}(A)$[1],

CC5) $\circ(A\nabla B) \supseteq \circ A\nabla \circ B$[1],

CC6) $A \subseteq \circ A$,

CC7) $\circ E^* = E^*$,

CC8) $\circ \circ A = \circ A$,

CC9) $\circ\, \mathcal{O}(A) \supseteq \mathcal{O}(\circ A)$[1].

We must mention that in some sense, the modal operators $\Diamond$ and $\Box$ are from "closure" and "interior" types, respectively. The reason for a such assertion will be discussed below.

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[1]In [3], the relation is "="

Theorem 5.1. *For each universe E, $\langle \mathcal{P}(E^*), \mathcal{Y}, *, \Diamond \rangle$ is a cl-cl-IFFMTS.*

Proof. Let the IFSs $A, B \in \mathcal{P}(E^*)$. The check of conditions CC1 - CC4 are analogous, but different from those in [3], while of CC6-CC8 are the same. We give them only here for completeness of the proof.

CC1.

$$\begin{aligned}\mathcal{Y}(A*B) &= \mathcal{Y}(\{\langle x, \max(\mu_A(x), \mu_B(x)), \nu_A(x)\nu_B(x)\rangle | x \in E\})\\ &= \{\langle x, \sup_{y\in E} \max(\mu_A(y), \mu_B(y)), \prod_{y\in E} \nu_A(y)\nu_B(y)\rangle | x \in E\}\\ &= \{\langle x, \max(\sup_{y\in E} \mu_A(y), \sup_{y\in E} \mu_A(y)\mu_B(y)), \prod_{y\in E} \nu_A(y) \prod_{y\in E} \nu_B(y)\rangle | x \in E\}\\ &= \{\langle x, \sup_{y\in E} \mu_A(y), \prod_{y\in E} \nu_A(y)\rangle | x \in E\} * \{\langle x, \sup_{y\in E} \mu_B(y), \prod_{y\in E} \nu_B(y)\rangle | x \in E\}\\ &= \mathcal{Y}(A) * \mathcal{Y}(B);\end{aligned}$$

CC2.

$$\begin{aligned}A &= \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}\\ &\subseteq \{\langle x, \sup_{y\in E} \mu_A(y), \prod_{y\in E} \nu_A(y)\rangle | x \in E\}\\ &= \mathcal{Y}(A);\end{aligned}$$

CC3.

$$\mathcal{Y}(O^*) = \mathcal{C}(\{\langle x, 0, 1\rangle | x \in E\}) = \{\langle x, \sup_{y\in E} 0, \prod_{y\in E} 1\rangle | x \in E\} = \{\langle x, 0, 1\rangle | x \in E\} = O^*;$$

CC4. Having in mind that $\sup_{y\in E} \mu_A(y)$ and $\prod_{y\in E} \nu_A(y)$ are constants, we obtain that:

$$\begin{aligned}\mathcal{Y}(\mathcal{Y}(A)) &= \mathcal{Y}(\{\langle x, \sup_{y\in E} \mu_A(y), \prod_{y\in E} \nu_A(y)\rangle | x \in E\})\\ &= \{\langle x, \sup_{z\in E} \sup_{y\in E} \mu_A(y), \prod_{z\in E} \prod_{y\in E} \nu_A(y)\rangle | x \in E\}\\ &= \{\langle x, \sup_{y\in E} \mu_A(y), \left(\prod_{y\in E} \nu_A(y)\right)^{\varepsilon}\rangle | x \in E\}\\ &\supseteq \{\langle x, \sup_{y\in E} \mu_A(y), \prod_{y\in E} \nu_A(y)\rangle | x \in E\}\\ &= \mathcal{Y}(A);\end{aligned}$$

CC5.

$$\begin{aligned}\Diamond(A*B) &= \Diamond\{\langle x, \max(\mu_A(x), \mu_B(x)), \nu_A(x)\nu_B(x)\rangle | x \in E\}\\ &= \{\langle x, 1-\nu_A(x)\nu_B(x), \nu_A(x)\nu_B(x)\rangle | x \in E\}\\ &\supseteq \{\langle x, 1-\min(\nu_A(x), \nu_B(x)), \nu_A(x)\nu_B(x)\rangle | x \in E\}\\ &= \{\langle x, \max(1-\nu_A(x), 1-\nu_B(x)), \nu_A(x)\nu_B(x)\rangle | x \in E\}\\ &= \{\langle x, 1-\nu_A(x), \nu_A(x)\rangle | x \in E\} * \{\langle x, 1-\nu_B(x), \nu_B(x)\rangle | x \in E\}\\ &= \Diamond A * \Diamond B;\end{aligned}$$

CC6.

$$\begin{aligned}A &= \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}\\ &\subseteq \{\langle x, 1-\nu_A(x), \nu_A(x)\rangle | x \in E\}\\ &= \Diamond A;\end{aligned}$$

CC7.

$$\Diamond E^* = \Diamond\{\langle x, 1, 0\rangle | x \in E\} = \{\langle x, 1-0, 0\rangle | x \in E\} = (\{\langle x, 1, 0\rangle | x \in E\}) = E^*;$$

CC8.

$$\Diamond\Diamond A = \Diamond\{\langle x, 1-\nu_A(x), \nu_A(x)\rangle | x \in E\} = \{\langle x, 1-\nu_A(x), \nu_A(x)\rangle | x \in E\} = \Diamond A;$$

CC9.

$$\begin{aligned}
\Diamond\mathcal{Y}(A) &= \Diamond\{\langle x, \sup_{y\in E}\mu_A(y), \prod_{y\in E}\nu_A(y)\rangle | x \in E\}) \\
&= \{\langle x, 1-\prod_{y\in E}\nu_A(y), \prod_{y\in E}\nu_A(y)\rangle | x \in E\}) \\
&\supseteq \{\langle x, 1-\inf_{y\in E}\nu_A(y), \prod_{y\in E}\nu_A(y)\rangle | x \in E\} \\
&= \{\langle x, \sup_{y\in E}(1-\nu_A(y)), \prod_{y\in E}\nu_A(y)\rangle | x \in E\} \\
&= \mathcal{Y}(\{\langle x, 1-\nu_A(x), \nu_A(x)\rangle | x \in E\}) \\
&= \mathcal{Y}(\Diamond A).
\end{aligned}$$

This completes the proof. □ □

The reason that operators $\mathcal{Y}$ and $\Diamond$ are from one type ("closure") is in the validity of conditions CC2 and CC6.

5.2 *in*-*in*-IFFMTS

The *in*-*in*-IFMTS is the object $\langle \mathcal{P}(E^*), \mathcal{Q}, \nabla, \bullet\rangle$, where E is a fixed universe, $\mathcal{Q} : \mathcal{P}(E^*) \to \mathcal{P}(E^*)$ is an operator of interior type, related to the operation ∇; $\bullet : \mathcal{P}(E^*) \to \mathcal{P}(E^*)$ is a modal operatorof an interior type, and for every two IFSs $A, B \in \mathcal{P}(E^*)$ the following nine conditions hold:

II1) $\mathcal{Q}(A\nabla B) = \mathcal{Q}(A)\nabla\mathcal{Q}(B)$,

II2) $\mathcal{Q}(A) \subseteq A$,

II3) $\mathcal{Q}(E^*) = E^*$,

II4) $\mathcal{Q}(\mathcal{Q}(A)) \subseteq \mathcal{Q}(A)$[2],

II5) $\bullet(A\Delta B) \subseteq \bullet A\Delta\bullet B$[2],

II6) $\bullet A \subseteq A$,

II7) $\bullet O^* = O^*$,

II8) $\bullet\bullet A = \bullet A$,

II9) $\bullet\ \mathcal{Q}(A) = \mathcal{Q}(\bullet A)$[2].

Theorem 5.2. *For each universe E, $\langle \mathcal{P}(E^*), \mathcal{Z}, \#, \Box\rangle$ is an in-in-IFFMTS.*

Proof. Let the IFSs $A, B \in \mathcal{P}(E^*)$ be given. We will sequentially prove the validity of the conditions II1–II5, and II9, because the checks of the validity of conditions II6–II8 are given in [3] and are similar to these in the proof of Theorem 5.1.

[2]In [3], the relation is "="

II1.

$$\begin{aligned}\mathcal{Z}(A\#B) &= \mathcal{Z}(\{\langle x, \mu_A(x)\mu_B(x), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\})\\ &= \{\langle x, \prod_{y\in E} \mu_A(y)\mu_B(y), \sup_{y\in E} \max(\nu_A(y), \nu_B(y))\rangle | x \in E\}\\ &= \{\langle x, \prod_{y\in E} \mu_A(y) \prod_{y\in E} \mu_B(y), \max(\sup_{y\in E} \nu_A(y), \sup_{y\in E} \nu_B(y))\rangle | x \in E\}\\ &= \{\langle x, \prod_{y\in E} \mu_A(y), \sup_{y\in E} \nu_A(y)\rangle | x \in E\}\#\{\langle x, \prod_{y\in E} \mu_B(y), \sup_{y\in E} \nu_B(y)\rangle | x \in E\}\\ &= \mathcal{Z}(A)\#\mathcal{Z}(B);\end{aligned}$$

II2.

$$\begin{aligned}A &= \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}\\ &\supseteq \{\langle x, \prod_{y\in E} \mu_A(y), \sup_{y\in E} \nu_A(y)\rangle | x \in E\}\\ &= \mathcal{Z}(A);\end{aligned}$$

II3.

$$\mathcal{Z}(E^*) = \mathcal{Z}(\{\langle x, 1, 0\rangle | x \in E\}) = \{\langle x, \sup_{y\in E} 1, \prod_{y\in E} 0\rangle | x \in E\} = \{\langle x, 1, 0\rangle | x \in E\} = E^*;$$

II4. Having in mind that $\prod_{y\in E} \mu_A(y)$ and $\sup_{y\in E} \nu_A(y)$ are constants, we obtain that:

$$\begin{aligned}\mathcal{Z}(\mathcal{Z}(A)) &= \mathcal{Z}(\{\langle x, \prod_{y\in E} \mu_A(y), \sup_{y\in E} \nu_A(y)\rangle | x \in E\})\\ &= \mathcal{Z}(\{\langle x, \prod_{z\in E}\prod_{y\in E} \mu_A(y), \sup_{z\in E}\sup_{y\in E} \nu_A(y)\rangle | x \in E\})\\ &= \mathcal{Z}(\{\langle x, \left(\prod_{y\in E} \mu_A(y)\right)^{\varepsilon}, \sup_{y\in E} \nu_A(y)\rangle | x \in E\})\\ &\subseteq \{\langle x, \prod_{y\in E} \mu_A(y), \sup_{y\in E} \nu_A(y)\rangle | x \in E\}\\ &= \mathcal{Z}(A);\end{aligned}$$

II5.

$$\begin{aligned}\Box(A\#B) &= \Box\{\langle x, \mu_A(x)\mu_B(x), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\}\\ &= \{\langle x, \mu_A(x)\mu_B(x), 1 - \mu_A(x)\mu_B(x)\rangle | x \in E\}\\ &\subseteq \{\langle x, \mu_A(x)\mu_B(x), 1 - \min(\mu_A(x), \mu_B(x))\rangle | x \in E\}\\ &= \{\langle x, \mu_A(x)\mu_B(x), \max(1 - \mu_A(x), 1 - \mu_B(x))\rangle | x \in E\}\\ &= \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\}\#\{\langle x, \mu_B(x), 1 - \mu_B(x)\rangle | x \in E\}\\ &= \Box A\#\Box B;\end{aligned}$$

II9.

$$\begin{aligned}\Box\mathcal{Z}(A) &= \Box\{\langle x, \prod_{y\in E} \mu_A(y), \sup_{y\in E} \nu_A(y)\rangle | x \in E\}\\ &= \{\langle x, \prod_{y\in E} \mu_A(y), 1 - \prod_{y\in E} \mu_A(y)\rangle | x \in E\}\\ &\subseteq \{\langle x, \prod_{y\in E} \mu_A(y), 1 - \inf_{y\in E} \mu_A(y)\rangle | x \in E\}\\ &= \{\langle x, \prod_{y\in E} \mu_A(y), \sup_{y\in E}(1 - \mu_A(y))\rangle | x \in E\}\\ &= \mathcal{Z}(\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\})\\ &= \mathcal{Z}(\Box A);\end{aligned}$$

This completes the proof. □

The reason that operators $\mathcal{Z}$ and $\square$ are from one type ("interior") is in the validity of conditions II2 and II6.

5.3 *cl-in*-IFFMTS

The *cl-in*-IFFMTS is the object $\langle \mathcal{P}(E^*), \mathcal{O}, \Delta, \bullet \rangle$, where E is a fixed universe, $\mathcal{O} : \mathcal{P}(E^*) \to \mathcal{P}(E^*)$ is an operator of a closure type related to operation Δ; $\bullet : \mathcal{P}(E^*) \to \mathcal{P}(E^*)$ is a modal operator of an interior type, and for every two IFSs $A, B \in \mathcal{P}(E^*)$ the following nine conditions hold:

CI1) $\mathcal{O}(A \Delta B) = \mathcal{O}(A) \Delta \mathcal{O}(B)$,

CI2) $A \subseteq \mathcal{O}(A)$,

CI3) $\mathcal{O}(O^*) = O^*$,

CI4) $\mathcal{O}(\mathcal{O}(A)) \supseteq \mathcal{O}(A)$,

CI5) $\bullet(A \nabla B) \subseteq \bullet A \nabla \bullet B$,

CI6) $\bullet A \subseteq A$,

CI7) $\bullet O^* = O^*$,

CI8) $\bullet \bullet A = \bullet A$,

CI9) $\bullet\ \mathcal{O}(A) \subseteq \mathcal{O}(\bullet A)$.

Now, we see that the first four conditions correspond to the conditions C1 – C4 for a topological operator "closure", next four conditions correspond to the conditions I1 – I4, but for a modal (instead of a topological) operator of an interior type and condition CI9 determines the relation between the two types of operators.

Theorem 5.3. *For each universe E, $\langle \mathcal{P}(E^*), \mathcal{Y}, *, \square \rangle$ is a cl-in-IFFMTS.*

Proof. Let the IFS $A \in \mathcal{P}(E^*)$ be given.

The checks of the conditions CI1–CI4 coincide with the proofs of conditions CC1–CC4 in Theorem 5.1. The checks of the conditions CI6– CI8 coincide with the proofs of conditions II6–II8 in Theorem .

Hence, it is enough only to show the validity of the conditions CI5 and CI9.

For the validity of condition (CI5) we obtain:

$$\begin{aligned}
\square\,(A * B) &= \square\,\{\langle x, \max(\mu_A(x), \mu_B(x)), \nu_A(x)\nu_B(x)\rangle | x \in E\} \\
&= \{\langle x, \max(\mu_A(x), \mu_B(x)), 1 - \max(\mu_A(x), \mu_B(x))\rangle | x \in E\} \\
&= \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(1 - \mu_A(x), 1 - \mu_B(x))\rangle | x \in E\} \\
&\subseteq \{\langle x, \max(\mu_A(x), \mu_B(x)), (1 - \mu_A(x))(1 - \mu_B(x))\rangle | x \in E\} \\
&= \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\} * \{\langle x, \mu_B(x), 1 - \mu_B(x)\rangle | x \in E\} \\
&= \square\, A * \square\, B.
\end{aligned}$$

For the validity of condition CI9, we obtain

$$\begin{aligned}
\square\,\mathcal{Y}(A)) &= \square\,\{\langle x, \sup_{y \in E} \mu_A(y), \prod_{y \in E} \nu_A(y)\rangle | x \in E\}) \\
&= \{\langle x, \sup_{y \in E} \mu_A(y), 1 - \sup_{y \in E} \mu_A(y)\rangle | x \in E\})
\end{aligned}$$

$$= \{\langle x, \sup_{y\in E} \mu_A(y), \inf_{y\in E}(1-\mu_A(y))\rangle | x \in E\})$$
$$\subseteq \{\langle x, \sup_{y\in E} \mu_A(y), \prod_{y\in E}(1-\mu_A(y))\rangle | x \in E\})$$
$$= \mathcal{Y}(\{\langle x, \mu_A(x), 1-\mu_A(x)\rangle | x \in E\})$$
$$= \mathcal{Y}(\Box A).$$

This completes the proof. □

5.4 *in-cl*-IFFMTS

The *in-cl*-IFLTS is the object $\langle \mathcal{P}(E^*), \mathcal{Q}, \nabla, \circ\rangle$, where E is a fixed universe, $\mathcal{Q} : \mathcal{P}(E^*) \to \mathcal{P}(E^*)$ is an operator of interior type, related to the operation ∇, $\circ : \mathcal{P}(E^*) \to \mathcal{P}(E^*)$ is a modal operator of a closure type, and for every two IFSs $A, B \in \mathcal{P}(E^*)$ the following nine conditions hold:

IC1) $\mathcal{Q}(A\nabla B) = \mathcal{Q}(A)\nabla\mathcal{Q}(B)$,

IC2) $\mathcal{Q}(A) \subseteq A$,

IC3) $\mathcal{Q}(E^*) = E^*$,

IC4) $\mathcal{Q}(\mathcal{Q}(A)) \subseteq \mathcal{Q}(A)$,

IC5) $\circ(A\Delta B) \supseteq \circ A\Delta \circ B$,

IC6) $A \subseteq \circ A$,

IC7) $\circ E^* = E^*$,

IC8) $\circ \circ A = \circ A$,

IC9) $\circ\ \mathcal{Q}(A) = \mathcal{Q}(\circ A)$.

Now, we see that the first four conditions correspond to the conditions I1 – I4 for a topological operator "interior", next four conditions correspond to the conditions C1 – C4, but for a modal operator of closure type, and condition IC9 determines the relation between the two types of operators.

Theorem 5.4. *For each universe E, $\langle \mathcal{P}(E^*), \mathcal{Z}, \#, \Diamond\rangle$ is an in-cl-IFFMTS.*

Proof. Let the IFS $A \in \mathcal{P}(E^*)$ be given.

The checks of the conditions IC1–IC4 coincide with the proofs of conditions II1–II4 in Theorem 5.3. The checks of the conditions IC6– IC8 coincide with the proofs of conditions CC6–CC8 from Theorem 5.1.

Hence, it remains that we check only the validity of the conditions IC5 and IC9.

For the validity of condition (IC5) we obtain:

$$\begin{aligned}
\Diamond(A\#B) &= \Diamond\{\langle x, \mu_A(x)\mu_B(x), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\} \\
&= \{\langle x, 1-\max(\nu_A(x), \nu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\} \\
&= \{\langle x, \min(1-\nu_A(x), 1-\nu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\} \\
&\supseteq \{\langle x, (1-\nu_A(x))(1-\nu_B(x)), \max(\nu_A(x), \nu_B(x)), \rangle | x \in E\} \\
&= \{\langle x, 1-\nu_A(x), \nu_A(x)\rangle | x \in E\}\#\{\langle x, 1-\nu_B(x), \nu_B(x)\rangle | x \in E\} \\
&= \Diamond A\#\Diamond B.
\end{aligned}$$

For the validity of condition IC9, we obtain

$$\begin{aligned}
\Diamond\mathcal{Z}(A)) &= \Diamond\{\langle x, \prod_{y\in E}\mu_A(y), \sup_{y\in E}\nu_A(y)\rangle | x \in E\}\\
&= \{\langle x, 1-\sup_{y\in E}\nu_A(y), \sup_{y\in E}\nu_A(y)\rangle | x \in E\}\\
&= \{\langle x, \inf_{y\in E} 1-\nu_A(y), \sup_{y\in E}\nu_A(y)\rangle | x \in E\}\\
&\supseteq \{\langle x, \prod_{y\in E}(1-\nu_A(y)), \sup_{y\in E}\nu_A(y)\rangle | x \in E\}\\
&= \mathcal{Z}(\{\langle x, 1-\nu_A(x), \nu_A(x)\rangle | x \in E\})\\
&= \mathcal{Z}(\Diamond A).
\end{aligned}$$

This completes the proof. □

6 Conclusion

The idea from [3] for IFMTSs has been discussed in a series of papers (e.g., [4, 5]), but a lot of them will be feeble structures, because some of the conditions, introduced above, are not valid.

In [3], it is stated: *"The described above idea opens some directions for future research...".* Really, in [6] the intuitionistic fuzzy modal operators were changed with temporal ones.

The present direction was not discussed there. On the other hand, the directions for the development of the IFMTS discussed in [3] are valid for the IFTTS, too. For example, one of the possible directions is related to the use of the extended topological operators, discussed in [2]. These operators will be applied over the elements of the universe E, as well as over the elements of the time-scale T.

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